Iranian Journal of Numerical Analysis and Optimization
Vol. 14, No. 4, 2024, pp 991–1015
https://doi.org/10.22067/ijnao.2024.86954.1395
https://ijnao.um.ac.ir/

Research Article

A numerical solution of parabolic quasi-variational inequality nonlinear using Newton-multigrid method

M. Bahi*, M. Beggas, N. Nesba and A. Imtiaz

Abstract

In this article, we apply three numerical methods to study the $\mathbb{L}^{\infty}$-convergence of the Newton-multigrid method for parabolic quasi-variational inequalities with a nonlinear right-hand side. To discretize the problem, we

*Corresponding author

Received 20 February 2024; revised 04 May 2024; accepted 13 May 2024

Mostafa Bahi
Department of Mathematics, Faculty of exact Sciences, University of EL-OUED, Algeria. e-mail: bahi-mostafa@univ-eloued.dz

Mohammed Beggas
Department of Mathematics, Faculty of exact Sciences, University of EL-OUED, Algeria. e-mail: mohammed-beggas@univ-eloued.dz

Nourelhouda Nesba
Department of Mathematics, Faculty of exact Sciences, University of EL-OUED, Algeria. e-mail: nesba-nourelhouda@univ-eloued.dz

Ahmad Imtiaz
Institute of Informatics and Computing in Energy (IICE), Universiti Tenaga Nasional, Kajang, Selangor, Malaysia. e-mail: imtiazkakakhil@gmail.com

How to cite this article

Bahi, M., Beggas, M., Nesba, N. and Imtiaz, A., A numerical solution of parabolic quasi-variational inequality nonlinear using Newton-multigrid method. *Iran. J. Numer. Anal. Optim.*, 2024; 14(4): 991-1015. https://doi.org/10.22067/ijnao.2024.86954.1395

utilize a finite element method for the operator and Euler scheme for the time. To obtain the system discretization of the problem, we reformulate the parabolic quasi-variational inequality as a Hamilton–Jacobi–Bellman equation. For linearizing the problem on the coarse grid, we employ Newton's method as an external interior iteration of the Jacobian system. On the smooth grid, we apply the multigrid method as an interior iteration on the Jacobian system. Finally, we provide a proof for the $\mathbb{L}^\infty$-convergence of the Newton-multigrid method for parabolic quasi-variational inequalities with a nonlinear right-hand, by giving a numerical example for this problem.

AMS subject classifications (2020): Primary 65M55, 65M60; Secondary, 65F10.

Keywords: Newton's method; Multigrid method; Parabolic variational inequality; Finite element method; Hamilton–Jacobi–Bellman equation.

1 Introduction

Before studying the $\mathbb{L}^\infty$-convergence of the Newton-multigrid method for parabolic quasi-variational inequalities (PQVIs) with a nonlinear right-hand side, we will define our problem. Let $\mathbb{A}$ be an elliptic operator, and let Ω be an open domain in $\mathbb{R}^d$, where $d \geq 1$, with sufficiently regular bounds Γ. We provide the definition of our proposed problem:
Find $\mathtt{u} \in L^2(H_0^1(\Omega); [0,T])$ solution of

$$\begin{cases} \dfrac{\partial \mathtt{u}}{\partial t} + \mathbb{A}\mathtt{u} \leq f(\mathtt{u}); \qquad \mathtt{u} - \psi \leq 0, \\ \left(\dfrac{\partial \mathtt{u}}{\partial t} + \mathbb{A}\mathtt{u} - f(\mathtt{u})\right)(\mathtt{u} - \psi) = 0 \ \text{ in } \mathbf{Q}_t, \\ \mathtt{u}(x,t) = \mathtt{u}_0 \qquad in \quad \Omega_h, \\ \mathtt{u}(x,t) = 0 \qquad in \quad \sum_h, \end{cases} \tag{1}$$

where $\mathbf{Q}_T$ is defined as $\mathbf{Q}_T = \Omega \times [0,T]$, $\sum = \Gamma \times [0,T]$, $T < \infty$, and the obstacle $\psi \in W^{2,\infty}$ such that $\psi \geq 0$.

We assumed that the function $f(\cdot)$ is nonlinear, continuous, and c-Lipschitz; that is,

$$|f(\mathtt{y}) - f(\mathtt{z})| \leq c\,|\mathtt{y} - \mathtt{z}| \quad \text{for all } \mathtt{y}, \mathtt{z} \in \mathbb{R}, \quad c > 0.$$

We have divided this research into six parts as follows:

Part one: We formulate our problem and provide the necessary and sufficient conditions for studying the $\mathbb{L}^\infty$-convergence of Newton-multigrid methods.

Part two: To facilitate understanding, we introduce important symbols and hypotheses that will be used in our research. Subsequently, we present a theorem that proves the existence and uniqueness of the solution for the continuous problem.

Part three: To study the discrete problem, we define the space $\mathbb{V}_h$ that

$$\mathbb{V}_h = \{\mathtt{v}_h \in L^2(H_0^1(\Omega); [0,T]) \cap C^1(H_0^1(\overline{\Omega}); [0,T]),\ \mathtt{v}_h(\cdot, t) \in P_1\}$$

with P_1 is a Lagrange polynomial with a degree ≤ 1.

To discretize, we employ the finite element approximation on the operator $\mathbb{A}$ and use the Euler scheme for time discretization, resulting in the following discrete problem. Find $\mathtt{u}_h^n \in \mathbb{V}_h$ as a solution of

$$\begin{cases} \langle \mathbb{B}_h \mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n \rangle \geq \langle F^n(\mathtt{u}_h^n), \mathtt{v}_h - \mathtt{u}_h^n \rangle, \\ \mathtt{u}_h^n \leq r_h \psi, \\ \mathtt{u}_h(x) = \mathtt{u}_h(x, 0) \quad in \quad \Omega_h, \\ \mathtt{u}(x,t) = 0 \quad in \quad \Sigma_h. \end{cases} \tag{2}$$

Part four: To solve the PQVI using Newton-multigrid methods, we discretize the domain through the finite element method and employ the Euler scheme for time discretization. Subsequently, we reformulate the PQVI as the Hamilton–Jacobi–Bellman (HJB) equation, expressed as follows:

$$\max_{1 \leq i \leq N} \left(\mathbb{B}_{k,i} \mathtt{u}_{k,i}^v - F(\mathtt{u}_{k,i}^v);\ \mathtt{u}_{k,i}^v - r_h \psi \right) = 0.$$

We employ the iterative procedure introduced by Hoppe (see [22, 23, 24]) to solve the obtained system. Subsequently, we apply Newton's method to address the nonlinear system by linearizing it to obtain the Jacobian system. Following this, we employ the multigrid method on the Jacobian system in all iterations.

