Iranian Journal of Numerical Analysis and Optimization
Vol 4, No. 1, (2014), pp 57-76

Solving an inverse problem for a parabolic equation with a nonlocal boundary condition in the reproducing kernel space

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Abstract

On the basis of a reproducing kernel space, an iterative algorithm for solving the inverse problem for heat equation with a nonlocal boundary condition is presented. The analytical solution in the reproducing kernel space is shown in a series form and the approximate solution v_n is constructed by truncating the series to n terms. The convergence of v_n to the analytical solution is also proved. Results obtained by the proposed method imply that it can be considered as a simple and accurate method for solving such inverse problems.

Keywords: Inverse problem; Parabolic equation; Nonlocal boundary conditions; Reproducing kernel space.

1 Introduction

The problem of finding the solution of partial differential equations with source control parameter has appeared increasingly in physical phenomena such as heat transfer, thermoelasticity, control theory, population dynamics, nuclear reactor dynamics, medical sciences, biochemistry, etc. [1, 2, 3]. The

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parameter determination in a parabolic partial differential equation from the over-specified data plays a crucial role in applied mathematics and physics. This technique has been widely used to determine the unknown properties of a region by measuring a specified location in the domain. These unknown properties such as the conductivity medium are important to the physical process but usually can not be measured directly, or very expensive to be measured [2, 3]. In general, these problems are ill-posed. Therefore a variety of numerical techniques based on regularization, finite differences, finite element and finite volume methods are given to approximate solutions of such problems [2, 3, 4].

In recent years all kinds of boundary conditions and over-specified conditions arise in the inverse problems which make them more and more difficult to solve. The integral over-specified condition arises from many important applications in heat transfer, termoelasticity, control theory, life sciences, etc. Some different partial differential equations with nonlocal boundary and over specified conditions can be found in [5, 6, 7, 8, 9, 10].

The theory of reproducing kernels [11], was used for the first time at the beginning of the 20th century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions. This theory has been successfully applied for solving a bunch of problems, see e.g. [12, 13, 14, 15, 16, 17, 18] and references cited therein. The book [19] provides excellent overviews of the existing reproducing kernel methods.
In this paper, a new algorithm for determining unknown solution and unknown control parameter of the parabolic inverse problem with nonlocal boundary and integral over-specified conditions based on the reproducing kernel space, is presented. The advantages of the approach must lie in the following facts. The approximate solution converges uniformly to the analytical solution. The method is mesh free, easily implemented and it needs no time discretization. Also we can evaluate the approximate solution $v_n(x,t)$ for fixed n once, and use it over and over.

The rest of the paper is organized as follows. In section 2 we describe the governing equation. Several reproducing kernel spaces are defined in Section 3. The method implementation and convergence analysis are prepared in Section 4. Numerical results are presented in section 5. The last section is a brief conclusion.

2 Governing equation

Consider the inverse problem of determination a pair of functions $\{v, p\}$ in the following parabolic equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = p(t)v + f(x,t) \qquad (x,t) \in \Omega = (0,1) \times (0,T] \tag{1}$$

with the initial condition

$$v(x,0)=\varphi(x), \qquad x\in[0,1] \tag{2}$$

nonlocal boundary conditions

$$v(0,t)=v(1,t), \qquad v_x(1,t)=0, \qquad 0\le t\le T \tag{3}$$

and the integral over-specified condition

$$\int_0^1 v(x,t)dx=E(t), \qquad t\in[0,T] \tag{4}$$

where $f(x,t)$, $\varphi(x)$, and $E(t)$ are known functions.

The existence, uniqueness, and continuous dependence of the solution upon the data for this problem are demonstrated in [20].

After taking integration from both sides of the equation (1) and using integral over-specified condition, we obtain

$$p(t)=\frac{E'(t)+v_x(0,t)-\int_0^1 f(x,t)dx}{E(t)}. \tag{5}$$

Then we have the following model problem

$$\begin{cases} \frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=\frac{E'(t)+v_x(0,t)-\int_0^1 f(x,t)dx}{E(t)}v+f(x,t), \quad (x,t)\in\Omega=(0,1)\times(0,T] \\ v(x,0)=\varphi(x), \quad v(0,t)=v(1,t), \quad v_x(1,t)=0. \end{cases}$$

After homogenizing the initial condition, we have

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+\frac{\int_0^1 f(x,t)dx-E'(t)}{E(t)}u=F(x,t,u,u_x) \quad (x,t)\in\Omega=(0,1)\times(0,T] \\ u(x,0)=0, \quad u(0,t)=u(1,t), \quad u_x(1,t)=0, \end{cases} \tag{6}$$

where

$$F(x,t,u,u_x)=\frac{(u(x,t)+\varphi(x))\left(u_x(0,t)+\varphi'(0)\right)}{E(t)} + \frac{E'(t)-\int_0^1 f(x,t)dx}{E(t)}\varphi(x)+\varphi''(x)+f(x,t).$$

3 Reproducing kernel spaces

Definition 1. *Let H be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. Denote by $\langle \cdot , \cdot \rangle$ the inner product and let $\| \cdot \| = \sqrt{\langle \cdot , \cdot \rangle}$ be the induced norm in H. A real valued function $K(x, y) : \Omega \times \Omega \to \mathbb{R}$ is called a reproducing kernel of H if the followings are satisfied:*

$$(i)\ K_y(x) = K(x, y) \in H \qquad \text{for all } y \in \Omega,$$
$$(ii)\ f(y) = \langle f(x), K_y(x) \rangle \qquad \text{for all } f \in H \text{ and for all } y \in \Omega.$$

Definition 2. *A Hilbert space H of functions on a set Ω is called a reproducing kernel Hilbert space if there exists a reproducing kernel K of H.*

Remark 1. *The existence of the reproducing kernel of a Hilbert space H is due to the Riesz Representation Theorem. It is known that the reproducing kernel is unique.*

Now, we define some useful reproducing kernel spaces. The corresponding reproducing kernels can be found by the usual technique in many articles in literature (see [13]).

