Homotopy perturbation and Elzaki transform for solving Sine-Gorden and Klein-Gorden equations

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Abstract

In this paper, the homotopy perturbation method (HPM) and Elzaki transform is employed to obtain the approximate analytical solution of the Sine Gorden and the Klein Gorden equations. The nonlinear terms can be handled by the use of homotopy perturbation method. The proposed homotopy perturbation method is applied to reformulate the first and the second order initial value problems which leads to the solution in terms of transformed variable, and the series solution that can be obtained by making use of the inverse transformation.

1 Introduction

In this paper, we have considered the Sine-Gorden (SG) and Klein-Gorden (KG) equations, as the following and also in [2] respectively:

$$u_{tt} - u_{xx} + \alpha g(u) = f(x,t), \tag{1}$$

and

$$u_{tt} - u_{xx} + \beta_1 u + \beta_2 g(u) = f(x,t), \tag{2}$$

where u is a function of x, t and g is a nonlinear function . The α parameter is so-called dissipative term, which is assumed to be a real number with $\alpha \geq 0$. When $\alpha = 0$, Eq. (1), reduces to the undamped SG equation, and when $\alpha > 0$, to the damped one. f is also a known analytic function. The Sine-Gorden and Klein-Gorden equations model many problems in classical and Quantum mechanics, solitons, and condensed matter physics

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[1, 5]. The SG equation arose in a strict mathematical context in differential geometry in the theory of surfaces of constant curvature [17]. For the SG equation, the exact soliton solution have been obtained in [16], using Hirotas method in [21], using Lambs method in [18], by Bäcklund transformation and painlevé transcendents in [18]. Numerical solutions for the undamped SG equation have been given among others by Guo et al. [8] by use of two diffirent schemes, Xin [24] who studied SG equation as an asymptotic reduction of the two-level dissipationless Max well-Bloch system, Christiansen and Lomdahl [3] used a generalized leapforg method and Argyris et al. found the accurate and efficient methods for solving such equations which is an active research undertaken by Herbst et al. The method [14], presented a numerical solution for the SG equation obtained by means of an explicit symplectic behavior of a double-discrete, completely integrable discretization of the Sine-Gorden equation, and they have illustrated their technique by numerical experiments. Wazwaz [23] has used the tanh method to obtain the exact solution of SG equation. Approximate analytical solution of Kline Gorden equation thrangh the Adomian Decomposition Method (ADM) was presented in [4, 6, 20]. Kaya has applied the modified ADM (MADM) for obtaining the approximate analytical solution of the Sine-Gorden equation in [19]. Another more powerful and convenient analytical technique, called the homotopy-perturbation method (HPM), was first developed by He [13]. Some part of this work on HPM can be found in [9, 10, 11]. HPM transformes a difficult problem into a set of problems which are easier to solve. Chowdhury and Hashim [2], have used the HPM to obtain the approximate analytical solution of Sine-Gorden and Klein-Gorden equations. Recently, Tarig Elzaki [7], has introduced a new integral transform, named the Elzaki transform, and it has further applied to the solution of ordinary and partial differential equations.

Now, we consider in this work the effectiveness of the homotopy-perturbation Elzaki transform method to obtain the exact and approximate analytical solution of the Sine-Gorden and the Klein-Gorden equations.

2 Elzaki Transform

The basic definition of modified form of Sumudo transform or Elzaki transform is defiend as follow, Elzaki transform of the function $f(t)$ is:

$$E[f(t)] = v \int_0^\infty f(t)e^{-t/v}dt, \qquad t > 0. \tag{3}$$

Tarig M. Elzaki showed the modified form of Sumudu transform or Elzaki transform in which it is applied to the differential equations, ordinary dif-

ferential equations, system of ordinary, partial differential equations, and integral equations.

