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Research Article

A discrete orthogonal polynomials approach for fractional optimal control problems with time delay

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Abstract

An efficient direct and numerical method has been proposed to approximate a solution of time-delay fractional optimal control problems. First, a class of discrete orthogonal polynomials, called Hahn polynomials, has been introduced and their properties are investigated. These properties are employed to derive a general formulation of their operational matrix of fractional integration, in the Riemann–Liouville sense. Then, the fractional derivative of the state function in the dynamic constraint of time-delay fractional optimal control problems is approximated by the Hahn polynomials with unknown coefficients. The operational matrix of fractional integration together with the dynamical constraints is used to approximate the control function directly as a function of the state function. Finally, these approximations were put in the performance index and necessary conditions for optimality transform the under consideration time-delay fractional optimal control problems into an algebraic system. Some illustrative examples are given and the obtained numerical results are compared with those previously published in the literature.

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Keywords: Delay fractional optimal control problems; Riemann–Liouville integration; Hahn polynomials; Operational matrix.

1 Introduction

It has been found that integer-order calculus is not an appropriate tool for modeling complex systems in science and engineering. Fractional calculus, as an extension of derivatives and integrals to noninteger orders, has been recently used to model many fundamental problems; see [32, 38]. Recently,

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fractional differential equations have found many applications in various fields of engineering and physics such as colored noise, signal processing, electromagnetism, electrochemistry, dynamic of viscoelastic materials, continuum and statistical mechanics, solid mechanics, and fluid-dynamic traffic model; see [3, 4, 9, 12, 40].

Contrary to ordinary differential equations (ODEs), delay differential equations (DDEs) contain derivatives that depend on the solution at previous times, thus making the mathematical model closer to the real-world phenomenon; see [26, 27]. Recently, many researches have been focused on DDEs and their numerical solution. For example, the finite difference method [22], Adams methods [41], Adomian decomposition method [13], variational iteration method [45], hybrid functions [29], spectral methods [19, 39, 44], and wavelet methods [35, 36] have been employed to approximate solution of DDEs.

Over the last decades, numerical methods based on orthogonal functions have been frequently used to approximate solution of differential and integral equations [7, 28, 31, 34, 39]. The main characteristic of these methods is that they reduce under consideration problems to those of solving a system of algebraic equations, thus greatly simplifying the problems. Depending on their structure, the orthogonal functions are classified into three families: the first class consists of sets of piecewise constant orthogonal functions such as the Walsh functions and block pulse functions. The second class consists of sets of sine-cosine functions, and the last class is the most widely used orthogonal polynomials, such as Laguerre, Legendre, and Chebyshev polynomials; see [28, 31]. It is well known that orthogonal polynomials that are solutions of singular Sturm–Liouville problems allow approximation of smooth functions, where truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation. This phenomenon is usually referred to as spectral accuracy; see [8, 30]. Moreover, according to the defined inner product in the solution space, orthogonal polynomials are classified into two main classes: continuous and discrete. For continuous orthogonal polynomials such as Legendre, Chebyshev, Hermite, and Laguerre, one has to evaluate an integral in the inner product, whereas discrete orthogonal polynomials come with a discrete scalar product and hence the integral becomes a sum. Although continuous orthogonal polynomials have been more frequently used to approximate solution of functional equations, there are some advantages of using discrete orthogonal polynomials. For example, by using discrete orthogonal polynomials, the Fourier coefficients can be calculated with the aid of a summation and the obtained coefficients are exact. Consequently, comparison to continuous cases, implementation of discrete orthogonal polynomials is more efficient and less complex; see [16, 43]. Moreover, there is a close connection between stochastic processes and discrete orthogonal polynomials that motivated many researchers to consider them to solve stochastic differential equations; see [2, 43].

The optimal control of a system is one of the most practical subjects in science and engineering [5, 6, 17, 21, 25, 33]. The optimal control problems involve minimization of a performance index subject to a dynamical system. As generalizations of the classical optimal control, FOCPs are problems in which fractional derivatives or integrals are used in the performance index or constraints. Similar to the other types of fractional functional equations, most of FOCPs do not have exact and analytic solutions [7, 34, 35, 37]. Furthermore, a time-delay fractional optimal control problem (TDFOCP) is a fractional optimal control problem in which the dynamic of system contains some time-delay equations. The control of time-delay systems frequently in electronic, age structure, biological, chemical, electronic, and transportation systems [14, 20, 23, 41]. Due to their variety of applications in the realistic models of phenomena, the control of time-delay systems has been investigated by many engineers and scientists. Also, considerable attention has been focused on the approximate and numerical solution of them [7,34,35,37]. Generally, there are two main approaches to the approximate solution of TDFOCPs. The first one is based on constructing a Hamiltonian system and then solve the arising two-point boundary value problem, which is the indirect approach, and the other involves facing directly the problems by discretizing or approximating the functions without constructing the Hamiltonian equations. Recently, more attention has been done to direct approaches [7,22,35].

The main purpose of this work is to introduce a new kind of discrete orthonormal polynomials. Some properties of these discrete polynomials are studied and a general formulation of their operational matrix of fractional integration, in the Riemann–Liouville sense, is derived. Make use of these orthogonal polynomials and their operational matrix, an efficient direct numerical method is proposed to approximate the solution of the following TDFOCP:

$$Min\ J = \frac{1}{2}\int_0^1 \left[r(t)x^2\left(t\right) + s(t)u^2\left(t\right)\right]dt, \tag{1}$$

subjected to

$$D^{\nu}x\left(t\right) = a\left(t\right)x\left(t\right) + b\left(t\right)u\left(t\right) + c\left(t\right)x\left(t-\tau\right) + e\left(t\right)u\left(t-\eta\right) + f(t), \tag{2}$$

$$t \in [0,1],\ 0 < \nu \leq 1, \tag{3}$$

$$\begin{array}{ll} x(t) = g_1(t), & t \in \left[-\tau, 0\right], \\ u(t) = g_2(t), & t \in \left[-\eta, 0\right], \end{array} \tag{4}$$

in which $x(t)$ and $u(t)$ are the state and control functions, respectively. Furthermore, $r(t)$ and $s(t)$ are positive functions and $a\left(t\right)$, $b\left(t\right)$, $c\left(t\right)$, $e\left(t\right)$, and $f\left(t\right)$ are continuous functions, $g_1(t)$ and $g_2(t)$ are arbitrary known functions defined on the intervals $[-\tau, 0]$ and $[-\eta, 0]$, respectively. Also, τ and $\eta > 0$

are given constant delays and $D^{\nu}x(t)$ is the fractional derivative of state function $x(t)$ in the Caputo sense.

This paper is organized as follows: In Section 2, some basic definitions and preliminary remarks on the fractional calculus are presented. The Hahn polynomials and their properties are introduced in Section 3. The operational matrix of fractional integration and product operational matrix for the Hahn polynomials have been derived in Section 4. In Section 5, an efficient direct method is proposed to solve TDFOCPs. To confirm the efficiency and accuracy of the presented method, some illustrative examples are given in Section 6. Finally, a conclusion is given in Section 7.