Definition 3. $W_0[0, 1] = \{u(x)|u(x), u'(x), u''(x)$ *are absolutely continuous in* $[0, 1], u^{(3)}(x) \in L^2[0, 1], u(0) = u(1), u'(1) = 0\}$. *The inner product and the norm in* $W_0[0, 1]$ *are defined respectively by*

$$\langle u, v \rangle_{W_0} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(x)v^{(3)}(x)dx, \quad u, v \in W_0[0, 1], \tag{7}$$

and

$$\|u\|_{W_0} = \sqrt{\langle u, u \rangle_{W_0}}, \quad u \in W_0[0, 1].$$

The space $W_0[0, 1]$ is a reproducing kernel space and its reproducing kernel function is called $R_y(x)$.

Definition 4. $W_1[0, T] = \{u(t)|u(t), u'(t)$ *are absolutely continuous in* $[0, T], u''(t) \in L^2[0, T],\ u(0) = 0\}$. *The inner product and the norm in* $W_1[0, T]$ *are defined respectively by*

$$\langle u, v \rangle_{W_1} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^T u''(t)v''(t)dt, \quad u, v \in W_1[0, T],$$

and

$$\|u\|_{W_1} = \sqrt{\langle u, u \rangle_{W_1}}, \quad u \in W_1[0, T].$$

The space $W_1[0, T]$ is a reproducing kernel space and its reproducing kernel function $r_s(t)$ is given by

$$r_s(t) = \begin{cases} st + \frac{s}{2}t^2 - \frac{1}{6}t^3 & t \le s, \\ st + \frac{s^2}{2}t - \frac{1}{6}s^3 & t > s. \end{cases}$$

Definition 5. $W_2[0,1] = \{u(x)|u(x), u'(x)$ *are absolutely continuous in* $[0,1], u''(x) \in L^2[0,1]\}$. *The inner product and the norm in* $W_2[0,1]$ *are defined respectively by*

$$\langle u, v\rangle_{W_2} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u''(x)v''(x)dx, \quad u, v \in W_2[0,1],$$

and

$$\|u\|_{W_2} = \sqrt{\langle u, u\rangle_{W_2}}, \quad u \in W_2[0,1].$$

The space $W_2[0,1]$ is a reproducing kernel space and its reproducing kernel function $Q_y(x)$ is given by

$$Q_y(x) = \begin{cases} 1 + yx + \frac{y}{2}x^2 - \frac{1}{6}x^3 & x \le y, \\ 1 + yx + \frac{y^2}{2}x - \frac{1}{6}y^3 & x > y. \end{cases}$$

Definition 6. $W_3[0,T] = \{u(t)|u(t)$ *is absolutely continuous in* $[0,T], u'(t) \in L^2[0,T]\}$. *The inner product and the norm in* $W_3[0,T]$ *are defined respectively by*

$$\langle u, v\rangle_{W_3} = u(0)v(0) + \int_0^T u'(t)v'(t)dt, \quad u, v \in W_3[0,T],$$

and

$$\|u\|_{W_3} = \sqrt{\langle u, u\rangle_{W_3}}, \quad u \in W_3[0,T].$$

The space $W_3[0,T]$ is a reproducing kernel space and its reproducing kernel function $q_s(t)$ is given by

$$q_s(t) = \begin{cases} 1 + t & t \le s, \\ 1 + s & t > s. \end{cases}$$

Definition 7. $W(\overline{\Omega}) = \{u(x,t)|\frac{\partial^3 u}{\partial x^2 \partial t}$ *is completely continuous in* $\overline{\Omega}$, $\frac{\partial^5 u}{\partial x^3 \partial t^2} \in L^2(\Omega), u(x,0) = 0, u(0,t) = u(1,t), u_x(1,t) = 0\}$. *The inner product and the norm in* $W(\overline{\Omega})$ *are defined respectively by*

$$\begin{aligned}\langle u,v\rangle_W = & \sum_{i=0}^{2}\int_0^T\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}u(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}v(0,t)\right]dt\\ & +\sum_{j=0}^{1}\langle\frac{\partial^j}{\partial t^j}u(x,0),\frac{\partial^j}{\partial t^j}v(x,0)\rangle_{W_0}\\ & +\int_0^T\int_0^1\left[\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}u(x,t)\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}v(x,t)\right]dxdt, \quad u,v\in W(\overline{\Omega}),\end{aligned}$$

and

$$\|u\|_W = \sqrt{\langle u,u\rangle_W}, \quad u\in W(\overline{\Omega}).$$

Theorem 1. $W(\overline{\Omega})$ *is a reproducing kernel space and its reproducing kernel function is*

$$K_{(y,s)}(x,t) = R_y(x)r_s(t),$$

such that for any $u(x,t)\in W(\overline{\Omega})$,

$$u(y,s) = \langle u(x,t), K_{(y,s)}(x,t)\rangle_W,$$

where $R_y(x)$, $r_s(t)$ *are the reproducing kernel functions of* $W_0[0,1]$ *and* $W_1[0,T]$, *respectively.*

Proof. see [19]. □

Definition 8. $\widetilde{W}(\overline{\Omega}) = \{u(x,t)|\frac{\partial u}{\partial x}$ *is completely continuous in* $\overline{\Omega}$, $\frac{\partial^3 u}{\partial x^2\partial t}\in L^2(\Omega)\}$. *The inner product and the norm in* $\widetilde{W}(\overline{\Omega})$ *are defined respectively by*

$$\begin{aligned}\langle u(x,t),v(x,t)\rangle_{\widetilde{W}} = & \sum_{i=0}^{1}\int_0^T\left[\frac{\partial}{\partial t}\frac{\partial^i}{\partial x^i}u(0,t)\frac{\partial}{\partial t}\frac{\partial^i}{\partial x^i}v(0,t)\right]dt\\ & +\langle u(x,0),v(x,0)\rangle_{W_2}\\ & +\int_0^T\int_0^1\left[\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}u(x,t)\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}v(x,t)\right]dxdt, \quad u,v\in\widetilde{W}(\overline{\Omega}),\end{aligned}$$

and

$$\|u\|_{\widetilde{W}} = \sqrt{\langle u,u\rangle_{\widetilde{W}}}, \quad u\in\widetilde{W}(\overline{\Omega}).$$

