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Research Article

Hybrid of Block-pulse and orthonormal Bernstein functions for fractional differential equations

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Abstract

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics, and engineering. In general, it is not easy to derive the analytical solutions to most of these equations. Therefore, it is vital to develop some reliable and efficient techniques to solve fractional differential equations. A numerical method for solving fractional differential equations is proposed in this paper. The method is based on a hybrid of Block-pulse and orthonormal Bernstein functions. Convergence analysis is given, and numerical examples are introduced to illustrate the effectiveness and simplicity of the method.

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1 Introduction

Fractional calculus is one of the most popular calculus types having a vast range of applications in many different scientific and engineering disciplines. The order of the derivatives in the fractional calculus might be any real number that separates the fractional calculus from the ordinary calculus where the order of derivatives are allowed to be only natural numbers. Fractional calculus is a highly efficient and useful tool in the modeling of many sorts of scientific phenomena including image processing, earthquake engineering, biomedical engineering, computational fluid mechanics, and physics. Fundamental concepts of fractional calculus and its applications to different research areas can be seen in the references [3, 4, 2, 7, 6, 5, 9, 12, 10, 16, 19, 20, 24, 29, 31, 32, 34], amongst many others. In recent years, many different basis functions have been used for solving fractional equations, such as operational matrix of Chebyshev polynomials, Block-pulse functions (BPFs), hybrid Bernoulli and Block-pulse functions, and hybrid Legendre [1, 8, 15, 23, 27, 26, 30, 33].

In this paper, we employ hybrid functions consisting of a combination of BPFs with orthonormal Bernstein polynomials (OBPs) to find the numerical solution of the fractional equation

$$F(x, y(x), D^{\alpha_1} y(x), \ldots, D^{\alpha_k} y(x)) = 0,\ x \in [0, X],\ \alpha_i \geq 0, \quad i = 1, \ldots, k, \tag{1}$$

with boundary or supplementary conditions

$$Z_i(y(\xi_i), y^{'}(\xi_i), \ldots, y^{(p)}(\xi_i)) = d_i, \quad i = 0, 1, \ldots, p, \tag{2}$$

where $0 \leq p < \max\{\alpha_i, i = 1, \ldots, k\} \leq p + 1,\ \xi_i \in [0, X],\ i = 0, \ldots, p$ and $Z_i, i = 0, \ldots, p$, are independent linear combinations of $y(\xi_i), y^{'}(\xi_i), \ldots, y^{(p)}(\xi_i)$ and $y \in L^2[0, X]$. It should be noted that F can be nonlinear in general.

The fractional equations are used in a variety of fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis, communication theory, mathematical economics, population genetics, the particle transport problem in astrophysics, and reactor theory. For the most part, it can be challenging to deduce the analytical solutions to most of these equations. As a result, it is crucial to develop some reliable and efficient ways to solve fractional differential equations. In this paper, we introduce a new method based on a hybrid of Block-pulse and orthonormal Bernstein functions. The present study's primary goal is to present and analyze a new stable algorithm for the numerical solution of the fractional differential equation based upon the hybrid of Block-pulse and orthonormal Bernstein functions (HBB method). To solve this problem, we propose a mathematical formulation that takes advantage of the orthogonality of BPFs and the OBPs properties to reduce the fractional differential equation to an algebraic system. Convergence analysis of the method is

discussed theoretically and numerically. Moreover, the applicability of the method is examined by solving some fractional differential equations and the Basset equation, which describes the unsteady motion of a sphere immersed in a Stokes fluid.

The outline of this paper is as follows: In section 2, we briefly overview some basic definitions for fractional calculus. In section 3, hybrid functions are introduced; therefore we approximate functions by using hybrid functions. In section 4, we present useful properties of hybrid functions such as coefficient matrix and operational matrix of derivative to solve fractional differential equations. In section 7, we present estimates for the error of the best approximation of smooth functions by hybrid functions. In section 6, we apply the proposed method to find an approximate solution to fractional differential equations. Finally, numerical examples are presented to show the effectiveness of the proposed method in section 7.