2 Definitions and preliminaries

Although, the fractional order operators enable to model a wider class of problems, there does not exist a unique definition of fractional derivative. Among the several formulations of the generalized derivative, the Riemann–Liouville and Caputo definitions are the most commonly used, which can be described as follows [32,38].

Definition 1. A real function $f(t), t > 0$, is said to belong the space $C_{\mu}, \mu \in \mathbb{R}$ if there exist a real number $p > \mu$ and a function $f_1(t) \in C[0,\infty)$ such that $f(t) = t^p f_1(t)$. Moreover, if $f^{(n)} \in C_{\mu}$, then $f(t)$ is said to belong the space $C_{\mu}^{n},\ n \in \mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integration of order $\nu \geq 0$ of a function $f \in C_{\mu}, \mu \geq -1$, is defined as

$$(I^{\nu}f)(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-\tau)^{\nu-1} f(\tau)d\tau, & \nu > 0, \\ f(t), & \nu = 0. \end{cases}$$

The Riemann–Liouville fractional operator I^{ν} has the following properties:

$$I^{\nu_1}\left(J^{\nu_2}f(t)\right) = I^{\nu_2}\left(I^{\nu_1}f(t)\right), \quad \nu_1, \nu_2 \geq 0,$$

$$I^{\nu_1}\left(I^{\nu_2}f(t)\right) = I^{\nu_1+\nu_2}f(t), \quad \nu_1, \nu_2 \geq 0,$$

$$I^{\nu}t^{\lambda} = \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}t^{\nu+\lambda}, \quad \nu \geq 0, \lambda > -1.$$

Definition 3. The Caputo fractional derivative of order $\nu > 0$ is defined as

$$\mathcal{D}^{\nu} f(t)=\begin{cases}\dfrac{d^n f(t)}{dt^n}, & \nu=n\in\mathbb{N},\\[2ex] \dfrac{1}{\Gamma(n-\nu)}\displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}}d\tau, & t>0,\ 0\le n-1<\nu<n,\end{cases}$$

where n is integer, $t>0$, and $f\in C_1^n$.

For $\mathbb{N}_0=\{0,1,2,\ldots\}$, $f\in C_{\mu},\ \mu,\lambda\ge -1$, and $n-1<\nu\le n$, some useful and practical properties of the Caputo fractional operators $\mathcal{D}^{\nu}$ are given by the following expressions:

$$I^{\nu}D^{\nu}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!},\quad t>0,$$

$$\mathcal{D}^{\nu}I^{\nu}f(t)=f(t),$$

$$\mathcal{D}^{\nu}t^{\lambda}=\begin{cases}0 & for\ \lambda\in\mathbb{N}_0\ and\ \lambda<\nu,\\[2ex] \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}t^{\lambda-\nu} & otherwise.\end{cases}$$

For more details on fractional calculus and their applications we refer the reader to [32, 38].

3 Hahn polynomials and their properties

The Hahn polynomials are a class of orthogonal polynomials, which are introduced and investigated by Wolfgang Hahn [18]. These polynomials can be considered as a discrete analog of the classical Jacobi polynomials [16, 42]. This section is devoted to a brief definition of Hahn polynomials and their properties. For more details and some applications of the Hahn polynomials, one can refer the reader to [16, 18, 24, 42, 43].

Definition 4. For arbitrary complex numbers $a_i,\ i=1,2,3$ and $b_j\neq 0,\ j=1,2$, the generalized hypergeometric series ${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;x\right)$ is defined as

$${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k}\frac{x^k}{k!}$$

in which a can be derived as follows:

$$(a)_0 = 1,$$
$$(a)_k = a(a+1)\cdots(a+k-1), \quad k \geq 1,$$

and the series terminates if one of the numbers a_i is zero or a negative integer.

Definition 5. Let $\alpha, \beta > -1$ be real numbers and let N be a positive integer number. The Hahn polynomial $H_n(t;\alpha,\beta,N)$ may be defined in terms of a hypergeometric series

$$\begin{aligned} Q_n(x;\alpha,\beta,N) &= {}_3F_2(-n, n+\alpha+\beta+1, -x; \alpha+1, -N; 1) \\ &= \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha+\beta+1)_k (-x)_k}{k!(\alpha+1)_k(-N)_k}, \quad n = 0,1,\ldots,N. \end{aligned}$$

Remark 1. Hereafter, the simple notation $Q_n(x)$ is used to denote the nth Hahn polynomial $Q_n(x;\alpha,\beta,N)$.

Orthogonality:
The set of Hahn polynomials $\{Q_n(x): \; n = 0,1,\ldots,N\}$ are orthogonal on the interval $[0,N]$ with respect to the following discrete norm:

$$\langle f,g\rangle_w = \sum_{r=0}^{N} f(r)g(r)w(r),$$

where $w(t)$ is the weight function and is defined by

$$\begin{aligned} w(x) &= \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x} \\ &= \frac{\Gamma(\alpha+x+1)\,\Gamma(\beta+N-x+1)}{\Gamma(x+1)\,\Gamma(N-x+1)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}. \end{aligned}$$

Moreover, the orthogonality condition for these polynomials reads as follows:

$$\langle Q_n(x), Q_m(x)\rangle_w = \sum_{r=0}^{N} Q_n(r)Q_m(r)w(r) = \pi(n)\,\delta_{mn},$$

where δ_{mn} is the Kronecker delta and $\pi(n)$ can be defined as

$$\pi(n) = \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n \, n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n \, N!}.$$

Recurrence relation:
The set of Hahn polynomials $\{Q_n(x): n = 0,1,\ldots,N\}$ can be determined with the aid of the following recurrence formulas:

$$\mu_n Q_{n+1}(x) = (\mu_n + \sigma_n - x)\,Q_n(x) - \sigma_n Q_{n-1}(x), \quad n = 0,1,\ldots,N-1,$$

where

$$\mu_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

and

$$\sigma_n = \frac{n(n+\beta)(n+\alpha+\beta+N)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Explicit formula:
By using the generalized hypergeometric series ${}_3F_2$, the explicit analytical form of the Hahn polynomials can be derived as follows:

$$Q_n(x) = \sum_{k=0}^{n} h_{k,n} x^k, \quad n = 0, 1, \ldots, N,$$

in which

$$h_{k,n} = \sum_{i=k}^{n} \frac{(-1)^i (-n)_i (n+\alpha+\beta+1)_i S(k,i)}{i! (\alpha+1)_i (-N)_i}, \tag{5}$$

and

$$S(k,i) = \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^k.$$

Shifted Hahn polynomial:
In order to use the Hahn polynomials on the interval $[0,1]$, we define the shifted Hahn polynomials (SHPs) by introducing the change of variable $x = Nt$. Let the nth shifted Hahn polynomial, $Q_n(Nt)$, be denoted by $H_n(t)$. Then, the set of shifted Hahn polynomials $\{H_n(t) : n = 0, 1, \ldots, N\}$ are orthogonal on the interval $[0,1]$ with respect to the shifted weight function $\bar{w}(t) = w(Nt)$ and the following discrete norm:

$$\langle f, g \rangle_{\bar{w}} = \sum_{r=0}^{N} f\left(\frac{r}{N}\right) g\left(\frac{r}{N}\right) \bar{w}\left(\frac{r}{N}\right). \tag{6}$$

Also, the orthogonality condition for SHPs can be defined as follows:

$$\langle H_n(t), H_m(t) \rangle_{\bar{w}} = \sum_{r=0}^{N} H_n\left(\frac{r}{N}\right) H_m\left(\frac{r}{N}\right) \bar{w}\left(\frac{r}{N}\right) = \pi(n)\,\delta_{mn}, \tag{7}$$

where $\pi(n)$ is defined by (10).