$\widetilde{W}(\overline{\Omega})$ is a reproducing kernel space and its reproducing kernel function is

$$G_{(y,s)}(x,t) = Q_y(x)q_s(t).$$

4 The method implementation

By defining the linear operator $L : W(\overline{\Omega}) \to \widetilde{W}(\overline{\Omega})$ as

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\int_0^1 f(x,t)dx - E'(t)}{E(t)}\, u,$$

model problem (6) changes to the following problem

$$\begin{cases} Lu(x,t) = F(x,t,u,u_x), \quad (x,t) \in \Omega, \\ u(x,0) = 0, \qquad u(0,t) = u(1,t), \qquad u_x(1,t) = 0. \end{cases} \tag{8}$$

Lemma 1. *L is a bounded linear operator.*

Proof. see [13]. $\square$

Now, we choose a countable dense subset $\{(x_1,t_1),(x_2,t_2),\ldots,\}$ in $\overline{\Omega}$, and define

$$\phi_i(x,t) = G_{(x_i,t_i)}(x,t), \qquad \psi_i(x,t) = L^*\phi_i(x,t),$$

where L^* is the adjoint operator of L. The orthonormal system $\{\bar{\psi}_i(x,t)\}_{i=1}^{\infty}$ of $W(\overline{\Omega})$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x,t)\}_{i=1}^{\infty}$ as

$$\bar{\psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t),$$

where the orthogonal coefficients β_{ik} are given by

$$\beta_{ik} = \begin{cases} \frac{1}{\|\psi_1\|}, & i = k = 1, \\ \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{j=1}^{i-1} c_{ij}^2}}, & i = k \neq 1, \\ -\frac{\sum_{j=k}^{i-1} c_{ij}\beta_{jk}}{\sqrt{\|\psi_i\|^2 - \sum_{j=1}^{i-1} c_{ij}^2}}, & i \neq k, \end{cases}$$

where

$$\begin{aligned} c_{ij} &= \langle \psi_i(x,t), \bar{\psi}_j(x,t) \rangle_W \\ &= \langle L^* \phi_i(x,t), \bar{\psi}_j(x,t) \rangle_W \\ &= \langle \phi_i(x,t), L_{(x,t)} \bar{\psi}_j(x,t) \rangle_W \\ &= \left(L_{(x,t)} \bar{\psi}_j(x,t) \right)_{(x,t)=(x_i,t_i)} \\ &= \left(\sum_{m=1}^{j} \beta_{jm} L_{(x,t)} \psi_m(x,t) \right)_{(x,t)=(x_i,t_i)}. \end{aligned}$$

Like in [13], we get the following theorems.

Theorem 2. *Suppose that* $\{(x_i,t_i)\}_{i=1}^{\infty}$ *is dense in* $\overline{\Omega}$, *then* $\{\psi_i(x,t)\}_{i=1}^{\infty}$ *is a complete system in* $W(\overline{\Omega})$ *and* $\psi_i(x,t) = L_{(y,s)} K_{(y,s)}(x,t)|_{(y,s)=(x_i,t_i)}$.

Theorem 3. *If* $\{(x_i,t_i)\}_{i=1}^{\infty}$ *is dense in* $\overline{\Omega}$, *then the analytical solution of (8) is*

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left[F(x_k, t_k, u(x_k,t_k), \partial_x u(0,t_k)) \right] \bar{\psi}_i(x,t). \tag{9}$$

By truncating the series in (9), we can obtain the approximate solution of (8). But, since the the series terms are not known, we need to construct an iterative method for obtaining the approximate solution. For this purpose, we choose nonnegative integer n and put the initial function $u_0(x,t) = 0$. Then the approximate solution is defined by

$$u_n(x,t) = \sum_{i=1}^{n} B_i \bar{\psi}_i(x,t), \tag{10}$$

where

$$B_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, t_k, u_{k-1}(x_k,t_k), \partial_x u_{k-1}(0,t_k)). \tag{11}$$

On account of (26), the approximate solution $p_n(t)$ can also be obtained by

$$p_n(t) = \frac{E'(t) + \partial_x u_n(0,t) + \varphi'(0) - \int_0^1 f(x,t)dx}{E(t)}. \tag{12}$$

4.1 Convergence analysis

The convergence of $u_n(x,t)$ can lead to that of $p_n(t)$, due to (26). So we only need to show that the approximate solution $u_n(x,t)$ converges to the analytical solution $u(x,t)$. At first, the following lemma is given.

Lemma 2. *Assume that u_n is a bounded sequence in $W(\overline{\Omega})$, $u_n \xrightarrow{\|\cdot\|} \bar{u}$, $(x_n, t_n) \to (y, s)$, as $n \to \infty$. If $F(x, t, u(x, t), u_x(0, t))$ is continuous, then $F(x_n, t_n, u_{n-1}(x_n, t_n), \partial_x u_{n-1}(0, t_n)) \to F(y, s, \bar{u}(y, s), \partial_x \bar{u}(0, s))$.*

Proof. Similar to proof of Lemma 2 in [13], we have

$$|u_{n-1}(x_n, t_n) - \bar{u}(y, s)| \to 0, \quad \text{as} \quad n \to \infty.$$

Since

$$|t_n - s| \leq \sqrt{|x_n - y|^2 + |t_n - s|^2},$$

if follows that

$$(0, t_n) \longrightarrow (0, s).$$

Thus in a same manner

$$|\partial_x u_{n-1}(0, t_n) - \partial_x \bar{u}(0, s)| \to 0, \quad \text{as} \quad n \to \infty.$$

The continuation of $F(x, t, u(x), v(x))$ implies that

$$F(x_n, t_n, u_{n-1}(x_n, t_n), \partial_x u_{n-1}(0, t_n)) \to F(y, s, \bar{u}(y, s), \partial_x \bar{u}(0, s)), \quad \text{as} \quad n \to \infty.$$

□

Theorem 4. *Suppose that u_n is a bounded sequence in $W(\overline{\Omega})$ and (8) has a unique solution. If $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in $\overline{\Omega}$, then the n-term approximate solution $u_n(x, t)$ derived from the above method converges to the analytical solution $u(x, t)$ of (8) in $W(\overline{\Omega})$, such that*

$$u(x, t) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x, t),$$

where B_i is given by (11).