To obtain the Elzaki transform of partial derivative, we use the integration by part, and then we have:

$$E[\frac{\partial f(x,t)}{\partial t}] = \frac{1}{\upsilon}T(x,\upsilon) - \upsilon f(x,0), \tag{4}$$

$$E[\frac{\partial^2 f(x,t)}{\partial t^2}] = \frac{1}{\upsilon^2}T(x,\upsilon) - f(x,0) - \upsilon\frac{\partial f(x,t)}{\partial t}, \tag{5}$$

$$E[\frac{\partial f(x,t)}{\partial x}] = \frac{d}{dx}T(x,\upsilon), \tag{6}$$

$$E[\frac{\partial^2 f(x,t)}{\partial x^2}] = \frac{d^2}{dx^2}T(x,\upsilon). \tag{7}$$

Proof: First, we assume

$$\upsilon\int_0^\infty \frac{\partial f}{\partial t}e^{\frac{-t}{\upsilon}}dt = T(x,\upsilon). \tag{8}$$

By using the integration by parts one obtain:

$$E[\frac{\partial f(x,t)}{\partial t}] = \int_0^\infty \upsilon\frac{\partial f}{\partial t}e^{\frac{-t}{\upsilon}}dt = \lim_{p\to\infty}\int_0^\infty \upsilon e^{\frac{-t}{\upsilon}}\frac{\partial f}{\partial t}dt$$

$$= \lim_{p\to\infty}\{[\upsilon e^{\frac{-t}{\upsilon}}f(x,t)]_0^p - \int_0^p e^{\frac{-t}{\upsilon}}f(x,t)dt\}$$

$$= \frac{T(x,\upsilon)}{\upsilon} - \upsilon f(x,0). \tag{9}$$

Assuming that f is a piecewise continuous function and $\frac{\partial f}{\partial x}$ exist, also it is of exponential order which is means that there exist nonegative constants M and T such that for all $t \geq T$, we have

$$|f(t)| \leq Me^{at}.$$

Now,

$$E[\frac{\partial f(x,t)}{\partial x}] = \int_0^\infty \upsilon\frac{\partial f(x,t)}{\partial x}e^{\frac{-t}{\upsilon}}dt, \tag{10}$$

using the Leibnitz rule to find:

$$E[\frac{\partial f(x,t)}{\partial x}] = \frac{d}{dx}T(x,\upsilon). \tag{11}$$

By using this method, we have:

$$E[\frac{\partial^2 f(x,t)}{\partial x^2}] = \frac{d^2}{dx^2}T(x,\upsilon). \tag{12}$$

To find:

$$E[\frac{\partial^2 f(x,t)}{\partial t^2}]. \tag{13}$$

Let:

$$\frac{\partial f(x,t)}{\partial t} = g, \tag{14}$$

then we have:

$$E[\frac{\partial^2 f(x,t)}{\partial t^2}] = E[\frac{\partial g(x,t)}{\partial t}] = E\frac{[g(x,t)]}{\upsilon} - \upsilon g(x,0), \tag{15}$$

$$E[\frac{\partial^2 f(x,t)}{\partial t^2}] = \frac{1}{\upsilon^2}T(x,\upsilon) - f(x,0) - \upsilon\frac{\partial f(x,t)}{\partial t}. \tag{16}$$

Therefore, one can easily extend this result to the *nth* partial derivative by using mathematical induction. We will see that, the Elzaki transform rivals the Laplace transform in solving the problem, its main advantage is the rivals that it may be used to solve problems without resorting to a new frequency domain because it preserve scales and units properties, the Elzaki transform may be used to solve intricate problems in engineering, mathematics and applied science without resorting to a new frequency domain.

