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Research Article

A novel integral transform operator and its applications

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Abstract

The proposed study is focused to introduce a novel integral transform operator, called Generalized Bivariate (GB) transform. The proposed transform includes the features of the recently introduced Shehu transform, ARA transform, and Formable transform. It expands the repertoire of existing Laplace-type bivariate transforms. The primary focus of the present work is to elaborate fashionable properties and convolution theorems for the proposed transform operator. The existence, inversion, and duality of the proposed transform have been established with other existing transforms. Implementation of the proposed transform has been demonstrated by applying it to different types of differential and integral equations. It validates the potential and trustworthiness of the GB transform as a mathematical tool. Furthermore, weighted norm inequalities for integral convolutions have been constructed for the proposed transform operator.

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1 Introduction

In real-life applications, the study of dynamic relations between individual components leads to different types of differential equations, integral equations, or integro-differential equations [21, 25, 36]. On account of extensive applications, these models crave for efficient techniques to construct their solutions. Integral transform techniques owing to the contribution of Heaveside to operational techniques have emerged as an alternative and bridge between analytic and numerical techniques in solving linear and nonlinear problems. Integral transform techniques are applicable over a wide class of problems, such as time-dependent boundary conditions, where the technique of separation of variables ceases to work. Even in a scenario of impossible analytic evaluation of transform or inverse transform, a wide variety of numerical and asymptotic techniques are now available for their evaluation [7, 10, 11, 19, 23, 28]. This hybrid mixture of techniques preserves some analytic aspects of the system that serves greater physical insight than a purely numerical procedure.

Integral transforms occur in a natural way by virtue of the principle of linear supposition in composing the integral form of the solution of linear differential equations. Integral transforms are one of the mathematical tools that have proved their worth not only for their theoretical interests but also for their accessible features to solve various problems in different fields of science and engineering. In recent work, the widely investigated subject of integral transforms has gained remarkable significance due to its demonstrated applications over quite challenging fractional operators [5, 15, 16, 30, 31]. The fundamental objective of integral transforms is to take one step forward to an easier form of the given problem. For example, an ordinary differential equation with constant coefficients transforms into an algebraic equation of transformed variable, and a partial differential equation (PDE) reduces to another PDE in one less variable. After the manipulation of the solution in the transformed domain, the inverse transform retracts the solution in the original domain. Different types of integral transforms are effectively utilized to obtain the solution of differential, difference, and integral equations. Indeed, the Fourier [9] and Laplace [29, 34] transforms are mostly applied and have been found to have a wide breadth of applications in mathematics, physics, statistics, and engineering sciences. Each of the existing integral transforms admits its strengths and deficiencies, which stimulates the interest to explore enhanced transforms with the arbitrariness of kernel function. Holding significance for centuries, the Fourier and Laplace transform even served as a generator for innumerable Laplace-type transforms with the imposition of specific conditions. The renowned Sumudu transform was introduced in the early 1990s by Watugala [33]. The natural transform [20] was devised in 2008. In 2011, Elzaki [12] framed a new integral transform known as Elzaki transform. In 2013, Atangana and Kilicman [6] established novel transforms for differential equations consisting of some kind of singularities. In recent years, the substantial interest of researchers resulted in many worth mentioning integral transforms, namely, Ramadan Group transform [24], Polynomial transform [8], Yang transform [35], Aboodh transform [1], Mohand transform [2], Rangaing transform [13], Sawi transform [18], HY-transform [4], Shehu transform [22], J-transform [37], ARA transform [27], Formable transform [26], and so on.

In the present work, a new integral transform operator, "Generalized Bivariate transform", has been proclaimed as a generalization of the recently introduced Shehu, ARA, and Formable transforms. Its harmony in the class of Laplace-type transforms marks it as a prime member with inherited advantages of allied integral transforms. By proving fashionable properties along with application to various differential and integral equations, this work is further enhanced by the construction of the weighted norm inequalities for integral convolutions using the proposed transform operator.

2 Formulation of the GB transform

The GB transform of order n of a function $f(\eta)$ is a semi-infinite convergent integral. It can be defined as

$$\mathbb{A}_n[f(\eta)] = \mathcal{P}_n(s,\gamma) = \frac{s}{\gamma^n}\int_0^\infty \eta^{n-1}\exp\left(\frac{-s}{\gamma}\eta\right)f(\eta)\,d\eta, \quad \gamma, s > 0. \tag{1}$$

Equation (1) is equivalent to

$$\mathbb{A}_n[f(\eta)] = \mathcal{P}_n(s,\gamma) = s\int_0^\infty \eta^{n-1}\exp\left(-s\eta\right)f(\gamma\eta)\,d\eta, \quad \gamma, s > 0, \tag{2}$$

over the set of functions

$$\mathcal{F} = \left\{ f(\eta)\colon \text{ there exist } N \in (0,\infty), \eta_i > 0 \text{ for } i = 1,2; |f(\eta)| < N\exp\left(\frac{|\eta|}{\eta_i}\right), \right.$$
$$\left. \text{if } \eta \in (-1)^i \times [0,\infty) \right\},$$

where s and γ are the variables of the GB transform.

In integral transforms theory, the recovery of a function from its transformed version is a more sophisticated subject than the evaluation of the transform itself, which is referred to as the inversion problem. For a given function, three questions arise at once: Does its inverse transform exist? is it unique? and how to find it? The uniqueness of Laplace-type transforms is determined by integration theory, which implies that a given function holds a unique continuous inverse transform. Moreover, in many cases, finding the inverse results in another transform with a different kernel function.

The inversion of the GB transform is given by

$$\begin{aligned} f(\eta) &= \mathbb{A}_{n+1}^{-1}\left[\mathbb{A}_{n+1}[f(\eta)]\right] \\ &= \frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\gamma}e^{\frac{s}{\gamma}\eta}\left[(-1)^n\left(\frac{1}{s\Gamma(n-1)}\int_0^s (s-x)^{n-1}\right.\right. \\ &\qquad \left.\left.\mathbb{A}_{n+1}[f(\eta)](x,\gamma)\,dx + \sum_{k=0}^{n-1}\frac{s^k}{k!}\frac{\partial^k \$(0,\gamma)}{\partial s^k}\right)\right]ds, \end{aligned} \tag{3}$$

where

$$\$(s,\gamma) = \int_0^\infty e^{\frac{s}{\gamma}\eta} f(\eta)\,d\eta$$

is the Shehu transform, which is $(n-1)$ times differentiable.