Proof. Similar to proof of Theorem 4 in [13], $u_n(x, t)$ converges to $\bar{u}(x, t)$ of the form

$$\bar{u}(x, t) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x, t),$$

such that

$$L\bar{u}(x_l, t_l) = F(x_l, t_l, u_{l-1}(x_l, t_l), \partial_x u_{l-1}(0, t_l)).$$

Since $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in $\overline{\Omega}$, for each $(y, s) \in \Omega$, there exist a subsequence $\{x_{n_j}, t_{n_j}\}_{j=1}^{\infty}$ such that

$$(x_{n_j}, t_{n_j}) \to (y, s) \quad (j \to \infty).$$

We know that $L\bar{u}(x_{n_j}, t_{n_j}) = F(x_{n_j}, t_{n_j}, u_{n_j-1}(x_{n_j}, t_{n_j}), \partial_x u_{n_j-1}(0, t_{n_j}))$. Let $j \to \infty$, by Lemma (2) and the continuity of F, we have

$$(L\bar{u})(y,s) = F(y, s, \bar{u}(y,s), \partial_x \bar{u}(0,s),$$

which indicates that $\bar{u}(x,t)$ satisfies (8). □

Theorem 5. *Under the conditions of Theorem 4, the approximate solution $u_n(x,t)$ and its derivatives $\partial_{xt}^{i+j} u_n(x,t)$, $i = 0,1,2$, $j = 0,1$, converge uniformly to exact solution $u(x,t)$ and its derivatives $\partial_{xt}^{i+j} u(x,t)$, $i = 0,1,2$, $j = 0,1$, respectively.*

Proof.

$$\begin{aligned}
|\partial_{xt}^{i+j} u_n(x,t) - \partial_{xt}^{i+j} u(x,t)| &= |\partial_{xt}^{i+j} \langle u_n(y,s) - u(y,s), K_{(x,t)}(y,s)\rangle_W| \\
&= |\langle u_n(y,s) - u(y,s), \partial_{xt}^{i+j} K_{(x,t)}(y,s)\rangle_W| \\
&\le \|\partial_{xt}^{i+j} K_{(x,t)}(y,s)\|_W \|u_n(y,s) - u(y,s)\|_W \\
&\le C_{i+j} \|u_n - u\|_W, \quad n \to \infty.
\end{aligned}$$

□

5 Numerical experiments

To test the accuracy of the proposed method, two examples are treated in this section. The results are compared with the exact solutions.

Example 1. *Consider problem (25)-(4) with*

$$\begin{aligned}
&\varphi(x) = 2 + \cos(2\pi x), \\
&E(t) = 1 + e^{-t}, \\
&f(x,t) = 1 + 4\pi^2 e^{-t} \cos(2\pi x).
\end{aligned}$$

It is easy to check that the exact solution is

$$\{v(x,t), p(t)\} = \{\mathrm{e}^{-t}(1 + \cos(2\pi x)), -1\}.$$

Using our method, we choose 81 points in the region $\overline{\Omega}$, and obtain the approximate solution $v_{81}(x,t)$. We have listed approximate versus exact solutions, along with the relative errors at some nodal points at time $T = \frac{1}{4}$ in Tables 1-2 and at time $T = \frac{1}{2}$ in Tables 3-4. Numerical results are in good agreement with the exact solutions. In Figs. 1-2, we display the exact and approximate solutions of v at times $T = \frac{1}{4}$, and $T = \frac{1}{2}$, respectively. In order to verify the convergence of the exact solution and its partial derivatives to the approximate solution and its partial derivatives, we depicted the relative errors graphs of v, v_{xt} and v_{xxt} at time $T = \frac{1}{4}$ for different values of n in Figs. 3-5, respectively. The results show that the errors becomes smaller as n increases.

Example 2. *Consider problem (25)-(4) with*

$$\begin{aligned}
&\varphi(x) = 1 + \cos^2(2\pi x),\\
&E(t) = \frac{1}{2}e^t + 1,\\
&f(x,t) = -8\pi^2 e^t + 16\pi^2 e^t \cos^2(2\pi x) - t - te^t \cos^2(2\pi x) - 1.
\end{aligned}$$

The exact solution is

$$\{v(x,t), p(t)\} = \{1 + \mathrm{e}^t \cos^2(2\pi x), 1 + t\}.$$

Taking $T = \frac{1}{4}$ and choosing 81 and 144 points in the region $\overline{\Omega}$, we have listed approximate versus exact solutions, along with the relative errors at some nodal points in Tables 5-6 and 7-8, respectively. Numerical results are in good agreement with the exact solutions and the accuracy of approximate solution is getting better as n increases. In Fig. 6, we display the exact and approximate solutions of v at time $T = \frac{1}{4}$. Relative error distribution of v at time $T = \frac{1}{2}$ is also given in Fig. 7a. It is clear that the numerical results are in good agreement with the exact solutions. Artificial errors 10^{-2} were introduced into the right end and conditional condition. It can be seen from Fig. 7b that the error never affects the results of the method.

Example 3. *Consider problem (25)-(4) with*

$$\begin{aligned}
&\varphi(x) = 1 + \cos(2\pi x),\\
&E(t) = \exp(-(2\pi)^2 t),\\
&f(x,t) = (2\pi)^2 \cos(2\pi x) \exp(-(2\pi)^2 t) + 2t(1 + \cos(2\pi x) \exp(-(2\pi)^2 t + 10t^2).
\end{aligned}$$

The exact solution is given by

$$\{v(x,t),\ p(t)\} = \{(1 + \cos(2\pi x) \exp(-(2\pi)^2 t), (2\pi)^2 + 2t \exp(10t^2)\}.$$

Relative error distribution of v at time $T = \frac{1}{2}$ is given in Fig. 8a. It can be noted from Fig. 8a that our results are in better accuracy than the results in [20]. In order to demonstrate the stability of our algorithm, we shall give a perturbation $\epsilon = 10^{-2}$ to the right side function $f(x,t)$ and over-specified condition $E(t)$. The relative error distribution of v at time $T = \frac{1}{2}$ depicted in Fig. 8b shows that the method is stable and gives excellent approximation to the solution.