3 Homotopy Perturbation Method:

The basic idea of the standard HPM was given by He [9, 12] and a new interpretation of this technique was presented by our research gorup [15]. To introduce HPM, considered the following general nonlinear differential equation:

$$Lu + Nu = f(x,t), \tag{17}$$

with initial conditions:

$$u(x,0) = k_1, \quad u_t(x,0) = k_2. \tag{18}$$

Where u is a function of x, t and k_1, k_2 are constants or functions of x. Also L and N are the linear and nonlinear operators respectively. According to HPM [2] we construct a homotopy which satisfies the following relation:

$$H(u,p) = Lu - L\nu_0 + p[L\nu_0 + Nu - f(x,t)] = 0, \tag{19}$$

where $p \in [0,1]$ is an embedded parameter and ν_0 is an arbitrary initial approximation satisfing the given initial condition.
By setting $p = 0$ and $p = 1$ in Eq. (19), one obtain:

$$H(u,0) = Lu - L\nu_0 = 0, \quad \text{and} \quad H(u,1) = Lu + Nu - f(x,t) = 0, \tag{20}$$

which are the linear and nonlinear original equations, respectively. In topology, this is called deformation and $Lu - L\nu_0$ and $Lu + Nu - f(x,t)$ are called homotopic. Here, the embedded parameter is introduced much more naturally, unaffected by artificial factor, further it can be considered as a small parameter for $0 \le p \le 1$.

Chowdhury and Hashim in [2], have presented an alternative way for choosing the initial approximation, that is:

$$\nu_0 = u(x,0) + tu_t(x,0) + L^{-1}f(x,t) = k_1 + tk_2 + L^{-1}f(x,t), \qquad (21)$$

where $L^{-1}(.) = \int_0^t \int_0^t ... \int_0^t (.)dt...dtdt$ depends on the order of the linear operator. With this assumption that the initial approximation ν_0 given in Eq. (21), in HPM, the solution of Eq. (19), is expressed as:

$$u(x,t) = u_0(x,t) + pu_1(x,t) + p^2 u_2(x,t) + ... \qquad (22)$$

Hence the approximate solution of Eq. (17), can be expressed as a power series of p, i. e.

$$u = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \qquad (23)$$

4 Homotopy Perturbation and Elzaki Transform Method:

Consider a general nonlinear partial differential equation with initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \qquad (24)$$

$$u(x,0) = c_1, \qquad\qquad u_t(x,0) = c_2. \qquad (25)$$

Where D is a linear differential operator of order two, R is a linear differential operator of less order than D , N is the general nonlinear differential operator, $f(x,t)$ is the source term and c_1, c_2 are constants or functions of x.

By taking Elzaki transform to both sides of Eq.(24), result in:

$$E[Du(x,t)] + E[Ru(x,t)] + E[Nu(x,t)] = E[f(x,t)]. \qquad (26)$$

Using the differentiation property of Elzaki transform and the initial condition in Eq.(24), one obtain:

$$E[u(x,t)] = \upsilon^2 E[f(x,t)] + \upsilon^2 c_1 + \upsilon^3 c_2 - \upsilon^2 E[R(x,t) + Nu(x,t)]. \qquad (27)$$

Applying the inverse Elzaki transform on both sides of Eq.(27), we get:

$$u(x,t) = F(x,t) - E^{-1}\{v^2 E[Ru(x,t) + Nu(x,t)]\}, \qquad (28)$$

where $F(x,t)$ represent the term arising from the source term and the prescribed initial conditions.

According to HPM we have:

$$u(x,t) = F(x,t) - pE^{-1}\{v^2 E[Ru(x,t) + Nu(x,t)]\}, \qquad (29)$$

now by substituting

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t), \qquad N[u(x,t)] = \sum_{i=0}^{\infty} p^i H_i(u), \qquad (30)$$

in Eq. (29) where $H_i(u)$ is He's polynomials that are given by:

$$H_i(u_0, u_1, ..., u_i) = \frac{1}{i!} \frac{\partial}{\partial p^i} [N(\sum_{i=0}^{\infty} p^i u_i)]_{p=0}, \qquad i = 0, 1, 2, ..., \qquad (31)$$

one obtain:

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = F(x,t)$$

$$- p\{E^{-1}[v^2 E[(R \sum_{i=0}^{\infty} p^i u_i(x,t)) + N(\sum_{i=0}^{\infty} p^i H_i(u))]]\}. \, (32)$$

