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Research Article

An optimal control approach for solving an inverse heat source problem applying shifted Legendre polynomials

T. Shojaeizadeh*and M. Darehmiraki

Abstract

This study addresses the inverse issue of identifying the space-dependent heat source of the heat equation, which is stated using the optimal control framework. For the numerical solution of this class of problems, an approach based on shifted Legendre polynomials and the associated operational matrix is presented. The approach turns the primary problem into the solution of a system of nonlinear algebraic equations. To do this, the temperature and heat source variables are enlarged in terms of the shifted Legendre polynomials with unknown coefficients employed in the objective function, inverse problem, and initial and Neumann boundary conditions. When paired with their operational matrix, these basis functions provide a quadratic optimization problem with linear constraints, which is then solved using the Lagrange multipliers approach. To assess the method's efficacy and precision, two examples are provided.

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*Corresponding author

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Tahereh Shojaeizadeh
Department of Mathematics, Qom Branch, Islamic Azad University, Qom, Iran.
e-mail: Ta.Shojaeizadeh@iau.ac.ir

Majid Darehmiraki
Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Khouzestan, Iran.
e-mail: darehmiraki@bkatu.ac.ir

1 Introduction

Many inverse problems for the heat equation are applied to many fields of physics and engineering, such as acoustics [19], medical imaging [10], signal processing [35], optic [6], and radar [7]. There are approximately five major classes of inverse heat diffusion equation problems.
(**i**) The problem of reverse time or conducting heat backward from the known last-minute distribution determines the initial temperature distribution.
(**ii**) Inverse heat conduction is the detection of temperature or temperature flux at one inaccessible boundary beyond the data available in the other case that is accessible.
(**iii**) Identify coefficients of over-posed data at the boundaries.
(**iv**) Determining the shape of unknown boundaries or cracks inside the heat conduction body.
(**v**) The identification of the heat source [18, 5].
The heat equation, in this research, treats the heat source as an uncertainty. Applications in the real world where these difficulties are useful include creating the end state of melting, and freezing processes and determining the contaminating source's intensity. Methods such as the generalized finite difference scheme [12], the radial basis function method [28], the sparse regularization approach [25], the meshless generalized finite difference scheme [13], the mollification regularization scheme [36], and the reproducing kernel space scheme [33], have all been applied to the solution of inverse heat source problems. In this paper, we propose a novel numerical method for obtaining the source parameter (or control parameter) in parabolic equations. Iterative methods and a variational approach have recently been proposed to numerically solve this problem [21]. These methods are computationally expensive because they solve a direct problem at each iteration. Tikhonov regularization is proposed in [37] as a stable optimal control solution to the inverse heat source problem. Parameter identification for a nonlinear heat equation in the 2D and 3D space-time domains was solved by Lin and Liu using homogenization functions as the basis [24]. The authors of [29] proposed a perfect method to investigate inverse heat source problems in functionally graded materials using the homogenization function. Due to the given conditions, a homogenization function for the boundary value problem is conceived, and a family of homogenization functions is further derived. Djennadi et al. [11] employed the expansion method and the overdetermination condition to solve the inverse source fractional diffusion problem that contains the Atangana–Baleanu–Caputo fractional derivative. In [20], for the stable reconstruction of the heat source in the parabolic heat equation, an iterative variable conjugate gradient algorithm is proposed based on a sequence of direct problems that are solved using the boundary element method of each iteration step. The gradient descent along with the finite difference method to find the solution nonlinear inverse heat transfer problem in [4]. Ciofalo [9] proposed using finite volume discretization to get a solution for

an inverse heat conduction problem. In which, with the assumption that the thermal boundary conditions in other walls are known, the steady state distribution of the displacement heat transfer coefficient on one slab wall is reconstructed from the temperature distribution in the plate embedded in the slab. With the increasing use of machine learning techniques, including neural networks, the use of these techniques in solving inverse problems has also attracted the attention of many researchers. Li and Hu [23] used a multi-layer neural network to solve the Cauchy inverse problem. Phisyc-informed neural network models are one of the powerful methods in deep learning. Authors in [27] applied it to solve a class of inverse problems related to partial differential equations (PDEs). The authors of [15] proposed a new method for solving large-scale inverse problems based on Bayesian inference, Markov chain Monte Carlo approach, and derivative-free algorithms. Bondarenko [8] presented a finite-difference-based method to investigate the discrete systems of the inverse of the Sturm–Liouville problem. Huntul [16] used the Tikhonov regularization and the nonlinear optimization for the first time in the third-order pseudo-parabolic equation with initial and nonlocal periodic boundary conditions derived from nonlocal integral observation for the inverse space-dependent heat problem. Huntul [17] recovered a source in a high-order pseudo parabolic equation using cubic spline functions. In [14], authors solved the two-dimensional inverse time-fractional diffusion problem with nonlocal boundary conditions using a-polynomials, collocation, and least squares methods. They calculated time using the L_1 method. Wen, Liu, and Wang [34] used the Fourier approach to find the source term and starting data in the time-fractional diffusion equation . Abbaszadeh and Dehghan [1] considered the inverse tempered fractional diffusion equation. They used Crank–Nicolson temporal discretization, a modified element-free Galerkin method, and a meshless method to solve the inverse problem.