6 Conclusion

In this paper, the reproducing kernel Hilbert space method was applied successfully for solving an inverse problem for a parabolic equation with nonlocal boundary condition. Proposed method is shown to be of good convergence, simple in principle, easy to program and easy to treat the boundary conditions. It seems that the method can also be applied to higher dimensional inverse problems. We leave this to our further works.

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Table 1: Relative errors of $v(x,t)$ for Example 1; $n = 81$, $T = \frac{1}{4}$

(x,t)	v_{exact}	v_{app}	Relative errors	(x,t)	v_{exact}	v_{app}	Relative errors
$(1, \frac{1}{1000})$	2.998001	2.998100	3.315676E-05	$(\frac{7}{8}, \frac{1}{6})$	2.445035	2.443523	6.182374E-04
$(\frac{1}{2}, \frac{12}{1000})$	1	1.000715	7.151300E-04	$(1, \frac{1}{6})$	2.692963	2.691261	6.321103E-04
$(1, \frac{1}{100})$	2.980099	2.980922	2.758099E-04	$(\frac{3}{4}, \frac{1}{5})$	1.818731	1.8169752	9.652627E-04
$(\frac{1}{10}, \frac{1}{10})$	2.636866	2.637919	3.995758E-03	$(\frac{1}{10}, \frac{1}{5})$	2.481098	2.479660	5.795225E-04
$(\frac{1}{4}, \frac{1}{10})$	1.904837	1.909188	2.283755E-03	$(\frac{2}{3}, \frac{1}{5})$	1.409365	1.408556	5.742847E-04
$(1, \frac{1}{10})$	2.809675	2.810196	1.853823E-04	$(\frac{1}{2}, \frac{2}{9})$	1	1.002348	2.347800E-03
$(\frac{1}{2}, \frac{1}{9})$	1	1.002317	2.317300E-03	$(\frac{3}{5}, \frac{2}{9})$	1.152927	1.152638	2.505188E-04
$(\frac{2}{3}, \frac{1}{9})$	1.447420	1.448317	6.202361E-04	$(\frac{1}{3}, \frac{2}{9})$	1.400369	1.399123	8.895521E-04
$(1, \frac{1}{8})$	2.764994	2.764715	1.009424E-04	$(\frac{3}{5}, \frac{1}{4})$	1.148738	1.148205	4.639945E-04
$(\frac{2}{3}, \frac{1}{8})$	1.441248	1.441835	4.073198E-04	$(\frac{1}{2}, \frac{1}{4})$	1	1.002458	2.458400E-03
$(\frac{3}{4}, \frac{1}{6})$	1.846482	1.845673	4.378733E-04	$(1, \frac{1}{4})$	2.557601	2.552739	1.901260E-03

Table 2: Relative errors of $p(t)$ for Example 1; $n = 81$, $T = \frac{1}{4}$

t	p_{exact}	p_{app}	Relative errors	t	p_{exact}	p_{app}	Relative errors
$\frac{1}{1000}$	-1	-1.000064	6.453900E-05	$\frac{1}{8}$	-1	-1.000684	6.838260E-04
$\frac{12}{1000}$	-1	-1.000669	6.689750E-04	$\frac{1}{6}$	-1	-0.999358	6.414128E-04
$\frac{1}{100}$	-1	-1.000585	5.854630E-04	$\frac{1}{5}$	-1	-0.9984953	1.504941E-03
$\frac{1}{10}$	-1	-1.000343	3.435430E-04	$\frac{2}{9}$	-1	-0.997999	2.000439E-03
$\frac{1}{9}$	-1	-1.000717	7.166060E-04	$\frac{1}{4}$	-1	-0.997132	2.868264E-03

Table 3: Relative errors of $v(x,t)$ for Example 1; $n = 81$, $T = \frac{1}{2}$

(x,t)	v_{exact}	v_{app}	Relative errors	(x,t)	v_{exact}	v_{app}	Relative errors
$(1, \frac{1}{1000})$	2.998001	2.998135	4.480185E-05	$(\frac{7}{8}, \frac{1}{6})$	2.445035	2.445058	9.422361E-06
$(\frac{1}{2}, \frac{12}{1000})$	1	1.000089	8.926900E-05	$(1, \frac{1}{6})$	2.692963	2.693930	3.587980E-04
$(1, \frac{1}{100})$	2.980099	2.981268	3.922174E-04	$(\frac{3}{4}, \frac{1}{5})$	1.818731	1.816356	1.305423E-03
$(\frac{1}{10}, \frac{1}{10})$	2.636866	2.640780	1.484401E-03	$(\frac{1}{10}, \frac{1}{5})$	2.481098	2.485261	1.677789E-03
$(\frac{1}{4}, \frac{1}{10})$	1.904837	1.912180	3.854839E-03	$(\frac{2}{3}, \frac{1}{5})$	1.409365	1.407343	1.434693E-03
$(1, \frac{1}{10})$	2.809675	2.809855	6.428644E-05	$(\frac{1}{2}, \frac{2}{9})$	1	1.002219	2.219280E-03
$(\frac{1}{2}, \frac{1}{9})$	1	1.004321	4.321410E-03	$(\frac{3}{5}, \frac{2}{9})$	1.152927	1.151739	1.030525E-03
$(\frac{2}{3}, \frac{1}{9})$	1.447420	1.447886	3.219329E-04	$(\frac{1}{3}, \frac{2}{9})$	1.400369	1.407778	5.291163E-03
$(1, \frac{1}{8})$	2.764994	2.765560	2.047726E-04	$(\frac{3}{5}, \frac{1}{4})$	1.148738	1.146675	1.795169E-03
$(\frac{2}{3}, \frac{1}{8})$	1.441248	1.441568	2.220845E-04	$(\frac{1}{2}, \frac{1}{4})$	1	1.001966	1.966360E-03
$(\frac{3}{4}, \frac{1}{6})$	1.846482	1.845236	6.746695E-04	$(1, \frac{1}{4})$	2.557601	2.556291	5.124707E-04