Proof. From the definition of the GB transform, we have

$$\mathbb{A}_{n+1}[f(\eta)](s,\gamma) = \frac{s}{\gamma^n}\int_0^\infty \eta^n e^{\frac{s}{\gamma}\eta} f(\eta)\,d\eta = s(-1)^n\frac{\partial \$(s,\gamma)}{\partial s^n}.$$

Thus

$$\frac{1}{s\Gamma(n-1)}\int_0^s (s-\eta)^{n-1}\mathcal{P}_{n+1}(\eta,\gamma)\,d\eta = (-1)^n\left(\$(s,\gamma) - \sum_{k=0}^{n-1}\frac{s^k}{k!}\$(0,\gamma)\right).$$

Therefore,

$$\frac{(-1)^n}{s\Gamma(n-1)}\int_0^s (s-\eta)^{n-1}\mathcal{P}_{n+1}(\eta,\gamma)\,d\eta + \sum_{k=0}^{n-1}\frac{s^k}{k!}\$(0,\gamma) = \$(s,\gamma).$$

It follows that

$$\begin{aligned} &\frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\gamma}e^{\frac{s}{\gamma}\eta}\left[(-1)^n\left(\frac{1}{s\Gamma(n-1)}\int_0^s (s-x)^{n-1}\mathcal{P}_{n+1}[f(\eta)](x,\gamma)\,dx + \sum_{k=0}^{n-1}\frac{s^k}{k!}\frac{\partial^k \$(0,\gamma)}{\partial s^k}\right)\right]ds \\ &= \frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\gamma}e^{\frac{s}{\gamma}\eta}\left[(-1)^n \$(s,\gamma)\right]ds \\ &= \frac{(-1)^{2n}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\gamma}e^{\frac{s}{\gamma}\eta}\$(s,\gamma)\Big]\,ds. \end{aligned}$$

Hence,

$$\mathbb{A}_{n+1}^{-1}\left[\mathbb{A}_{n+1}[f(\eta)]\right] = \frac{(-1)^{2n}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\gamma}e^{\frac{s}{\gamma}\eta}\$(s,\gamma)\Big]\,ds = f(\eta).$$

□

Theorem 1 (Sufficient condition for the existence of the GB transform)**.** If the function $f(\eta)$ is piecewise continuous on every finite interval $0<\eta<\xi$ and satisfies

$$|\eta^{n-1}f(\eta)|\leq Ke^{\beta\eta}, \tag{4}$$

then the GB transform exists for all $\frac{s}{\gamma}>\beta$.

Proof. Let ξ be any positive number. This will give

$$\frac{s}{\gamma^n}\int_0^\infty \eta^{n-1}e^{\frac{-s}{\gamma}\eta}f(\eta)d\eta=\frac{s}{\gamma^n}\int_0^\xi \eta^{n-1}e^{\frac{-s}{\gamma}\eta}f(\eta)d\eta+\frac{s}{\gamma^n}\int_\xi^\infty \eta^{n-1}e^{\frac{-s}{\gamma}\eta}f(\eta)d\eta.$$

Since the function is continuous on finite intervals, the first integral on the right-hand side exists. Also, by the hypothesis in (4), the latter integral on the right-hand side converges

$$\begin{aligned}\left|\frac{s}{\gamma^n}\int_\xi^\infty \eta^{n-1}e^{\frac{s}{\gamma}\eta}f(\eta)\,\eta\right| &\leq \frac{s}{\gamma^n}\int_\xi^\infty e^{\frac{-s}{\gamma}\eta}Ke^{\beta\eta}\,d\eta\\ &=\lim_{\alpha\to\infty}-\frac{sK}{\gamma^n}\frac{e^{-\xi(\frac{s}{\gamma}-\beta)}}{(\frac{s}{\gamma}-\beta)}\Bigg|_0^\alpha=\frac{sK}{\gamma^{n-1}(\frac{s}{\gamma}-\beta)}.\end{aligned}$$

Thus the GB transform $\mathbb{A}_n[f(\eta)]$ exists for all $\frac{s}{\gamma}>\beta$. □

There are many functions for which most of variants of the Laplace transform do not exist. The GB transform expands the repertoire of Laplace-type transforms by its applicability over the following functions:

$$\mathbb{A}_2\left[\frac{1}{\eta}\right]=\frac{s}{\gamma^2}\int_0^\infty \eta^{2-1}e^{\frac{-s}{\gamma}\eta}\frac{1}{\eta}\,d\eta=\frac{1}{\gamma},$$

$$\begin{aligned}&\mathbb{A}_2\left[2e^{\eta^2}\cos e^{\eta^2}\right]\\ &\quad=\frac{s}{\gamma^2}\int_0^\infty 2\eta e^{\frac{-s}{\gamma}\eta}e^{\eta^2}\cos\left(e^{\eta^2}\right)d\eta\\ &\quad=\frac{s}{\gamma^2}\left[e^{\frac{-s}{\gamma}\eta}\sin\left(e^{\eta^2}\right)\Big|_0^\infty+\frac{s}{\gamma}\int_0^\infty e^{\frac{-s}{\gamma}\eta}\sin\left(e^{\eta^2}\right)d\eta\right]\quad\text{(Integration by parts)}\\ &\quad=\frac{s}{\gamma^2}\left[-\sin(1)+\mathbb{A}_1[\sin\left(e^{\eta^2}\right)]\right]. \end{aligned}\tag{5}$$

and the latter GB transform exists by Theorem 1. Similarly, $\mathbb{A}_2\left[2e^{\eta^2}\sin e^{\eta^2}\right]$ can be obtained as per (5).

Theorem 2 (Uniqueness of the GB transform)**.** Let $f(\eta)$ and $g(\eta)$ be the continuous functions, defined for $\eta\geq 0$ and having the GB transform for order n, $\mathcal{P}_n(s,\gamma)$, and $\mathcal{Q}_n(s,\gamma)$, respectively. If $\mathcal{P}_n(s,\gamma) = \mathcal{Q}_n(s,\gamma)$, then $f(\eta) = g(\eta)$.

Proof. From the definition of the GB transform of order n, if c is sufficiently large, then the integral expression can be obtained as

$$f(\eta) = \frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{u} e^{\frac{s}{\gamma}\eta}[(-1)^n \mathcal{P}_n(s,\gamma)]\, ds.$$

By hypothesis, $\mathcal{P}_n(s,\gamma) = \mathcal{Q}_n(s,\gamma)$ and

$$f(\eta) = \frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{u} e^{\frac{s}{\gamma}\eta}[(-1)^n \mathcal{Q}_n(s,\gamma)]\, ds = g(\eta).$$

□

3 Dualities between the GB transform and some integral transforms

In this section, the associative nature of the GB transform with other well-known transforms is illustrated. This association of the GB transform enhances to exploration of other transforms simultaneously under the study of the GB transform.