Part five: To establish the $\mathbb{L}^\infty$-convergence of the Newton-multigrid method, we rely on the approximation and smoothing suggestions proposed by Huckbach (see[19, 20]). In this research, we demonstrate the $\mathbb{L}^\infty$-convergence in the addressed problem based on the techniques presented in Huckbach's work.

Part six: Here, we give a numerical example that confirms the results obtained in the theoretical part.

2 Newton-multigrid method

2.1 Assumptions and symbols

Consider $\mathbb{A}$ as an elliptic operator defined by

$$\mathbb{A} = -\sum_{1\leq i,j\leq d}^{d} \mathtt{a}_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} \mathtt{a}_j(x)\frac{\partial}{\partial x_j} + \mathtt{a}_0(x). \tag{3}$$

The coefficients $\mathtt{a}_{ij}(x),\ \mathtt{a}_j(x),\ \mathtt{a}_0(x) \in L^\infty(\Omega) \cap C^2(\bar{\Omega})$ meet the following conditions:

$$\mathtt{a}_{ij}(x) = \mathtt{a}_{ji}(x), \quad \mathtt{a}_0(x) \geq \alpha > 0, \quad \alpha\ is\ a\ constant, \quad x \in \bar{\Omega}, \tag{4}$$

$$\sum_{1\leq i,j\leq d}^{d} \mathtt{a}_{ij}(x)\zeta_i\zeta_j \geq \gamma|\zeta|^2; \quad \zeta \in \mathbb{R}^d \quad \gamma > 0,$$

and $\mathtt{a}(\cdot,\cdot)$ is associated with the elliptic operator $\mathbb{A}$, where $\mathtt{a}(\cdot,\cdot)$ is a continuous and coercive bilinear form, as given by

$$\mathtt{a}(\mathtt{u},\mathtt{v}) = \int_\Omega \left[\sum_{1\leq i,j\leq d}^{d} \mathtt{a}_{ij}(x)\frac{\partial \mathtt{u}}{\partial x_i}\frac{\partial \mathtt{v}}{dx_j} + \sum_{j=1}^{d} \mathtt{a}_j(x)\frac{\partial \mathtt{u}}{\partial x_j} + \mathtt{a}_0(x)\mathtt{u}\mathtt{v}\right] dx, \tag{5}$$

$$\mathtt{a}(\mathtt{v},\mathtt{v}) + \omega\|\mathtt{v}\|^2_{L^2(\Omega)} \geq \gamma\|\mathtt{v}\|^2_{H^1_0(\Omega)}, \quad \gamma > 0, \qquad \omega > 0, \qquad \text{for all } \mathtt{v} \in H^1_0(\Omega). \tag{6}$$

Let $f(\mathtt{u})$ be a nonlinear right-hand function and f is c-Lipschitzian given by

$$f(\mathtt{u}) \in L^\infty(H^1_0(\Omega);[0,T] \cap C^1(H^1_0(\Omega);[0,T])), \quad f > 0, \quad and \quad c < \alpha.$$

2.2 Continuous problem

After applying Green's formulation, we obtain the weak formulation of our proposed problem, $\mathtt{v}$ in $H^1_0(\Omega)$ and $\mathtt{v} \leqslant \psi$ and find $\mathtt{u}$ in $L^2(H^1_0(\Omega);[0,T])$ a solution to the following problem:

$$\begin{cases} \left(\dfrac{\partial \mathtt{u}}{\partial t}, \mathtt{v} - \mathtt{u}\right) + \mathtt{a}(\mathtt{u}, \mathtt{v} - \mathtt{u}) \geq (f(\mathtt{u}), \mathtt{v} - \mathtt{u}), \\ \mathtt{u} \leq \psi, \\ \mathtt{u}(x,t) = \mathtt{u}_0 \qquad in \quad \Omega, \\ \mathtt{u}(x,t) = 0 \quad in \quad \Sigma. \end{cases} \tag{7}$$

Certainly, the problem (7) has a unique solution, as supported by fixed-point theory and in accordance with the previously established assumptions; see [7, 10, 12].

3 Discretization

3.1 Discretize

To establish a multigrid loop, consider a decreasing sequence $\{h_k\}_{k=0}^{l}$ such that

$$0 < h_k < h_{k+1} < 1, \quad 0 \leq k \leq m-1,$$

that $\{T_h\}$ is a family of uniform triangulations, where $\Omega_h = \underset{T \in T_h}{\cup} T_h$, and that for all T_h, we have

$$\Omega_h \subset \Omega_{h+1} \subset \Omega, \quad dist(\Gamma_h, \Gamma) \leq ch_k^2, \quad h_{k+1}h_k \leq c_1.$$

We associate each h_k with the following symbols to facilitate our work:

$$\Omega_{h_k} = \Omega_h, \qquad \mathbb{V}_{h_k} = \mathbb{V}_h, \qquad \mathbb{A}_{h_k} = \mathbb{A}_h.$$

We consider ϕ_h^i, $i = 1, 2, \ldots, m(h)$ as the conventional basis for affine functions defined as follows:

$$\phi_h^i(K_h^j) = \delta_{ij},$$

where K_h^j denotes a vertex of the triangulation T_h. Let $\mathbb{U}_h = R^{mh}$. We consider the restriction operator defined as

$$r_h : \mathbb{U}_h \longrightarrow \mathbb{V}_h,$$

$$r_h \mathtt{v} = \sum_{i=1}^{m(h)} \mathtt{v}(K_h^i)\phi_h^i(x). \tag{8}$$

We provide a definition for the following adjoint operator $r_h^* : \mathbb{V}_h \longrightarrow \mathbb{U}_h$ satisfying

$$\langle r_h \mathtt{u}, \mathtt{u} \rangle_{L^2} = \langle \mathtt{u}, r_h^* \mathtt{v} \rangle, \qquad \text{for all } \mathtt{u} \in \mathbb{U}_h, \mathtt{v} \in \mathbb{V}_h.$$

The norm $\|\cdot\|_\infty$ (on $\mathbb{U}_k$) and the norm $\|\cdot\|_{L^\infty}$ (on $\mathbb{V}_k$) are equivalent, which are indicated by $\|\cdot\|_\infty$.

Lemma 1. [3] There exist λ_1 and λ_2 independent of h such that

1. $\|r_h(\mathtt{u})\|_\infty = \|\mathtt{u}\|_\infty$, for all $\mathtt{u} \in \mathbb{U}_h$,

2. $\lambda_1 \|\mathtt{v}\|_\infty \leq \|r^*(\mathtt{v})\|_\infty \leq \lambda_2 \|\mathtt{v}\|_\infty$, for all $\mathtt{v} \in \mathbb{V}_h$.