Function approximation:
A function $f(t)$, defined over the interval $[0,1]$, may be expanded by the SHPs

as

$$f(t) \simeq \sum_{i=0}^{N} c_i H_i(t) = C^T \Phi(t), \tag{8}$$

where C and $\Phi(x)$ are $(N+1)$ vectors given by

$$C^T = [c_0, c_1, \ldots, c_N], \quad \Phi(t) = [H_0(t), H_1(t), \ldots, H_N(t)]^T, \tag{9}$$

and the coefficients c_i can be derived as follows:

$$c_i = \frac{\langle f(t), H_i(t)\rangle_{\bar{w}}}{\pi(i)}, \quad i = 0, 1, \ldots, N. \tag{10}$$

Convergence analysis:

Convergence analysis and spectral accuracy of the Hahn polynomials expansion are investigated thoroughly in [15, 16]. The following Theorem provides conditions that ensure that the series expansion of a function by the discrete Hahn polynomials converges.

Theorem 1. *The series expansion* $\sum_{k=0}^{n} \frac{\langle f, Q_k\rangle}{\langle Q_k, Q_k\rangle} Q_k(x)$ *of a function* f *by the discrete Hahn polynomials converges pointwise, if the series expansion of the function* f *by the Jacobi polynomials converges pointwise and if* $\frac{n^{3+\alpha+max\{1,\alpha\}}}{N} \to 0$ *as* $n, N \to \infty$*.*

Proof. The proof follows directly from [15, Theorems 1.1, 2.1 and 2.2]. □

4 Operational matrices

In this section, the operational matrix of fractional integration in the Riemann–Liouville sense and product operational matrix for the SHPs vector $\Phi(t)$ will be derived. To this end, the inner product of the SHP $H_n(t)$ and t^s is derived explicitly. Then, by using this explicit formulation, the operational matrix of fractional integration and product operational matrix can be obtained easily.

Lemma 1. *Suppose that, for a positive real number* s*, the inner product of the* n*th Hahn polynomial* $H_n(t)$ *and* t^s *is denoted by* $\lambda(n, s)$*. It can be derived as*

$$\begin{aligned}\lambda(n, s) &= \langle H_n(t), t^s\rangle_{\bar{w}} \\ &= \frac{1}{N^s} \sum_{r=0}^{N} \sum_{k=0}^{n} \frac{h_{k,n} r^{s+k} \Gamma(\alpha + r + 1) \Gamma(\beta + N - r + 1)}{\Gamma(r+1) \Gamma(N - r + 1) \Gamma(\alpha + 1) \Gamma(\beta + 1)},\end{aligned}$$

in which $h_{k,n}$ is defined in relation (5).

Proof. From the definition of the discrete inner product $\langle .,. \rangle_{\bar{w}}$ in (7), we have

$$\begin{aligned}\lambda(n,s) &= \sum_{r=0}^{N} H_n\left(\frac{r}{N}\right)\frac{r^s}{N^s}\bar{w}\left(\frac{r}{N}\right) = \frac{1}{N^s}\sum_{r=0}^{N} Q_n(r)\, r^s w(r) \\ &= \frac{1}{N^s}\sum_{r=0}^{N}\left(\sum_{k=0}^{n} h_{k,n} r^k\right) r^s \binom{r+\alpha}{r}\binom{\beta+N-r}{N-r} \\ &= \frac{1}{N^s}\sum_{r=0}^{N}\sum_{k=0}^{n}\frac{h_{k,n} r^{s+k}\Gamma(\alpha+r+1)\Gamma(\beta+N-r+1)}{\Gamma(r+1)\Gamma(N-r+1)\Gamma(\alpha+1)\Gamma(\beta+1)}. \end{aligned} \tag{11}$$

□

Theorem 2. *Let $\Phi(t)$ be the $(N+1)$ SHPs vector defined in (9). The fractional integration of order ν for this vector can be expressed as*

$$I^{\nu}\Phi(t) \simeq P^{(\nu)}\Phi(t), \tag{12}$$

where $P^{(\nu)}$ is an $(N+1)\times(N+1)$ matrix and its (i,j)th element can be defined as

$$P_{i,j}^{(\nu)} = \sum_{k=0}^{i-1}\frac{h_{k,i-1}N^k\Gamma(k+1)\lambda(j,k+\nu)}{\Gamma(k+\nu+1)\pi(j)}, \quad i,j = 0,1,\ldots,N.$$

Proof. The ith element of the vector $\Phi(t)$ is $H_{i-1}(t)$ and its fractional integral of order ν can be derived as

$$\begin{aligned} I^{\nu}H_{i-1}(t) &= I^{\nu}Q_{i-1}(Nt) \\ &= I^{\nu}\left(\sum_{k=0}^{i-1} h_{k,i-1}N^k t^k\right) = \sum_{k=0}^{i-1}\frac{h_{k,i-1}N^k\Gamma(k+1)}{\Gamma(k+\nu+1)} t^{\nu+k}. \end{aligned} \tag{13}$$

By expanding $t^{k+\nu}$ in terms of SHPs, we get

$$t^{k+\nu} \simeq \sum_{j=0}^{N}\beta_{j,k}H_j(t), \tag{14}$$

in which $\beta_{r,j}$ can be derived by using Lemma 1 as follows:

$$\beta_{j,k} = \frac{1}{\pi(j)}\left\langle H_j(t), t^{k+\nu}\right\rangle_{\bar{w}} = \frac{\lambda(j,k+\nu)}{\pi(j)}. \tag{15}$$

