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Groups with soluble minimax conjugate classes of subgroups

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Abstract

A classical result of Neumann characterizes the groups in which each subgroup has finitely many conjugates only as central-by-finite groups. If $\mathfrak{X}$ is a class of groups, a group G is said to have $\mathfrak{X}$-conjugate classes of subgroups if $G/core_G(N_G(H)) \in \mathfrak{X}$ for each subgroup H of G. Here we study groups which have soluble minimax conjugate classes of subgroups, giving a description in terms of $G/Z(G)$. We also characterize FC-groups which have soluble minimax conjugate classes of subgroups.

Keywords and phrases: Conjugacy classes; soluble minimax groups, FC-groups, polycyclic groups.

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1 Introduction

Following [11], the class of all *abelian minimax* groups is the class of all *max*-by-*min* abelian groups. A group G is called *soluble minimax* if it has a finite

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characteristic series $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$ whose factors are abelian minimax groups. Moreover a soluble minimax group is said to be *reduced minimax* if it has no nontrivial normal Chernikov radicable subgroups. Fundamental properties of soluble minimax and reduced minimax groups are described in [11].

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an $\mathfrak{X}C$-group, if $G/C_G(x^G) \in \mathfrak{X}$ for all $x \in G$. If $\mathfrak{X}$ is the class of all finite groups, we obtain the class of FC-groups; Baer in [1] introduced this class of groups. If $\mathfrak{X}$ is the class of all polycyclic-by-finite groups, then the class of PC-groups are obtained which are introduced in [2]. If $\mathfrak{X}$ is the class of all Chernikov groups, then one obtains the class of CC-groups and introduced in [9].

If $\mathfrak{X}$ is the class of all (soluble minimax)-by-finite groups, we obtain the class of MC-groups and when $\mathfrak{X}$ is the class of all (reduced minimax)-by-finite groups, then the class of M_rC-groups is obtained. These classes of groups are introduced in [4].

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an $\mathfrak{X}CS$-group, or a group with $\mathfrak{X}$-conjugate classes of subgroups, if $G/core_G(N_G(H)) \in \mathfrak{X}$ for each subgroup H of G.

If $\mathfrak{X}$ is the class of all finite groups, we obtain the class of FCS-groups. Neumenn in [8] has investigated FCS-groups with a different approach. The current approach can be found in [6]. If $\mathfrak{X}$ is the class of all polycyclic-by-finite groups, one obtains the class of PCS-groups, which are studied in [6]. If $\mathfrak{X}$ is the class of all Chernikov groups, we obtain the class of CCS-groups, which are described in [7] and [10].

If $\mathfrak{X}$ is the class of all (soluble minimax)-by-finite groups, we obtain the class of MCS-groups. In particular, if $\mathfrak{X}$ is the class of all (reduced minimax)-by-finite groups, then the class of M_rCS-groups are obtained.

The present paper is devoted to the studying the classes of MCS and M_rCS-groups. We prove the following description of the groups with soluble minimax conjugate classes of subgroups.

2 Main Theorem

(i) *Let G be a periodic group. Then G is an MCS-group if and only if it is central-by-Chernikov*;

(ii) *Let G be an MCS-group. If $InnG$ has finite abelian subgroup rank, then G is central-by-(soluble minimax)-by-finite*;

(iii) *Let G be an MCS-group. If G contains proper maximal abelian normal subgroups, then G is (soluble minimax)-by-finite-by-abelian.*

Our group-theoretic notation is standard and refered to [11]. Section 2 contains the preparatory results, which are used in Section 3 to prove the Main Theorem. Section 3 is devoted to give the proof of Main Theorem. In section 4, we describe some special classes of MCS-groups.

3 Preliminary results

By defintion each PCS-group is an MCS-group and each CCS-group is an MCS-group. In [6] and [7] some classes of MCS-groups are studied, giving a first answer to Main Theorem.

We omit the elementary proofs of the next two results.