- GB-ARA duality:

$$\begin{aligned}\mathcal{P}_n(s,\gamma) &= \frac{1}{\gamma^{n-1}} G\left(n, \frac{s}{\gamma}\right), \\ \mathcal{P}_n(s,1) &= G\left(n,s\right),\end{aligned} \tag{6}$$

  where [27]

$$\mathcal{G}_n[f(\eta)](s) = G\left(n,s\right) = s\int_0^\infty \eta^{n-1} e^{-s\eta} f(\eta)\, d\eta.$$

- GB-Formable duality:

$$\begin{aligned}\mathcal{P}_n(s,\gamma) &= \frac{1}{\gamma^{n-1}} \mathcal{R}[\eta^{n-1} f(\eta)], \\ \mathcal{P}_1(s,\gamma) &= B(s,\gamma),\end{aligned} \tag{7}$$

  where [26]

$$\mathcal{R}[f(\eta)] = B(s,\gamma) = \frac{s}{\gamma}\int_0^\infty e^{-\frac{s}{\gamma}\eta} f(\eta)\, d\eta.$$

- GB-Shehu duality:

$$\mathcal{P}_n(s,\gamma) = \frac{s}{\gamma^n}\mathbb{S}[\eta^{n-1}f(\eta)], \tag{8}$$
$$\mathcal{P}_1(s,\gamma) = \frac{s}{\gamma}V(s,\gamma),$$

where [22]

$$\mathbb{S}[f(\eta)] = V(s,\gamma) = \int_0^\infty e^{-\frac{s}{\gamma}\eta} f(\eta)\, d\eta.$$

- GB-Natural duality:

$$\mathcal{P}_n(s,\gamma) = \frac{s}{\gamma^{n-1}}\mathcal{N}[\eta^{n-1}f(\eta)], \tag{9}$$
$$\mathcal{P}_1(s,\gamma) = sR(s,\gamma),$$

where [20]

$$\mathcal{N}[f(\eta)] = R(s,\gamma) = \frac{1}{\gamma}\int_0^\infty e^{-\frac{s}{\gamma}\eta} f(\eta)\, d\eta.$$

- GB-Aboodh duality:

$$\mathcal{P}_n(s,\gamma) = \frac{s^2}{\gamma^{n+1}}\mathcal{A}[\eta^{n-1}f(\eta)]\left(\frac{s}{\gamma}\right), \tag{10}$$
$$\mathcal{P}_1(s,1) = s^2\mathcal{A}[f(\eta)](s),$$

where [1]

$$\mathcal{A}[f(\eta)](s) = \frac{1}{s}\int_0^\infty e^{-s\eta} f(\eta)\, d\eta.$$

- GB-J duality:

$$\mathcal{P}_n(s,\gamma) = \frac{s}{\gamma^{n+1}}\mathcal{J}[\eta^{n-1}f(\eta)], \tag{11}$$
$$\mathcal{P}_1(s,\gamma) = \frac{s}{\gamma^2}J(s,\gamma),$$

where [37]

$$\mathcal{J}[f(\eta)] = J(s,\gamma) = u\int_0^\infty e^{\frac{-s}{\gamma}\eta} f(\eta)\, d\eta.$$

- GB-Laplace Carson duality:

$$\mathcal{P}_n(s,\gamma) = \frac{1}{\gamma^{n-1}}\mathcal{L}_*[\eta^{n-1}f(\eta)]\left(\frac{s}{\gamma}\right), \tag{12}$$
$$\mathcal{P}_1(s,1) = \mathcal{L}_*[f(\eta)](s),$$

where [3]

$$\mathcal{L}_*[f(\eta)](s) = s\int_0^\infty e^{-s\eta} f(\eta)\, d\eta.$$

- GB-Elzaki duality:

$$\mathcal{P}_n(s,\gamma) = \frac{s^2}{\gamma^{n+1}}\mathcal{E}[\eta^{n-1}f(\eta)]\left(\frac{\gamma}{s}\right), \tag{13}$$
$$\mathcal{P}_1(1,\gamma) = \frac{1}{\gamma^2}\mathcal{E}[f(\eta)](\gamma),$$

where [12]

$$\mathcal{E}[f(\eta)](\gamma) = \gamma\int_0^\infty e^{\frac{-\eta}{\gamma}} f(\eta)\, d\eta.$$

- GB-Sumudu duality:

$$\mathcal{P}_n(s,\gamma) = \frac{1}{\gamma^{n-1}}\mathcal{S}[\eta^{n-1}f(\eta)]\left(\frac{\gamma}{s}\right), \tag{14}$$
$$\mathcal{P}_1(1,\gamma) = \mathcal{S}[f(\eta)](\gamma),$$

where [33]

$$\mathcal{S}[f(\eta)](\gamma) = \frac{1}{\gamma}\int_0^\infty e^{\frac{-\eta}{\gamma}} f(\eta)\, d\eta.$$

4 Properties of the GB transform

In this section, some basic properties such as the linearity property, the shifting in domains, the derivative property, and the convolution property are presented, which enable us to determine the GB transform in applications.

Property 1 (Linearity property)**.** Suppose that $f(\eta)$ and $g(\eta)$ are two functions for which the GB transform exists. Then

$$\mathbb{A}_n\left[\alpha f(\eta) + \beta g(\eta)\right] = \alpha\mathbb{A}_n\left[f(\eta)\right] + \beta\mathbb{A}_n\left[g(\eta)\right], \tag{15}$$

where α and β are nonzero arbitrary constants.

Property 2 (Change of scale)**.** Suppose that the GB transform exists for the given function $f(\alpha\eta)$. Then

$$\mathbb{A}_n\left[f(\alpha\eta)\right] = \mathcal{P}_n(s,\alpha\gamma), \tag{16}$$

where α is an arbitrary constant.

Property 3 (Shifting in s-domain)**.**

$$\mathbb{A}_n[e^{-\alpha\eta}f(\eta)] = \frac{s}{s+\gamma\alpha}\mathcal{P}_n(s+\gamma\alpha,\gamma), \tag{17}$$

where α is an arbitrary constant.

Property 4 (Shifting in η-domain)**.**

$$\mathbb{A}_n[u_\alpha f(\eta-\alpha)] = \frac{e^{\frac{-s}{\gamma}\alpha}}{\gamma^{n-1}} \mathbb{A}_1[(\eta+\alpha)^{n-1} f(\eta)], \tag{18}$$

where u_α is the unit function and α is an arbitrary constant.

Property 5 (Shifting in n-domain)**.**

$$\mathbb{A}_n[\eta^m f(\eta)] = \gamma^m \, \mathbb{A}_{n+m}[f(\eta)], \tag{19}$$

where $m \geq 0$ or $n-1 \geq m$.

The proofs of Properties 1–5 can be easily proved by usual calculus.