Now, by inserting (14) and (15) in (13), we get

$$I^{\nu}\Phi_i(t) \simeq \sum_{j=0}^{N}\left(\sum_{k=0}^{i-1}\frac{h_{k,i-1}N^k\Gamma(k+1)\lambda(j,k+\nu)}{\Gamma(k+\nu+1)\pi(j)}\right)H_j(t),$$

and this leads to the desired results. □

Theorem 3. *Let $\Phi(t)$ be the $(N+1)$ Hahn polynomials vector defined in (9) and let V be an arbitrary $(N+1)$ vector. Then, it is convenient to write*

$$\Phi(t)\Phi^T(t)V = \tilde{V}\Phi(t), \tag{16}$$

where $\tilde{V}$ is the $(N+1)\times(N+1)$ product operational matrix and its (i,j)th element can be defined as

$$\tilde{V}_{i+1,j+1} = \frac{1}{\pi(j)}\sum_{k=0}^{N}V_k\left\langle H_k(t)H_i(t), H_j(t)\right\rangle_{\bar{w}}, \quad i,j=0,1,\ldots,N.$$

Proof. The product of two Hahn polynomial vectors $\Phi(t)$ and $\Phi^T(t)$ is an $(N+1)\times(N+1)$ matrix as follows

$$\Phi(t)\Phi^T(t) = \begin{bmatrix} H_0(t)H_0(t) & H_0(t)H_1(t) & \ldots & H_0(t)H_N(t) \\ H_1(t)H_0(t) & H_1(t)H_1(t) & \ldots & H_1(t)H_N(t) \\ \vdots & \vdots & \ddots & \vdots \\ H_N(t)H_0(t) & H_N(t)H_1(t) & \ldots & H_N(t)H_N(t) \end{bmatrix}.$$

As a result, relation (16) can be rewritten as:

$$\sum_{k=0}^{N}H_k(t)H_i(t)V_{k+1} = \sum_{k=0}^{N}H_k(t)\tilde{V}_{i+1,k+1}, \quad i=0,1,\ldots,N.$$

Now, by multiplying both sides of the above equation by $H_j(t)$ and using the defined inner product in (6), we get

$$\sum_{k=0}^{N}\left\langle H_k(t)H_i(t), H_j(t)\right\rangle_{\bar{w}}V_{k+1} = \left\langle H_j(t), H_j(t)\right\rangle_{\bar{w}}\tilde{V}_{i+1,j+1}, \quad i,j=0,1,\ldots,N.$$

Therefore, the matrix $\tilde{V}$ may be given by

$$\tilde{V}_{i+1,j+1} = \frac{\sum_{k=0}^{N}V_k\left\langle H_k(t)H_i(t), H_j(t)\right\rangle_{\bar{w}}}{\left\langle H_j(t), H_j(t)\right\rangle_{\bar{w}}} = \frac{1}{\pi(j)}\sum_{k=0}^{N}V_k\left\langle H_k(t)H_i(t), H_j(t)\right\rangle_{\bar{w}}.$$

□

5 Description of the proposed method

In this section, SHPs along with their operational matrix of fractional integration will be applied to solve TDFOCPs. Consider the optimal control of time-varying linear system (1)–(4) with delays in state and control and with quadratic performance. First, we approximate all functions involved in the problem as follows:

$$\mathcal{D}^{\nu}x(t) \simeq X^T\Phi(t), \tag{17}$$

$$u(t) \simeq U^T\Phi(t), \tag{18}$$

$$r(t) \simeq R^T\Phi(t),\ \ s(t) \simeq S^T\Phi(t), \tag{19}$$

$$a(t) \simeq A^T\Phi(t),\ \ c(t) \simeq C^T\Phi(t), \tag{20}$$

$$b(t) \simeq B^T\Phi(t),\ \ e(t) \simeq E^T\Phi(t), \tag{21}$$

$$f(t) \simeq F^T\Phi(t),\ \ g_2(t) \simeq G_1^T\Phi(t),\ \ g_1(t) \simeq G_2^T\Phi(t), \tag{22}$$

where X and U are unknown vectors to be determined, while R, S, A, C, B, E, G_1, G_2, and F are known SHPs coefficient vectors that can be computed as explained in (8)–(10). By applying the fractional Riemann–Liouville operator I^{ν} on both sides of the relation (17), we have

$$x(t) \simeq I^{\nu}X^T\Phi(t) + g_1(0) = X^TP^{(\nu)}\Phi(t) + G_1^T\Phi(0), \tag{23}$$

in which $P^{(\nu)}$ is the operational matrix of fractional integration of the SHPs vector $\Phi(t)$ derived in Theorem 2. Hence, by using (23) and (18), the delay state function $x(t-\tau)$ and control function $u(t-\eta)$ can be derived as

$$x\left(t-\tau\right) = \begin{cases} G_1^T\Phi\left(t-\tau\right), & 0 \le t \le \tau, \\ \left(X^TP^{(\alpha)} + d^T\right)\Phi(t-\tau), & \tau \le t \le 1, \end{cases}$$

and

$$u\left(t-\eta\right) = \begin{cases} G_2^T\Phi\left(t-\eta\right), & 0 \le t \le \eta, \\ U^T\Phi(t-\eta),\ \ \eta \le t \le 1. \end{cases}$$

These delay functions can be expanded by the Hahn polynomials as

$$x\left(t-\tau\right)=X^{T}P^{(\nu)}\Omega_{\tau}\Phi\left(t\right)+V_{\tau}^{T}\Phi\left(t\right), \tag{24}$$

and

$$u\left(t-\eta\right)=U^{T}\Omega_{\eta}\Phi\left(t\right)+V_{\eta}^{T}\Phi\left(t\right), \tag{25}$$

where V_τ and V_η are known $(N+1)$ vectors, whereas Ω_η and Ω_η are known $(N+1)\times(N+1)$ matrices, which are depend on the delay parameters τ and η, respectively (please, see details in Appendix A). Now, by substituting (17)–(25) in dynamical system (4), we get

$$\begin{aligned}X^{T}\Phi(t) = {} & X^{T}P^{(\nu)}\Phi(t)\Phi^{T}(t)A+G_{1}^{T}\Phi(0)A^{T}\Phi(t)+U^{T}\Phi(t)\Phi^{T}(t)B\\ & +X^{T}P^{(\nu)}\Omega_{\tau}\Phi(t)\Phi^{T}(t)C+V_{\tau}^{T}\Phi(t)\Phi^{T}(t)C+U^{T}\Omega_{\eta}\Phi(t)\Phi^{T}(t)E\\ & +V_{\eta}^{T}\Phi(t)\Phi^{T}(t)E+F^{T}\Phi(t).\end{aligned}$$

By using the product operational matrix as described in Theorem 3, it is convenient to rewrite this relation as follows:

$$\begin{aligned}X^{T}\Phi(t)=\Big(&X^{T}P^{(\nu)}\tilde{A}+G_{1}^{T}\Phi(0)A^{T}+U^{T}\tilde{B}+X^{T}P^{(\nu)}\Omega_{\tau}\tilde{C}\\ &+V_{\tau}^{T}\tilde{C}+U^{T}\Omega_{\eta}\tilde{E}+V_{\eta}^{T}\tilde{E}+F^{T}\Big)\Phi(t).\end{aligned}$$