Lemma 2.1. *Let G be a central-by-(soluble minimax)-by-finite group. If H is a subgroup of G, then $H/core_G(H)$ is (soluble minimax)-by-finite group.*

Lemma 2.2. *Let G be an MCS-group. If $L \lhd H \leq G$, then H/L is an MCS-group.*

Lemma 2.3. *Let G be a periodic group. If G is an MCS-group, then G is a CCS-group.*

Proof. For each subgroup H of G, $G/core_G(N_G(H))$ is periodic (soluble minimax)-by-finite, so it is Chernikov by [11, vol.II,p.166].

The following lemma extends [6, Corollary 2.7] and [7, Lemma 2.3].

Lemma 2.4. *If G is an MCS-group, then G is an MC-group.*

Proof. If G is periodic, then the result follows by Lemma 2.3 and [7, Lemma 2.3]. If G is a PCS-group, then the result follows by [6, Corollary 2.7]. Let G be neither periodic nor a PCS-group. Take $g \in G$ and assume $H = core_G(N_G(< g >))$, $H_1 = C_H(< g >)$, $H_2 = core_G(H_1) = C_H(g^G)$. We have that G/H is (soluble minimax)-by-finite, $H \leq N_G(< g >)$ and $H/C_H(< g >)$ is finite abelian. It is sufficient to prove that G/H_2 is (soluble minimax)-by-finite.

Since

$$H_2 = \bigcap_{x \in G} (C_H(< g >))^x = \bigcap_{x \in G} C_H(< g >^x) = \bigcap_{x \in G} C_H(< g^x >)$$

and $H/(C_H(< g >))^x \simeq H/C_H(< g >)$ for every $x \in G$, we obtain the embedding

$$H/H_2 \hookrightarrow \prod_{x \in G} H/H_1^x.$$

In particular we deduce that H/H_2 is a bounded abelian group. Lemmas 2.2 and 2.3 imply that G/H_2 is an MCS-group such that H/H_2 is a periodic normal CCS-subgroup of G/H_2. H/H_2 has no nontrivial Chernikov normal subgroups, so [7, Lemma 2.5] implies that H/H_2 is central-by-finite. By definition we can find a subgroup A/H_2 of $Z(H/H_2) \leq Z(G/H_2)$ such that $(H/H_2)/(A/H_2) \simeq H/A$ is finite. Obviously G/A is (soluble minimax)-by-finite, so G/H_2 is central-by-(soluble minimax)-by-finite. By Lemma 2.1, H/H_2 is (soluble minimax)-by-finite, and so is G/H_2.

There are MC-groups which are not MCS-groups, improving [3, Proposition 2.2].

Example 2.5. Here exibit a metabelian 2-nilpotent MC-group G which is not an MCS-group. Let p be a prime number and C a nontrivial subgroup of the additive group of rational numbers, whose denominators are p-numbers. Let $Q = Dr_{n \in \mathbb{N}} < x_n >$ be a free abelian group of countably infinite rank. Denote multiplicatively the operation in C and let $C = \{c_n | n \in \mathbb{N}\} \cup \{1\}$, where $c_n \neq 1$ for all n and $c_n \neq c_m$ if $n \neq m$. A central extension $C \rightarrowtail G \twoheadrightarrow Q$ can be defined by putting $[x_{2i-1}, x_{2i}] = c_i$ for all $i \in \mathbb{N}$ and $[x_i, x_j] = 1$, otherwise. Given $z \in G \setminus C$, $z = cx_{i_1}^{k_1} \dots x_{i_t}^{k_t}$, where $c \in C$, $i_1 < \dots < i_t$ and $k_{i_1} \neq 0$. Put $y = x_{i_1 - 1}$ if i_1 is even and $y = x_{i_1+1}$ if i_1 is odd. Then $[x_{i_j}, y] = 1$ if $j > 1$, so that $[z, y] = [x_{i_1}^{k_1}, y] = [x_{i_1}, y]^{k_1} \neq 1$ and $Z(G) = G' = C$.