This our method is infact a coupling technique of Elzaki transform and the homotopy perturbation method. Comparing the coefficients of the like powers of p the following approximations are resulted:

$$p^0 : u_0(x,t) = F(x,t),$$
$$p^1 : u_1(x,t) = -E^{-1}\{v^2 E[Ru_0(x,t) + H_0(u)]\},$$
$$p^2 : u_2(x,t) = -E^{-1}\{v^2 E[Ru_1(x,t) + H_1(u)]\},$$
$$\vdots$$
$$etc.$$

Therefore the solution will be obtained as:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \qquad (33)$$

5 Numerical Applications

In this section, we apply the homotopy-perturbation and Elzaki transform method for solving Sine-Gorden and Klein-Gorden equations. Numerical results are very encouraging.

Example 5.1 First, consider the following Sine-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} + \sin u = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 4\text{sech(x)}. \quad (34)$$

The exact solution is given in [2] as:

$$u(x,t) = 4\arctan[t\text{sech(x)}]. \quad (35)$$

To solve the example by this method, we take $\sin u \simeq u - \frac{u^3}{6} + \frac{u^5}{120}$.

After taking Elzaki transform of (34), subjected to the initial conditions, one obtain:

$$E[(u(x,t)] = 4v^3\text{sech(x)} + v^2\frac{\text{d}2}{\text{dx}2}E[(u(x,t)] - v^2E[u - \frac{u3}{6} + \frac{u5}{120}]. \quad (36)$$

The inverse Elzaki transform implies that:

$$u(x,t) = 4t\text{sech(x)} + E^{-1}\{v^2\frac{\text{d}2}{\text{dx}2}E[(u(x,t)] - v^2E[u - \frac{u3}{6} + \frac{u5}{120}]\}. \quad (37)$$

Now applying the homotopy perturbation method, we get:

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = 4t\text{sech(x)} + p\{v^2\frac{\text{d}2}{\text{dx}2}E[\sum_{i=0}^{\infty} p^i u_i(x,t)] \quad (38)$$

$$- v^2E[\sum_{i=0}^{\infty} p^i u_i(x,t) - \frac{(\sum_{i=0}^{\infty} p^i u_i(x,t))^3}{6} + \frac{(\sum_{i=0}^{\infty} p^i u_i(x,t))^5}{120}]\}.$$

By comparing the coefficients of the same powers of p, result in:

$$p^0 : u_0(x,t) = 4t\text{sech(x)},$$

$$p^1 : u_1(x,t) = \frac{d^2}{dx^2}E^{-1}\{4t\text{sech(x)}v^2E(t)\} - E^{-1}\{4t\text{sech(x)}v^2E(t)\}$$

$$+ E^{-1}\{\frac{64}{6}\text{sech}^3(x)v^2E(t^3)\} - E^{-1}\{\frac{1024}{120}\text{sech}^5(x)v^2E(t^5)\}.$$

Then, we get:

$$p^1 : u_1(x,t) = \frac{4}{315}\text{sech}^5(x) - (-105t^3 \cosh^2(x) + 42t^5 \cosh^2(x) - 16t^7),$$

$$p^2 : u_2(x,t) = \frac{4}{2027025}t^5 \text{sech}^9(x)(7040t^8 - 33696t^9 \cosh^2(x) - 4290t^4 \cosh^4(x)$$
$$+ 143000t^4 \cosh^2(x) - 205920t^2 \cosh^4(x) + 51480t^2 \cosh^6(x)$$
$$- 270270 \cosh^6(x) + 405405 \cosh^4(x),$$

therefore, the 3-terms Elzaki-HPM solution is:

$$u(x,t) = \frac{4}{2027025}t\text{sech}^9(x)(7040t^{12} - 33696t^{10} \cosh^2(x) - 4290t^8 \cosh^4(x)$$
$$+ 143000t^8 \cosh^2(x) - 308880t^6 \cosh^4(x) + 51480t^6 \cosh^6(x)$$
$$+ 405405t^4 \cosh^4(x) - 675675t^2 \cosh^6(x) - 270270 \cosh^8(x).$$