2 Fractional calculus

We give some basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 1 (see [32])**.** The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x), \quad \alpha > 0, x > 0, \tag{3}$$

$$J^0 f(x) = f(x). \tag{4}$$

It has the following property:

$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} x^{\gamma+\alpha}, \quad \gamma > -1.$$

Definition 2 (see [32])**.** The Caputo definition of fractional derivative operator is given by

$$D^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

where $m-1 < \alpha \leq m,\ m \in \mathbb{N},\ x > 0$. For the Caputo derivative, we have

$$D^{\alpha} C = 0 \qquad (C \textit{ is a constant}),$$

$$D^{\alpha} x^j = \begin{cases} 0 & \text{for } j \in \mathbb{N} \cup 0 \textit{ and } j < \lceil \alpha \rceil, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha} & \text{for } j \in \mathbb{N} \cup 0 \textit{ and } j \geq \lceil \alpha \rceil \textit{ or } j \in \mathbb{N}. \end{cases} \tag{5}$$

3 Hybrid of Block-pulse and orthonormal Bernstein functions

The orthonormal set of hybrids $h_{ji}(x)$, $j = 1, 2, \ldots, m$ and $i = 0, 1, \ldots, n$, where j is the order for BPFs, i is the order for OBPs, and x is the normalized time, is defined on the interval $[0, X)$ as

$$h_{ji}(x) = \begin{cases} \sqrt{m}\, b_{in}(mx - \gamma_j) & \text{if } \frac{X(j-1)}{m} \le x < \frac{Xj}{m}, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $\gamma_j = X(1-j)$ and $b_{in}, i = 0, \ldots, n$ are OBPs defined on $[0, X]$. Then

$$H(x) = [h_{10}(x), \ldots, h_{1n}(x), \ldots, h_{j0}(x), \ldots, h_{jn}(x), \ldots, h_{m0}(x), \ldots, h_{mn}(x)]^T$$
$$= [H_1(x), \ldots, H_j(x), \ldots, H_m(x)]^T,$$

is the $m(n+1)$ hybrid function vector. Here, the Bernstein polynomials (BPs) of the nth degree are defined on the interval $[0, X]$ as

$$B_{in}(x) = \frac{1}{X^n} \binom{n}{i} x^i (X - x)^{n-i}, \quad i = 0, 1, 2, \ldots, n,$$

by using expansion of $(X - x)^{n-i}$, we have

$$B_{in}(x) = \sum_{j=i}^{n} (-1)^{j-i} \left(\frac{1}{X^j}\right) \binom{n}{i} \binom{n-i}{j-i} x^j, \quad i = 0, 1, 2, \ldots, n,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

According to [25], BPs form a complete basis over the interval $[0, X]$.

The explicit representation of the OBPs, denoted by $b_{in}(x)$ here, was recently discovered by analyzing the resulting orthonormal polynomials after applying the Gram–Schmidt process; see [13]

$$b_{in}(x) = (\sqrt{2(n-i)+1})(X-x)^{n-i} \sum_{k=0}^{i} (-1)^k \frac{1}{X^{n-k}} \binom{2n+1-k}{i-k} \binom{i}{k} x^{i-k}.$$

In addition, it can be written in terms of the non-orthonormal Bernstein basis functions as

$$b_{in}(x) = (\sqrt{2(n-i)+1}) \sum_{k=0}^{i} (-1)^k \frac{\binom{2n+1-k}{i-k}\binom{i}{k}}{\binom{n-k}{i-k}} B_{i-kn-k}(x),$$

for $i = 0, 1, \ldots, n$. By using the Taylor expansion, $b_{in}(x)$ can be represented as [21]

$$b_{in} = Z_i T_n(x), \quad i = 0, \ldots, n,$$

where $T_n(x) = [1, x, x^2, \ldots, x^n]^T$, Z_i is a row vector of Taylor coefficients as

$$(Z_i)_j = \sqrt{2(n-i)+1} \sum_{k=\max\{0, j-n+i\}}^{\min\{i,j\}} \rho_{ij-k}\beta_{ik}, \quad j = 0, \ldots, n,$$

where

$$\rho_{ir} = (-1)^r \binom{n-i}{r}, \quad r = 0, \ldots, n-i,$$

$$\beta_{ij} = (-1)^{i-j} \binom{2n+1-i+j}{j} \binom{i}{i-j}, \quad j = 0, \ldots, i.$$

Thus $b(x) = [b_{0n}(x), b_{1n}(x), \ldots, b_{nn}(x)]^T$ can be defined by

$$b(x) = ZT_n(x), \tag{7}$$

where Z is an $(n+1) \times (n+1)$ matrix whose ith row is $Z_i, i = 0, \ldots, n$.