3.2 Discrete problem

We apply the semi-implicit Euler scheme to discretize problem (7) with respect to time. Consequently, we are in seek of a sequence of elements $\mathtt{u}_h^n \in \mathbb{V}_h$ that approaches $\mathtt{u}(t_n)$, $t_n = n\delta t$, with initial data $\mathtt{u}_0$ and $\dfrac{\partial \mathtt{u}_h}{\partial t} = \frac{\mathtt{u}_h^n - \mathtt{u}_h^{n-1}}{\delta t}$. Thus, we have for $n = 1, \ldots, N$, and for all $\mathtt{v}_h \in \mathbb{V}_h$,

$$\begin{cases} (\frac{\mathtt{u}_h^n - \mathtt{u}_h^{n-1}}{\delta t}, \mathtt{v}_h - \mathtt{u}_h^n) + \mathtt{a}(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) \geq (f(\mathtt{u}_h^n), \mathtt{v}_h - \mathtt{u}_h^n), \\ \mathtt{u}_h^n \leq r_h\psi; \quad \mathtt{v}_h^n \leq r_h\psi. \end{cases} \tag{9}$$

We can write the problem (9) as for all $\mathtt{v}_h \in \mathbb{V}_h$,

$$\begin{cases} \mathtt{a}(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) + \frac{1}{\delta t}(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) \geq (f(\mathtt{u}_h^n) + \frac{\mathtt{u}_h^{n-1}}{\delta t}, \mathtt{v}_h - \mathtt{u}_h^n), \\ \mathtt{u}_h^n \leq r_h\psi; \quad \mathtt{v}_h^n \leq r_h\psi. \end{cases} \tag{10}$$

When we put $\omega = \frac{1}{\delta t}$ and $F(u_h^n) = f(\mathtt{u}_h^n) + \omega \mathtt{u}_h^{n-1}$, we obtain

$$\begin{cases} \omega(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) + \mathtt{a}(\mathtt{u}_h^n, \mathtt{v}_h - u_h^n) \geq (F_h^n(\mathtt{u}_h^n), \mathtt{v}_h - \mathtt{u}_h^n), \\ \mathtt{u}_h^n \leq r_h\psi. \end{cases} \tag{11}$$

We can express the problem (11) in the following form:
Find $\mathtt{u}_h^n \in \mathbb{V}_h$, for all $\mathtt{v}_h \in \mathbb{V}_h$

$$\begin{cases} \mathtt{b}(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) \geq (F_h^n(\mathtt{u}_h^n), \mathtt{v}_h - \mathtt{u}_h^n), \\ \mathtt{u}_h^n \leq r_h\psi, \\ \mathtt{u}(x,t) = \mathtt{u}_0 \qquad in \quad \Omega_h, \\ \mathtt{u}(x,t) = 0 \quad in \quad \Sigma_h, \end{cases} \tag{12}$$

where

$$\mathtt{b}(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) = \omega(\mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n) + \mathtt{a}(\mathtt{u}_h^n, \mathtt{u}_h - \mathtt{u}_h^n).$$

The bilinear form $\mathtt{b}(\cdot,\cdot)$ exhibits strong coerciveness. Consequently, the formulation (12) denotes a coercive and continuous issue concerning elliptic quasi-variational inequalities (see[8]).

Denote by $\mathbb{B}_h$ the outcome obtained by addressing problem (12) through a finite element method, resulting in the solution to the subsequent problem ascertain $\mathtt{u}_h^n \in \mathbb{V}_h$ such that

$$\begin{cases} \langle \mathbb{B}_h \mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n \rangle \ \geq \ \langle F^n(\mathtt{u}_h^n), \mathtt{v}_h - \mathtt{u}_h^n \rangle, \\ \mathtt{u}_h^n \leq r_h\psi, \\ \mathtt{u}_h(x) = \mathtt{u}_h(x,t) \qquad in \quad \Omega_h, \\ \mathtt{u}(x,t) = 0. \quad in \quad \Sigma_h, \end{cases} \tag{13}$$

where $(\mathbb{B}_h)_{i,j}$ denotes the finite element matrix defined by

$$(\mathbb{B}_h)_{i,j} = \mathtt{b}(\phi_i, \phi_j) = \mathtt{a}(\phi_i, \phi_j) + \omega(\phi_i, \phi_j),$$

and $\{\phi_i\}$ is basis of $\mathbb{V}_h,\ 1 \leq i, 1 \leq j$.

The discretization matrices $\mathbb{B}_h$ and the generic coefficient matrices $\mathtt{a}(\phi_h^i, \phi_h^j)$ are introduced in a natural progression, where the customary basic functions are denoted by $\phi_h = 1, 2, \ldots, m(h_h)$. Now that these definitions have been established, we may state the discrete problem as follows: Determine $\mathtt{u}_h^n \in \mathbb{V}_h$, representing the solution for

$$\begin{cases} \langle \mathbb{B}_h \mathtt{u}_h^n, \mathtt{v}_h - \mathtt{u}_h^n \rangle \ \geq \ \langle F^n(\mathtt{u}_h^n), \mathtt{v}_h - \mathtt{u}_h^n \rangle\,, \\ \mathtt{u}_h^n \leq r_h \psi, \\ \mathtt{u}_h(x) = \mathtt{u}_h(x,t) \qquad in \quad \Omega_h, \\ \mathtt{u}(x,t) = 0 \quad in \quad \Sigma_h. \end{cases} \tag{14}$$

The matrix $(\mathbb{B}_h)$ with coefficients $\mathtt{b}(\phi_i, \phi_j)$ is an M-matrix (see([16, 17, 18]). The problem (14) has a unique solution (as is well known), and we obtain the following regularity result.

Theorem 1. [10] Let $\mathtt{u}^\infty$ and $\mathtt{u}_h^n$ be the solutions of (7) and (14), respectively. Then

$$\|\mathtt{u}_h^n - \mathtt{u}^\infty\|_\infty \leq C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right], \tag{15}$$

where $\mathtt{u}^\infty$ is the asymptotic solution of the problem (12), and $\mathtt{u}_h^n$ is the discrete solution calculated at the moment $T = n\delta t, C > \ 0$.

3.3 HJB-formulation of discrete problem

Before commencing work in this section, we will use the following symbols:

$$\mathbb{B}_h = \mathbb{B}_k^v, \quad \mathtt{u}_h^n = \mathtt{u}_k^*, \quad \mathbb{U}_h = \mathbb{U}_k, \quad \mathbb{V}_h = \mathbb{V}_k.$$

The problem (14) can be expressed as the following HJB equation. Let $\mathtt{u}_k^v$ denote the unique solution of the discrete HJB equation,

$$\max_{1\leq i\leq N}\left(\mathbb{B}_{k,i}\mathtt{u}^v_{k,i}-F(\mathtt{u}^v_{k,i});\ \mathtt{u}^v_{k,i}-r_k\psi\right)=0.$$

We put

$$\mathtt{u}^v_{k,i}-r_k\psi=\lambda^v_{k,i}.$$

Iterative steps are as follows:

Step 1: We select an initial vector $\mathtt{u}^0_k\in\mathbb{R}^{n_k}$.