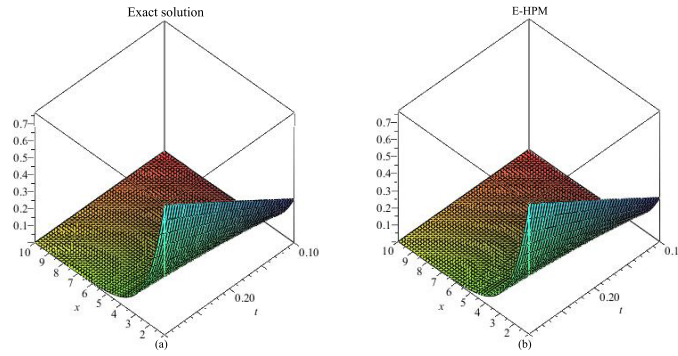The behavior of the solution (34), by 4-terms of EHPM and its exact solution are shown in Figures 1 and 2.

Figure 1: These surfaces show the approximate solutions obtained by 4-terms of EHPM and the exact solution of $u(x,t)$, respectively .(a) Exact plot; (b) EHPM plot(Eq. (35))

Example 5.2 Considering the following Sine-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} + \sin u = 0, \qquad u(x,0) = \pi + \varepsilon \cos(\mu x), \quad u_t(x,0) = 0. \quad (39)$$

Where $\mu = \frac{\sqrt{2}}{2}$ and ε is a constant.

We take $\sin u \simeq u - \frac{u^3}{6} + \frac{u^5}{120}$ to solve this example.

By taking Elzaki transform of (39), subjected to the initial conditions,we have:

$$E[u(x,t)] = v^2(\pi + \varepsilon \cos(\mu x)) + v^2 \frac{d^2}{dx^2}E[u(x,t)] - v^2 E[u - \frac{u^3}{6} + \frac{u^5}{120}](40)$$
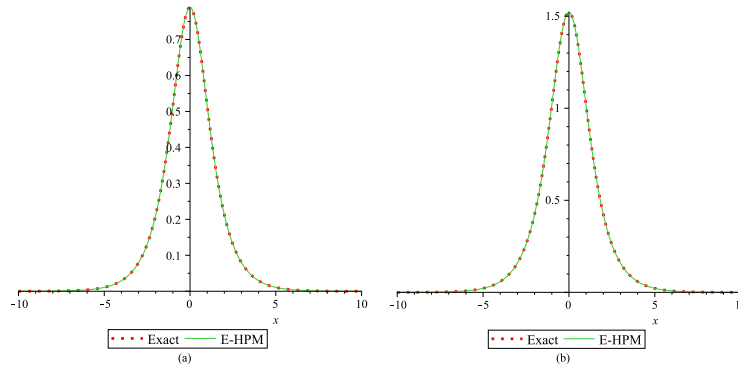
Figure 2: Comparison of the results by EHPM with the exact solution for example 5.1.

Table 1: Comparison of the exact and numerical solutions of example 5.1

x	t	Exact solution	EHPM-4term	Absolute error
0	0	0	0	0
0.062	0.162	0.6412071928	0.6412071885	4.3×10^{-9}
0.156	0.156	0.6116706532	0.6116706503	2.9×10^{-9}
0.469	0.469	1.5964486815	1.596478333	$.8482 \times 10^{-7}$
0.539	0.403	1.349539812	1.349538454	$.1358 \times 10^{-6}$
0.620	0.685	2.077035544	2.076989415	$.0590 \times 10^{-9}$
0.781	0.781	2.136014353	2.135871203	$.1431 \times 10^{-7}$
0.896	0.893	2.234155992	2.233473035	$.6829 \times 10^{-7}$
0.975	0.992	2.319920856	2.317982103	$.1938 \times 10^{-7}$
1.000	1.000	2.300024730	2.297874130	$.2150 \times 10^{-7}$

The inverse Elzaki transform implies that:

$$u(x,t) = (\pi + \varepsilon \cos(\mu x))E^{-1}[v^2] + E^{-1}\{v^2 \frac{d^2}{dx^2} E[(u(x,t)] - v^2 E[u]$$
$$+ v^2 E[\frac{u^3}{6}] - v^2 \frac{[u^5]}{120}\}.$$

