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Research Article

Regularization of the generalized auto-convolution Volterra integral equation of the first kind

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Abstract

In this paper, a generalized version of the auto-convolution Volterra integral equation of the first kind as an ill-posed problem is studied. We apply the piecewise polynomial collocation method to reduce the numerical solution of this equation to a system of algebraic equations. According to the proposed numerical method, for $n = 0$ and $n = 1, \dots, N-1$, we obtain a nonlinear and linear system, respectively. We have to distinguish between two cases, nonlinear and linear systems of algebraic equations. A double iteration process based on the modified Tikhonov regularization method is considered to solve the nonlinear algebraic equations. In this process, the outer iteration controls the evolution path of the unknown vector U_0^δ in the selected direction $\tilde{u}_0$, which is determined from the inner iteration process. For the linear case, we apply the Lavrentiev $\tilde{m}$ times iterated regularization method to deal with the ill-posed linear system. The validity and efficiency of the proposed method are demonstrated by several numerical experiments.

AMS subject classifications (2020): 65R20, 65Q20, 45G10.

Keywords: Auto-convolution Volterra integral equations; Collocation methods; Tikhonov regularization method; Lavrentiev regularization method.

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1 Introduction

Auto-convolution equations have been investigated by mathematicians in various fields. Some of the applications of auto-convolution equations are in spectroscopy, stochastic, and probability theory. In 1990, for a deeper understanding of mathematics and applied numerical analysis of stochastic [19], inverse deconvolution problems were made as a class of problems in continuous spaces. Auto-convolution Volterra integral equations (AVIEs) of the first kind,

$$g(t) = \int_0^t u(t-s)u(s)ds, \tag{1}$$

as an ill-posed problem, have been studied in [3, 5, 6, 9]. Conditions for the compactness, injectivity, and weakly closed of the associated integral operator in (1) have been studied in [9]. Also, the authors applied the Tikhonov regularization for the nonlinear ill-posed problem (1) together with an approach to defining different levels and degrees of ill-posedness in Hilbert spaces. In [3], the authors have developed a local regularization theory for the nonlinear inverse auto-convolution problem (1), which provides regularization methods to preserve the causal nature of the auto-convolution problem. Also, they investigated the convergence of the regularized solutions to the true solution as the noise level in the data shrank to zero and supplied convergence rates for the cases of both L^2 and continuous data. For confluent hypergeometric functions, a class of auto-convolution equations of the first kind was derived in [23] and a further class of auto-convolution equations of the first kind with Mittag-Leffler type functions as solutions was treated. In [5], the authors have presented new results concerning the ill-posedness character of the nonlinear auto-convolution equation (1) and discussed quasi-solutions restricted to specific (relatively) compact subsets together with limitations of Fourier transform techniques for analyzing the auto-convolution problem. Some regularization methods and specific numerical approximations applied to the first kind of integral equations as well as the equation (1) were also discussed in [10, 11, 12, 7, 18, 2, 22, 3, 1, 24, 26, 28, 29, 30, 4, 13, 25, 16]. The auto-convolution equation (1) is an interesting example of a nonlinear ill-posed operator equation with interesting properties. We can consider important properties from [9] in the following form:

The auto-convolution operator $F : L^2(0,1) \longrightarrow L^2(0,1)$ is defined by

$$[F(u)](t) := \int_0^t u(t-s)u(s)ds, \qquad 0 \le t \le 1,$$

with a domain

$$D(F) := \{u \in L^2(0,1) : u(t) \ge 0 \text{ a.e. } on\ [0,1]\}. \tag{2}$$

The operator F is continuous, weakly closed but not compact; see [9]. On the other hand, for all $u^* \in D(F)$, the Fréchet derivative of F as a compact linear operator can be defined by

$$[F'(u^*)u](t) := 2\int_0^t u^*(t-s)u(s)ds, \qquad 0 \le t \le 1,$$

and also, we have

$$\|F(u) - F(u^*) - F'(u^*)(u-u^*)\|_{L^2(0,1)} \le \|u-u^*\|^2_{L^2(0,1)}.$$

Using the Titchmarsh convolution theorem [20], the injectivity of the auto-convolution operator F and solutions of (1) can be given by the following theorem.

Theorem 1. [6] For any $u \in L^2(0,1)$, let

$$\varepsilon(u) := \sup\{0 \le \epsilon \le 1 : u(t) = 0 \text{ a.e. } \ on\ [0,\epsilon]\}.$$

Then the auto-convolution equation (1) together with (2) has a unique solution if and only if $\varepsilon(g) = 0$. If $u^* \in D(F)$ is the uniquely determined solution, then it fulfills the condition $\varepsilon(u^*) = 0$.

In this paper, we consider the generalized AVIEs of the first kind as

$$\int_0^t K(t,s)u(t-s)u(s)ds = g(t), \qquad t \in I := [0,1], \tag{3}$$

where g, K are given functions and $u(t)$ is unknown function. Recently, in [8], the authors have studied an ill-posed inverse problem of auto-convolution equation such that this inverse problem occurs in nonlinear optics in the context of ultrashort laser pulse characterization. They applied an iterative regularization approach, which is specifically adapted to the physical situation in pulse characterization, using a nonstandard stopping rule for the iteration process of computing regularized solutions. In [27], an AVIE of the second kind was solved by using the collocation method according to piecewise polynomials, and the (super) convergence of the mentioned method has been studied. However, as far as we know, it seems to be an open problem under what conditions we can consider for the kernel in the integral equation (3) to investigate the solutions of this equation similar to Theorem 1.