Step 2: Let $\mathtt{u}^v_k\in\mathbb{R}^{n_k}, v\geq 0$, and we compute $\mathtt{u}^{(v+1)}_k$, where $\mathtt{u}^{(v+1)}_k$ is a solution of the following equation:

$$\mathbb{B}^v_k\mathtt{u}^{v+1}_k-F^v_k(\mathtt{u}^v_k)=0, \tag{16}$$

where

$$\mathbb{B}^v_k=\begin{cases}\mathbb{B}_{k,i}(\mathtt{u}^v_k)\ if\ \mathbb{B}_{k,i}\mathtt{u}^v_{k,i}-F(\mathtt{u}^v_{k,i})\ >\ \lambda^v_{k,i}\\ \quad I_{k,i}\quad if\ \mathbb{B}_{k,i}\mathtt{u}^v_{k,i}-F(\mathtt{u}^v_{k,i})\ \leq\ \lambda^v_{k,i}\end{cases} \tag{17}$$

$$F^v_k(\mathtt{u}^v_k)=\begin{cases}F^v_k(\mathtt{u}^v_k)\ if\ \mathbb{B}_{k,i}\mathtt{u}^v_{k,i}-F(u^v_{k,i})\ >\ \lambda^v_{k,i}\\ \quad\psi_k\quad if\ \mathbb{B}_{k,i}\mathtt{u}^v_{k,i}-F(\mathtt{u}^v_{k,i})\ \leqslant\lambda^v_{k,i}\end{cases} \tag{18}$$

Consider $\mathtt{u}^*_k$ as the unique solution to the discrete HJB equation,

$$\max_{1\leq i\leq N}\left(\mathbb{B}_{k,i}\mathtt{u}^*_{k,i}-F(\mathtt{u}^*_{k,i});\ \mathtt{u}^*_{k,i}-\psi_k\right)=0.$$

The following theorem represents the equivalence between the HJB equation and the PQVI by using the finite elements method, and we will rely on its proof of the work done by Hoppe on the elliptic quasi-variational inequality using the finite differences method.

Theorem 2. Let $\mathtt{u}^v_k$ be the iterate obtained by the previous iterative scheme, satisfying the HJB-equation above. Additionally, we suppose that $\mathbb{B}_k$ is monotone. Then, the sequence $\mathtt{u}^v_k$, where $(v\geq 0)$, exhibits monotonically decreasing convergence toward the unique solution $\mathtt{u}^*_k$ of (13).

Proof. Let $\mathtt{u}^v_k$ represent an iteration, and let $\mathtt{u}^*_k$ denote a solution of the HJB equation. To prove that $\mathtt{u}^v_k$ converges towards $\mathtt{u}^*_k$, it is adequate to show that $(\mathtt{u}^v_k,\mathtt{v}\geq 0)$ consistently decreases towards $\mathtt{u}^*_k$, as outlined below:

$$\mathtt{u}^*_k\leq\mathtt{u}^{v+1}_k\leq\mathtt{u}^v_k.$$

To begin, let us consider $\mathtt{u}^0_k$ satisfying

$$\max(\mathbb{B}_k^0 \mathtt{u}_k^0 - F_k, \mathtt{u}_k^0 - \psi_k) \geq 0,$$

for all $v \geq 0$,

$$\max(\mathbb{B}_k^v \mathtt{u}_k^v - F_k, \mathtt{u}_k^v - \psi_k \geq (\mathbb{B}_k^v \mathtt{u}_k^v - F_k) \geq 0.$$

Using (16), we get

$$(\mathbb{B}_k^v \mathtt{u}_k^v - \mathbb{B}_k^v u_k^{v+1}) \geq 0.$$

We have $\mathbb{B}_k^v$, which is linear and monotone,

$$\mathbb{B}_k^v (\mathtt{u}_k^v - \mathtt{u}_k^{v+1}) \geq 0.$$

Then

$$(\mathtt{u}_k^v - \mathtt{u}_k^{v+1}) \geq 0,$$

$$\mathtt{u}_k^{v+1} \leq \mathtt{u}_k^v. \tag{19}$$

We know that $\mathtt{u}_k^*$ is a solution of (14). We have

$$\max(\mathbb{B}_k^v \mathtt{u}_k^* - F_k, \mathtt{u}_k^* - \psi_k) = 0,$$

again, using (16)

$$\mathbb{B}_k^v \mathtt{u}_k^* - \mathbb{B}_k^v \mathtt{u}_k^{v+1} \leq 0,$$

$$\mathbb{B}_k^v \mathtt{u}_k^* \leq \mathbb{B}_k^v \mathtt{u}_k^{v+1}.$$

We have $\mathbb{B}_k^v$ is a monotone matrix; that is, $\mathbb{B}_k^v$ is an invertible matrix ($(\mathbb{B}_k^v)^{-1}$ exists). Thus

$$(\mathbb{B}_k^v)^{-1} \mathbb{B}_k^v \mathtt{u}_k^* \leq (\mathbb{B}_k^v)^{-1} \mathbb{B}_k^v \mathtt{u}_k^{v+1}.$$

Then,

$$\mathtt{u}_k^* \leq \mathtt{u}_k^{v+1}. \tag{20}$$

From (19) and (20), we obtain

$$\mathtt{u}_k^* \leq \mathtt{u}_k^{v+1} \leq \mathtt{u}_k^v.$$

The uniqueness: To demonstrate uniqueness, we suppose that there are two solutions $\mathtt{u}_1^*$ and $\mathtt{u}_2^*$ of (14).

First, we put $\mathtt{u}_1^* = \mathtt{u}_k^*$ and $\mathtt{u}_2^* = \mathtt{u}_k^{v+1}$,

$$\max(\mathbb{B}_k^v \mathtt{u}_1^* - F_k, \mathtt{u}_1^* - \psi_k) = 0 \geq (\mathbb{B}_k^v u_1^* - \mathbb{B}_k^v u_k^{v+1}),$$

$$\mathbb{B}_k^v \mathtt{u}_1^* - \mathbb{B}_k^v \mathtt{u}_2^* \leq 0, \qquad \mathbb{B}_k^v \mathtt{u}_1^* \leq \mathbb{B}_k^v \mathtt{u}_2^*.$$

Therefore

$$\mathtt{u}_1^* - \mathtt{u}_2^* \leq 0. \tag{21}$$

Secondly, we put $u_2^* = u_k^*$ and $u_1^* = u_k^{v+1}$,

$$\max(\mathbb{B}_k^v \mathtt{u}_2^* - F_k, \mathtt{u}_2^* - \psi_k) = 0 \geq (\mathbb{B}_k^v \mathtt{u}_2^* - \mathbb{B}_k^v \mathtt{u}_k^{v+1}).$$

$$\mathbb{B}_k^v \mathtt{u}_2^* - \mathbb{B}_k^v \mathtt{u}_1^* \leq 0 \quad then \quad \mathbb{B}_k^v \mathtt{u}_2^* \leq \mathbb{B}_k^v \mathtt{u}_1^*.$$