Now applying the homotopy perturbation method, result in:

$$u(x,t) = (\pi + \varepsilon \cos(\mu x))E^{-1}[v^2] + p\{E^{-1}\{v^2 \frac{d^2}{dx^2} E[(u(x,t)] - v^2 E[u]$$
$$+ v^2 E[\frac{u^3}{6}] - v^2 \frac{[u^5]}{120}\}\}.$$

By taking $u(x, t) = \sum_{i=0}^{\infty} p^i u_i(x, t)$ and comparing the coefficients of the same powers of p, one obtain:

$$p^0 : u_0(x, t) = (\pi + \varepsilon \cos(\mu x)),$$

$$p^1 : u_1(x, t) = \frac{t^2}{2}\{-\mu^2 \varepsilon \cos(\mu x) + \frac{1}{6}(\pi + \varepsilon \cos(\mu x))^3 - \frac{1}{120}(\pi + \varepsilon \cos(\mu x))^5$$
$$- (\pi + \varepsilon \cos(\mu x))\}.$$
$$\vdots$$
$$etc.$$

Therefore, the 3-terms Elzaki-HPM solution is:

$$u(x, t) = \pi + \varepsilon \cos(\mu x) + \frac{t^2}{2}\{-\mu^2 \varepsilon \cos(\mu x) + \frac{1}{6}(\pi + \varepsilon \cos(\mu x))^3 - \frac{1}{120}$$
$$(\pi + \varepsilon \cos(\mu x))^5 - (\pi + \varepsilon \cos(\mu x))\} + \frac{1}{69120}[\varepsilon^2 \cos^2(\mu x) + 9\pi\varepsilon^8 \cos^8(\mu x)$$
$$+ (36\pi^2 - 32)\varepsilon^7 \cos^7(\mu x) + (84\pi^3 - 224\pi)\varepsilon^6 \cos^6(\mu x)$$
$$+ 720(\mu^2 + \frac{7}{40}\pi^4 - \frac{14}{15}\pi^2 + \frac{8}{15})\varepsilon^5 \cos^5(\mu x)$$
$$+ 2400\pi(\frac{21}{400}\pi^4 + \frac{4}{5} + \mu^2 - \frac{7}{15}\pi^2)\varepsilon^4 \cos^4(\mu)(x)$$
$$- 480(\varepsilon^2\mu^2 + (-6\pi^2 + 12)\mu^2 - 8\pi^2 + \frac{7}{3}\pi^4 - \frac{7}{40}\pi^6 + \varepsilon)\varepsilon^3 \cos^3(\mu x)$$
$$- 1440\pi(\varepsilon^2\mu^2 + (-\pi^2 + 6)\mu^2 + \frac{7}{15}\pi^4 - \frac{1}{40}\pi^6 - \frac{8}{3}\pi^2 + \varepsilon)\varepsilon^2 \cos^2(\mu x)$$
$$- 1440(\mu^2\varepsilon^2(\pi^2 - 2) - 2\mu^4 + \{\frac{-1}{6}\pi^4 - 4 + 2\pi^2\}\mu^{24} + \frac{7}{45}\pi^6$$
$$+ 4\pi^2 - \frac{1}{160}\pi^8) - \frac{4}{3}\pi^4 - 2)\varepsilon \cos(\mu x) - 480\pi(\pi^2 - 6)\mu^2\varepsilon^2$$
$$+ 384\pi^5 + 2880\pi - 1920\pi^3 + \pi^9 - 32\pi^7]t^4.$$

The behavior of the solution (39), by 4-terms of EHPM and its HPM solution are shown in Figure 3.