Moreover, $[z, G] =< [z, x_j] : i_1 - 1 \leq j \leq i_t + 1 >$, so that $[z, G]$ is finitely generated and hence it is cyclic. By construction we have that z^G is (infinite cyclic)-by-cyclic and G is an M_rC-group (precisely G is a PC-group). The subgroup $H = Dr_{i \in \mathbb{N}} < x_{2i} >$ of G has $K = N_G(H) = core_G(N_G(H)) = CH$, so that $G/K \geq Dr_{i \in \mathbb{N}} < x_{2i-1}K >$ and G/K has infinite abelian rank.

To convenience the reader, we recall two properties of MC-groups.

Lemma 2.6. *Let G be an MC-group and $x_1, \dots, x_n \in G$. If $X =< x_1, \dots, x_n >$, then X^G is (soluble minimax)-by-finite. Moreover, if G is an M_rC-group then X^G and $G/C_G(X^G)$ are reduced minimax.*

Proof. It follows by [4, Theorem 2].

Lemma 2.6 shows that an MC-group can be covered by normal (soluble minimax)-by-finite subgroups (see [4, p.161-162]). [2, Theorem 2.2] and [11, Theorem 4.36] give the corresponding condition for PC-groups and CC-groups.

Proposition 2.7. *If G is an MC-group, then it is locally-(normal and (soluble minimax)-by-finite). Moreover if G is an MC-group then G' is locally-(normal and (soluble minimax)-by-finite).*

Proof. It follows by Lemma 2.6.

4 Proof of the Main Theorem

Proof. (i) By Lemma 2.3, G is a periodic CCS-group and so [10, Main Theorem] implies that G is central-by-Chernikov. Conversely, let G be central-by-

Chernikov, H be a subgroup of G such that $H \not\leq Z(G)$ and $K = core_G(N_G(H))$. If $K \geq Z(G)$, then the result obviously is obtained. If $K \cap Z(G) = 1$, then K is isomorphic with $KZ(G)/Z(G)$, so it is Chernikov and G/K is isomorphic with $(G/Z(G))/(KZ(G)/Z(G))$, which is again Chernikov.

(ii) Since $InnG \simeq G/Z(G)$, we may suppose that $G/Z(G)$ has finite abelian subgroup rank. Lemma 2.2 implies that $G/Z(G)$ is an MCS-group, so it is an MC-group, by Lemma 2.4. Thanks to Proposition 2.7, $G/Z(G)$ can be covered by (soluble minimax)-by-finite normal subgroups $S_\lambda/Z(G)$, where λ is an ordinal, indiciated in Λ. Without loss of generality assume $Z(G) = 1$. We exibit a covering of G with subgroups T_α such that $\alpha \in A \leq \Lambda$, $T_\alpha < T_{\alpha+1}$, T_α is (soluble minimax)-by-finite and $T_\beta = T_{\beta+1} = \ldots$ for an ordinal $\beta \in A$.

$G =< S_\lambda : \lambda \in \Lambda >$ and we obviously conclude when λ is a limit ordinal, so let λ be not a limit ordinal. By induction the chain

$$T_1 = \bigcap_{\lambda \in \Lambda} S_\lambda,$$

$$T_\alpha =< T_{\alpha-1}, x_{\alpha-1} >, \quad \text{where } \mathrm{x}_{\alpha-1} \notin \mathrm{T}_{\alpha-1}$$

has $T_\alpha < T_{\alpha+1}$, T_α is (soluble minimax)-by-finite, $A \leq \Lambda$. $H = Dr_{\alpha \in A} < x_\alpha >$ has infinite abelian rank which is a contradiction. It follows that G can be covered by finitely many (soluble minimax)-by-finite normal subgroups T_α, so that G is (soluble minimax)-by-finite.

(iii) Let A be a proper maximal abelian normal subgroup of G. By Lemma 2.4 and [5, Corollary 3], A has finite index in G. It is enough to verify that G' is (soluble minimax)-by-finite. If G is periodic the result follows Lemma 2.4 and by [7, Lemma 3.7]. A similar situation happens when G is a PCS-group by [6, Lemma 3.1]. Let G be an MCS-group which is neither periodic nor a PCS-group. Put $X = \{x_1, \ldots, x_n\}$ a transversal to A in G, $G/A = \{x_1A, \ldots, x_nA\}$ and $G = XA$. Lemmas 2.4 and 2.6 imply that $X^G = Y$ is (soluble minimax)-by-finite, in particular $G' = [G,G] = [YA, YA] = Y'[Y,A]$. Now Y' is (soluble minimax)-by-finite and $[Y, A] \leq Y^A = (X^G)^A = Y$ is (soluble minimax)-by-finite, and so G' is.