We have

$$\mathtt{u}_1^* - \mathtt{u}_2^* \geq 0. \tag{22}$$

From (21) and (22), we get

$$\mathtt{u}_1^* = \mathtt{u}_2^*.$$

Then, the solution of (14) ($\mathtt{u}_k^v$), approximates towards the solution of problem (13), $\mathtt{u}_k^v \mapsto \mathtt{u}_k^*$. □

3.4 Description of the Newton-multigrid method for PQVIs

In addressing the nonlinear system (16), we utilize the Newton-multigrid method, a hybrid technique that merges Newton's method for nonlinear systems with the multigrid method for linear systems. The application of the Newton-multigrid approach involves the following procedural steps:

First step: By utilizing Newton's method, we obtain the nonlinear system, recognizing that f is a Lipschitz function.

Second step: In every linear step of the system (16), we seek the Jacobian system to derive a linear system connected with the Jacobian matrix.

Third step: We use the multigrid method to solve the linear system resulting from the Jacobian matrix.

Let $\mathtt{u}_k^v$ be the exact solution of the system (16), and let $\mathtt{w}_k^v$ be an approximation to $\mathtt{u}_k^v$. We define the residual as follows:

$$\mathfrak{R}_k(\mathtt{u}_k^v) = F_k^v(\mathtt{u}_k^v) - \mathbb{B}_k^v \mathtt{w}_k^v. \tag{23}$$

Substituting (16) in (23), we find

$$\mathfrak{R}_k(\mathtt{u}_k^v) = \mathbb{B}_k^v \mathtt{u}_k^v - \mathbb{B}_k^v \mathtt{w}_k^v, \tag{24}$$

$$\mathfrak{R}_k(\mathtt{u}_k^v) = \mathbb{B}_k^v \ (\mathtt{u}_k^v - \mathtt{w}_k^v) = \mathbb{B}_k^v(e_k^v), \tag{25}$$

where $\mathtt{e}_k^v$ is the error with $\mathtt{e}_k^v = \mathtt{u}_k^v - \mathtt{w}_k^v$. In the context of (25), determining the error $\mathtt{e}_k^v$ for the linear equation (25) on the coarse grid, as commonly done in the multigrid method, is not feasible. Consequently, we adopt (25) as a residual equation, given the nonlinear characteristics of f.

To facilitate access to the solution, we choose $\mathcal{H}$ as a nonlinear operator, where

$$\mathcal{H}_k(\mathtt{u}_k^v) = \mathbb{B}_k^v \mathtt{u}_k^v - F_k^v(\mathtt{u}_k^v) = 0. \tag{26}$$

The residual in the fine grid can be rewritten as follows:

$$\mathfrak{R}_k(\mathtt{u}_k^v) = -\mathcal{H}_k(\mathtt{u}_k^v). \tag{27}$$

To solve (26), we employ the Newton iteration method as follows:

$$\mathtt{u}_k^{v+1} = \mathtt{u}_k^v + \frac{\mathfrak{R}_k(\mathtt{u}_k^v)}{\mathcal{J}_k(\mathtt{u}_k^v)}, \tag{28}$$

where $\mathcal{J}_k(\mathtt{u}_k^v)$ represents the Jacobian matrix of the nonlinear system, and $\mathcal{J}_k(\mathtt{u}_k^v) = \mathcal{H}_k^{'}(\mathtt{u}_k^v)$.

From (28), we can derive the following Jacobian linear system for $\mathtt{e}_k^v$:

$$\mathcal{J}_k(\mathtt{u}_k^v)\mathtt{e}_k^v = \mathfrak{R}_k(\mathtt{u}_k^v), \tag{29}$$

where $\mathtt{u}_k^{v+1} - \mathtt{u}_k^v = \mathtt{e}_k^v$.

The solution to the linear system (29) is utilized as an approach to solve the nonlinear system (26). Therefore, to find the solution for the nonlinear system (26), we employ the multigrid method to solve the associated linear system (29).

3.5 Multigrid technique

In addressing the linear system (29), we employ the multigrid technique. By selecting an iteration $\mathtt{e}_k^v$, where $v > 0$, within the multigrid method, we derive $\overline{\mathtt{e}}_k^v$ through the application of an iterative method. This iterative process is utilized to solve the system (29), utilizing α as the coefficient expression as follows:

$$\overline{\mathtt{e}}_k^v = \mathcal{S}_k^v(\mathtt{e}_k^v). \tag{30}$$

Here, $\mathcal{S}_k^v$ stands for the iteration or smoothing operator, and α represents the number of iterations executed. The solution to (29) is denoted as $\mathtt{e}^\star$.

The error is defined as $\mathtt{E}_k^v = \overline{\mathtt{e}}_k^v - \mathtt{e}_k^\star$, and the residual is also considered

$$\mathtt{d}_k^v = \mathfrak{R}_k(\mathtt{u}_k^v) - \mathcal{J}_k(\mathtt{u}_k^v)\overline{\mathtt{e}}_k^v.$$

We can write (29) as

$$\mathcal{J}_k(\mathtt{u}_k^v)(\overline{\mathtt{e}}_k^v + \mathtt{E}_k^v) = \mathfrak{R}_k(\mathtt{u}_k^v). \tag{31}$$

We derive the subsequent residual equation,

$$\begin{aligned} &\mathcal{J}_k(\mathtt{u}_k^v)\mathtt{E}_k^v = \mathfrak{R}_k(\mathtt{u}_k^v) - \mathcal{J}_k(\mathtt{u}_k^v)\overline{\mathtt{e}}_k^v = \mathtt{d}_k^v,\\ &\mathcal{J}_k(\mathtt{u}_k^v)\mathtt{E}_k^v = \mathtt{d}_k^v. \end{aligned}$$

To fully determine $\mathtt{E}_k^v$, it is necessary to compute $\mathtt{E}_k^v$ at the level $(k-1)$ as the solution to the coarse grid system,

$$\mathcal{J}_{k-1}(\mathtt{e}_{k-1}^v)\mathtt{E}_{k-1}^v = \mathtt{d}_{k-1}^v. \tag{32}$$

In the context, $\mathtt{E}_{k-1}^v$, $\mathcal{J}_{k-1}(\mathtt{e}_{k-1}^v)$, $\mathtt{d}_{k-1}^v$ represent approximations at the $(k-1)$ of $\mathtt{E}_k^v$, $\mathcal{J}_k(\mathtt{e}_k^v)$, $\mathtt{d}_k^v$, respectively.