The above expression is satisfied for all t in the interval $[0,1]$; therefore

$$\begin{aligned}X^{T}={} & X^{T}P^{(\nu)}\tilde{A}+G_{1}^{T}\Phi(0)A^{T}+U^{T}\tilde{B}+X^{T}P^{(\nu)}\Omega_{\tau}\tilde{C}\\ & +V_{\tau}^{T}\tilde{C}+U^{T}\Omega_{\eta}\tilde{E}+V_{\eta}^{T}\tilde{E}+F^{T}.\end{aligned}$$

Now, from this relation, the unknown vector U can be derived as a function of unknown vector X, by solving the following linear system:

$$U^{T}\Lambda=Y, \tag{26}$$

where

$$\Lambda=-\left(\tilde{B}+\Omega_{\eta}\tilde{E}\right), \tag{27}$$

and

$$\begin{aligned}Y={} & X^{T}P^{(\nu)}\tilde{A}+G_{1}^{T}\Phi(0)A^{T}+X^{T}P^{(\nu)}\Omega_{\tau}\tilde{C}\\ & +S_{\tau}^{T}\tilde{C}+S_{\eta}^{T}\tilde{E}+F^{T}-X^{T}.\end{aligned} \tag{28}$$

By inserting the obtained vector U in (18), the unknown control function $u(t)$ can be derived as a function of the unknown vector X. Therefore, the performance index can be derived as:

$$J\left[x_{0},x_{1},\ldots,x_{N}\right]\simeq\int_{0}^{1}\Theta\left(t,X\right)dt, \tag{29}$$

where

$$\Theta\left(t, X\right) = \left[\left(X^T P^{(\nu)}\Phi(t) + d^T\Phi(t)\right)^2 \Phi^T(t)R + \left(U^T\Phi(t)\right)^2 \Phi^T(t)S\right].$$

Moreover, by applying the Gauss–Legendre quadrature formula in the interval $[0,1]$, the performance index $J\left[x_0, x_1, \ldots, x_N\right]$ can be approximated as

$$J\left[x_0, x_1, \ldots, x_N\right] \simeq \sum_{k=1}^{r} w_k \Theta\left(t_k, X\right), \tag{30}$$

where w_k and t_k are Gauss–Legendre quadrature nodes and weights, respectively. Finally, the necessary conditions for the optimality of the performance index are

$$\frac{\partial J}{\partial x_i} = 0, \quad i = 0, 1, \ldots, N, \tag{31}$$

which constitute a system of algebraic equations for unknown vector X. By solving this system and determining the vector X, the optimal state function $x(t)$ and control function $u(t)$ can be approximated by inserting vector X in (18) and (23), respectively.

Algorithm 1 Algorithm in pseudo-code format

Inputs: N, α, β, ν, r, τ, η, the functions $r(t), s(t), f(t), g_1(t), g_2(t)$, $a(t), c(t)$, and $b(t), e(t)$.

Step 1: Construct the Hahn polynomials vector $\Phi(t)$ and weight function $w(t)$ from (5) and (9).

Step 2: Define an unknown $(N+1)$ vector X and approximate $D^{\nu}x(t)$ in (17).

Step 3: Compute the fractional operational matrix $P^{(\nu)}$ form Theorem 2.

Step 4: Compute the coefficient vectors R, S, A, C, B, E, G_1, G_2, and F from (19)–(22).

Step 5: Compute the product operational matrices $\tilde{A}, \tilde{C}$ and $\tilde{B}, \tilde{E}$ from Theorem 3.

Step 6: Compute the matrices Ω_τ and Ω_η and vectors V_τ and V_η from Lemma 2 in Appendix A.

Step 7: Compute the matrix Λ and vector Y from (27) and (28).

Step 7: Compute the vectors U from (26).

Step 8: Insert the obtained vectors U in (18) to approximate the control function $u(t)$.

Step 9: Compute the performance index J in (29).

Step 10: Apply the Gauss–Legendre quadrature formula to approximate J in (30).

Step 11: Constitute the system of algebraic equations in (31).

Step 12: Solve the obtained system of algebraic equations in Step 11 to get the unknown vector X.

Outputs: The approximate state and control functions $x(t) \simeq X^T\Phi(t)$ and $u(t) \simeq U^T\Phi(t)$.

5.1 Main features of the proposed method

A direct numerical method based on discrete orthogonal polynomials is proposed to approximate solution of time-delay fractional optimal control problems. As these polynomials are orthogonal with respect to a discrete norm, implementation of the proposed numerical method is more efficient and less complex in comparison to similar methods that in which continuous polynomials are used. In the proposed method, the fractional derivative of an unknown state function is approximated by using the discrete polynomials. Then, the operational matrix of fractional integration together with the dynamical system is used to approximate the control function as a function of state function. Consequently, the need for the direct approximation of control function and Lagrange multiplier method are eliminated. The numerical algorithm of the proposed method in pseudo-code format is designed in the following section.

6 Illustrative examples

In this section, some illustrative examples are considered to investigate the efficiency and accuracy of the presented SHPs method. In all numerical examples, the orthogonal SHPs over the interval $[0, 1]$ with $\alpha = \beta = 1$ are used to approximate the solution of (1)–(4). For the problems defined in different intervals, first, the interval has been transformed into $[0, 1]$. All the numerical simulation have been done by using Maple 17 with 20 digits precision.

Example 1. In the first example, we consider the following TDFOCP [20]:

$$J = \frac{1}{2}\int_0^2 \left[x^2(t) + u^2(t)\right] dt$$

subjected to the dynamical system

$$\begin{aligned} &D^\nu x(t) = a_1(t)x\,(t-1) + u\,(t)\,, \quad 0 \le t \le 2,\ 0 < \nu \le 1,\\ &x(t) = 1, \quad -1 \le t \le 0, \end{aligned}$$

where $a_1(t)$ is a known continuous function. For $a_1(t) = 1$, $\nu = 1$, and $N = 12$, Table 1 provides a comparison between the obtained optimal value J and those derived by Walsh series [33], average approximation [5], Bernstein polynomials [37], Boubaker functions [34], shifted Legendre polynomials [7], Bernoulli wavelet method [35], and linear programming [20]. From the presented results of Table 1, it is clear that the numerical results obtained by the presented SHPs method have a good agreement with the true solution given by the average approximation [5], Walsh series [33], and linear programming [20]. Furthermore, it is easy to conclude that the reported values in [7, 34, 35, 37] are not in good agreement with the other numerical results. In order to investigate the convergence issue of the proposed algorithm, the obtained optimal values of J for different values of N and ν are presented in Table 2 and compared with the reported values in [7]. For $a_1(t) = t$, the proposed method is employed to solve these TDFOCPs for various values of ν. In Table 3, a comparison is made between the obtained optimal values of J and the derived values by linear programming method [20]. From the presented results in Table 3, we can conclude that the derived optimal values J are in a good agreement with the numerical solutions reported in [20]. Finally, the graph of approximate state function $x(t)$ and control function $u(t)$ with $N = 12$ are plotted in Figures 1 and 2, for $a_1(t) = 1$ and $a_1(t) = t$, respectively. From the obtained results, it is clear that the proposed SHPs method is efficient and accurate in solving TDFOCP and that, as the fractional order ν approaches 1, the approximate solution converges to that of integer-order problem. Moreover, the presented results in Tables 2 and 3 indicate that numerical results converge well as the number of basis function N increases.