Now, from (7) and (4), we have

$$H_j(x) = \sqrt{m} Z T_n(mx - \gamma_j), \tag{8}$$

where

$$T_n(mx - \gamma_j) = [1, mx - \gamma_j, (mx - \gamma_j)^2, \ldots, (mx - \gamma_j)^n]^T,$$

by using the binomial expansion of $(mx - \gamma_j)^k$, (8) can be expressed as

$$H_j(x) = Z\Lambda_j T_n(x),$$

where

$$\Lambda_j = \sqrt{m} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \gamma_j & m & 0 & 0 & \cdots & 0 & 0 \\ \binom{2}{0}(\gamma_j)^2 & \binom{2}{1}(\gamma_j)m & \binom{2}{2}m^2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ \binom{n}{0}(\gamma_j)^n & \binom{n}{1}(\gamma_j)^{n-1}m & \binom{n}{2}(\gamma_j)^{n-2}m^2 & \binom{n}{3}(\gamma_j)^{n-3}m^3 & \cdots & \binom{n}{n-1}(\gamma_j)m^{n-1} & \binom{n}{n}m^n \end{pmatrix}.$$

Therefore the hybrid function vector $H(x)$ can be defined by

$$H(x) = \tilde{\mathbf{\Gamma}}\tilde{T}(x), \tag{9}$$

where $\tilde{T}(x) = [T_n(x), T_n(x), \ldots, T_n(x)]^T$ is an $m(n+1)$ vector, and $\tilde{\mathbf{\Gamma}} = diag[\Gamma_1, \Gamma_2, \ldots, \Gamma_m]$ is an $m(n+1) \times m(n+1)$ coefficient matrix of the $(n+1) \times (n+1)$ coefficient submatrix $\Gamma_j = Z\Lambda_j$.

3.1 Function approximation

Suppose that $\Omega = L^2[0, X]$ and

$$\{h_{10}, \ldots, h_{1n}, \ldots, h_{j0}, \ldots, h_{jn}, \ldots, h_{m0}, \ldots, h_{mn}\} \subset \Omega$$

are the set of hybrid functions and that

$$\Omega_{mn} = span\{h_{10}, \ldots, h_{1n}, \ldots, h_{j0}, \ldots, h_{jn}, \ldots, h_{m0}, \ldots, h_{mn}\}.$$

Let f_{mn} be the best approximation of an arbitrary function $f \in L^2(\Omega)$ out of Ω_{mn}, that is,

$$\|f - f_{mn}\|_2 \leq \| f - g \|_2 \qquad \text{for all } g \in \Omega_{mn}.$$

Moreover, since $f_{mn} \in \Omega_{mn}$, there exist unique coefficients $c_{10}, c_{11}, \ldots, c_{mn}$ such that

$$f(x) \simeq f_{mn}(x) = \sum_{j=1}^{m}\sum_{i=0}^{n} c_{ji}h_{ji}(x) = \sum_{j=1}^{m} C_j H_j(x) = C^T H(x) = H^T(x)C,$$

where C and $H(x)$ are $m(n+1)$ vectors as

$$C = [c_{10}, \ldots, c_{1n}, \ldots, c_{j0}, \ldots, c_{jn}, \ldots, c_{m0}, \ldots, c_{mn}]^T = [C_1, \ldots, C_j, \ldots, C_m]^T,$$

$$H = [h_{10}, \ldots, h_{1n}, \ldots, h_{j0}, \ldots, h_{jn}, \ldots, h_{m0}, \ldots, h_{mn}]^T \quad = [H_1, \ldots, H_j, \ldots, H_m]^T,$$

and the hybrid function coefficients c_{ji} are obtained by

$$c_{ji} = \frac{\langle f, h_{ji}\rangle}{\|h_{ji}\|_2^2}.$$

Since the set of hybrid functions is a complete orthonormal system in Ω, then

$$\|h_{ji}\|_2 = \int_0^X h_{ji}(x)h_{ji}(x)dx = 1.$$

Therefore

$$c_{ji} = \langle f, h_{ji} \rangle = \int_0^X f(x) h_{ji}(x) dx. \tag{10}$$

4 Operational matrix of differentiation

We construct the operational matrices of differentiation for $H(x)$. First, we consider integer order derivatives of $H(x)$. Hence

$$\frac{dH(x)}{dx} = \mathbf{D}^{(1)} H(x),$$

where $\mathbf{D}^{(1)}$ is the $m(n+1) \times m(n+1)$ operational matrix of first derivative of Hybrid function. From (9), we have

$$\frac{dH(x)}{dx} = \tilde{\mathbf{\Gamma}} \frac{d\tilde{T}(x)}{dx} = \tilde{\mathbf{\Gamma}} \tilde{\mathbf{Q}} \tilde{T}(x) = \tilde{\mathbf{\Gamma}} \tilde{\mathbf{Q}} \tilde{\mathbf{\Gamma}}^{-1} H(x),$$

where $\tilde{\mathbf{Q}} = diag[Q, Q, \ldots, Q]$ is an $m(n+1) \times m(n+1)$ matrix that consists the m submatrix $Q_{(n+1)\times(n+1)}$ as