Here, we consider the numerical collocation method based on piecewise polynomials to solve (3). According to the proposed numerical method, we have to distinguish between two cases, nonlinear and linear systems. Since the auto-convolution problem (3) is ill-posed, for given g, the solutions u need not be uniquely determined and mainly small perturbations in the right-hand side g caused by noisy data may lead to arbitrarily large errors in the solution. To overcome the negative consequences of ill-posedness, we consider a novel double iteration process from [21] and the Lavrentiev $\tilde{m}$ times iterated regularization method [14] to deal with the ill-posed nonlinear and linear algebraic systems, respectively.

2 Collocation methods

Let Π_h be a uniform partition of the interval I with grid points

$$t_n = nh, \qquad n = 0, \dots, N,$$

and let h be the stepsize. We consider polynomial spline approximations $u_h(t)$ of the exact solution $u(t)$ in the spline space

$$S_{m-1}^{-1}(\Pi_h) = \{v : v|_{\sigma_n} \in \pi_{m-1}(0 \le n \le N-1)\}, \tag{4}$$

where $v|_{\sigma_n}$ is the restriction of v to the subinterval $\sigma_n = [t_n, t_{n+1}]$, and $\pi_{m-1}(m \ge 1)$ denotes the set of real polynomials of degree not exceeding $m-1$. Let the collocation parameters be $0 < c_1 < c_2 < \cdots < c_m \le 1$ and let the collocation points be $t_{nj} = t_n + c_j h, j = 1, \dots, m, n = 0, \dots, N-1$. Hence, our collocation solution $u_h \in S_{m-1}^{-1}(\Pi_h)$ is determined by

$$g(t) = \int_0^t K(t,s) u_h(t-s) u_h(s) ds, \tag{5}$$

where the collocation equation is satisfied for $t = t_{nj}, j = 1, \dots, m, n = 0, \dots, N-1$. We consider the approximate solution on the subinterval σ_n as

$$u_h|_{\sigma_n} = u_h^n(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \qquad s \in (0,1], \tag{6}$$

where $U_{n,j} := u_h(t_n + c_j h)$ and

$$L_j(s) := \prod_{i=1, i \ne j}^{m} \frac{s - c_i}{c_j - c_i}, \qquad j = 1, \dots, m.$$

Inserting (6) into (5), we obtain

$$\begin{aligned} g(t_{n,i}) &= \int_0^{t_{n,i}} K(t_{n,i}, s) u_h(t_{n,i} - s) u_h(s) ds \\ &= \int_{t_n}^{t_{n,i}} K(t_{n,i}, s) u_h^0(t_{n,i} - s) u_h^n(s) ds \\ &\quad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k,i}} K(t_{n,i}, s) u_h^{n-k}(t_{n,i} - s) u_h^k(s) ds \end{aligned}$$

$$+\sum_{k=0}^{n-1}\int_{t_{k,i}}^{t_{k+1}} K(t_{n,i},s)u_h^{n-k-1}(t_{n,i}-s)u_h^k(s)ds. \tag{7}$$

System (7) for $n=0$ is a nonlinear system, and for $n=1,\dots,N-1$, it is a linear system. Then we have to distinguish between two cases, $n=0$ and $n=1,\dots,N-1$.

2.1 Double iteration process to solve ill-posed nonlinear algebraic systems

Case I: $n=0$. Now (7) becomes

$$\begin{aligned}
g(c_ih) =& \int_0^{c_ih} K(c_ih,s)u_h^0(c_ih-s)u_h^0(s)ds\\
=&h\int_0^{c_i} K(c_ih,sh)u_h^0((c_i-s)h)u_h^0(sh)ds\\
=&h\sum_{j,k=1}^{m}\int_0^{c_i} K(c_ih,sh)L_j(c_i-s)L_k(s)ds\ u_h^0(c_jh)u_h^0(c_kh)\\
=&h\sum_{j,k=1}^{m}\alpha_{jk}^{(i)}u_h^0(c_jh)u_h^0(c_kh) = h\sum_{j,k=1}^{m}\alpha_{jk}^{(i)}U_{0,j}U_{0,k},
\end{aligned} \tag{8}$$

where

$$\alpha_{jk}^{(i)} = \int_0^{c_i} K(c_ih,sh)L_j(c_i-s)L_k(s)ds.$$

Nonlinear system (8) can be written as

$$f(U_0)-G_0=0, \quad or \quad F(U_0)=0, \tag{9}$$

where

$$U_0=(U_{0,1},\dots,U_{0,m})^T\in\mathbb{R}^m, \qquad G_0=(g(c_1h),\dots,g(c_mh))^T.$$

Let us assume that g^δ is an available approximation of g satisfying

$$\|g^\delta-g\|\le\delta,$$

where δ is a given noise level.

Now, we consider a novel double iteration process from [21] to deal with the ill-posed nonlinear algebraic equation system (9). The proposed method combines the residual norm based algorithm and the modified Tikhonov's regularization method. For the outer iteration process, the evolution path of the unknown vector follows the searching direction determined from the inner iteration process, and the process requires the path falls on the space-time manifold such that the convergence rate can be guaranteed. For the inner iteration, the direction of evolution $\tilde{u}$ is determined by solving a linear algebraic equation

$$B^T B\tilde{u} = B^T F,$$

where B is the Jacobian matrix. For an ill-posed system, this linear algebraic equation is very difficult to solve since the resulting leading coefficient matrix is ill-posed in nature. The modified Tikhonov's regularization method [15] is adapted to solve the ill-posed linear algebraic equation. However, for ill-posed problems to really find the solution of $\tilde{u}$ may require too many iteration steps for the modified Tikhonov's regularization method, which makes the whole numerical process not economical at all. Therefore, the inner iteration process stops while the direction $\tilde{u}$ makes the value of $\tilde{a}_0$ smaller than the selected margin $\tilde{a}_c$ or when the number of inner iteration steps exceeds the maximum tolerance $\tilde{I}_{\max}$. For the outer iteration process, it terminates once the root mean square error for the residual is less than the convergence criterion ε or when the number of inner iteration steps exceeds the maximum tolerance $\tilde{I}_{\max}$.