In this research, we provide a numerical solution for solving the inverse heat source issue in an optimal control setting by using orthogonal polynomials. This piece is an attempt to provide a fresh strategy for addressing the issue of the mysterious heat source. The optimal control issue is reduced to a set of algebraic equations in the suggested approach [26, 32, 30]. This is achieved by approximating the temperature y and the heat source f in $\mathbf{P_I}$ (see (1)) with the help of shifted Legendre polynomials (SLPs) and their operational matrix with unknown coefficients. By substituting these approximations for the objective function in the inverse problem, we are able to determine not only the unknown coefficients but also the initial and boundary conditions. To conclude, we utilize Lagrange multipliers to connect the algebraic equation produced from the objective function to the algebraic equations derived from the inverse system and the starting and boundary conditions. Then, we can use the constrained extremum method to solve the resulting algebraic system of equations to find the best solution. The authors of [3] investigated the inverse heat equation problem with variable boundary conditions using a weak solution strategy. The Legendre spectral collocation method was used to solve a fractional inverse heat conduction problem in [2], where both the temperature function and the boundary heat fluxes were unknown. Following the introduction, the article will focus on five primary sections that together address this inverse problem. In Section 2, we present the optimal control issue and the inverse heat source problem. In Section 3, we describe the SLPs and their characteristics. The problem is resolved in Section 4. In Section 5, we provide numerical examples that demonstrate the effectiveness and precision of the suggested approach. The last part explains the results.

2 Problem statement

Suppose the following inverse problem:
Let us suppose $\Theta := (0,1)\times(0,T),\ T \geq 1$, one is going to find the temperature z and the heat source f that satisfy (1); that is,

$$\mathbf{P_I}: \begin{cases} z_t(x,t) - z_{xx}(x,t) = f(x), & (x,t) \in \ \Theta, \\ z(x,0) = \nu(x), & x \in \ (0,1), \\ z_x(0,t) = g_0(t), \quad z_x(1,t) = g_1(t), & t \in \ (0,T). \end{cases} \tag{1}$$

The second-order parabolic equation $\mathbf{P_I}$ with sufficiently smooth functions $\nu(x)$ (the initial condition), (Neumann boundary conditions) $g_0(t)$ and $g_1(t)$, forms the governing equations.

Assume that the desired function measured data $h_\epsilon(x)$ (desired function) and the actual data $z(x,T) := h(x)$ meet the following relation:

$$\|h(x) - h_\epsilon(x)\|_{L^2[0,1]} \leqslant \epsilon, \tag{2}$$

where ϵ is the known noise level and the norm $\|\cdot\|_{L^2[0,1]}$ of a function $z(x)$ is determined by

$$\|z(x)\|_{L^2[0,1]} = \Big(\int_0^1 z^2(x)dx \Big)^{\frac{1}{2}}.$$

In the following part, we convert the problem $\mathbf{P_I}$ into an optimal control problem of $\mathbf{P_{II}}$ and solve it using the suggested approach. The following is a consideration of the optimal control problem:

$$\mathbf{P_{II}}: \min_{f\in F_{ad}} \quad J(z,f) := \frac{1}{2}\|z(x,T) - h_\varepsilon(x)\ \|^2_{L^2[0,1]} + \frac{\sigma}{2}\|\nabla f\ \|^2_{L^2[0,1]}, \tag{3}$$

where F_{ad} has defined the set of admissible controls of the objective function J as

$$F_{ad} = \{f(x) : 0 \leqslant a \leq f \leq b, \nabla f \in L^2[0,1]\}, \tag{4}$$

with the constant bounds, $a, b \in \mathbb{R}$. Moreover, $z(x,t)$ is the solution of (1) for a given heat source $f(x) \in F_{ad}$, and σ is the regularization parameter. For noisy data $h_\varepsilon(x)$, the purpose of the optimal control problem is to find functions $f(x)$ and $z(x,t)$ that minimize the objective function $\mathbf{P_{II}}$ and satisfy $\mathbf{P_{I}}$.

3 Shifted Legendre Polynomials (SLPs)

The orthogonal polynomials with regard to the weight function $W(x) = 1$ on $[-1,1]$ are known as Legendre polynomials of degree m and are denoted by $L_m(x)$ $(m = 0,1,\ldots)$. The following recurrence formula can be used to create these polynomials:

$$L_m(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), \qquad m = 1,2,\ldots, \tag{5}$$

where $L_0(x) = 1$ and $L_1(x) = x$. The well-known SLPs in $[0,1]$ can be created by changing the variable $x = 2t-1$, which is expressed as $\mathcal{L}_m(t)$ $(m = 0,1,2,\ldots,)$, by

$$\mathcal{L}_m(t) = \frac{(2m+1)(2t-1)}{m+1} \mathcal{L}_m(t) - \frac{m}{m+1} \mathcal{L}_{m-1}(t), \qquad m = 1,2,\ldots, \tag{6}$$

where $\mathcal{L}_0(t) = 1$ and $\mathcal{L}_1(t) = 2t-1$. The explicit formula of the SLPs is as follows [31]:

$$\mathcal{L}_m(t) = \sum_{i=0}^{m} b_{mi} t^i, \tag{7}$$

where $\mathcal{L}_m(0) = (-1)^m$, $\mathcal{L}_m(1) = 1$, and

$$b_{mi} = (-1)^{m+i} \frac{(m+i)!}{(m-i)!(i!)^2}. \tag{8}$$

The orthogonality condition of the SLPs with respect to the weight function $w(t) = 1$ is given by

$$\int_0^1 \mathcal{L}_m(t)\mathcal{L}_n(t)dt = h_m \delta_{mn}, \tag{9}$$

where δ_{mn} is Kronecker's delta function and $h_m = \dfrac{1}{2m+1}$. Any given function $z(t) \in L^2[0,1]$ can be represented in $(n+1)$ terms of the SLPs as

$$z(t) \simeq \sum_{i=0}^{n} z_i \mathcal{L}_i(t) \triangleq Z^T \Phi_n(t), \tag{10}$$

where

$$Z = [z_0 \ z_1 \ \ldots \ z_n]^T,$$

$$\Phi_n(t) \triangleq [\mathcal{L}_0(t) \ \mathcal{L}_1(t) \ \ldots \ \mathcal{L}_n(t)]^T, \tag{11}$$

and

$$z_i = \frac{1}{h_i} \int_0^1 z(t) \mathcal{L}_i(t) dt, \qquad i = 0, 1, \ldots, n.$$

In a similar way, a two-variable function $z(x,t) \in L^2(\Theta)$ can be expanded by the SLPs as

$$z(x,t) \simeq \sum_{i=0}^{m} \sum_{j=0}^{n} z_{ij} \mathcal{L}_i(x) \mathcal{L}_j(t) \triangleq \Phi_m^T(x) Z \Phi_n(t), \tag{12}$$

where $Z = [z_{ij}]$ is the matrix of coefficients with dimensions $(m+1) \times (n+1)$ whose entries are unknown and obtained from the following equation:

$$z_{ij} = \frac{1}{h_i h_j} \int_0^1 \int_0^1 z(x,t) \mathcal{L}_i(x) \mathcal{L}_j(t) dx dt, \tag{13}$$

for $i = 0, 1, \ldots, m, \ j = 0, 1, \ldots, n$. Suppose that $\Phi_n(t)$ is the vector introduced in (11). Then the derivative of this vector is as follows: [31]

$$\frac{d\Phi_n(t)}{dt} = D_t^{(1)} \Phi_n(t), \tag{14}$$

where $D_t^{(1)} = [d_{ij}^{(1)}]$ is called the derivative operational matrix of SLPs of $(n+1)$-order, whose structure is as follows:

$$d_{ij}^{(1)} = \begin{cases} 2(2j+1), & j = i-k, \begin{cases} k = 1, 3, \ldots, n & if \ n \ odd, \\ k = 1, 3, \ldots, n-1 & if \ n \ even, \end{cases} \\ \\ 0 & \text{otherwise}. \end{cases} \tag{15}$$

Remark 1. Generally, the r-derivative operational matrix of SLPs of $\Phi_n(t)$ can be given by [31]

$$\frac{d^r \Phi_n(t)}{dt^r} = D_t^{(r)} \Phi_n(t), \tag{16}$$

in which $D_t^{(r)}$ is obtained by r times multiplying $D_t^{(1)}$ in itself.

4 Convergence analysis

In this section, the convergence analysis of SLPs expansion in two dimensions is investigated.

Theorem 1. Suppose that $z : \Theta \longrightarrow \mathbb{R}$ is $(n+m+1)$ times continuously differentiable. If $\Phi_m^T(x) Z \Phi_n(t)$ is a unique best approximation of z, then the following inequality holds:

$$\|z(x,t) - \Phi_m^T(x) Z \Phi_n(t)\|_{L^2(\Theta)} \leqslant \frac{\Delta\sqrt{\Gamma}(n+m+2)}{r!(n+m+1-r)!}, \tag{17}$$

where

$$\Delta = \max_{\Theta}\{|\ \frac{\partial^{n+m+1}}{\partial x^{n+m+1-i}\partial t^i} z(x,t)\ |\quad i = 0, 1, \ldots, n+m+1\},$$
$$\Gamma = \max_{T \geqslant 1}\{T^{2n+2m-i+3}, \quad i = 0, 1, \ldots, 2(n+m+1)\}.$$

Proof. Maclaurin's expansion for $z(x,t)$ reads as follows:

$$z(x,t) = p(x,t) + \frac{1}{(n+m+1)!}(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t})^{n+m+1} z(\xi_0 x, \xi_0 t), \quad \xi_0 \in (0,1),$$

where

$$p(x,t) = \sum_{k=0}^{n+m} \frac{1}{k!}(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t})^k z(0,0).$$

Thus

$$|z(x,t) - p(x,t)| = |\frac{1}{(n+m+1)!}(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t})^{n+m+1} z(\xi_0 x, \xi_0 t)|, \quad \xi_0 \in (0,1).$$