We have

$$\begin{aligned} \mathtt{E}_{k-1}^v &= \mathtt{R}_k\mathtt{E}_k^v,\\ \mathtt{d}_{k-1}^v &= \mathtt{R}_k\mathtt{d}_k^v,\\ \mathcal{J}_{k-1}^v(\mathtt{e}_{k-1}^v) &= \mathcal{R}_k\mathcal{J}_k^v(e_k^v)\mathcal{P}_k, \end{aligned}$$

where $\mathcal{R}_k$ is the restriction matrix and $\mathcal{P}_k$ is the prolongation matrix. Owing to the nested structure, we employ the clearly defined identity operator,

$$\Psi : \mathbb{V}_{k-1} \longrightarrow \mathbb{V}_k,$$

for defining the prolongation and restriction operators, that is,

$$\begin{aligned} \mathcal{R}_k &= \mathcal{P}_k^t \\ \mathcal{P}_k &= r_k r_{k-1}^{-1}. \end{aligned} \tag{33}$$

Note: To solve the linear system (29) for the two sequences Ω_k and Ω_{k-1}, we employ the multigrid technique as an internal iteration. Additionally, to address system (26), we utilize Newton's method as an external iteration. This results in the iterative solution of the system (32), achieved by applying a two-grid iteration repeatedly to the sequence $\Omega_k\{k = 0, \dots, m_k\}$ until the iteration process is halted.

4 $\mathbb{L}^\infty$-convergence of multigrid method

This paragraph explores the assessment of uniform convergence for the multigrid algorithm (inner iteration) and the convergence characteristics of Newton's method (outer iteration). The assumptions employed in these convergence analyses closely resemble those utilized in multigrid methods specifically crafted for addressing nonlinear equations. We now present the main hypotheses:

Hypothesis 1:

$$\text{There exists } \mathtt{u}_k^\star \in V_k \quad \textit{such that} \quad \mathcal{H}_k(\mathtt{u}_k^\star)^{-1} = 0.$$

Hypothesis 2 (inverse):

$$\mathcal{H}_k^{'}(\mathtt{u}_k^\star)^{-1} \textit{ is exist and } \|\mathcal{H}_k^{'}(\mathtt{u}_k^\star)^{-1}\|_\infty \leq k \quad \textit{with } k > 0.$$

Hypothesis 3 (continuous):

$$\text{for all } \epsilon > 0, \quad \text{there exists } \eta > 0, \quad \|I - \mathcal{H}_k^{'}(\mathtt{u}_k^\star)\mathcal{H}_k^{'}(\mathtt{u}_k)^{-1}\|_\infty \leq \epsilon,$$

whenever

$$\|(\mathtt{u}_k - \mathtt{u}_k^\star)\|_\infty \leq \eta.$$

Hypothesis 4: For any $\mathtt{u}_k$ in the neighborhood of $\mathtt{u}_k^\star$, there is a linear mapping denoted as $\mathcal{H}_k^{'}(\mathtt{u}_k)$ such that for all $\epsilon > 0$ there exists $\eta > 0$ such that

$$\|\mathcal{H}_k(\mathtt{u}_k) - \mathcal{H}_k(\mathtt{u}_k^\star) - \mathcal{H}_k^{'}(\mathtt{u}_k^\star)\|(\|(\mathtt{u}_k - \mathtt{u}_k^\star)\|_\infty \leq \epsilon\|(\mathtt{u}_k - \mathtt{u}_k^\star)\|_\infty$$

hold, whenever

$$\|(\mathtt{u}_k - \mathtt{u}_k^\star)\|_\infty \leq \eta.$$

4.1 Matrix associated with the MGHJB algorithm

The two-grid method's iteration matrix, which includes α_1 presmoothing and α_2 postsmoothing at level $k-1$, can be represented as follows:

$$\mathbb{TG}_k(\alpha_1, \alpha_2) = \mathcal{S}_k^{\alpha_2}\, (\mathcal{J}_k^{-1} - \mathcal{P}_k\, \mathcal{J}_{k-1}^{-1}\mathcal{R}_k)\mathcal{J}_k\, \mathcal{S}_k^{\alpha_1}, \qquad k = 1, 2, \ldots. \tag{34}$$

Theorem 3. [26, 27] The multigrid approach is a linear iterative technique characterized by the iteration matrix $\mathbb{MG}_k$, which is expressed as follows:

$$\mathbb{MG}_0 = 0 \quad if \quad k = 0,$$

$$\mathbb{MG}_k = \mathcal{S}_k^{\alpha_2}\left(\mathbb{I}_k - \mathcal{P}_k\,(\mathbb{I}_k - \mathbb{MG}_{k-1})\,(\mathcal{J}_{k-1})^{-1}\mathcal{R}_k\right)\mathcal{J}_k\,\mathcal{S}_k^{\alpha_1} \quad if \quad k = 1, 2, \ldots, \tag{35}$$

$$\begin{aligned}\mathbb{MG}_k &= \mathcal{S}_k^{\alpha_2}\left(\mathbb{I}_k - \mathcal{P}_k\,(\mathbb{I}_k - \mathbb{MG}_{k-1})\,(\mathcal{J}_k^{-1}\mathcal{R}_k\right)\mathcal{J}_k\,\mathcal{S}_k^{\alpha_1}\\ &= \mathbb{TG}_k + \mathcal{S}_k^{\alpha_2}\mathcal{P}_k(\mathbb{MG}_{k-1})\mathcal{J}_k^{-1}\mathcal{R}_k\mathcal{J}_k\mathcal{S}_k^{\alpha_1} \quad if \quad k = 1, 2, \ldots.\end{aligned}$$

4.2 Approximation property

The following approximation property is based on the inequality (15) in Theorem 1 and in Lemma 1. The demonstration of the theorem is adapted from the one in [21], originally presented for the problem of variational inequality.

4.3 The main result

The following theorems give us the convergence of the multigrid matrix of the parabolic quasi variational, in its proof we will rely on the work done by Haiour [21] in the elliptic quasi-variational inequality.

Theorem 4. $\mathbb{X}$ is the iteration matrix of the multigrid, given by

$$\mathbb{X} = \left[\mathcal{J}_k^{-1} - \mathcal{P}_k\, \mathcal{J}_{k-1}^{-1}\mathcal{R}_k\right]. \tag{36}$$

Under the previous assumptions, the matrix $\mathbb{X}$ satisfies the following approximation properties:

$$\|\mathbb{X}\|_\infty \leq C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]. \tag{37}$$

Proof. According to Theorem 1, let $\mathtt{u} \in H_0^1(\Omega)$. We have

$$\|\mathtt{u}_k^n - \mathtt{u}^\infty\|_\infty \leq C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty.$$

Then

$$\|\mathtt{u}_k^\star - \mathtt{u}^\infty\|_\infty \leq C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty.$$

Let $\mathtt{u}_k^\star$ and $\mathtt{u}_{k-1}^\star$ be solutions to the problem (13). Then by applying Theorem 1,

$$\begin{aligned}
\|\mathtt{u}_k^* - \mathtt{u}_{k-1}^*\|_\infty &\leq \|\mathtt{u}_k^* - \mathtt{u}^\infty\|_\infty + \|\mathtt{u}_{k-1}^* - \mathtt{u}^\infty\|_\infty \\
&\leq C_1\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty \\
&\quad + C_2\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty \\
&\leq (C_1 + C_2)\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty. \qquad (38)
\end{aligned}$$

We obtain

$$\|\mathtt{u}_k^* - \mathtt{u}_{k-1}^*\|_\infty \leq C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty, \tag{39}$$

where $C = (C_1 + C_2)$.