Now, first, we consider a perturbed version of the system (9) as

$$F^\delta(U_0^\delta) = 0, \tag{10}$$

where $F^\delta(U_0^\delta) = f(U_0^\delta) - G_0^\delta$. Then, we proposed a double iteration process for system (9) based on the following algorithm.

Remark 1. Let $\theta_1 = (x_1, \ldots, x_n)^T$ and let $\theta_2 = (y_1, \ldots, y_n)^T$. Then the root mean squared error (RMSE) is defined as

$$REMS(\theta_1, \theta_2) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - y_i)^2}.$$

Remark 2. From [21], the value of $\tilde{a}_c$ is suggested to be in the range of 2.5 to 4, and the value of $\tilde{I}_{\max}$ actually depends on the selection of $\tilde{a}_c$ such that it is suggested to be in the range of 30,000 to 80,000. The selection of ε depends on the system we want to solve. If the system is a well-posed system, then the value of ε can be very small, and also, if the system is an ill-posed system, then the value of ε should not be very big.

Algorithm 1 Double Iteration Process (DIP)

Step 1. Give initial guess $U_0^{\delta,0}$ and ε , α, $\tilde{I}_{\max}$, $\tilde{a}_c$.

Step 2.Outer Iteration: For $k = 0, 1, 2, \ldots$ Repeat

Compute $F_k^\delta = F^\delta(U_0^{\delta,k})$ and the Jacobian matrix $B_k = B(U_0^{\delta,k})$.

Step 3. Inner Iteration: Give the initial guess of $\tilde{u}$ as $\tilde{u}_0 = \dfrac{B_k^T F_k^\delta}{||B_k^T F_k^\delta||}$,

For $p = 1, 2, \ldots, \tilde{I}_{\max}$, solve $\tilde{u}_{p+1}$ by $(B_k^T B_k + \alpha I)\tilde{u}_{p+1} = B_k^T F_k^\delta + \alpha \tilde{u}_p$,

Compute $\tilde{v}_{p+1} = B_k \tilde{u}_{p+1}$,

$\eta_{p+1} = \frac{||F_k^\delta||^2 ||\tilde{v}_{p+1}||^2}{((F_k^\delta)^T \tilde{v}_{p+1})^2}$,

(a) If $\eta_{p+1} \leq \tilde{a}_c$, then $\eta_k = \eta_{p+1}$, $\tilde{u}_k = \tilde{u}_{p+1}$, $\tilde{v}_k = \tilde{v}_{p+1}$ and one terminates the inner iteration; otherwise, continue.

(b) If $p = \tilde{I}_{\max}$, then terminate the whole process.

Step 4. End of Inner Iteration:

Calculate $r_k = ||1 - \dfrac{\eta_k}{2}||$,

$U_0^{\delta,k+1} = U_0^{\delta,k} - (1 - r_k)\dfrac{(F_k^\delta)^T \tilde{v}_k}{||\tilde{v}_k||^2}\tilde{u}_k.$

If $RMSE \leq \varepsilon$ or (b) is true, then the outer iteration process stop; otherwise, continue.

End of Iteration Process.

2.2 The Lavrentiev regularization method to solve ill-posed linear algebraic system

Case I: For $n = 1, \ldots, N-1$, we can rewrite system (7) as

$$\begin{aligned} g(t_{n,i}) = & h \int_0^{c_i} K(t_{n,i}, t_n + sh) u_h^0((c_i - s)h) u_h^n(t_n + sh) ds \\ & + h \int_0^{c_i} K(t_{n,i}, sh) u_h^n(t_n + (c_i - s)h) u_h^0(sh) ds \end{aligned} \tag{11}$$

$$+ h\sum_{l=1}^{n-1}\int_0^{c_i} K(t_{n,i}, t_l + sh)u_h^{n-l}(t_{n-l} + (c_i - s)h)u_h^l(t_l + sh)ds$$
$$+ h\sum_{l=0}^{n-1}\int_{c_i}^{1} K(t_{n,i}, t_l + sh)u_h^{n-l-1}(t_{n-l} + (c_i - s)h)u_h^l(t_l + sh)ds.$$

Using (6), we have

$$\begin{aligned} & h\sum_{j,k=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(c_i - s)L_k(s)dsU_{0,j}U_{n,k} \\ & + h\sum_{j,k=1}^{m}\int_0^{c_i} K(t_{n,i}, sh)L_j(s)L_k(c_i - s)dsU_{0,j}U_{n,k} \\ & \quad = g(t_{n,i}) - h\sum_{l=1}^{n-1}\sum_{j,k=1}^{m}\int_0^{c_i} K(t_{n,i}, t_l + sh)L_j(c_i - s)L_k(s)dsU_{n-l,j}U_{l,k} \\ & \qquad - h\sum_{l=0}^{n-1}\sum_{j,k=1}^{m}\int_{c_i}^{1} K(t_{n,i}, t_l + sh)L_j(1 + c_i - s)L_k(s)dsU_{n-l-1,j}U_{l,k}. \end{aligned} \tag{12}$$

Linear system (12) can be written as

$$A_nU_n = Y_n, \tag{13}$$

where

$$U_n = (U_{n,1}, \dots, U_{n,m})^T \in \mathbb{R}^m, \qquad G_n = (g(t_{n,1}), \dots, g(t_{n,m}))^T,$$

and $Y_n = \hat{f}(U_0, \dots, U_{n-1}) + G_n$. Since $U_0, \dots, U_{n-1}$ have been determined in the previous steps, then Y_n is a definite m-dimensional vector. A perturbed version of the discrete system (13) may be considered as follows:

$$(A_n + \bar{\alpha}_nI)U_n^\delta = Y_n^\delta. \tag{14}$$

Now , we apply the Lavrentiev $\tilde{m}$ times iterated regularization method [14] for the fixed integer $\tilde{m} \geq 1$, and the regularization parameter $\bar{\alpha}_n > 0$ for the equation (13), which determines U_n^δ based on the following algorithm.