On the other hand, since $\Phi_m^T(x) Z \Phi_n(t)$ is the best approximation of $z(x,t)$, we obtain

$$\| z - \Phi_m^T Z \Phi_n\|^2{}_{L^2(\Theta)} \leq \| z - p\|^2{}_{L^2(\Theta)}.$$

By the definition of the L^2-norm and expand $(x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t})^{n+m+1}$, we have

$$
\begin{aligned}
&\| z(x,t)-p(x,t) \|^2_{L^2(\Theta)}\\
&=\int_0^T\int_0^1[\frac{1}{(n+m+1)!}(x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t})^{n+m+1}z(\xi_0 x,\xi_0 t)]^2dxdt\\
&=\int_0^T\int_0^1[\frac{1}{(n+m+1)!}\sum_{i=0}^{n+m+1}\binom{n+m+1}{i}x^{n+m+1-i}t^i\\
&\frac{\partial^{n+m+1}}{\partial x^{n+m+1-i}\partial t^i}z(\xi_0 x,\xi_0 t)]^2dxdt\\
&\leq\frac{\Delta^2}{(n+m+1)!^2}\int_0^T\int_0^1[\sum_{i=0}^{n+m+1}\binom{n+m+1}{r}x^{n+m+1-i}t^i]^2dxdt,
\end{aligned}
$$

where

$$
\binom{n+m+1}{r}=\max\left\{\binom{n+m+1}{i};i=0,1,\dots,n+m+1\right\}.
$$

To find the upper bound for the above expression, we calculate the following terms:

$$
\begin{aligned}
\int_0^T\int_0^1 x^{2(n+m+1-i)}t^{2i}dxdt=&\frac{T^{1+2i}}{(1+2i)(2n+2m-2i+3)},\\
&\qquad i=0,1,\dots,n+m+1,\\
\int_0^T\int_0^1 x^{(2n+2m+1-i)}t^{i+1}dxdt=&\frac{T^{2+i}}{(2+i)(2n+2m-i+2)},\\
&\qquad i=0,1,\dots,n+m,\\
&\vdots\\
\int_0^T\int_0^1 x^{(2+i)}t^{2n+2m-i}dxdt=&\frac{T^{2n+2m-i+1}}{(3+i)(2n+2m-i+1)},\qquad i=0,1,\\
\int_0^T\int_0^1 x^{(1+i)}t^{2n+2m-i+1}dxdt=&\frac{T^{2n+2m-i+2}}{(2+i)(2n+2m-i+2)},\qquad i=0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\| z-p\|^2{}_{L^2(\Theta)}\leq&\frac{\Delta^2}{r!^2(n+m+1-r)!^2}\int_0^T\int_0^1[\sum_{i=0}^{n+m+1}x^{n+m+1-i}t^i]^2dxdt\\
=&\frac{\Delta^2}{r!^2(n+m+1-r)!^2}\Big[\sum_{i=0}^{n+m+1}\frac{T^{1+2i}}{(1+2i)(2n+2m-2i+3)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{n+m}\frac{T^{2+i}}{(2+i)(2n+2m-i+2)}+\cdots\\
&+\sum_{i=0}^{1}\frac{T^{2n+2m-i+1}}{(3+i)(2n+2m+1)}+\frac{T^{2n+2m-i+2}}{2(2n+2m+2)}\Big]\\
\le\ &\frac{\Delta^2\Gamma}{r!^2(n+m+1-r)!^2}\\
&\times\big[(n+m+2)+(n+m+1)+\cdots+2+1\big]\\
\le\ &\frac{\Delta^2\Gamma}{r!^2(n+m+1-r)!^2}(n+m+2)^2,
\end{aligned}
$$

which is the desired result. □

5 Description of the presented method

Now in this section we will use numerical methods to address the problem raised in (1) and (3). We will use numerical methods to address the problem raised in (1) and (3) in this section. SLPs approximate the temperature and heat source for this purpose as follows:

$$z(x,t) \simeq \Phi_m(x)^T Z \Phi_n(t), \tag{18}$$

$$f(x) \simeq F^T \Phi_m(x), \tag{19}$$

where Z and F are the following unknown matrices of coefficients with dimensions $(m+1)\times(n+1)$ and $(m+1)\times 1$, respectively, while $\Phi_m(x)$ and $\Phi_n(t)$ in (11) have been expressed:

$$Z=\begin{pmatrix} z_{00} & z_{01} & \dots & z_{0n}\\ z_{10} & z_{11} & \dots & z_{1n}\\ \vdots & \vdots & & \vdots\\ z_{m0} & z_{m1} & \dots & z_{mn}\end{pmatrix}, \qquad F=\begin{pmatrix} f_0\\ f_1\\ \vdots\\ f_m\end{pmatrix}. \tag{20}$$

Set

$$\mathcal{P}(x,t) \triangleq [\mathcal{L}_0(x)\mathcal{L}_0(t),\ \dots,\ \mathcal{L}_m(x)\mathcal{L}_0(t)\ \mid\ \cdots\ \mid\ \mathcal{L}_0(x)\mathcal{L}_n(t),\ \dots,\ \mathcal{L}_m(x)\mathcal{L}_n(t)]. \tag{21}$$