Table 1: Comparison of the performance index values obtained by different methods for $a_1(t) = 1$ and $\nu = 1$ (Example 1).

Method	Performance index value J
Presented method	1.64731019
Linear programming [20]	1.64886527
Average approximation method [5]	1.6419
Walsh series [33]	1.6497
Legendre polynomials [7]	0.4727464
Bernstein polynomials [37]	0.6381
Boubaker functions [34]	0.00002674
Bernoulli wavelet [35]	0.3048

Table 2: The performance index J for $a_1(t) = 1$ and different values of ν (Example 1).

	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$	$\nu = 0.95$	$\nu = 0.99$
$N = 6$	1.5581796	1.5964603	1.6266313	1.6384862	1.6464170
$N = 8$	1.5487500	1.5907820	1.6239023	1.6368166	1.6454020
$N = 12$	1.5410926	1.5862331	1.6239023	1.6357471	1.6448406
[7]	−	0.4985242	0.5021900	−	0.4778890

Table 3: The performance index J for $a_1(t) = t$ and different values of ν (Example 1).

	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$	$\nu = 0.95$	$\nu = 1$
$N = 6$	0.92467418	0.97005408	1.01321336	1.03283445	1.05078685
$N = 8$	0.90796836	0.95993974	1.00820712	1.02968701	1.04905258
$N = 12$	0.90014337	0.95537457	1.00625088	1.02863607	1.04863866
[20]	–	1.08069403	1.06577389	–	1.05137240

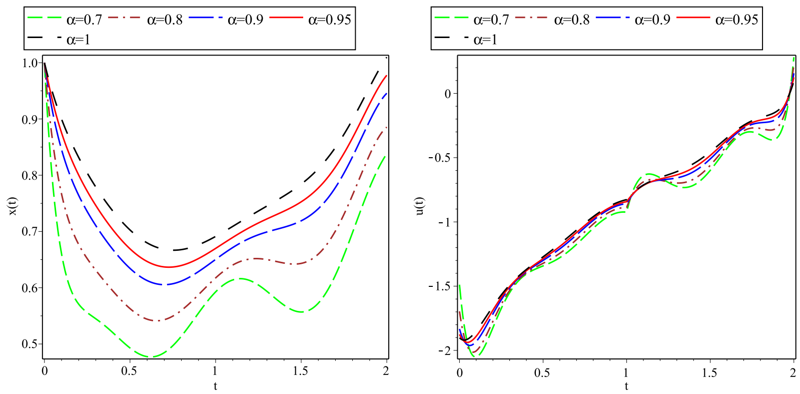

Figure 1: The approximate solutions $x(t)$ and $u(t)$ for $a_1(t) = 1$ and $N = 12$ (Example 1).

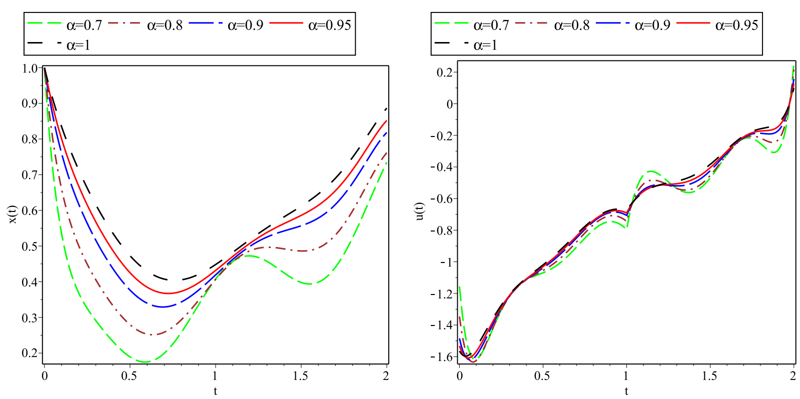

Figure 2: The approximate solutions $x(t)$ and $u(t)$ for $a_1(t) = t$ and $N = 12$ (Example 1).

Example 2. In this example, we consider the following TDFOCP [34, 35]

$$J = \frac{1}{2}\int_0^2 \left[x^2(t) + u^2(t)\right] dt$$

subjected to the dynamical system

$$D^\nu x(t) = tx\,(t) + x\,(t-1) + u\,(t)\,, \quad 0 \le t \le 2,\ 0 < \nu \le 1,$$
$$x(t) = 0, \quad -1 \le t \le 0.$$

The SHPs and presented technique in section 5 are used to solve this problem for various values of ν. For $\nu = 1$, Table 4 provides a comparison between the results of our proposed method and those derived by FD method [22], recursive shooting method [21], line-up competition algorithm [10], Legendre multiwavelets [25], Walsh series [33] and Bernoulli wavelet [35]. From Table 4 we can see that there is good agreement between the obtained results and the those reported in Refs. [10, 21, 22]. To investigate the convergence issue of the proposed algorithm, the obtained optimal values of J for different values of N and ν are presented in Table 5. Moreover, Figure 3 shows the approximate state and control functions $x(t)$ and $u(t)$ for various values of ν and $N = 12$. Based on the obtained results, it is clear that proposed SHPs method is efficient and accurate for solving such problems and the approximate solutions converge to the exact solution as fractional order ν approaches integer order 1. Moreover, it is easy to conclude that numerical results converge well as the number of basis function N increases.

Table 4: Comparison of the performance index values obtained by different methods for $\nu = 1$ (Example 2).

Method	Performance index value J
Presented method	4.792110320
FD method [22]	4.79678
Recursive shooting method [21]	4.79682
Line-up competition algorithm [10]	4.7976
Legendre multiwavelets [25]	5.1713
Iterative dynamic programming [11]	5.0674
Walsh series [33]	6.0079
Bernoulli wavelet [35]	2.0481

Table 5: The performance index J for different values of ν (Example 2).

	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$	$\nu = 0.95$	$\nu = 0.99$
$N = 6$	4.04799977	4.32580075	4.58104543	4.69543943	4.77884109
$N = 8$	4.01176960	4.30195764	4.56856743	4.68746981	4.77382051
$N = 12$	3.98516242	4.28472972	4.56062884	4.68299652	4.77143293

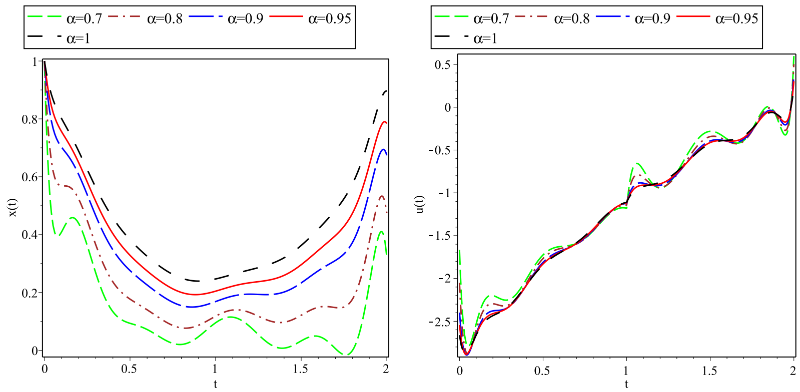

Figure 3: The approximate solutions $x(t)$ and $u(t)$ for $N = 12$ (Example 2).