Further issues related to theoretical analysis of the Lavrentiev-iterated method can be found in the references [14, 17].

3 Numerical results

Two examples are considered in this section, which provides evidence of the effectiveness of the proposed numerical methods applied to the first-kind

Algorithm 2 The Lavrentiev $\tilde{m}$ times iterated regularization method

Step 1. Give N, $\tilde{m}$.

Step 2.Outer Iteration: For $n = 1, 2, \ldots, N-1$ Repeat

Determine A_n and Y_n^δ
Choose initial guess $U_n^{\delta,0}$ and $\bar{\alpha}_n$.

Step 3. Inner Iteration: For $p = 1, 2, \ldots, \tilde{m}$,

solve $U_n^{\delta,p}$ by $(A_n + \bar{\alpha}_n I)U_n^{\delta,p} = \bar{\alpha}_n U_n^{\delta,p-1} + Y_n^\delta$

If $RMSE(U^{\delta,p}, U^{\delta,p-1}) \nrightarrow 0$, then stop the iteration, change $\bar{\alpha}_n$, and consider the inner iteration with the new $\bar{\alpha}_n$.

Step 4. End of Inner Iteration:

put $U_n^\delta = U_n^{\delta,\tilde{m}}$

End of Iteration Process.

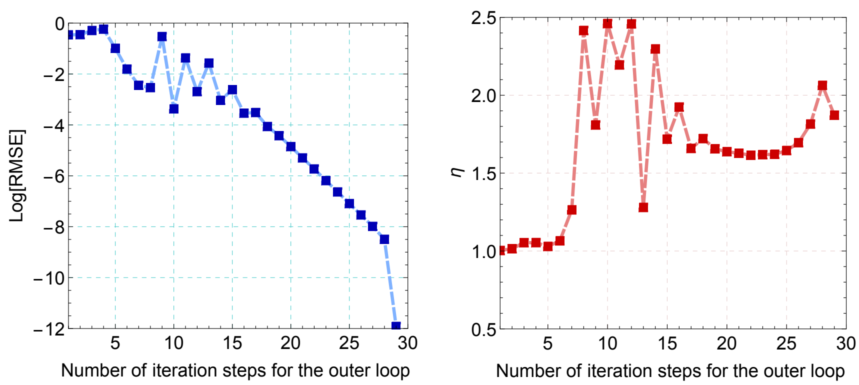

Figure 1: $Log[RMSE]$ versus the number of iteration steps with $N = 4, \delta = 0$ for the outer loop in Example 1 (left). The value of η never exceeds the critical value $\tilde{a}_c = 2.5$ in Example 1 (right).

auto-convolution equation. We produce the perturbed data g^δ by adding uniformly distributed noise from the interval $[-\delta g, \delta g]$ to the discrete values of $g(t)$ for $t = t_{n,i}, n = 0, \ldots, N-1, i = 1, \ldots, m$. All the computations were performed by Wolfram Mathematica 10.0.

Example 1. Consider the AVIE of the first kind as follows:

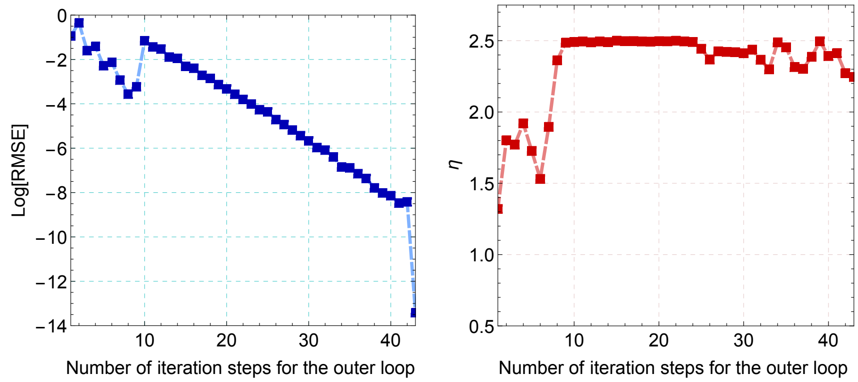

Figure 2: $Log[RMSE]$ versus the number of iteration steps with $N = 8, \delta = 0$ for the outer loop in Example 1 (left). The value of η never exceeds the critical value $\tilde{a}_c = 2.5$ in Example 1 (right).