According to (21), we can express (18) as

$$z(x,t) \simeq \Phi_m(x)^T Z \Phi_n(t) = \mathcal{P}(x,t)\,\mathrm{vec}(Z), \tag{22}$$

where

$$\mathrm{vec}(Z) = [z_{00},\dots,z_{m0} \mid \dots \mid z_{0n},\dots,z_{mn}]^T.$$

From (14), (22), and Remark 1, the result will be as follows:

$$z_x(x,t) \simeq \Phi_m(x)^T D_x^{(1)T} Z \Phi_n(t) = \mathcal{P}(x,t)(I_{n+1} \otimes D_x^{(1)T}) \operatorname{vec}(Z), \tag{23}$$

$$z_{xx}(x,t) \simeq \Phi_m(x)^T D_x^{(2)T} Z \Phi_n(t) = \mathcal{P}(x,t)(I_{n+1} \otimes D_x^{(2)T}) \operatorname{vec}(Z), \tag{24}$$

$$z_t(x,t) \simeq \Phi_m(x)^T Z D_t^{(1)} \Phi_n(t) = \mathcal{P}(x,t)(D_t^{(1)T} \otimes I_{m+1}) \operatorname{vec}(Z), \tag{25}$$

so that I_{m+1} and I_{n+1} are identity matrices of orders $m+1$ and $n+1$, respectively. Additionally, $\otimes$ refers to the Kronecker product [22]. Now, (19), (24), and (25) are substituted into the first subequation of (1), and the result is

$$\mathcal{K}(x,t) \operatorname{vec}(Z) - F^T \Phi_m(x) = 0, \tag{26}$$

in which

$$\mathcal{K}(x,t) \triangleq \mathcal{P}(x,t) \left[(D_t^{(1)T} \otimes I_{m+1}) - (I_{n+1} \otimes D_x^{(2)T}) \right].$$

Thus, as to (22) and (23) and with regards to initial and Neumann boundary conditions in (1), we have

$$\begin{aligned} &\mathcal{P}(x,0) \operatorname{vec}(Z) = \nu(x), \\ &\mathcal{P}(0,t)(I_{n+1} \otimes D_x^{(1)T}) \operatorname{vec}(Z) = g_0(t), \\ &\mathcal{P}(1,t)(I_{n+1} \otimes D_x^{(1)T}) \operatorname{vec}(Z) = g_1(t). \end{aligned} \tag{27}$$

We follow the suggested procedure by constructing an $(m+1)\times(n+1)$ algebraic system of equations. For this reason, we derive the following equations from (26) and (27):

$$\begin{cases} \mathcal{K}(\xi_i,\eta_j) \operatorname{vec}(Z) - F^T \Phi_m(\xi_i) = 0, & i = 2,\dots,m, \quad j = 2,\dots,n+1, \\ \mathcal{P}(\xi_i,0) \operatorname{vec}(Z) = \nu(\xi_i) & i = 1,\dots,m+1, \\ \mathcal{P}(0,\eta_j)(I_{n+1} \otimes D_x^{(1)T}) \operatorname{vec}(Z) = g_0(\eta_j), & j = 2,\dots,n+1, \\ \mathcal{P}(1,\eta_j)(I_{n+1} \otimes D_x^{(1)T}) \operatorname{vec}(Z) = g_1(\eta_j), & j = 2,\dots,n+1, \end{cases} \tag{28}$$

where a collocation scheme is defined by evaluating the outcome at the points (ξ_i, η_j) in (28). We employ the shifted Legendre–Gauss–Lobatto nodes ξ_i $(1 \leq i \leq m+1)$ and the shifted Legendre roots η_j $(1 \leq j \leq n+1)$ of $\mathcal{L}_n(t)$ to find suitable collocation points. It is possible to write (28) as follows:

$$\mathcal{M}\,\mathrm{vec}(Z) - \mathcal{N}\hat{\mathcal{F}} = Q, \tag{29}$$

in which

$$\mathcal{M} = \begin{pmatrix} \mathcal{T}(2:n+1,:) \otimes \mathcal{X}(2:m,:) \\ \mathcal{T}(1,:) \otimes \mathcal{X}(1:m+1,:) \\ \mathcal{T}(2:n+1,:) \otimes \mathcal{X}(1,:) \\ \mathcal{T}(2:n+1,:) \otimes \mathcal{X}(m+1,:) \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} \mathcal{S} \otimes \mathcal{X}(2:m,:) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{\mathcal{F}} = \Big[f_0, \dots, f_m | \underbrace{0, \dots, 0}_{m+1} | \cdots | \underbrace{0, \dots, 0}_{m+1} \Big]^T,$$

$$Q = \Big[\overbrace{\underbrace{0, \dots, 0}_{m-1} | \cdots | \underbrace{0, \dots, 0}_{m-1}}^{n} | \nu(\xi_1), \dots, \nu(\xi_{m+1}) \\ | g_0(\eta_2), \dots, g_0(\eta_{n+1}) | g_1(\eta_2), \dots, g_1(\eta_{n+1}) \Big]^T,$$

where $\hat{F}$ and Q are $(m+1)(n+1)$-order vectors and

$$\mathcal{S} \triangleq \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}_{n \times (n+1)},$$