Property 6 (GB transform for derivatives)**.** Suppose that $f(\eta), f'(\eta), \ldots,$ $f^{m-1}(\eta)$ are continuous and of exponential order on $[0,\infty)$ while $f^m(\eta)$ is piecewise continuous on $[0,\infty)$. Then

$$\mathbb{A}_n[f^m(\eta)] = (-1)^{n-1} \frac{s}{\gamma} \frac{\partial^{n-1}}{\partial s^{n-1}} \left[\left(\frac{s}{\gamma}\right)^{m-1} \mathbb{A}_1[f(\eta)] - \sum_{k=0}^{m-1} \left(\frac{s}{\gamma}\right)^{m-(k+1)} f^k(0) \right]. \tag{20}$$

Proof. We have

$$\begin{aligned}
&\mathbb{A}_n[f^m(\eta)] \\
&\quad= \frac{s}{\gamma^n} \int_0^\infty \eta^{n-1} e^{\frac{-s}{\gamma}\eta} f^m(\eta)\, d\eta \\
&\quad= \frac{1}{\gamma^{n-1}} \mathbb{A}_1[\eta^{n-1} f^m(\eta)] \\
&\quad= \frac{1}{\gamma^{n-1}} \mathcal{R}[\eta^{n-1} f^m(\eta)] \quad \text{(Duality between GB and Formable transforms)} \\
&\quad= (-1)^{n-1} s \frac{\partial^{n-1}}{\partial s^{n-1}} \left[\frac{\mathcal{R}[f^m(\eta)]}{s} \right] \\
&\quad= (-1)^{n-1} s \frac{\partial^{n-1}}{\partial s^{n-1}} \left[\frac{s^{m-1}}{\gamma^{m-1}} \mathcal{R}[f(\eta)] - \sum_{k=0}^{m-1} \frac{s^{m-(k+1)}}{\gamma^{m-(k+1)}} f^k(0) \right] \\
&\quad= (-1)^{n-1} \frac{s}{\gamma} \frac{\partial^{n-1}}{\partial s^{n-1}} \left[\left(\frac{s}{\gamma}\right)^{m-1} \mathbb{A}_1[f(\eta)] - \sum_{k=0}^{m-1} \frac{s^{m-(k+1)}}{\gamma^{m-(k+1)}} f^k(0) \right]. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Using the GB-Formable duality)}
\end{aligned}$$

□

Property 7 (GB transform of the convolution)**.** Suppose that $\mathcal{P}_1(s,\gamma)$ and $\mathcal{Q}_1(s,\gamma)$ are the GB transform of order one of the functions $f(\eta)$ and $g(\eta)$, respectively. Then

$$\mathbb{A}_n[f(\eta)*g(\eta)] = (-1)^{n-1}s\gamma\sum_{r=0}^{n-1}C_r^{n-1}\frac{\partial^{n-1-r}}{\partial s^{n-1-r}}\left(\frac{1}{s^2}\right) \sum_{k=0}^{r}C_k^r\frac{\partial^{r-k}}{\partial s^{r-k}}\mathcal{P}_1(s,\gamma).\frac{\partial^k}{\partial s^k}\mathcal{Q}_1(s,\gamma), \tag{21}$$

where the convolution (i.e., $f(\eta)*g(\eta)$) of the functions $f(\eta)$ and $g(\eta)$ is given by the integral

$$f(\eta)*g(\eta) = \int_0^\eta f(\eta)g(\eta-\zeta)\,d\zeta.$$

Proof. We have

$$\begin{aligned}
&\mathbb{A}_n[f(\eta)*g(\eta)]\\
&\quad= \frac{1}{\gamma^{n-1}}\mathbb{A}_1[\eta^{n-1}(f(\eta)*g(\eta))]\\
&\quad= (-1)^{n-1}s\gamma\frac{\partial^{n-1}}{\partial s^{n-1}}\left[\frac{\mathcal{P}_1(s,\gamma).\mathcal{Q}_1(s,\gamma)}{s^2}\right] \quad \text{(Using the GB-Formable duality)}\\
&\quad= (-1)^{n-1}s\gamma\sum_{r=0}^{n-1}C_r^{n-1}\frac{\partial^{n-1-r}}{\partial s^{n-1-r}}\left(\frac{1}{s^2}\right)\frac{\partial^r}{\partial s^r}\left(\mathcal{P}_1(s,\gamma).\mathcal{Q}_1(s,\gamma)\right)\\
&\quad= (-1)^{n-1}s\gamma\sum_{r=0}^{n-1}C_r^{n-1}\frac{\partial^{n-1-r}}{\partial s^{n-1-r}}\left(\frac{1}{s^2}\right)\sum_{k=0}^{r}C_k^r\frac{\partial^{r-k}}{\partial s^{r-k}}\mathcal{P}_1(s,\gamma).\frac{\partial^k}{\partial s^k}\mathcal{Q}_1(s,\gamma)
\end{aligned}$$

where C_i^j is the binomial coefficient. □

Now, computational simplicity of the GB transform is presented by its evaluation for some elementary functions.

Example 1. Consider

$$\mathbb{A}_n[\eta^m e^{\alpha\eta}] = (m+n-1)!\frac{s\gamma^m}{(s-\alpha\gamma)^{m+n}}, \tag{22}$$

where $m \geq 0$ or $n-1 \geq m$ and α is an arbitrary constant.

Proof. We have

$$\begin{aligned}
\mathbb{A}_{n+1}[\eta^m e^{\alpha\eta}] &= \frac{s}{\gamma^{n+1}}\int_0^\infty \eta^{m+n}e^{-\left(\frac{s-\alpha\gamma}{\gamma}\right)\eta}\,d\eta\\
&= \frac{(m+n)}{\gamma^n}\frac{s}{s-\alpha\gamma}\int_0^\infty \eta^{m+n-1}e^{-\left(\frac{s-\alpha\gamma}{\gamma}\right)\eta}\,d\eta
\end{aligned}$$

$$= \frac{(m+n)(m+n-1)}{\gamma^n}\frac{s\gamma}{(s-\alpha\gamma)^2}\int_0^\infty \eta^{m+n-2}e^{-\left(\frac{s-\alpha\gamma}{\gamma}\right)\eta}\,d\eta$$

$$\vdots$$

$$= \frac{(m+n)!}{\gamma^n}\frac{s\gamma^{m+n}}{(s-\alpha\gamma)^{m+n+1}} = (m+n)!\frac{s\gamma^m}{(s-\alpha\gamma)^{m+n+1}}.$$

$\square$

Example 2. Consider

$$\mathbb{A}_n[\sin\alpha\eta] = \frac{s(n-1)!}{2i}\left[\frac{(s+i\alpha\gamma)^n-(s-i\alpha\gamma)^n}{(s^2+\alpha^2\gamma^2)^n}\right], \tag{23}$$

$$\mathbb{A}_n[\cos\alpha\eta] = \frac{s(n-1)!}{2}\left[\frac{(s+i\alpha\gamma)^n+(s-i\alpha\gamma)^n}{(s^2+\alpha^2\gamma^2)^n}\right], \tag{24}$$

$$\mathbb{A}_n[\sinh\alpha\eta] = \frac{s(n-1)!}{2}\left[\frac{(s+\alpha\gamma)^n-(s-\alpha\gamma)^n}{(s^2-\alpha^2\gamma^2)^n}\right], \tag{25}$$

$$\mathbb{A}_n[\cosh\alpha\eta] = \frac{s(n-1)!}{2}\left[\frac{(s+\alpha\gamma)^n+(s-\alpha\gamma)^n}{(s^2-\alpha^2\gamma^2)^n}\right], \tag{26}$$

where α is an arbitrary constant.