Table 4: Relative errors of $p(t)$ for Example 1; $n = 81$, $T = \frac{1}{2}$

t	p_{exact}	p_{app}	Relative errors	t	p_{exact}	p_{app}	Relative errors
$\frac{1}{1000}$	-1	-0.999794	2.063958E-04	$\frac{1}{8}$	-1	-0.977383	2.261733E-02
$\frac{12}{1000}$	-1	-0.997471	2.528598E-03	$\frac{1}{6}$	-1	-0.979758	2.024151E-02
$\frac{1}{100}$	-1	-0.997903	2.096714E-03	$\frac{1}{5}$	-1	-0.975690	2.430969E-02
$\frac{1}{10}$	-1	-0.976715	2.328518E-02	$\frac{2}{9}$	-1	-0.971106	2.889415E-02
$\frac{1}{9}$	-1	-0.976639	2.336095E-02	$\frac{1}{4}$	-1	-0.964804	3.519553E-02

Table 5: Relative errors of $v(x,t)$ for Example 2; $n = 81$, $T = \frac{1}{4}$

(x,t)	v_{exact}	v_{app}	Relative errors	(x,t)	v_{exact}	v_{app}	Relative errors
$(1, \frac{1}{1000})$	2.001000	2.000890	5.523337E-05	$(\frac{7}{8}, \frac{1}{6})$	1.590680	1.591406	4.562790E-04
$(\frac{1}{2}, \frac{12}{1000})$	2.012072	2.001377	5.315484E-03	$(1, \frac{1}{6})$	2.181360	2.164421	7.765664E-03
$(1, \frac{1}{100})$	2.010050	2.008924	5.601139E-04	$(\frac{3}{4}, \frac{1}{5})$	1	1.002636	2.636500E-03
$(\frac{1}{10}, \frac{1}{10})$	1.723344	1.715849	4.348790E-03	$(\frac{1}{10}, \frac{1}{5})$	1.799418	1.801355	1.076466E-03
$(\frac{1}{4}, \frac{1}{10})$	1	0.996456	3.544300E-03	$(\frac{2}{3}, \frac{1}{5})$	1.305351	1.276493	2.210723E-02
$(1, \frac{1}{10})$	2.105171	2.089205	7.583953E-03	$(\frac{1}{2}, \frac{2}{9})$	2.248849	2.171894	3.421967E-02
$(\frac{1}{2}, \frac{1}{9})$	2.117519	2.073065	2.099328E-02	$(\frac{3}{5}, \frac{2}{9})$	1.817382	1.751413	3.629875E-02
$(\frac{2}{3}, \frac{1}{9})$	1.279380	1.256824	1.762984E-02	$(\frac{1}{3}, \frac{2}{9})$	1.312212	1.303871	6.356607E-03
$(1, \frac{1}{8})$	2.133148	2.116888	7.622889E-03	$(\frac{3}{5}, \frac{1}{4})$	1.840405	1.768371	3.914031E-02
$(\frac{2}{3}, \frac{1}{8})$	1.283287	1.260171	1.801297E-02	$(\frac{1}{2}, \frac{1}{4})$	2.284025	2.199602	3.696255E-02
$(\frac{3}{4}, \frac{1}{6})$	1	0.999525	4.751000E-04	$(1, \frac{1}{4})$	2.284025	2.266547	7.652462E-03

Table 6: Relative errors of $p(t)$ for Example 2; $n = 81$, $T = \frac{1}{4}$

t	p_{exact}	p_{app}	Relative errors	t	p_{exact}	p_{app}	Relative errors
$\frac{1}{1000}$	1.001000	1.000778	2.216753E-04	$\frac{1}{8}$	1.125000	1.177465	4.663580E-02
$\frac{12}{1000}$	1.012000	1.009609	2.363016E-03	$\frac{1}{6}$	1.166667	1.138380330	2.424571E-02
$\frac{1}{100}$	1.010000	1.007937	2.042468E-03	$\frac{1}{5}$	1.200000	1.157718084	3.523493E-02
$\frac{1}{10}$	1.100000	1.141458	3.768888E-02	$\frac{2}{9}$	1.222222	1.170887564	4.200091E-02
$\frac{1}{9}$	1.111111	1.156422	4.077993E-02	$\frac{1}{4}$	1.250000	1.211209275	3.103258E-02

Table 7: Relative errors of $v(x,t)$ for Example 2; $n = 144$, $T = \frac{1}{4}$

(x,t)	v_{exact}	v_{app}	Relative errors	(x,t)	v_{exact}	v_{app}	Relative errors
$(1, \frac{1}{1000})$	2.001000	2.000941	2.983507E-05	$(\frac{7}{8}, \frac{1}{6})$	1.590680	1.595549	3.060781E-03
$(\frac{1}{2}, \frac{12}{1000})$	2.012072	2.008807	1.622859E-03	$(1, \frac{1}{6})$	2.181360	2.180958	1.846293E-04
$(1, \frac{1}{100})$	2.010050	2.009449	2.992512E-04	$(\frac{3}{4}, \frac{1}{5})$	1	1.013353	1.335274E-02
$(\frac{1}{10}, \frac{1}{10})$	1.723344	1.722972	2.159116E-04	$(\frac{1}{10}, \frac{1}{5})$	1.799418	1.810589	6.207763E-03
$(\frac{1}{4}, \frac{1}{10})$	1	1.001805	1.805350E-03	$(\frac{2}{3}, \frac{1}{5})$	1.305351	1.309086	2.861377E-03
$(1, \frac{1}{10})$	2.105171	2.101083	1.941670E-03	$(\frac{1}{2}, \frac{2}{9})$	2.248849	2.230536	8.143059E-03
$(\frac{1}{2}, \frac{1}{9})$	2.117519	2.102535	7.076129E-03	$(\frac{3}{5}, \frac{2}{9})$	1.817382	1.808653	4.803084E-03
$(\frac{2}{3}, \frac{1}{9})$	1.279380	1.276005	2.637737E-03	$(\frac{1}{3}, \frac{2}{9})$	1.312212	1.325054	9.786674E-03
$(1, \frac{1}{8})$	2.133148	2.129999	1.476139E-03	$(\frac{3}{5}, \frac{1}{4})$	1.840405	1.833820	3.578477E-03
$(\frac{2}{3}, \frac{1}{8})$	1.283287	1.280545	2.136414E-03	$(\frac{1}{2}, \frac{1}{4})$	2.284025	2.266462	7.689720E-03
$(\frac{3}{4}, \frac{1}{6})$	1	1.008052	8.052170E-03	$(1, \frac{1}{4})$	2.284025	2.293042	3.947540E-03