$$\mathcal{T} \triangleq \begin{pmatrix} \mathcal{L}_0(\eta_1) & \mathcal{L}_1(\eta_1) & \dots & \mathcal{L}_n(\eta_1) \\ \mathcal{L}_0(\eta_2) & \mathcal{L}_1(\eta_2) & \dots & \mathcal{L}_n(\eta_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_0(\eta_{n+1}) & \mathcal{L}_1(\eta_{n+1}) & \dots & \mathcal{L}_n(\eta_{n+1}) \end{pmatrix}_{(n+1) \times (n+1)},$$

$$\mathcal{X} \triangleq \begin{pmatrix} \mathcal{L}_0(\xi_1) & \mathcal{L}_1(\xi_1) & \dots & \mathcal{L}_m(\xi_1) \\ \mathcal{L}_0(\xi_2) & \mathcal{L}_1(\xi_2) & \dots & \mathcal{L}_m(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_0(\xi_{m+1}) & \mathcal{L}_1(\xi_{m+1}) & \dots & \mathcal{L}_m(\xi_{m+1}) \end{pmatrix}_{(m+1) \times (m+1)}.$$

Next, we approximate $\mathbf{P_{II}}$ by SLPs. First, we approximate the desired function $h_\varepsilon(x)$ as

$$h_\varepsilon(x) \simeq H^T \Phi_m(x). \tag{30}$$

From (14) and (19), we have

$$\nabla f(x) \simeq F^T D_x^{(1)} \Phi_m(x). \tag{31}$$

Inserting (18), (30), and (31) into (3) yields that

$$\begin{aligned} J(z,f) &\simeq J_{m,n}(Z,F) \\ &= \frac{1}{2}\int_0^1 (\Phi_m(x)^T Z \Phi_n(T) - \Phi_m(x)^T H)(\Phi_m(x)^T Z \Phi_n(T) - \Phi_m(x)^T H)^T dx \\ &+ \frac{\sigma}{2}\int_0^1 (F^T D_x^{(1)} \Phi_m(x))(F^T D_x^{(1)} \Phi_m(x))^T dx. \end{aligned}$$

The value $\int_0^1 (\phi_m(x)^T H)^2 dx$ is positive, meaning it has no influence on minimization and according to (9), the equation can be expressed as follows:

$$\begin{aligned} J_{m,n}(Z,F) = &\frac{1}{2}\mathrm{vec}(Z)^T (\Phi_n(T)\Phi_n(T)^T \otimes \Upsilon_m)\,\mathrm{vec}(Z) \\ &- H^T(\Phi_n(T)^T \otimes \Upsilon_m)\,\mathrm{vec}(Z) \\ &+ \frac{\sigma}{2} F(I_{n+1} \otimes (D_x^{(1)} \Upsilon_m D_x^{(1)T}))F, \end{aligned} \tag{32}$$

where

$$\Upsilon_m = diag(h_0, \dots, h_m).$$

The problem of optimal control in the discussion has now become a finite dimension optimization. We use the Lagrangian multipliers method to solve the ensuing optimization problem. Let us clarify

$$J^*(z,f) \simeq J^*(Z,F,\Omega) = J_{m,n}(Z,F) + \Lambda^T(\mathcal{M}\,\mathrm{vec}(Z) - \mathcal{N}\hat{\mathcal{F}} - Q), \tag{33}$$

where

$$\Lambda = \left[\, \lambda_1 \ \lambda_2 \ \dots \ \lambda_{(m+1)\times(n+1)} \,\right]^T,$$

which shows the vector of Lagrange multipliers as Λ. The following equations lead to the following optimality conditions:

$$\begin{cases} \dfrac{\partial J^*(z,f)}{\partial\,\mathrm{vec}(Z)} = 0, \\[2ex] \dfrac{\partial J^*(z,f)}{\partial F} = 0, \\[2ex] \dfrac{\partial J^*(z,f)}{\partial \Lambda} = 0. \end{cases}$$

The Newton iterative approach or MATLAB software tools can be used to solve the aforementioned algebraic equation system. We can get the approximate solutions $z(x,t)$ and $f(x)$ from (18) and (19), respectively, by figuring out Z and F.

6 Numerical examples

This section gives two examples along with figures to illustrate how the recommended technique may be implemented successfully and to show its potential. The results of the existing plan are analyzed and compared to the solution that was found analytically and method of [37]. The rand function is used by the MATLAB software to generate noisy data, and the value of h_ε for $0 \leq \delta \leq 1$ in the collocation points $\{\xi_j\}_{j=1}^{m+1}$ is calculated as follows:

$$h_\varepsilon = h + \delta.rand(size(h)), \tag{34}$$

$$\varepsilon = \|h_\varepsilon - h\|_{l^2} = (\frac{1}{m+1}\sum_{j=1}^{m+1}|h_\varepsilon - h|^2)^{\frac{1}{2}}. \tag{35}$$