Now, using the linearity property given in 1 and basic calculus, above results can be proved easily.

Further applications of the GB transform over some elementary and special functions are given in Table 1.

5 Weighted norm inequalities for integral convolution of the GB transform

Theorem 3. Suppose that $f(\eta)$ and $g(\eta)$ are complex valued continuous functions on $[0,\infty)$ such that the GB transforms

$$\mathcal{P}_\alpha(s,\gamma) = \mathbb{A}_\alpha\left[\frac{f(\eta)}{\Gamma\alpha}\right] \quad and \quad \mathcal{Q}_\beta(s,\gamma) = \mathbb{A}_\beta\left[\frac{g(\eta)}{\Gamma\beta}\right] \tag{27}$$

exist for some $\alpha,\beta > 0$ and $\frac{s}{\gamma} > s_0 \geq 0$.

Then, for any arbitrary $\lambda > 0, p > 1$ $\left(\frac{1}{p}+\frac{1}{q}=1\right), \delta \in [0, min(p,q)]$, and $\xi \in [0,\infty)$, the following inequality holds [14]:

Table 1: GB transform of some special functions

$f(\eta)$	GB transform
1	$\Gamma(n)s^{1-n}$
η	$\Gamma(n+1)\frac{\gamma}{s^n}$
η^m	$\Gamma(m+n)\frac{\gamma^m}{s^{n+m-1}}$
$e^{\alpha\eta}$	$\Gamma(n)\frac{s}{(s-\gamma\alpha)^n}$
$e^{-\alpha\eta}$	$\Gamma(n)\frac{s}{(s+\gamma\alpha)^n}$
$\eta^m e^{\alpha\eta}$	$\Gamma(m+n)\frac{s\gamma^m}{(s-\gamma\alpha)^{m+n}}$
$\sin(\alpha\eta)$	$\left(1+\frac{(\alpha\gamma)^2}{s^2}\right)^{\frac{n}{2}} s^{1-n}\Gamma(n)\sin\left(n\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)$
$\cos(\alpha\eta)$	$\left(1+\frac{(\alpha\gamma)^2}{s^2}\right)^{\frac{n}{2}} s^{1-n}\Gamma(n)\cos\left(n\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)$
$\eta\sin(\alpha\eta)$	$\Gamma(n+1)\frac{\gamma s}{(s^2+\alpha^2\gamma^2)^{\frac{1}{2}(n+1)}}\sin\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)$
$\eta\cos(\alpha\eta)$	$\Gamma(n+1)\frac{\gamma s}{(s^2+\alpha^2\gamma^2)^{\frac{1}{2}(n+1)}}\cos\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)$
$\sin(\alpha\eta)-\alpha\eta\cos(\alpha\eta)$	$\Gamma(n)\frac{\alpha s}{(s^2+\alpha\gamma)^{\frac{1}{2}(n+1)}}\left[-\alpha n\cos\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)+\frac{1}{\gamma}\sqrt{s^2+\alpha^2\gamma^2}\sin\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)\right]$
$\sin(\alpha\eta)+\alpha\eta\cos(\alpha\eta)$	$\Gamma(n)\frac{\alpha s}{(s^2+\alpha\gamma)^{\frac{1}{2}(n+1)}}\left[\alpha n\cos\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)+\frac{1}{\gamma}\sqrt{s^2+\alpha^2\gamma^2}\sin\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)\right]$
$\cos(\alpha\eta)-\alpha\eta\sin(\alpha\eta)$	$\Gamma(n)\frac{\alpha s}{(s^2+\alpha\gamma)^{\frac{1}{2}(n+1)}}\left[\frac{s}{\gamma}\cos\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)+\alpha(n-1)\sin\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)\right]$
$\cos(\alpha\eta)+\alpha\eta\sin(\alpha\eta)$	$\Gamma(n)\frac{\alpha s}{(s^2+\alpha\gamma)^{\frac{1}{2}(n+1)}}\left[\frac{s}{\gamma}\cos\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)+\alpha(n+1)\sin\left((1+n)\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)\right]$
$\sin(\alpha\eta+\beta)$	$\frac{s}{(s^2+\alpha^2\gamma^2)^{\frac{n}{2}}}\Gamma(n)\sin\left(\beta+n\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)$
$\cos(\alpha\eta+\beta)$	$\frac{s}{(s^2+\alpha^2\gamma^2)^{\frac{n}{2}}}\Gamma(n)\cos\left(\beta+n\tan^{-1}\left(\frac{\alpha\gamma}{s}\right)\right)$
$e^{\alpha\eta}\sin(\beta\eta)$	$\Gamma(n)\frac{n}{(s-\alpha\gamma)^n}\left(1+\frac{\beta^2\gamma^2}{(\alpha\gamma-s)^2}\right)^{\frac{-n}{2}}\sin\left(n\tan^{-1}\left(\frac{\beta\gamma}{s-\alpha\gamma}\right)\right)$
$e^{\alpha\eta}\cos(\beta\eta)$	$\Gamma(n)\frac{n}{(s-\alpha\gamma)^n}\left(1+\frac{\beta^2\gamma^2}{(\alpha\gamma-s)^2}\right)^{\frac{-n}{2}}\cos\left(n\tan^{-1}\left(\frac{\beta\gamma}{s-\alpha\gamma}\right)\right)$
$\sinh(\alpha\eta)$	$\frac{1}{2}\frac{s\Gamma(n)}{(s^2-\alpha^2\gamma^2)^n}\left[-(s-\gamma\alpha)^n+(s+\gamma\alpha)^n\right]$
$\cosh(\alpha\eta)$	$\frac{1}{2}\frac{s\Gamma(n)}{(s^2-\alpha^2\gamma^2)^n}\left[(s-\gamma\lvert\alpha\rvert)^n+(s+\gamma\lvert\alpha\rvert)^n\right]$
$e^{\alpha\eta}\sinh(\beta\eta)$	$\Gamma(n)\frac{n}{(s-\alpha\gamma)^n}\left(1-\frac{\beta^2\gamma^2}{(\alpha\gamma-s)^2}\right)^{-n}\left[-\left(1+\frac{\beta\gamma}{\alpha\gamma-s}\right)^n+\left(1-\frac{\beta\gamma}{\alpha\gamma-s}\right)^n\right]$
$e^{\alpha\eta}\cosh(\beta\eta)$	$\Gamma(n)\frac{n}{(s-\alpha\gamma)^n}\left[\left(1+\frac{\beta\gamma}{\alpha\gamma-s}\right)^{-n}+\left(1-\frac{\beta\gamma}{\alpha\gamma-s}\right)^{-n}\right]$
$\delta(\eta-\alpha)$	$\alpha^{n-1}\frac{s}{\gamma^n}\exp\left(\alpha\frac{s}{\gamma}\right)$
$J_0(\alpha\eta)$	$(-1)^{n-1}s\frac{\partial^{n-1}}{\partial s^{n-1}}\left[\frac{1}{\sqrt{s^2-\alpha^2\gamma^2}}\right]$
$\mathcal{U}(\eta-\alpha)$	$\sum_{k=0}^{n-1}\frac{1}{s^k}\left(\frac{\alpha}{\gamma}\right)^{n-k-1}\exp\left(-\frac{\alpha}{\gamma}s\right)$