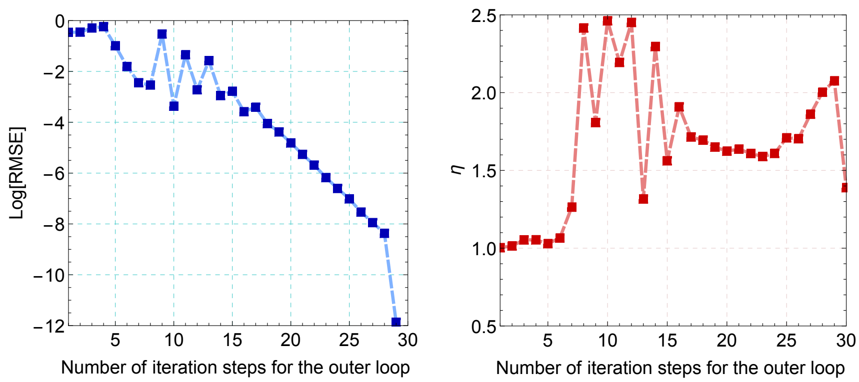

Figure 3: $Log[RMSE]$ versus the number of iteration steps with $N = 4, \delta = 10^{-6}$ for the outer loop in Example 1 (left). The value of η never exceeds the critical value $\tilde{a}_c = 2.5$ in Example 1 (right).

$$g(t) = \int_0^t \cos(t-s)u(t-s)u(s)ds, \qquad t \in [0,1], \tag{15}$$

where $g(t) = -2500e^{-t}(t + t\cos t - 2\sin t)$ and the exact solution $u(t) = 50te^{-t}$.

Let $m = 3$, let $N = 4, 8$ and let the collocation parameters be $c_1 = 0.4, c_2 = 0.6, c_3 = 1$. For $n = 0$, we have the following nonlinear system:

$$g^\delta(c_i h) = h \sum_{j,k=1}^{3} \alpha_{jk}^{(i)} U_{0,j}^\delta U_{0,k}^\delta, \qquad i = 1, 2, 3, \tag{16}$$

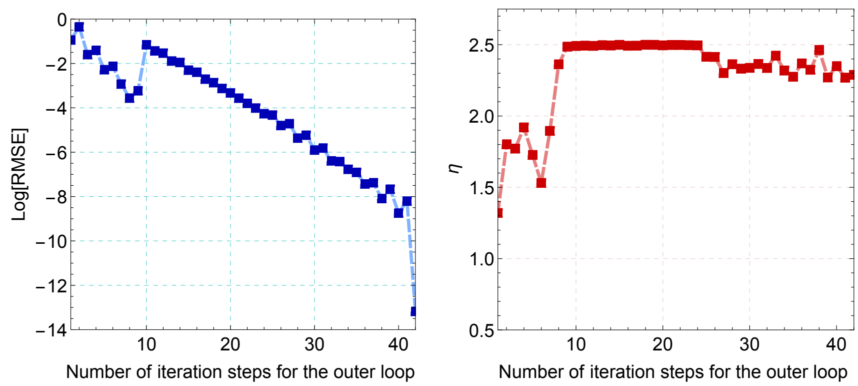

Figure 4: $Log[RMSE]$ versus the number of iteration steps with $N = 8, \delta = 10^{-6}$ for the outer loop in Example 1 (left). The value of η never exceeds the critical value $\tilde{a}_c = 2.5$ in Example 1 (right).

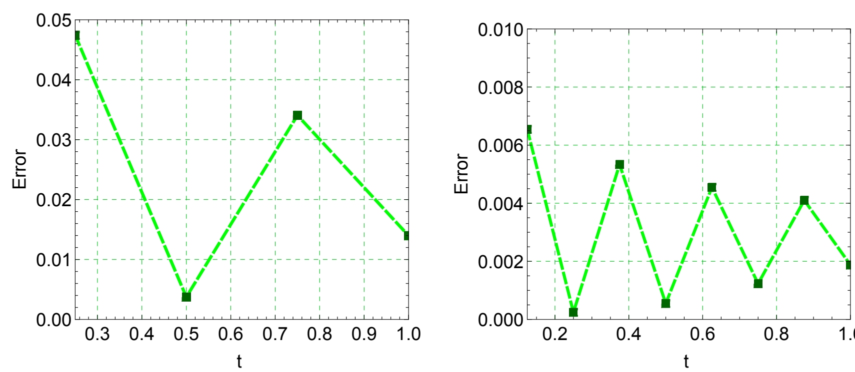

Figure 5: The absolute errors with $N = 4, \delta = 0$ at the grid-points in Example 1 (left). The absolute errors with $N = 8, \delta = 0$ at the grid-points in Example 1 (right).

where

$$\alpha_{jk}^{(i)} = \int_0^{c_i} \cos(c_i h - sh) L_j(c_i - s) L_k(s) ds, \qquad i, j, k = 1, 2, 3.$$

For $\delta = 0, 10^{-6}$, we use the double iteration process (Algorithm 1) to solve this problem with the initial guess $U_0^{\delta,0} = (1, 1, 1)^T$ and the parameters $\alpha = 0.1, \tilde{I}_{\max} = 30000, \tilde{a}_c = 2.5$ and $\epsilon = 10^{-4}$. Figures 1–4 show that for $N = 4$ and $N = 8$ with $\delta = 0, 10^{-6}$ the process terminates after 29 and 43 steps for the outer loop, respectively. Also, we observe that η never exceeds the critical value $\tilde{a}_c = 2.5$, which once more shows that this method really can guarantee the trajectory of the solution vector falls on the manifold. We report the solutions of system (16) in Table 1 and use these values to solve

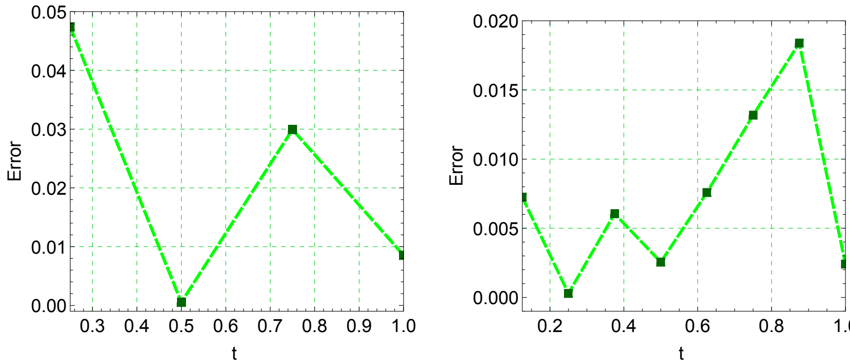

Figure 6: The absolute errors with $N = 4, \delta = 10^{-6}$ at the grid-points in Example 1 (left). The absolute errors with $N = 8, \delta = 10^{-6}$ at the grid-points in Example 1 (right).