5 Special classes of MCS-groups

The Example in [7] shows that there is a CCS-group G such that $G/Z(G)$ has infinite abelian rank. The consideration of this group does not yield to characterize an MCS-group G without restrictions on the rank of $G/Z(G)$. On the other hand, the restriction on the size of Frattini subgroup of an MCS-group gives rise the structural informations.

Corollary 4.1. *Let G be an MCS-group. If G contains a subgroup H such that $N_G(H)$ has a non-generator element g of G, then G is (soluble minimax)-by-(radicable nilpotent of class at most 2).*

Proof. By Lemma 2.4, G is an MC-group such that $FratG \geq N_G(H)$, but $FratG = core_G(FratG) \geq core_G(N_G(H))$ and [5, Theorem 4] complete the proof.

Given a group G, a subgroup H of G is said to be $\mathfrak{F}$-perfect if H has no proper subgroups of finite index (in H). The subgroup $\mathfrak{F}(G)$ of G generated by all normal $\mathfrak{F}$-perfect subgroups of G is clearly $\mathfrak{F}$-perfect. This subgroup is called the $\mathfrak{F}$-perfect part of G and if $D(G)$ is the subgroup of G generated by all periodic radicable abelian normal subgroups of G, then $D(G) \leq \mathfrak{F}(G)$.

Corollary 4.2. *If G is an $\mathfrak{F}$-perfect MCS-group, then G is metabelian.*

Proof. Put $R = \mathfrak{F}(G)$ and $D = D(G)$, then Lemma 2.4 and [5, Lemma 2] imply that the series $1 \triangleleft D \triangleleft R = G$ has abelian factors.

The notion of Fitting subgroup allows us to characterize an M_rCS-group.

Proposition 4.3. *Let G be an M_rCS-group and H a subgroup of G. Then G is central-by-polycyclic-by-finite if and only if $Fit(G/core_G(N_G(H))$ is finitely generated.*

Proof. Let $G/Z(G)$ be polycyclic-by-finite and $H \leq G$. Put $K = core_G(N_G(H))$, we may assume that $K \not\leq Z(G)$. If $K \geq Z(G)$ then the result follows immediately. If $K \cap Z(G) = 1$, then $K \simeq KZ(G)/Z(G)$ is polycyclic-by-finite, and hence so is G. It follows that $Fit(G/K)$ is finitely generated. Conversely, if G is an M_rCS-group, then $Fit(G/K)$ is nilpotent by [11, Theorem 10.33]. $Fit(G/K)$ is finitely generated so that G is a PCS-group. Now the main Theorem of [6] completes our proof.

A special situation happens for the class of FC-groups.

Proposition 4.4. *Let G be an FC-group. Then the following conditions are equivalent:*

(i) G is FCS-group;

(ii) G is CCS-group;

(iii) G is PCS-group;

(iv) G is MCS-group;

(v) G is central-by-finite.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious. (v) $\Rightarrow$ (i) is described in [7, Proposition 2.4].

(ii) $\Rightarrow$ (iii). By [7, Proposition 2.4], the class of CCS-groups coincide with the class of FCS-groups, but each FCS-group is a PCS-group, which gives the result.

(iv) $\Rightarrow$ (v). Put U to be the maximal torsion-free subgroup of $Z(G)$ and $G/Z(G)$ is periodic (see [11, Theorem 4.32]), so it implies that G/U is also periodic. If T is the periodic part of G and G/T is torsion-free abelian, then $T \cap U = 1$ and $G \hookrightarrow G/T \times G/U$. By Lemma 2.2 and the Main Theorem of [10] implies that G/U is central-by-finite. Since G/T is abelian and $G/T \times G/U$ is central-by-finite, we conclude that G is central-by-finite.

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