Example 3. In this example, we consider the following TDFOCP with delays in state and control [7,34,35]

$$J = \frac{1}{2}\int_0^1 \left[x^2(t) + \frac{1}{2}u^2(t)\right] dt,$$

subjected to the dynamical system

$$\begin{array}{l} D^{\nu}x(t) = -x\,(t) + x\left(t-\frac{1}{3}\right) + u\,(t) - \frac{1}{2}u\left(t-\frac{2}{3}\right),\ 0 \le t \le 1, \quad 0 < \nu \le 1, \\ x(t) = 1,\ -\frac{1}{3} \le t \le 0, \\ u(t) = 0,\ -\frac{2}{3} \le t \le 0, \end{array}$$

The proposed SHPs method has been employed to approximate the solution of this TDFOCP for different values of ν. For $\nu = 1$, this problem has been solved by using the Hybrid functions [17, 28] and Bezier curve method [14]. Table 6 provides a comparison between the results of our proposed method and those reported in [7,14,17,28,34,35]. From Table 6, we can see that there is a good agreement between our obtained results and the previously reported numerical results in [17, 28]. Also, this Table indicates that the reported values by using Boubaker polynomials [34], Shifted Legendre polynomials [7], Bernoulli wavelet [35], and Bezier curve method [14] are not in a good agreement with our proposed method and the valid values given in [17, 28]. To investigate the convergence issue of the proposed algorithm, the obtained optimal values of J for different values of N and ν are presented in Table 7 and compared with the presented results in [7]. Moreover, Figure 4 shows the approximate state and control functions $x(t)$ and $u(t)$ for various values of and $N = 12$. Based on the obtained results, it is clear that the proposed SHPs method is efficient and accurate for solving such problems and that the approximate solutions converge to the exact solution as fractional order

ν approaches integer order 1. Moreover, it is easy to conclude that numerical results converge well as the number of basis function N increases.

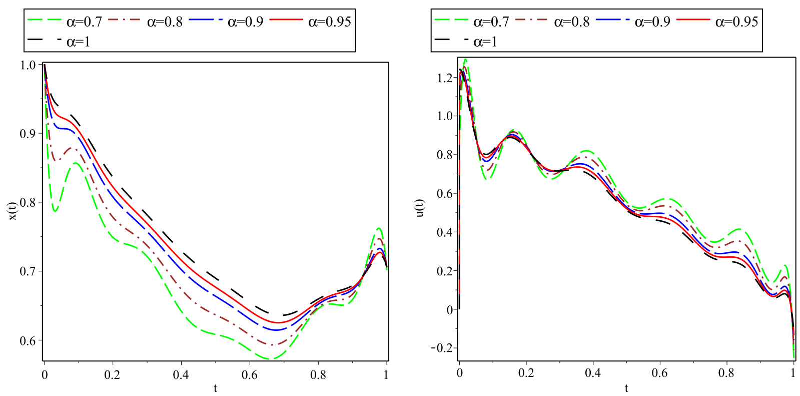

Figure 4: The approximate solutions $x(t)$ and $u(t)$ for $N = 12$ (Example 3).

Table 6: Comparison of the performance index values obtained by different methods for $\nu = 1$ (Example 3).

Method	Performance index value J
Presented method method	0.373230152
Hybrid Legendre functions [28]	0.37311241
Hybrid Bernoulli functions [17]	0.37310517
Bernoulli wavelet [35]	0.1027
Bezier curve method [14]	0.4220497643
Boubaker polynomials [34]	0.04553
Legendre polynomials [7]	0.01451

Table 7: The performance index J for different values of ν (Example 3).

	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$	$\nu = 0.95$	$\nu = 0.99$
$N = 8$	0.3393198	0.3496035	0.3600181	0.3652627	0.3694686
$N = 12$	0.3415913	0.3515520	0.3613160	0.3661582	0.3700268
$N = 16$	0.3549220	0.3549220	0.3637609	0.3683158	0.3720233
Ref. [7]	–	0.0126951	0.0116573	–	0.0139046

Example 4. Finally, consider the following TDFOCP [7, 14]

$$J = \frac{1}{2}\int_0^{\frac{1}{4}} \left[x^2(t) + u^2(t)\right] dt$$

subjected to the dynamical system

$$\begin{aligned}
&D^\nu x(t) = x(t) + u\left(t - \tfrac{1}{10}\right) + u(t), \ 0 \le t \le \tfrac{1}{4}, \quad 0 < \nu \le 1,\\
&u(t) = 0, \quad -\tfrac{1}{10} \le t \le 0,\\
&x(0) = 1.
\end{aligned}$$

Basin and Gonzalez [6] and Ghomanjani et al. [14] solved this problem for $\nu = 1$ by using the Bezier curve and Linear-quadratic method, respectively. Here, this TDFOCP has been solved by using the proposed Hahn polynomial method for variose values of ν. In Table 8, a comparison is made between the optimal values of J obtained by the SHPs method and those derived by the Bezier curve method [14], shifted Legendre polynomials [7], and Linear-quadratic method [6]. From this Table, we can realize that the SHPs method is efficient and that the obtained results are in good agreement with the presented results in [6, 14]. However, the reported results in [7] are not in good agreement with other methods. For noninteger values of ν, the obtained optimal values of J are presented in Table 9 and compared with the value of J derived by shifted Legendre polynomials [7]. Finally, Figure 5 shows the approximate state function $x(t)$ and the control function $u(t)$ for various values of ν and $N = 12$. Based on the obtained results, it is clear that the proposed SHPs method is efficient and accurate for solving such problems and that the approximate solutions converge to the exact solution as fractional order α approaches integer order 1. Moreover, it is easy to conclude that numerical results converge well as the number of basis function N increases.

Table 8: Comparison of the performance index values obtained by different methods for $\nu = 1$ (Example 4).

Method	Performance index value J
Presented method method	0.1554640279
Bezier curve method [14]	0.1565866913
Linear-quadratic [6]	0.1563
Legendre polynomials [7]	0.0143671

Table 9: The performance index J for different values of N and ν (Example 4).