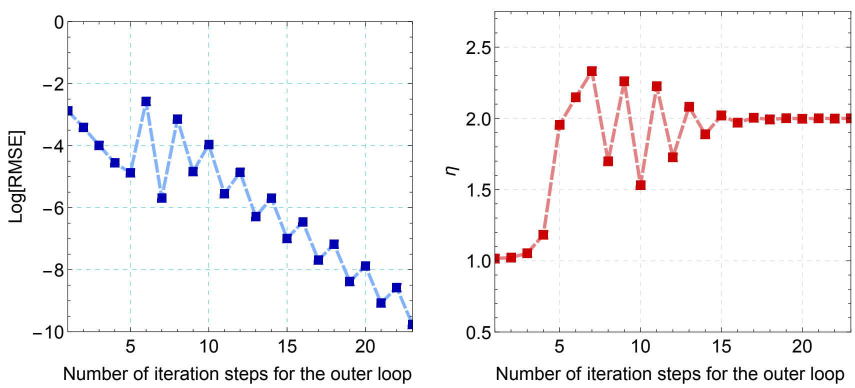

Figure 7: $Log[RMSE]$ versus the number of iteration steps with $N = 4, \delta = 0$ for the outer loop in Example 2 (left). The value of η never exceeds the critical value $\tilde{a}_c = 2.5$ in Example 2 (right).

the following linear system by the Lavrentiev $\tilde{m}$ times iterated regularization method, for $n = 1, \ldots, N-1$:

$$\begin{aligned} & h \sum_{j,k=1}^{3} \int_0^{c_i} \cos[t_{n,i} - (t_n + sh)] L_j(c_i - s) L_k(s) ds U_{0,j}^{\delta} U_{n,k}^{\delta} \\ & + h \sum_{j,k=1}^{3} \int_0^{c_i} \cos[t_{n,i} - sh] L_j(s) L_k(c_i - s) ds U_{0,j}^{\delta} U_{n,k}^{\delta} \\ & \quad = g^{\delta}(t_{n,i}) - h \sum_{l=1}^{n-1} \sum_{j,k=1}^{3} \int_0^{c_i} \cos[t_{n,i} - (t_l + sh)] L_j(c_i - s) L_k(s) ds U_{n-l,j}^{\delta} U_{l,k}^{\delta} \end{aligned} \tag{17}$$

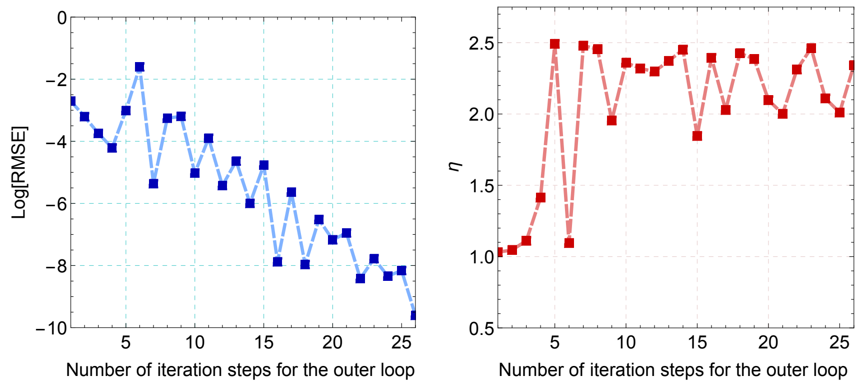

Figure 8: $Log[RMSE]$ versus the number of iteration steps with $N=8, \delta=0$ for the outer loop in Example 2 (left). The value of η never exceeds the critical value $\tilde{a}_c=2.5$ in Example 2 (right).

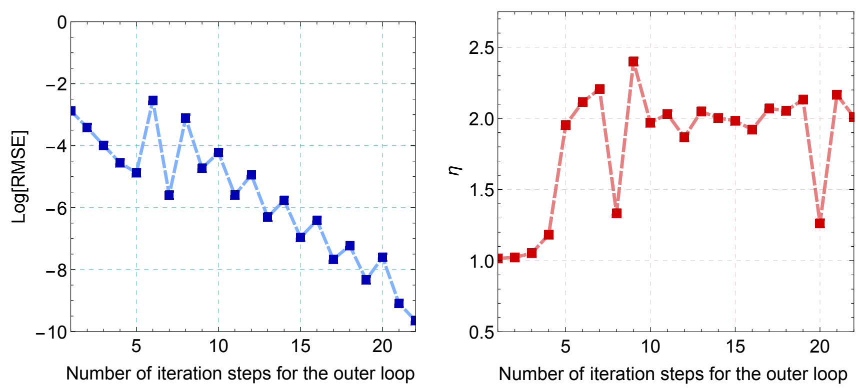

Figure 9: $Log[RMSE]$ versus the number of iteration steps with $N=4, \delta=10^{-4}$ for the outer loop in Example 2 (left). The value of η never exceeds the critical value $\tilde{a}_c=2.5$ in Example 2 (right).

$$-h\sum_{l=0}^{n-1}\sum_{j,k=1}^{3}\int_{c_i}^{1}\cos[t_{n,i}-(t_l+sh)]L_j(1+c_i-s)L_k(s)ds U^{\delta}_{n-l-1,j}U^{\delta}_{l,k},$$

$$i=1,2,3.$$

In Algorithm 2, we choose initial guess $U_n^{\delta,0}=(0,0,0)^T, \bar{\alpha}_n=0.01$ and $\tilde{m}=20$. The maximum absolute errors for $N=4,8$ and $\delta=0,10^{-6}$ have been reported in Table 2. Also, Figures 5 and 6 show the absolute errors at the grid-points for $N=4,8$ and $\delta=0,10^{-6}$.