$$\left|\left|h(\xi\eta)\right|\right|_{[\delta;\,\alpha+\beta,\,\lambda]} \leq \left|\left|f(\xi\eta)\right|\right|_{[p;\,\alpha,\,\beta+\lambda]}\left|\left|g(\xi\eta)\right|\right|_{[q;\,\beta,\,\alpha+\lambda]}, \tag{28}$$

where

$$||(\cdot)||_{[p,\,\alpha,\,\beta]} = \left[\int_0^1 \eta^{\alpha-1}(1-\eta)^{\beta-1}|(\cdot)|^p \frac{d\eta}{B(\alpha,\beta)}\right]^{\frac{1}{p}} \tag{29}$$

and $h(\eta)$, $0 \leq \eta < \infty$, is a continuous solution to the integral equation:

$$\mathbb{A}_{\alpha+\beta}\left[\frac{h(\eta)}{\Gamma(\alpha+\beta)}\right] = \frac{1}{s}\,\mathcal{P}_\alpha(s,\gamma)\,\mathcal{Q}_\beta(s,\gamma). \tag{30}$$

Equivalently, $h(\eta)$ is such function that $\eta^{\alpha+\beta-1}h(\eta)/\Gamma(\alpha+\beta)$ is the convolution of the functions $\eta^{\alpha-1}f(\eta)/\Gamma\alpha$ and $\eta^{\beta-1}g(\eta)/\Gamma\beta$.

Proof. Consider a continuous function $h(\eta)$, such as

$$h(\eta) = \langle f(\eta\delta)g(\eta(1-\delta))\rangle_{(\alpha,\,\beta)}, \qquad \eta \in [0,\infty]. \tag{31}$$

Indeed, the left-hand side of (30) with this function $h(\eta)$ and $\frac{s}{\gamma} > s_0$ yields

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{s}{\gamma^{\alpha+\beta}}\int_0^\infty \eta^{\alpha+\beta-1}e^{\frac{s}{\gamma}\eta}\int_0^1 \delta^{\alpha-1}(1-\delta)^{\beta-1}f(\eta\delta)g(\eta(1-\delta))\,d\delta\,d\eta. \tag{32}$$

Substituting $\nu = \eta\delta \in [0,\eta]$, the change of the order of integration assembles (32) as □

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{s}{\gamma^{\alpha+\beta}}\int_0^\infty \nu^{\alpha-1}f(\nu)\int_\nu^\infty e^{\frac{s}{\gamma}\eta}(\eta-\nu)^{\beta-1}g(\eta-\nu)\,d\eta\,d\nu. \tag{33}$$

Another substitution $\mu = \eta - \nu \in [0,\infty)$ reformulates (33) as

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{s}{\gamma^{\alpha+\beta}}\int_0^\infty \nu^{\alpha-1}f(\nu)\int_\nu^\infty e^{-(\mu+\nu)\frac{s}{\gamma}}(\mu)^{\beta-1}g(\mu)\,d\mu\,d\nu\\ &\quad= \frac{1}{s}\mathcal{P}_\alpha(s,\gamma)\,\mathcal{Q}_\beta(s,\gamma),\end{aligned} \tag{34}$$

where $\mathcal{P}_\alpha(s,\gamma)$ and $\mathcal{Q}_\beta(s,\gamma)$ are defined in (27).

6 Applications of the GB transform

In this section, the application of the GB transform is demonstrated for the purpose to solve various Lane–Emden type differential equations, wave-like partial differential equations, and convolution-type integral equations. The success of the newly proposed transform with simplified computation suggests its further implementation to physical problems in sciences and engineering.

Problem 1. Consider the linear Lane–Emden differential equation as

$$u''(\eta) + \frac{2}{\eta}u'(\eta) + u(\eta) = 0, \tag{35}$$

subject to initial conditions

$$u(0) = 1, \quad u'(0) = 0. \tag{36}$$

Solution 1. Application of $\mathbb{A}_2$ on the both sides of (35) implies

$$\begin{aligned}&-\frac{s^2}{\gamma^2}\mathbb{A}'_1[u(\eta)] - \frac{s}{\gamma^2}\mathbb{A}'_1[u(\eta)] + \frac{s}{\gamma^2}u(0) + \frac{1}{\gamma}\left[\frac{s}{\gamma}\mathbb{A}_1[u(\eta) - \frac{s}{\gamma}u(0)]\right]\\ &\quad+ \frac{\mathbb{A}_1[u(\eta)]}{s} - \mathbb{A}'_1[u(\eta)] = 0.\end{aligned}$$

The use of given initial conditions and reordering of terms give

$$\mathbb{A}_1'[u(\eta)] + \frac{\mathbb{A}_1[u(\eta)]}{s} = -\frac{s}{(s^2+\gamma^2)},$$
$$-\gamma s \frac{d}{ds}\left[\frac{\mathbb{A}_1[u(\eta)]}{s}\right] = \frac{s\gamma}{(s^2+\gamma^2)},$$
$$\mathbb{A}_1[\eta u(\eta)] = \frac{s\gamma}{(s^2+\gamma^2)}.$$

The utilization of the inverse GB transform yields

$$u(\eta) = \frac{\sin\eta}{\eta},$$

which is an exact solution.

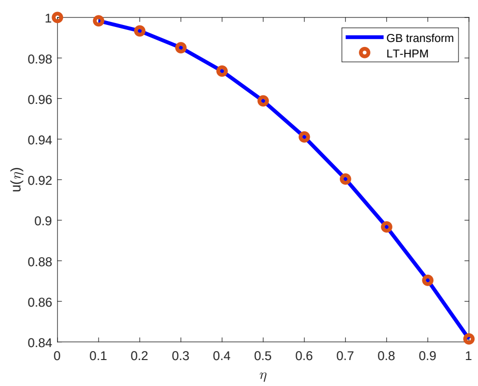

Figure 1: Comparison between the GB transform and LT-HPM solution

In Figure 1, the comparison of solution profiles for problem 1 reveals well agreement of the solution obtained by the GB transform with a series solution of LT-HPM [32].