	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$	$\nu = 0.95$	$\nu = 0.99$
$N = 6$	0.1594198	0.1574747	0.1553910	0.1543186	0.1534526
$N = 8$	0.1607079	0.1584826	0.1561770	0.1550124	0.1540808
$N = 12$	0.1616864	0.1592375	0.1567619	0.1555286	0.1545484
Ref. [7]	–	0.0126951	0.0116573	–	0.0139046

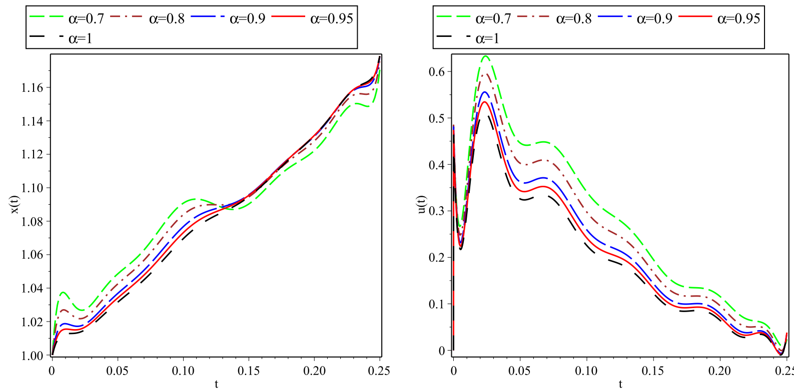

Figure 5: The approximate solutions $x(t)$ and $u(t)$ for $N = 12$ (Example 4).

7 Conclusion

A direct numerical method based on the discrete SHPs was proposed to approximate solution of TDFOCPs with quadratic performance index. First, the operational matrix of fractional integration in the Riemann–Liouville sense and product operational matrix for the shifted Hahn polynomials were derived. The fractional derivative of unknown state function was approximated by using the SHPs. Then, the operational matrix of fractional integration together with the dynamical system were used to approximate the control function as a function of state function. Finally, these approximations were put in the performance index and necessary conditions for optimality transform the under consideration TDFOCPs into an algebraic system. Numerical results confirm that the presented SHPs method is accurate and effective.

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Appendix A

Lemma 2. *Consider the function*

$$x(t-\tau)=\begin{cases} G_1^T\Phi(t-\tau),\ 0\le t\le\tau,\\ \\ X^TP^{(\nu)}\Phi(t-\tau)+d^T\Phi(t-\tau),\ \tau\le t\le 1,\end{cases}$$

it can be expanded into SHPs as follows:

$$x(t-\tau)=X^TP^{(\nu)}\Omega_\tau\Phi(t)+V_\tau\Phi(t), \tag{32}$$

in which Ω_τ *and* S_τ *are, respectively,* $(N+1)\times(N+1)$ *and* $(N+1)\times 1$ *matrices defined as follows:*

Proof. By expanding the function $x(t-\tau)$ expanded by the SHPs, we get

$$x(t-\tau)=\sum_{i=0}^{N} z_iH_i(t)=Z^T\Phi(t),$$

where ith element of the coefficient vector Z can be derived as

$$z_i=\frac{\langle x(t-\tau),H_i(t)\rangle_{\bar w}}{\langle H_i(t),H_i(t)\rangle_{\bar w}}=\frac{1}{\pi(i)}\sum_{k=0}^{N}x\left(\frac{k}{N}-\tau\right)H_i\left(\frac{k}{N}\right)\bar w\left(\frac{k}{N}\right),\quad i=0,1,\dots,N.$$

Let l be the greatest integer such that $l\le\tau N$. Then

$$z_i=\frac{G_1^T}{\pi(i)}\sum_{k=0}^{l}\Phi\left(\frac{k}{N}-\tau\right)H_i\left(\frac{k}{N}\right)\bar w\left(\frac{k}{N}\right)+\frac{X^TP^{(\nu)}}{\pi(i)}\sum_{k=l+1}^{N}\Phi\left(\frac{k}{N}-\tau\right)H_i\left(\frac{k}{N}\right)\bar w\left(\frac{k}{N}\right)$$

$$+\frac{d^T}{\pi(i)}\sum_{k=l+1}^{N}\Phi\left(\frac{k}{N}-\tau\right)H_i\left(\frac{k}{N}\right)\bar w\left(\frac{k}{N}\right)$$

Now, let us define

$$\chi_i = \frac{1}{\pi(i)} \begin{bmatrix} \sum_{k=0}^{l} H_0\left(\frac{k}{N}-\tau\right) H_i\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \\ \sum_{k=0}^{l} H_1\left(\frac{k}{N}-\tau\right) H_i\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \\ \vdots \\ \sum_{k=0}^{l} H_N\left(\frac{k}{N}-\tau\right) H_i\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \end{bmatrix},$$

and

$$\Pi_i = \frac{1}{\pi(i)} \begin{bmatrix} \sum_{k=l+1}^{N} H_0\left(\frac{k}{N}-\tau\right) H_i\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \\ \sum_{k=l+1}^{N} H_1\left(\frac{k}{N}-\tau\right) H_i\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \\ \vdots \\ \sum_{k=l+1}^{N} H_N\left(\frac{k}{N}-\tau\right) H_i\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \end{bmatrix},$$

then, the element z_i can be obtained as

$$z_i = G_1^T \chi_i + d^T \Pi_i + X^T P^{(\nu)} \Pi_i.$$

Consequently, we get

$$x(t-\tau) = X^T P^{(\nu)} \Omega_\tau \Phi(t) + V_\tau^T \Phi(t),$$

where V_τ is an $(N+1)\times 1$ vector as

$$V_\tau^T = \left[G_1^T \chi_0 + d^T Y_0, G_1^T \chi_1 + d^T Y_1, \ldots, G_1^T \chi_N + d^T Y_N\right]$$

and Ω_τ is an $(N+1)\times(N+1)$ matrix which its ith column is Π_i, that is,

$$\Omega_\tau = \frac{1}{\pi(i)} \begin{bmatrix} \sum_{k=l+1}^{N} H_0\left(\frac{k}{N}-\tau\right) H_0\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) & \ldots & \sum_{k=l+1}^{N} H_0\left(\frac{k}{N}-\tau\right) H_N\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \\ \sum_{k=l+1}^{N} H_1\left(\frac{k}{N}-\tau\right) H_0\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) & \ldots & \sum_{k=l+1}^{N} H_1\left(\frac{k}{N}-\tau\right) H_N\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \\ \vdots & \ddots & \\ \sum_{k=l+1}^{N} H_N\left(\frac{k}{N}-\tau\right) H_0\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) & \ldots & \sum_{k=l+1}^{N} H_N\left(\frac{k}{N}-\tau\right) H_N\left(\frac{k}{N}\right) \bar{w}\left(\frac{k}{N}\right) \end{bmatrix}.$$

Remark 2. In the same way, it can be proved that the

$$u\left(t-\eta\right) = U^T \Omega_\eta \Phi\left(t\right) + V_\eta^T \Phi\left(t\right),$$

where Ω_η and V_η are $(N+1)\times(N+1)$ and $(N+1)\times 1$ matrices, respectively.
$\square$