For noisy data $h_\varepsilon(x)$, the goal of the optimal control problem is to find functions $f(x)$ and $z(x,t)$ that minimize the following objective function and satisfy (1):

$$\begin{aligned} \min J(z,f) &= \frac{1}{2} \parallel z(x,1) - h_\varepsilon(x) \parallel^2_{L^2[0,1]} + \frac{\sigma}{2} \parallel f^{'}(x) \parallel^2_{L^2[0,1]} \\ &= \frac{1}{2}\int_0^1 |z(x,1) - h_\varepsilon(x)|^2 dx + \frac{\sigma}{2}\int_0^1 |f^{'}(x)|^2 dx. \end{aligned} \tag{36}$$

We take the regularization parameter $\sigma = \varepsilon^2$, and, in order to observe the convergence of the method described in numerical experiments, we calculate the approximate error resulting from the following equation:

$$e(f) = \parallel \widetilde{f} - f\|_{L_\infty}, \tag{37}$$

where $\widetilde{f}$ is the numerical approximation of the exact solution f in the collocation points $\{\xi_i\}_{i=1}^{m+1}$.

Example 1. Consider the inverse problem with $\Theta = (0,1) \times (0,1)$ [37]

$$\begin{cases} z_t(x,t) - z_{xx}(x,t) = f(x), & (x,t) \in \Theta, \\ z(x,0) = 0, & x \in (0,1), \\ z_x(0,t) = z_x(1,t) = 0, & t \in (0,1). \end{cases} \tag{38}$$

We attempt to approximate the heat source defined by

$$f(x) = \pi^2 \cos(\pi x). \tag{39}$$

Then with f given by (39), the forward problem presented by (38) has an analytical solution as follows:

$$z(x,t) = \sum_{n=1}^{\infty} \frac{1 - e^{-(n\pi)^2 t}}{(n\pi)^2} f_n \cos(n\pi x), \tag{40}$$

where f_n is the Fourier coefficient as follows:

$$f_n = 2 \int_0^1 f(x) \cos(n\pi x) dx. \tag{41}$$

From (40), we have

$$h(x) = z(x,1) = \sum_{n=1}^{\infty} \frac{1 - e^{-(n\pi)^2}}{(n\pi)^2} f_n \; cos(n\pi x). \tag{42}$$

Table 1: Comparison of errors estimate obtained for functions f in Example 1 over a range of σ values between the proposed method and [37]

	$\sigma = 10^{-4}$	$\sigma = 10^{-5}$	$\sigma = 10^{-6}$
Proposed method	$6.1387e-01$	$6.8907e-02$	$1.2413e-02$
Method of [37]	$2.511e-01$	$4.81e-02$	$3.26e-02$

Table 1 analyzes the error behavior of the proposed method in here and the presented method in [37] by varying the value of σ.

Table 2: Errors estimate for the functions f and z in Example 1 over a range of σ values

	$\sigma = 10^{-4}$	$\sigma = 10^{-5}$	$\sigma = 10^{-6}$	$\sigma = 10^{-7}$
Eroor(f)	$6.1387e-01$	$6.8907e-02$	$1.2413e-02$	$8.3175e-03$
Eroor(z)	$5.5543e-02$	$6.0304e-03$	$6.0868e-04$	$6.3734e-05$

The approximate solutions for the functions f and z are shown in Figure 1. The approximation inaccuracy are shown in Table 2. Figure 2 depicts the convergence of the suggested approach.

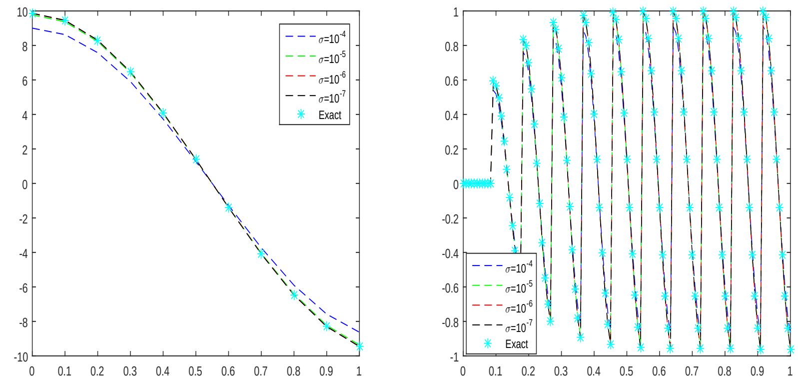

Figure 1: Results of Example 1's numerical solutions for functions f (left) and z (right) for a range of σ values

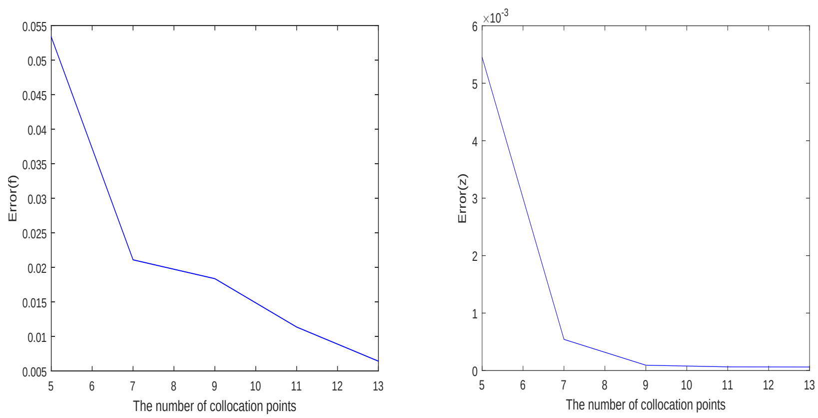

Figure 2: Convergence of the numerical solutions of Example 1 for functions f (left) and z (right) for a range of collocation point values