Now, we consider a special case of (3) with $K(t,s)=1$ from [3] as follows.

Example 2. [3] Consider the AVIE of the first kind as follows:

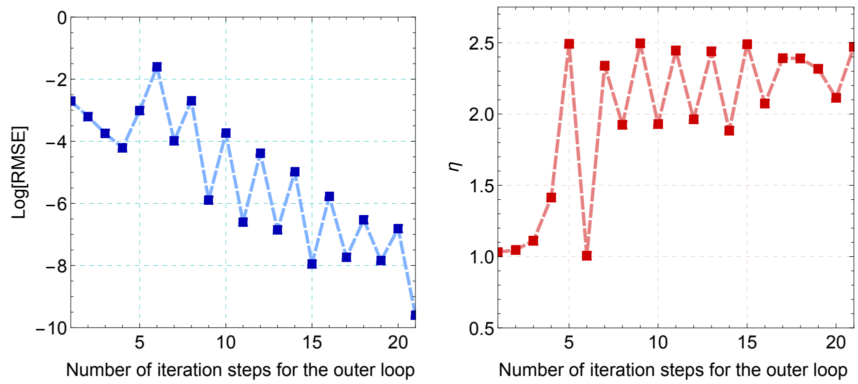

Figure 10: $Log[RMSE]$ versus the number of iteration steps with $N = 8, \delta = 10^{-4}$ for the outer loop in Example 2 (left). The value of η never exceeds the critical value $\tilde{a}_c = 2.5$ in Example 2 (right).

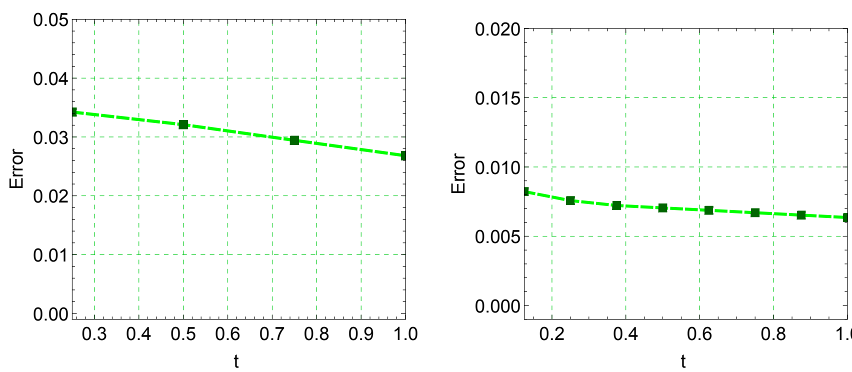

Figure 11: The absolute errors with $N = 4, \delta = 0$ at the grid-points in Example 2 (left). The absolute errors with $N = 8, \delta = 0$ at the grid-points in Example 2 (right).

	$N = 4$ $\delta = 0$	$N = 4$ $\delta = 10^{-6}$	$N = 8$ $\delta = 0$	$N = 8$ $\delta = 10^{-6}$
$U^{\delta}_{0,1}$	4.52533	4.52527	2.37834	2.37832
$U^{\delta}_{0,2}$	6.46103	6.46100	3.47984	3.47980
$U^{\delta}_{0,3}$	9.68703	9.68729	5.50843	5.50838

Table 1: The solution of system (16) in Example 1

$$g(t) = \int_0^t u(t-s)u(s)ds, \qquad t \in [0,1], \tag{18}$$

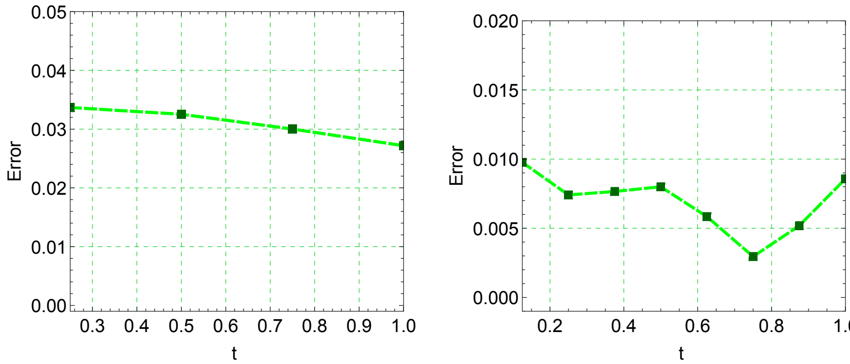

Figure 12: The absolute errors with $N = 4, \delta = 10^{-4}$ at the grid-points in Example 2 (left). The absolute errors with $N = 8, \delta = 10^{-4}$ at the grid-points in Example 2 (right).