$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \end{pmatrix}_{(n+1)\times(n+1)}.$$

Therefore we have

$$\mathbf{D}^{(1)} = \tilde{\mathbf{\Gamma}} \tilde{\mathbf{Q}} \tilde{\mathbf{\Gamma}}^{-1}. \tag{11}$$

In general, we obtain

$$\frac{d^n H(x)}{dx^n} = (\mathbf{D}^{(1)})^n H(x),$$

where $n \in \mathbb{N}$ and the superscript, in $\mathbf{D}^{(1)}$, denotes the matrix powers. Thus

$$\mathbf{D^{(n)}} = (\mathbf{D}^{(1)})^n, \qquad n = 1, 2, \ldots.$$

For the Caputo fractional derivative introduced in section 2, we denote fractional order derivative of $H(x)$ by

$$\frac{d^\alpha H(x)}{dx^\alpha} = \mathbf{D}^{(\alpha)} H(x),$$

where $\mathbf{D}^{(\alpha)}$ is an $m(n+1) \times m(n+1)$ operational matrix of fractional order derivative of Hybrid function. From (9), we have

$$\frac{d^\alpha H(x)}{dx^\alpha} = \tilde{\mathbf{\Gamma}} \frac{d^\alpha \tilde{T}(x)}{dx^\alpha},$$

where

$$\frac{d^\alpha \tilde{T}(x)}{dx^\alpha} = [\frac{d^\alpha T(x)}{dx^\alpha}, \frac{d^\alpha T(x)}{dx^\alpha}, \ldots, \frac{d^\alpha T(x)}{dx^\alpha}]^T,$$

and by using (3), we can specify the elements of $\frac{d^\alpha T(x)}{dx^\alpha}$ as

$$\frac{d^\alpha x^k}{dx^\alpha} = \begin{cases} 0, & 0 \le k < \lceil \alpha \rceil, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}, & \lceil \alpha \rceil \le k < n. \end{cases}$$

We can also approximate $x^{k-\alpha}$ for $k = \lceil \alpha \rceil, \ldots, n$ by hybrid functions as

$$x^{k-\alpha} \simeq \sum_{j=1}^{m} \sum_{i=0}^{n} p_{kji} h_{ji}(x) = P_k^T H(x),$$

where P_k is an $m(n+1)$ vector that we can obtain by (10) and (9) as

$$P_{kji} = \int_0^X x^{k-\alpha} h_{ji}(x) dx = \tilde{\Gamma}_{ji} \int_{\frac{X(j-1)}{m}}^{\frac{X_j}{m}} x^{k-\alpha+i} dx.$$

Therefore

$$\mathbf{D}^{(\alpha)} = \tilde{\mathbf{\Gamma}} \tilde{\mathbf{\Psi}} \tilde{\mathbf{P}}, \tag{12}$$

where $\tilde{\mathbf{\Psi}} = diag[\Psi, \Psi, \ldots, \Psi]$ is an $m(n+1) \times m(n+1)$ matrix with m submatrix $\Psi_{(n+1)\times(n+1)}$ as

$$\Psi = diag[0, \ldots, 0, \frac{\Gamma(\lceil \alpha \rceil + 1)}{\Gamma(\lceil \alpha \rceil + 1 - \alpha)}, \ldots, \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}],$$

and $\tilde{\mathbf{P}} = [P, P, \ldots, P]^T$ is the $m(n+1) \times m(n+1)$ matrix with the $(n+1) \times m(n+1)$ submatrix P, defined as

$$P = \begin{pmatrix} p_{\lceil \alpha \rceil 10} & p_{\lceil \alpha \rceil 11} & p_{\lceil \alpha \rceil 12} & \cdots & p_{\lceil \alpha \rceil m(n-1)} & p_{\lceil \alpha \rceil mn} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ p_{n10} & p_{n11} & p_{n12} & \cdots & p_{nm(n-1)} & p_{nmn} \end{pmatrix}.$$

5 Convergence analysis

The purpose of this section is to obtain an estimate of the error norm of the best approximation of a smooth function of two variables, on a certain domain $[0, X]$, by a bivariate polynomial. This estimate will be used for comparisons in section 6, when analyzing the error of the numerical results.