Example 2. Consider the inverse problem with $\Theta = (0, 1) \times (0, 1)$ [28]

$$\begin{cases} z_t(x,t) - z_{xx}(x,t) = f(x), & (x,t) \in \ \Theta, \\ z(x,0) = \sin(\pi x), & x \in \ (0,1), \\ z_x(1,t) = -z_x(0,t) = \pi(e^{-\pi^2 t} - 2), & t \in \ (0,1). \end{cases} \tag{43}$$

We attempt to approximate the heat source defined by

$$f(x) = 2\pi^2 \sin(\pi x). \tag{44}$$

Then with f given by (44), the forward problem presented by (43) has an analytical solution as follows:

$$z(x,t) = -(e^{-\pi^2 t} - 2)\sin(\pi x). \tag{45}$$

From (45), we have

$$h(x) = z(x,1) = -(e^{-\pi^2} - 2)\sin(\pi x). \tag{46}$$

Table 3: Values of errors for the functions f and z with different values of σ in Example 2

	$\sigma = 10^{-5}$	$\sigma = 10^{-6}$	$\sigma = 10^{-7}$	$\sigma = 10^{-8}$
Eroor(f)	$4.3462e-00$	$1.2586e-00$	$7.4287e-01$	$6.8760e-01$
Eroor(z)	$9.7406e-02$	$1.6259e-02$	$1.7443e-03$	$1.7581e-04$

Figure 3 shows the approximate solutions for the functions f and z. The approximation error is presented in Table 3. The convergence of the proposed method can be seen in Figure 4.

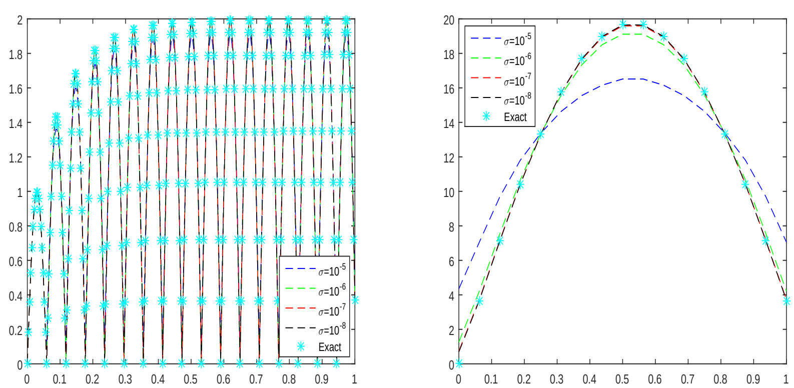

Figure 3: Behavior of the numerical solutions for the functions f (right) and z (left) at some different values of σ in Example 2

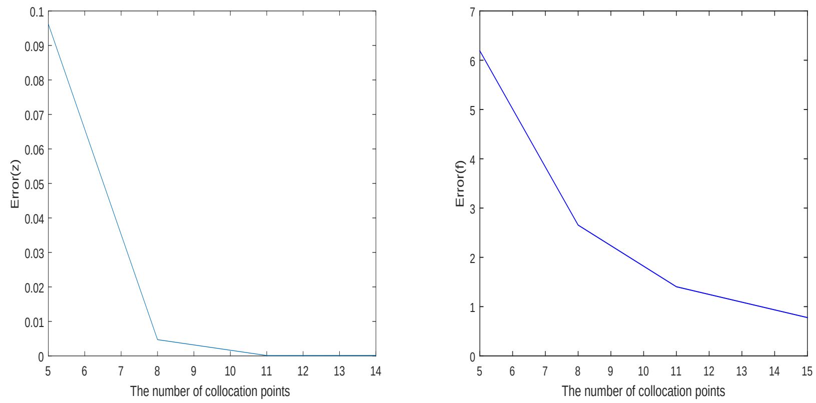

Figure 4: Convergence of the numerical solutions for the functions f (right) and z (left) at some different values of collocation points in Example 2

7 Conclusion

When it comes to finding a regular and stable solution, inverse problems that are related to PDEs provide a significant computing challenge since it is very difficult to do so. The scope of this work is an investigation into an inverse space-dependent source issue for a heat equation. A shifted Legendre polynomial and an optimum control strategy were used in the process of creating a heat source. One of the most popular and efficient tools for resolving computing problems is the Legendre polynomial. The shifted Legendre polynomials operational matrix was utilized to resolve this optimal control problem. By applying the suggested collocation method and using an operational matrix, the issue was converted into a set of equations that can be solved using algebra. When utilizing this method to solve an inverse problem, as demonstrated by the examples provided in the paper, a high level of precision was achieved in the solution. The method presented here was based on the optimal control problem and shifted Legendre polynomials. In future work, we will try to use machine learning techniques, including deep neural networks, to solve this problem.

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