Problem 2. Consider the linear, nonhomogeneous Lane–Emden differential equation

$$u''(\eta) + \frac{2}{\eta}u'(\eta) + u(\eta) = 6 + 12\eta + \eta^2 + \eta^3 \tag{37}$$

with initial conditions

$$u(0) = 0, \qquad u'(0) = 0. \tag{38}$$

Solution 2. Application of $\mathbb{A}_2$ on the both sides of (37) implies

$$-\frac{s^2}{\gamma^2}\mathbb{A}'_1[u(\eta)] - \frac{s}{\gamma^2}\mathbb{A}'_1[u(\eta)] + \frac{s}{\gamma^2}u(0) + \frac{1}{\gamma}\left[\frac{s}{\gamma}\mathbb{A}_1[u(\eta)] - \frac{s}{\gamma}u(0)\right]$$
$$+\frac{\mathbb{A}_1[u(\eta)]}{s} - \mathbb{A}'_1[u(\eta)] = \frac{6}{s} + \frac{244}{s^2} + 6\frac{\gamma^2}{s^3} + 24\frac{\gamma^3}{s^4}.$$

Use of given initial conditions and reordering of terms give

$$\mathbb{A}_1'[u(\eta)] + \frac{\mathbb{A}_1[u(\eta)]}{s} = -\frac{\gamma^2}{(s^2+\gamma^2)}\left[\frac{6}{s} + 24\frac{\gamma}{s^2} + 6\frac{\gamma^2}{s^3} + 24\frac{\gamma^3}{s^4}\right],$$
$$-\gamma s\frac{d}{ds}\left[\frac{\mathbb{A}_1[u(\eta)}{s}\right] = \frac{\gamma^3}{(s^2+\gamma^2)}\left[\frac{6}{s} + 24\frac{\gamma}{s^2} + 6\frac{\gamma^2}{s^3} + 24\frac{\gamma^3}{s^4}\right],$$
$$\mathbb{A}_1[\eta u(\eta)] = 6\frac{\gamma^3}{s^3} + 24\frac{\gamma^4}{s^4}.$$

Utilization of the inverse GB transform yields

$$u(\eta) = \eta^2 + \eta^3,$$

which is an exact solution.

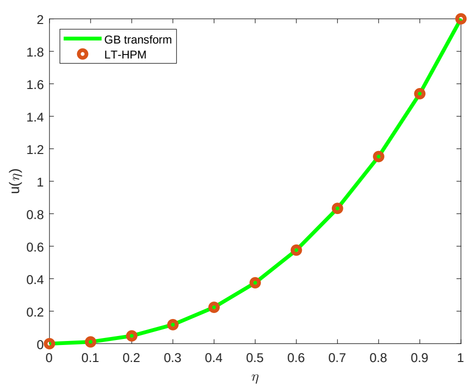

Figure 2: Comparison between the GB transform and LT-HPM solution

In Figure 2, the graphical comparison of the solution for Problem 2 reflects the good agreement of the solution obtained by the GB transform and the series solution of LT-HPM [32].

In Problems 1 and 2, the GB transform has been found to be independently efficient in constructing the exact solution of linear Lane–Emden type equations, whereas Laplace transform demands modification of governing equations or other collaborative techniques to drive exact or approximate solutions. The same fact can be stated for the Shehu transform and Formable transform in comparison with the GB transform.

Problem 3. Consider the Bessel differential equation with polynomial coefficients as

$$u''(\eta) + \frac{1}{\eta}u'(\eta) + u(\eta) = 0 \tag{39}$$

with initial data

$$u(0) = 1, \quad u'(0) = 1. \tag{40}$$

Solution 3. Application of $\mathbb{A}_2$ on the both sides of (39) implies

$$\begin{aligned} &-\frac{s^2}{\gamma^2}\mathbb{A}'_1[u(\eta)] - \frac{s}{\gamma^2}\mathbb{A}'_1[u(\eta)] + \frac{s}{\gamma^2}u(0) \\ &+ \frac{1}{\gamma}\left[\frac{s}{\gamma}\mathbb{A}_1[u(\eta)] - \frac{s}{\gamma}u(0)\right] + \frac{\mathbb{A}_1[u(\eta)]}{s} - \mathbb{A}'_1[u(\eta)] = 0. \end{aligned}$$

Use of (40) and reordering of terms give

$$\frac{(-s^2 - \gamma^2)}{\gamma^2}\mathbb{A}'_1[u(\eta)] + \frac{\mathbb{A}_1[u(\eta)]}{s} = 0.$$

The solution of the above equation yields

$$\mathbb{A}_1[u(\eta)] = \frac{\alpha s}{\sqrt{s^2 + \gamma^2}}.$$

Utilization of the inverse GB transform yields

$$u(\eta) = \alpha J_0(\eta).$$

Use of the initial data provides

$$u(\eta) = J_0(\eta),$$

where J_0 is the Bessel function.

Problem 4. Consider the wave-like partial differential equation

$$\frac{\partial^2 u(\xi,\eta)}{\partial \eta^2} = \frac{\partial^2 u(\xi,\eta)}{\partial \xi^2} + u(\xi,\eta) \tag{41}$$

with boundary conditions

$$u(0,\eta) = \cosh(\eta) \quad \text{and} \quad \frac{\partial u(0,\eta)}{\partial \xi} = 1 \tag{42}$$

and initial conditions

$$u(\xi,0) = \sin(\xi) + 1 \quad \textit{and} \quad \frac{\partial u(\xi,0)}{\partial \eta} = 1. \tag{43}$$

Solution 4. Application of the GB transform of order one and the given initial conditions to (41) and (42) give

$$\frac{\partial^2}{\partial \xi^2}\mathcal{P}_1(\xi,s,\gamma)+\mathcal{P}_1(\xi,s,\gamma)\left(1-\frac{s^2}{\gamma^2}\right)+(\sin(\xi)+1)\frac{s^2}{\gamma^2}=0 \tag{44}$$

with boundary conditions

$$\mathcal{P}_1(0,s,\gamma)=\frac{s}{s^2-\gamma^2} \quad \text{and} \quad \frac{\partial}{\partial \xi}\mathcal{P}_1(0,s,\gamma)=1. \tag{45}$$

Here to obtain the solution of (44), the homotopy perturbation technique proposed by He [17] has been applied. In the view of (44), a perturbation equation can be readily constructed by embedding homotopy parameter $\theta \in [0,1]$ as

$$(1-\theta)\left[\frac{\partial^2\mathcal{P}_1}{\partial \xi^2}-\frac{\partial^2\mathcal{P}*_{1,0}}{\partial \xi^2}\right]+\theta\left[\frac{\partial^2\mathcal{P}_1}{\partial \xi^2}+\left(1-\frac{s^2}{\gamma^2}\right)\mathcal{P}_1+\frac{s^2}{\gamma^2}(\sin(\xi)+1)\right]=0. \tag{46}$$

Assume that the solution of the (46) can be expanded as

$$\mathcal{P}_1(\xi,s,\gamma)=\mathcal{P}_{1,0}(\xi,s,\gamma)+\theta\mathcal{P}_{1,1}(\xi,s,\gamma)+\theta^2\mathcal{P}_{1,2}(\xi,s,\gamma)+\cdots. \tag{47}$$