Table 8: Relative errors of $p(t)$ for Example 2; $n = 144$, $T = \frac{1}{4}$

t	p_{exact}	p_{app}	Relative errors	t	p_{exact}	p_{app}	Relative errors
$\frac{1}{1000}$	1.001000	1.000929	7.116752E-05	$\frac{1}{8}$	1.125000	1.148678	2.104750E-02
$\frac{12}{1000}$	1.012000	1.010731951	1.253013E-03	$\frac{1}{6}$	1.166667	1.145867427	1.782820E-02
$\frac{1}{100}$	1.010000	1.008977553	1.012324E-03	$\frac{1}{5}$	1.200000	1.196111194	3.240672E-03
$\frac{1}{10}$	1.100000	1.118037	1.639767E-02	$\frac{2}{9}$	1.222222	1.264996	3.499686E-02
$\frac{1}{9}$	1.111111	1.131717	1.854514E-02	$\frac{1}{4}$	1.250000	1.245162209	3.870233E-03

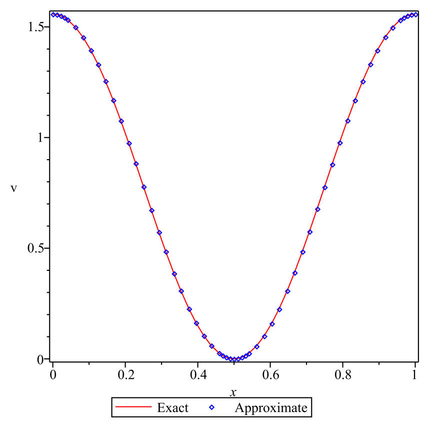


Figure 1: Exact and approximate solution of v for Example 1 at $T = \frac{1}{4}$

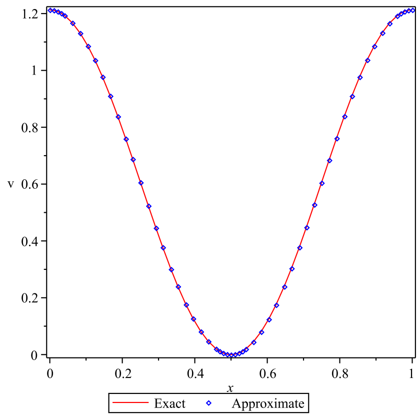


Figure 2: Exact and approximate solution of v for Example 1 at $T = \frac{1}{2}$

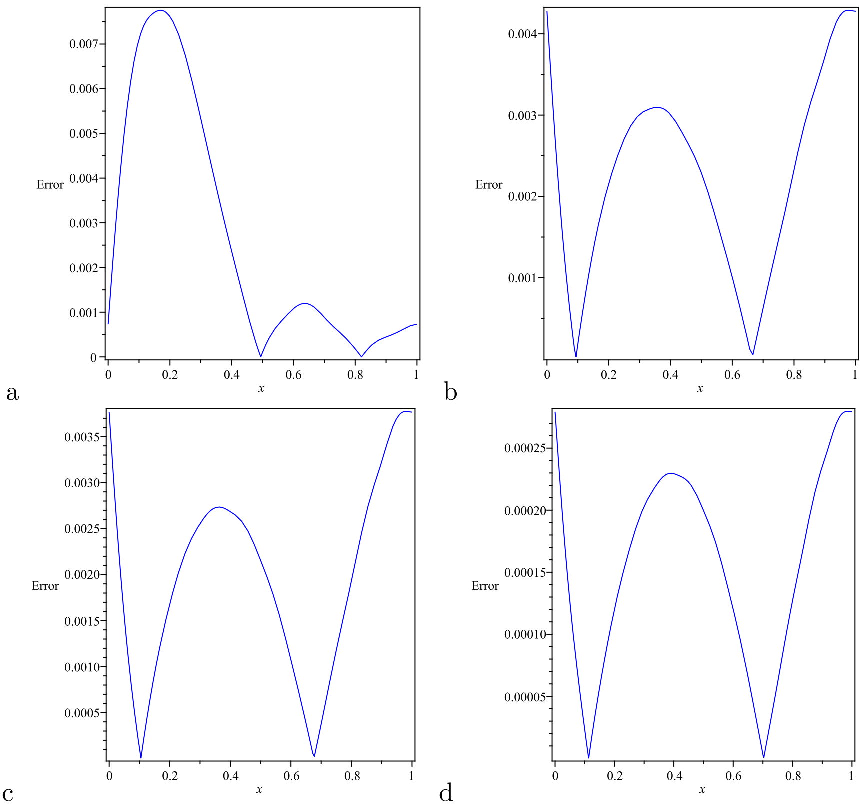


Figure 3: Relative errors graphs of v for Example 1 at time $T = \frac{1}{4}$; a($n = 36$), b($n = 64$), c($n = 81$), d($n = 100$)