Example 5.3 Considering the following Klein-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} = u, \qquad u(x, 0) = 1 + \sin x, \qquad u_t(x, 0) = 0. \qquad (41)$$

Taking Elzaki transform of (41), subjected to the initial condition, one obtain:

$$E[(u(x, t)] = (1 + \sin x)v^2 + v^2 E[u] + v^2 \frac{d^2}{dx^2} E[u]. \qquad (42)$$
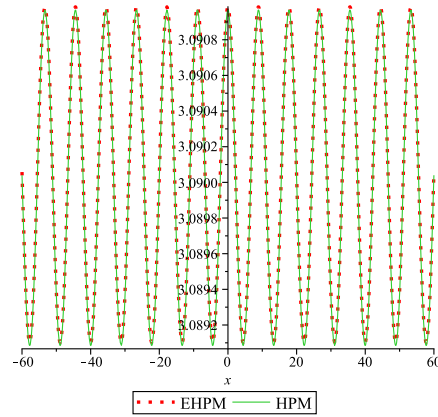
The inverse Elzaki transform implies that:

Figure 3: Comparison of the results by EHPM with the HPM solution for example 5.2.

$$u(x,t) = 1 + \sin x + E^{-1}\{v^2 E[u] + v^2 \frac{d^2}{dx^2} E[u]\}. \tag{43}$$

Now, applying the homotopy perturbation method, we get:

$$u(x,t) = 1 + \sin x + p\{E^{-1}\{v^2 E[u] + v^2 \frac{d^2}{dx^2} E[u]\}\}. \tag{44}$$

By taking $u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t)$ and comparing the coefficients of the same powers of p, result in:

$$p^0 : u_0(x,t) = 1 + \sin x,$$
$$p^1 : u_1(x,t) = \frac{t^2}{2},$$
$$p^2 : u_2(x,t) = \frac{t^4}{24},$$
$$p^3 : u_3(x,t) = \frac{t^6}{720},$$
$$\vdots$$
$$etc.$$

Therefore, the 4-terms approximate series solution is:

$$u(x,t) = 1 + \sin x + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720}, \tag{45}$$

and this will, in the limit of infinitely many terms, yield the closed form solution [6],

$$u(x,t) = \sin x + \cosh t. \tag{46}$$

Example 5.4 Consider the following Klein-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} = -u, \qquad u(x,0) = 0, \qquad u_t(x,0) = x. \tag{47}$$

Taking Elzaki transform of (47), subjected to the initial conditions, one obtain:

$$E[(u(x,t)] = xv^3 + v^2 \frac{d^2}{dx^2} E[u] - v^2 E[u]. \tag{48}$$

The inverse Elzaki transform implies that:

$$u(x,t) = tx + E^{-1}\{v^2 \frac{d^2}{dx^2} E[u] - v^2 E[u]\}. \tag{49}$$

Now applying the homotopy perturbation method, we get:

$$u(x,t) = tx + p\{E^{-1}\{v^2 \frac{d^2}{dx^2} E[u] - v^2 E[u]\}\}. \tag{50}$$

By taking $u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t)$ and comparing the coefficients of the same powers of p, one obtain:

$$p^0 : u_0(x,t) = tx,$$
$$p^1 : u_1(x,t) = \frac{-xt^3}{3!},$$
$$p^2 : u_2(x,t) = \frac{+xt^5}{5!},$$
$$p^3 : u_3(x,t) = \frac{-xt^7}{7!},$$
$$\vdots$$
$$etc.$$

Thus, the 4-terms approximate series solution is:

$$u(x,t) = tx - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!}, \tag{51}$$

and this will, in the limit of infinitely many terms, yield the closed form solution [22],

$$u(x,t) = x \sin t. \tag{52}$$

6 Conclusion:

In this paper, the Elzaki Homotopy-Perturbation method (EHPM) has been successfully employed to obtain the approximate analytical solutions of the Sine-Gorden and the Klein-Gorden equations. In example 5.3, the obtained result by this method is almost accurate and very near to the exact solution, also in example 5.4, it is observed that the (EHPM) solution yields the exact solution in only few iterations. Therefore, this novel iterative method has a bright aspect in future to obtain the approximate analytical solutions of ordinary and partial differential equations.