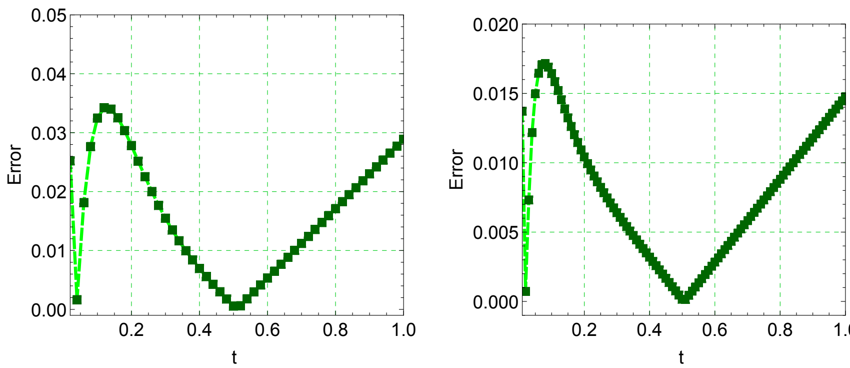

Figure 13: The absolute errors obtained by method 1 [3] with $N = 50, r = 4, \delta = 0$ at the grid-points in Example 2 (left). The absolute errors obtained by method 1 [3] with $N = 100, r = 4, \delta = 0$ at the grid-points in Example 2 (right).

N	$\delta = 0$	$\delta = 10^{-6}$
4	4.74×10^{-2}	4.74×10^{-2}
8	6.54×10^{-3}	1.83×10^{-2}

Table 2: The maximum absolute errors in Example 1

where $g(t) = \frac{3}{10}t^5 - \frac{3}{2}t^4 + t^3 + \frac{3}{4}t^2 + \frac{1}{16}t$ and the exact solution $u(t) = 1 - 3(t - \frac{1}{2})^2$.

In Example 2, assume that $m = 2, N = 4, 8, \delta = 0, 10^{-4}$, and the collocation parameters as $c_1 = 0.6, c_2 = 1$. For $n = 0$, we solve the obtained nonlinear system by the double iteration process (Algorithm 1) with the ini-

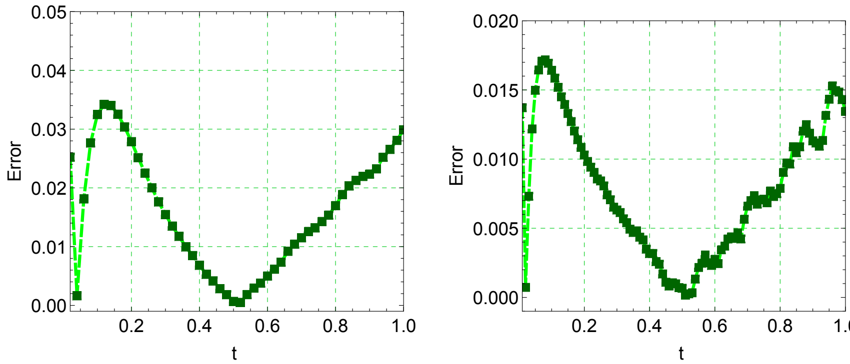

Figure 14: The absolute errors obtained by method 1 [3] with $N = 50, r = 4, \delta = 10^{-4}$ at the grid-points in Example 2 (left). The absolute errors obtained by method 1 [3] with $N = 100, r = 4, \delta = 10^{-4}$ at the grid-points in Example 2 (right).

tial guess $U_0^{\delta,0} = (1,1)^T$ and the parameters $\alpha = 0.1, \tilde{I}_{\max} = 30000, \tilde{a}_c = 2.5$ and $\epsilon = 10^{-4}$. In what follows, for $n = 1, \dots, N-1$, we solve the obtained linear system by the Lavrentiev $\tilde{m}$ times iterated regularization method with initial guess $U_n^{\delta,0} = (0,0)^T, \bar{\alpha}_n = 0.01$ and $\tilde{m} = 20$. The maximum absolute errors for $N = 4, 8$ and $\delta = 0, 10^{-4}$ have been reported in Table 3.

N	$\delta = 0$	$\delta = 10^{-4}$
4	3.42×10^{-2}	3.56×10^{-2}
8	8.22×10^{-3}	9.76×10^{-3}

Table 3: The maximum absolute errors in Example 2

We also consider a local regularization technique from [3] by method 1 for the auto-convolution integral equation of the first kind (18) and report the maximum absolute errors for $N = 50, 100$, $r = 4$ and $\delta = 0, 10^{-4}$ in Table 4.

N	$\delta = 0$	$\delta = 10^{-4}$
50	3.42×10^{-2}	3.43×10^{-2}
100	1.71×10^{-2}	3.01×10^{-2}

Table 4: The maximum absolute errors obtained by method 1 [3] in Example 2

Similar to Example 1, in Figures 7–10, we show the number of the termination of the process in the outer loop and observe that η never exceeds the critical value $\tilde{a}_c = 2.5$. Also, Figures 11 and 12 represent the absolute errors at the grid-points for $N = 4, 8$ and $\delta = 0, 10^{-4}$. The absolute errors obtained by method 1 [3] with $N = 50, 100, r = 4, \delta = 0, 10^{-4}$ at the grid-points are shown in Figures 13 and 14.

4 Conclusion

In this paper, we considered the numerical collocation method based on piecewise polynomials to solve the generalized version of the AVIE of the first kind as an ill-posed problem. We obtained two ill-posed nonlinear and linear systems. The double iteration process and the Lavrentiev $\tilde{m}$ times iterated regularization method have been considered nonlinear and linear systems, respectively. Numerical examples demonstrated the validity and efficiency of the proposed method. In our future work, we will investigate the existence, uniqueness, and structure of the solution related to the generalized version of the first-kind AVIEs similar to Theorem 1.

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