Substitution of (47) into (46) and equating the coefficients of identical powers of θ serve the system as

$$\begin{aligned}
&\theta^0: \quad \frac{\partial^2\mathcal{P}_{1,0}}{\partial \xi^2}-\frac{\partial^2\mathcal{P}*_{1,0}}{\partial \xi^2}=0,\\
&\mathcal{P}_{1,0}(0,s,\gamma)=\frac{s^2}{s^2-\gamma^2}, \quad \textit{and} \quad \frac{\partial}{\partial \xi}\mathcal{P}_{1,0}(0,s,\gamma)=1,\\
&\theta^1: \frac{\partial^2\mathcal{P}_{1,1}}{\partial \xi^2}+\left(1-\frac{s^2}{\gamma^2}\right)\mathcal{P}_{1,0}+(1+\sin(\xi))\frac{s^2}{\gamma^2}=0,\\
&\mathcal{P}_{1,1}(0,s,\gamma)=0, \quad \textit{and} \quad \frac{\partial}{\partial \xi}\mathcal{P}_{1,1}(0,s,\gamma)=0,\\
&\vdots\\
&\theta^j: \frac{\partial^2\mathcal{P}_{1,1}}{\partial \xi^2}+\left(1-\frac{s^2}{\gamma^2}\right)\mathcal{P}_{j-1}=0,\\
&\mathcal{P}_{1,j}(0,s,\gamma)=0, \quad \text{and} \quad \frac{\partial}{\partial \xi}\mathcal{P}_{1,j}(0,s,\gamma)=0.
\end{aligned} \tag{48}$$

Utilizing the freedom of initialization, set initial approximation as

$$\mathcal{P}_{1,0}(\xi, s, \gamma) = \mathcal{P}*_{1,0}(\xi, s, \gamma) = \xi + \frac{s^2}{s^2 - \gamma^2}, \tag{49}$$

which satisfies the obtained boundary conditions in (45).

Substitution of preceding components will drive the rest of components of the expanded solution in (47) as

$$\begin{aligned}
\mathcal{P}_{1,1}(\xi, s, \gamma) &= \left(\frac{\xi^3}{6} - \xi + \sin(\xi)\right)\frac{s^2}{\gamma^2} - \frac{\xi^3}{6},\\
\mathcal{P}_{1,2}(\xi, s, \gamma) &= \left(\frac{\xi^5}{120} - \frac{\xi^3}{6} + \xi - \sin(\xi)\right)\frac{s^4}{\gamma^4} - \left(\frac{-\xi^5}{60} + \frac{\xi^3}{6} - \xi + \sin(\xi)\right)\frac{s^2}{\gamma^2} + \frac{\xi^5}{120},\\
&\vdots
\end{aligned} \tag{50}$$

Therefore, the expression for the expanded solution can be written as

$$\begin{aligned}
\mathcal{P}_1(\xi, s, \gamma) = \xi &+ \frac{s^2}{s^2 - \gamma^2} + \left(\frac{\xi^3}{6} - \xi + \sin(\xi)\right)\frac{s^2}{\gamma^2} - \frac{\xi^3}{6}\\
&+ \left(\frac{\xi^5}{120} - \frac{\xi^3}{6} + \xi - \sin(\xi)\right)\frac{s^4}{\gamma^4}\\
&- \left(\frac{-\xi^5}{60} + \frac{\xi^3}{6} - \xi + \sin(\xi)\right)\frac{s^2}{\gamma^2} + \frac{\xi^5}{120}.
\end{aligned} \tag{51}$$

Utilizing the inverse GB transform over (51) with observation that $\mathbb{A}_1^{-1}\left[u(\eta)\frac{s^{n+1}}{\gamma^{n+1}}\right] = u(\eta)\frac{d^n\delta(\eta)}{d\eta^n}$ in which $\delta(\eta)$ is the Dirac delta function, that is, universally zero except at the origin. Thus, corresponding terms will get vanished, and the solution will reduce to

$$u(\xi, \eta) = \xi + \cosh(\eta) - \frac{\xi^3}{6} + \frac{\xi^5}{120} + \cdots \simeq \cosh(\eta) + \sin(\xi),$$

which is an exact solution.

Problem 5. Consider Abel's integral equation as

$$t = \int_0^\eta \frac{1}{\sqrt{\eta - \xi}} u(\xi)\, d\xi. \tag{52}$$

Solution 5. Application of the GB transform of order one and its convolution property to (52) give

$$\frac{\gamma}{s} = \frac{\gamma}{s}\Gamma\left(\frac{-1}{2}+1\right)\left(\frac{u}{s}\right)^{\frac{-1}{2}}\mathbb{A}_1[u(\eta)],$$

$$\frac{1}{\sqrt{\pi}}\left(\frac{u}{s}\right)^{\frac{-1}{2}} = A_1[u(\eta)].$$

Utilization of the inverse GB transform yields

$$\frac{2}{\pi}\eta^{\frac{1}{2}} = u(\eta),$$

which is an exact solution.

Problem 6. Consider the convolution type Volterra integral equation of first kind:

$$\sin\eta = \int_0^\eta u(\eta-\xi)u(\eta)\,d\xi. \tag{53}$$

Solution 6. Application of the GB transform of order one and its convolution property to (53) give

$$\frac{s\gamma}{s^2+\gamma^2} = \frac{\gamma}{s}\mathbb{A}_1^2[u(\eta)],$$
$$\mathbb{A}_1[u(\eta)] = \frac{s}{\sqrt{s^2+\gamma^2}}.$$

Utilization of the inverse GB transform yields

$$u(\eta) = J_0(\eta),$$

which is an exact solution.

Problem 7. Consider convolution type Volterra integral equation of second kind:

$$u(\eta) = \eta + \int_0^\eta u(\xi)\sin(\eta-\xi)\,d\xi. \tag{54}$$

Solution 7. Application of the GB transform of order one and its convolution property to (54) will derive

$$\mathbb{A}_1[u(\eta)] = \frac{\gamma}{s} + \frac{\gamma}{s}\mathbb{A}_1[u(\eta)]\mathbb{A}_1[\sin\eta].$$

Thus,

$$\mathbb{A}_1[u(\eta)] = \frac{\gamma}{s} + \frac{\gamma^3}{s^3}.$$

Utilization of the inverse GB transform yields

$$u(\eta) = \eta + \frac{\eta^3}{6},$$

which is an exact solution.

7 Conclusion

In this paper, a new integral transform operator called the GB transform has been presented along with sufficient conditions for its existence. The explanation of the duality of the GB transform with other transforms enhanced it as a generalized version of members in the class of Laplace-type transforms. For theoretical interest, the present work proved the worth of the GB transform with essential properties, viz., uniqueness, linearity, convolution, and so on. In view of applicability, the accessible features of the proposed transform are demonstrated in solving Lane–Emden type, wave-like, and convolution-type equations. In addition, the construction of weighted norm inequalities for integral convolution with the GB transform extends the scope of its study for the future also. In the future, we intend to apply the GB transform over fractional equations and will propose their bounds.

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