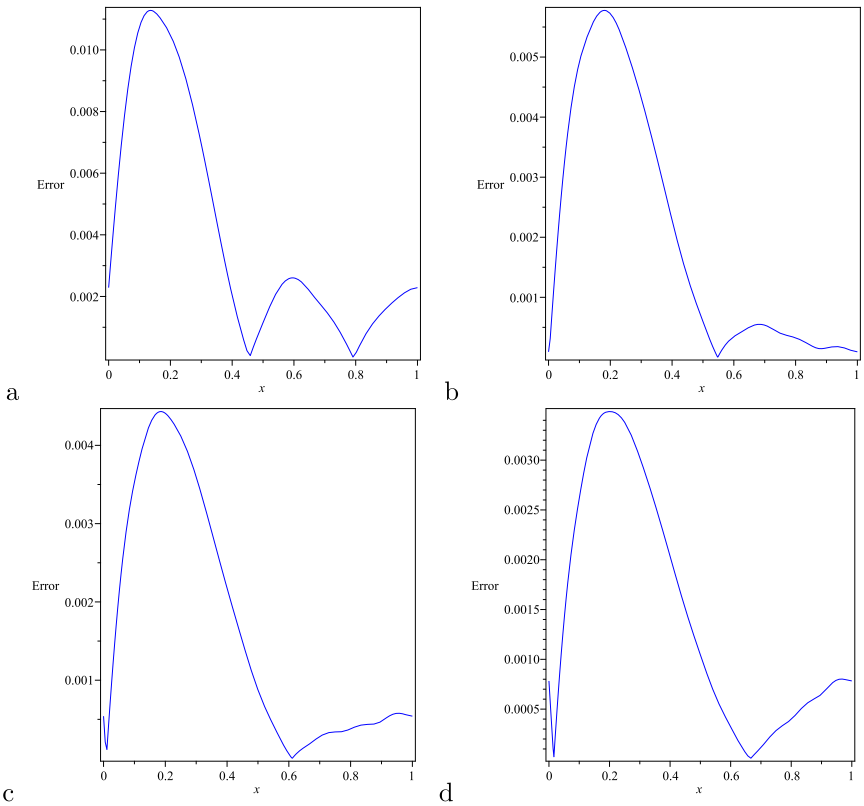


Figure 4: Relative errors graphs of v_{xt} for Example 1 at time $T = \frac{1}{4}$; a($n = 36$), b($n = 64$), c($n = 81$), d($n = 100$)

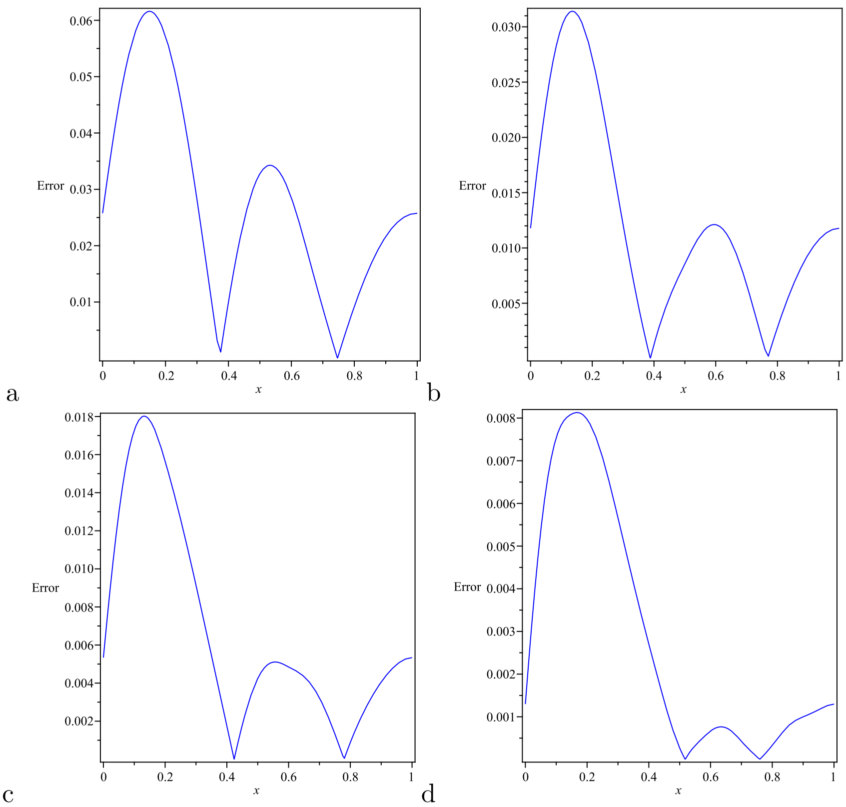


Figure 5: Relative errors graphs of v_{xxt} for Example 1 at time $T = \frac{1}{4}$; a($n = 36$), b($n = 64$), c($n = 81$), d($n = 100$)

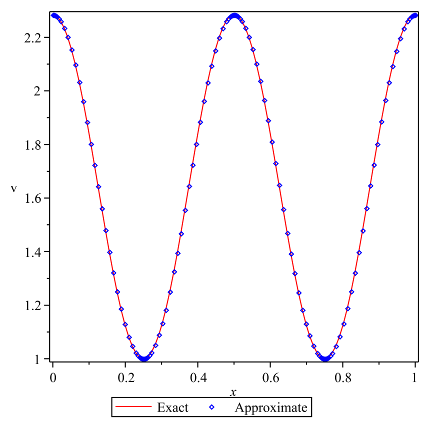
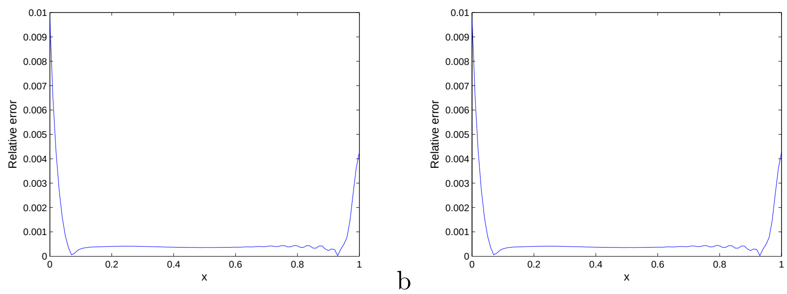
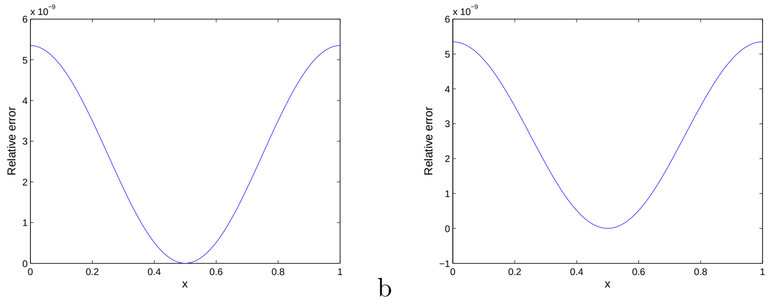


Figure 6: Exact and approximate solution of v for Example 2 at $T = \frac{1}{4}$



a b

Figure 7: Relative error graphs of v for Example 2 at time $T = \frac{1}{2}$; a(without noise), b(with noise)



a b

Figure 8: Relative error graphs of v for Example 3 at time $T = \frac{1}{2}$; a(without noise), b(with noise)