References

1. Caudrey P., Elibeck I. and Gibbon J. *The sine-gorden equation as a model classical field theory*, Nuovo cimento, 25 (1975), 496–511.

2. Chowdhury, M. S. H. and Hashim, I. *Application of homotopy-perturbation method to Kline-Gorden and Sine-Gorden equations*, Chaos, Solitons Fractals, 39 (2009), 1928-1935.

3. Christiansen P. L. and Lomdahl, P. S. *Numerical solution of 2+1 dimensional Sine-Gordon solitons*, Phys 2D (1981), 482-494.

4. Deeba, E. and Khuri, S. *A decomposition method for solving the nonlinear Kline-gorden equation*, J Comput Phys, 124 (1996), 442-448.

5. Dodd P., Elibeck I. and Gibbon J. *salitons and nonlinear wave equation*, London Academic, 1982.

6. El-Sayad S. *The decomposition method for studying the Kline-Gordon equation*, Chaos, Soliton Fractals, 18 (2003), 1025–1030.

7. Elzaki T. M. and Hilal, E. M. A. *Homotopy perturbation and Elzaki transform for solving nonlinear partial differential equations*, Mathematical Theory and Modling, 2 (2012), 3, 33-42.

8. Guo, B. Y., Pascual, P. J., Rodriguez, M. J. and Vzquez, L. *Numerical solution of the Sine-Gorden equation,* Appl Math comput, 18 (1986), 1–14.

9. He J. *A copuling method of homotopy technique and perturbation technique for nonlinear problem*, Int J Nonlinear Mech, 35 (2000), 37-43.

10. He J. *Homotopy-perturbation method for solving boundary value problem*, Phys Lett A, 350 (2006), 87-88.

11. He J. *Non-perturbative methods for strongly nonlinear problems*, Germanay: Die Deutsche bibliothek, 2006.

12. He J. *Some asymptotic method for strongly nonlinear equations*, Int J Mod Phys B, 20 (2006), 1141-1199.

13. He J. *Varuational iteration method-a kind of non-linear analytical technique: some examples*, Int J Nonlinear mech, 34 (1999), 699–708.

14. Herbst B. and Ablowitz M. *Numerical homoclinic instabilities in the Sine-Gorden equation*, Quaest Math, 15 (1992), 345–630.

15. Hesameddini, E. and Latifizadeh, H. *A new vision to the He's homotopy perturbation method*, Nonlinear sci. simul., 10 (2009), 1389–1398.

16. Hirota, R. *Exact three-soliton solution of the two-dimensional Sine-Gorden equation*, J Phys Soc Jpn, 35 (1973), 15–66.

17. Hu H. C. and Lou, S. Y. *New guasi-periodic waves of the (2+1)dimensional Sine-Gorden system*, Phys Lett A, 341 (2005), 422–426.

18. Kaliappan, P. and Lakshmanan, M *Kadomstev-Petviashvili two-dimensional Sine-Gorden equations: reduction to painleve transcendents*, J Phys A, 12 (1979), 249–252.

19. Kaya D. *A numerical solution of the Sine-Gorden equation using the modified decomposition method*, Appl Math Comput, 143 (2003, 309–317.

20. Kaya D. and El-Sayed S. *A numerical solution of the Kline-Gordon equation and convergence of the decomposition method*, Appl Math Comput, 156 (2004), 341–353.

21. Leibbrandt, G. *New exact solutions of the classical Sine-Gorden equation in 2+1 and 3+1 dimensionals*, Phys Rev Lett, 41 (1978), 435–438.

22. Mohyud-Din, S. T. and Yildirim A. *Variatonal Iteration Method for solving Kline-Gordon equation*, JAMSI, 6 (2010), 1, 99–106.

23. Wazwaz A. *The tanh method: exact solution of Sine-Gordon and the sinh-Gordon equations*, Appl Math Comput, 167 (2005), 1196–1210.

24. Xin, J. X. *Modeling light bullets with the two-dimensional Sine-Gordon equation*, Phys D, 135 (2000), 345–368.