We use the Galerkin discretization to obtain

$$\mathtt{b}(r_k\mathtt{u},\ r_k\mathtt{v}) = \langle(\mathbb{B}_k),\ \mathtt{v}\rangle_{L^2(\Omega)}\,;\ \text{for all } \mathtt{u},\mathtt{v}\in\mathbb{U}_k.$$

So

$$(\mathtt{b}(\mathtt{u},\ \mathtt{v}))' = \langle(\mathcal{J}_k),\ \mathtt{v}_k\rangle_{L^2(\Omega)}\,;\ \text{for all } \mathtt{u},\mathtt{v}\in\mathbb{U}_k.$$

Also, we have for all $\mathtt{v}\in\mathbb{V}$

$$(\mathtt{b}((r_k)^{-1}\,(\mathcal{J}_k)^{-1}(F_k),\ \mathtt{v}))' = \left\langle(r^*_{k-1})^{-1}\,\nabla F\ \ \mathtt{v}\right\rangle_{L^2(\Omega)}.$$

Let $\mathtt{u}_k\in\mathbb{V}_k$ and $\mathtt{u}_{k-1}\in\mathbb{V}_{k-1}$. Then

$$\mathtt{b}(\mathtt{u}_k,\ \mathtt{v}) = \left\langle\left(r^*_k\right)^{-1}F,\ \mathtt{v}\right\rangle_{L^2(\Omega)},$$
$$\mathtt{b}(\mathtt{u}_{k-1},\ \mathtt{v}) = \left\langle\left(r^*_{k-1}\right)^{-1}F,\ \mathtt{v}\right\rangle_{L^2(\Omega)},$$

which implies $\mathtt{u}^\star_k = r_k^{-1}\mathcal{J}_k^{-1}\nabla(F)$ and $\mathtt{u}^\star_{k-1} = r_{k-1}^{-1}\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\nabla(F)$. Using (39) and (33) and Lemma 1, we get

$$\|r_k^{-1}\mathcal{J}_k^{-1}\nabla(F) - r_{k-1}^{-1}\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\nabla(F)\|_\infty \le C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty.$$

Then

$$\|\mathcal{J}_k^{-1} - r_k r_{k-1}^{-1}\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\|_\infty\|\nabla F\|_\infty \le C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right]\|\nabla F\|_\infty.$$

The proof is now concluded from

$$\|\mathcal{J}_k^{-1} - \mathcal{P}_k\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\|_\infty \le C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right],$$
$$\|\mathbb{X}\|_\infty \le C\left[h_k^2|\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right].$$

□

4.4 Smoothing property

To establish a smoothing property, we form the matrix $\mathcal{J}_k^v = \mathcal{L}_k - \mathcal{N}_k$, and rely on the subsequent assumptions:
$\mathcal{L}_k$ is considered regular, where

$$\|\mathcal{L}_k^{-1}\, \mathcal{N}_k\|_\infty \leq 1 \qquad \text{for all } k, \tag{40}$$

$$\|\mathcal{L}_k^{-1}\|_\infty \leq \frac{C}{h^2}, \quad \text{for all } k, \text{ with } C \text{ independent } k. \tag{41}$$

We utilized a relaxation method with an iteration matrix as a smoother,

$$\mathcal{S}_k = I_k - \omega \mathcal{J}_k^{-1} \mathcal{N}_k, \qquad \omega \in [0; 1].$$

Theorem 5. [26, 27] Given that the preceding assumptions and notations are met, there exists a constant C, independent of k, such that the following smoothing property holds:

$$\|\mathcal{J}_k\, \mathcal{S}_k^\alpha\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h_k^2}. \tag{42}$$

Following the approximation and smoothing properties, it is essential to establish the subsequent stability bound:

$$\text{There exists } C_s \quad \|\mathcal{S}_k^\alpha\|_\infty \leq C_s, \qquad \text{for all k and } \alpha. \tag{43}$$

The convergence analysis relies on the following decomposition of the iteration matrix for the two-grid method, where $\alpha_2 = 0$:

$$\begin{aligned}\|\mathbb{TG}_k(\alpha_1, 0)\|_\infty &= \|\left[(\mathcal{J}_k)^{-1} - \mathcal{P}_k\, (\mathcal{J}_{k-1})^{-1}\, \mathcal{R}_k\right]\, \mathcal{J}_k\, \mathcal{S}_k^{\alpha_1}\, \|_\infty \\ &\leq \|(\mathcal{J}_k)^{-1} - \mathcal{P}_k\, (\mathcal{J}_{k-1})^{-1}\, \mathcal{R}_k\|_\infty\, \|\mathcal{J}_k\, \mathcal{S}_k^{\alpha_1}\|_\infty.\end{aligned}$$

Now, choosing more than two grids for a hierarchy comprising, we can formulate the matrix $\mathbb{X}$ in (36) by recursively utilizing the matrix $\mathbb{MG}_k$ in (35) for all levels. Assuming the validity of condition (36), convergence outcomes can be easily inferred from the preceding findings.

Theorem 6. [25] Examine a multigrid approach for a given iterative matrix (35). Then, based on the earlier assumption, for the specified parameter value $\alpha_1 = \alpha,\ \ \alpha_2 = 0,\ $ for all $\varepsilon \in [0,1],\ $ there exists $\alpha^* \leq \alpha :$

$$\|\mathbb{MG}_k\|_\infty \leq \varepsilon.$$

Combining the approximation and smoothness properties with (3.5) allows us to utilize identical parameters, as outlined in the subsequent theorem. This theorem constitutes the primary outcome of our study.

Theorem 7. Given the preceding assumptions and notations, the iterated u_k^v, $v > 0$, for two meshes k and $k-1$, adhere to the following relationship: There exists $C > 0$:

$$\|\mathtt{u}_k^{v+1} - \mathtt{u}_k^*\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h_k^2}\left[h_k^2 |\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right] \|\mathtt{u}_k^v - \mathtt{u}_k^*\|_\infty.$$

Proof. We have

$$\begin{aligned}\|\mathtt{u}_k^{v+1} - \mathtt{u}_k^*\|_\infty &= \|(\mathcal{J}_k\, \mathcal{S}_k^\alpha)(I_k - \mathcal{P}_k(I_k - \mathbb{MG}_{k-1})(\mathcal{J}_{k-1}^{-1})\mathcal{R}_k(\mathtt{u}_k^v - \mathtt{u}_k^*)\|_\infty \\ &\leq \|(\mathcal{J}_k\, \mathcal{S}_k^\alpha)\|_\infty \|(I_k - \mathcal{P}_k(I_k - \mathbb{MG}_{k-1})(\mathcal{J}_{k-1}^{-1})\mathcal{R}_k\|_\infty \|\mathtt{u}_k^v - \mathtt{u}_k^*\|_\infty \\ &\leq \frac{C_1 C_2}{\sqrt{\alpha}\, h_k^2}\left[h_k^2 |\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right] \|\mathtt{u}_k^v - \mathtt{u}_k^*\|_\infty \\ &\leq \frac{C}{\sqrt{\alpha}\, h_k^2}\left[h_k^2 |\log(h_k)|^3 + \left(\frac{1+nc}{1+n\alpha}\right)^N\right] \|\mathtt{u}_k^v - \mathtt{u}_k^*\|_\infty,\end{aligned}$$

where $C = C_1 \times C_2$. □

5 Numerical example

Example 1. In this numerical example, we find the solution of the following problem (44) by using multigrid method and Gauss–Siedel method:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au \leq f(u) \ in \ [0,1] \times \Omega, \\ \\ \left\langle \dfrac{\partial u}{\partial t} + Au - f(u)\, ; \mathtt{u} - \psi \right\rangle = 0, \\ \\ u(x,t) = 0, \ \text{ in } [0,1] \times \Gamma, \\ \\ u(x,0) = 0, \ \ \text{in } \Omega, \end{cases} \tag{44}$$

where

$$\Omega = \{(x_1, x_2 | x_1 + x_2 \leq 1)\}, \quad \Delta t = 0.01, \quad A\mathtt{u} = -\Delta \mathtt{u}, \quad f(u) = \cos 2u, \quad \psi = 0.$$

For discretization in space, we have used PDE toolbox in MATLAB (r2018) to generate the mesh, and semi-discretization on time, and start-iterate $u_k^0 = (0, \dots, 0)^t \in R^{1024}$.

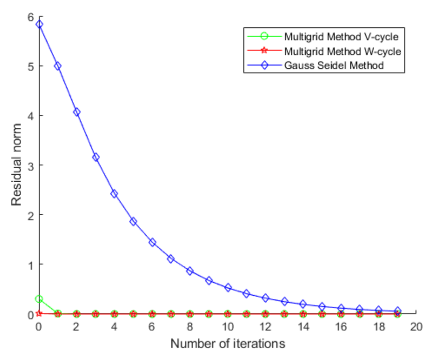


Figure 1: Comparison between the convergence of multigrid method and Gauss–Siedel methods.

We note from Figure 1 that the multigrid method (V-cycle, W-cycle) gives us the solution with the least number of iterations, that is, after one iteration, while the Gauss–Siedel method gives after more than 20 iterations. From here, we conclude that the multigrid method gives us the solution to the sets of large linear equations with the least number of iterations.

Figures 2 and 3 represent the solution to the problem using MATLAB 2018 using the Gauss–Siedel method and the multigrid method (V-cycle and W-cycle).

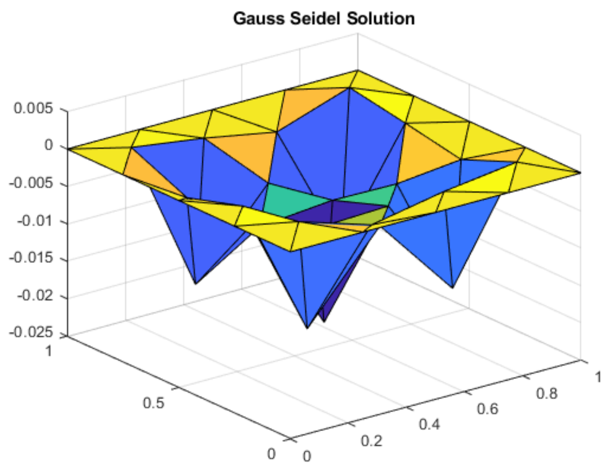
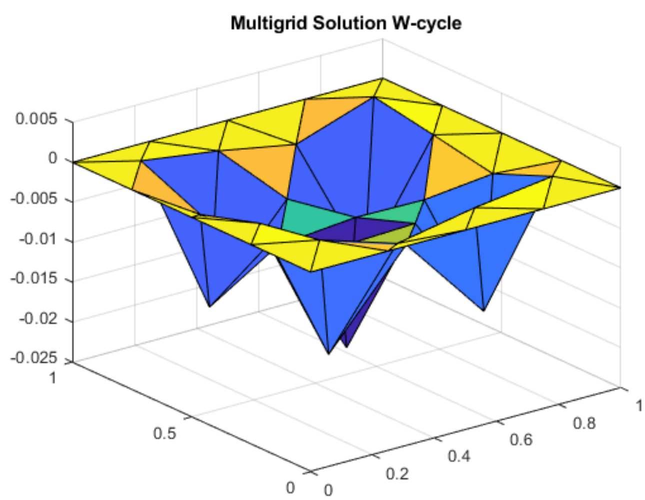


Figure 2: Solution of P.Q.V.I using Gauss–Siedel method (after 20 iterations).



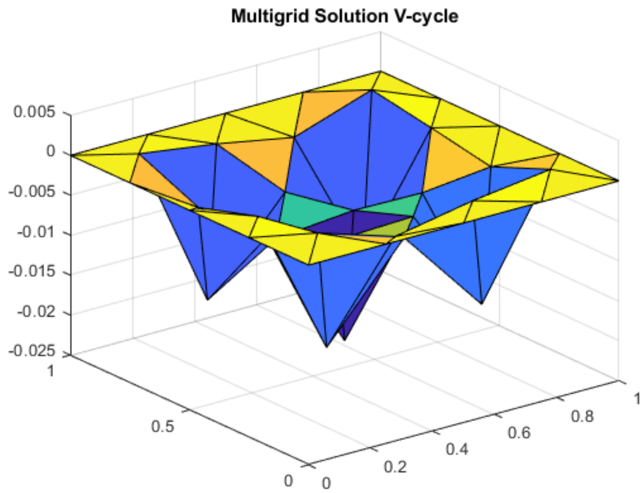


Figure 3: Solution of parabolic quasi-variational inequality after two iterations of multigrid method.

Acknowledgments

Greetings and appreciation are extended to the members of the journal. We also welcome all directions and inquiries submitted by reviewers.

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