We assume that $f = \sum_{j=1}^{m} f_j$ is a sufficiently smooth function on $[0, X]$ and that $p \simeq \sum_{j=1}^{m} p_j$ is the interpolating polynomial to f at points $x_i, i = 0, 1, \ldots, n$, which are the roots of the $(n+1)$-degree shifted Chebyshev polynomial on $[0, X]$. Then we have [17]

$$f(x) - P(x) = \sum_{j=1}^{m}(f_j - P_j) = \sum_{j=1}^{m}(\frac{f_j^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)),$$

where $\xi \in I_j = [\frac{(j-1)X}{m}, \frac{jX}{m}]$. Therefore

$$\mid f(x) - P(x) \mid \leq \sum_{j=1}^{m} \max_{x \in I_j} \mid f_j^{(n+1)}(x) \mid \frac{\prod_{i=0}^{n}(x - x_i)}{(n+1)!}. \tag{13}$$

We assume that there is a real number γ such that

$$\max_{x \in I_j} \mid f_j^{(n+1)}(x) \mid \leq \gamma. \tag{14}$$

By replacing (14) in (13) and taking into account the estimates for Chebyshev interpolation nodes [28], we obtain

$$\mid f(x) - P(x) \mid \leq \gamma \frac{X^{n+1}}{2^{2n+1} m^n (n+1)!}. \tag{15}$$

With the help of (15) we obtain the following result.

Theorem 1. Let $f_{mn}(x) = C^T H(x)$ be the hybrid functions expansion of the sufficiently smooth real function f in Ω, where

$$C = [c_{10}(x), \ldots, c_{1n}(x), \ldots, c_{j0}(x), \ldots, c_{jn}(x), \ldots, c_{m0}(x), \ldots, c_{mn}(x)]^T,$$

and

$$c_{ji} = \int_0^X f(x) h_{ji}(x) dx.$$

Then, there exists a real number γ' such that

$$\| f - f_{mn} \|_2 \leq \gamma' \frac{X^{n+1}}{2^{2n+1} m^n (n+1)!}. \tag{16}$$

Proof. Let Ω_{mn} be the space of bivariate polynomials of degree $\leq n$. As shown in section 3, f_{mn} is the best approximation of f in Ω_{mn}, that is,

$$\| f - f_{mn} \|_2 \leq \| f - g \|_2,$$

where g is any arbitrary polynomial in Ω_{mn}. In particular, we have

$$\|f-f_{mn}\|_2^2 dx \leq \int_0^X \mid f(x)-p_{mn}(x) \mid^2 dx, \tag{17}$$

where p_{mn} is the interpolating polynomial of f. From (15) and (17), we obtain

$$\|f-f_{mn}\|_2^2 \leq \int_0^X [\gamma \frac{X^{n+1}}{2^{2n+1}m^n(n+1)!}]^2 dx = X[\gamma \frac{X^{n+1}}{2^{2n+1}m^n(n+1)!}]^2. \tag{18}$$

From (18), we conclude that (16) is valid with

$$\gamma' = \gamma\sqrt{X}.$$

□

Theorem 2. Suppose $f \in L^2[0,1], n-1 < \alpha \leq n$. Then

$$\| D^\alpha f - D^\alpha f_{mn} \| \leq \frac{1}{\Gamma(n-\alpha+1)} \gamma' \frac{X^{n+1}}{2^{2n+1}m^n(n+1)!}. \tag{19}$$

Proof. By using (18) and [11], we have

$$\begin{aligned} \| D^\alpha f - D^\alpha f_{mn} \|^2 &= \| I^{n-\alpha}(D^n f - D^n f_{mn}) \|^2 \\ &= \| \frac{1}{x^{1+\alpha-n}\Gamma(n-\alpha)}(D^n f - D^n f_{mn}) \|^2 \\ &\leq (\frac{1}{(n-\alpha)\Gamma(n-\alpha)})^2 \| D^n f - D^n f_{mn} \|^2 \\ &\leq (\frac{1}{\Gamma(n-\alpha+1)})^2 \| f - f_{mn} \|^2 . \end{aligned}$$

From (18), we get (19). □

6 Solving fractional differential equations

In this section, in order to show the high importance of the proposed method, we apply it to solve fractional differential equation (1) and (2). In order to use hybrid functions for this problem, we approximate by hybrid functions as

$$y(x) \simeq \sum_{j=1}^{m}\sum_{i=0}^{n} c_{ji}h_{ji}(x) = C^T H(x), \tag{20}$$

where $C = [c_{10}, \ldots, c_{1n}, \ldots, c_{m0}, \ldots, c_{mn}]^T$ is an unknown vector. Using (11) and (12), we have

$$D^{\alpha_j} y(x) \simeq C^T D^{\alpha_j} H(x) \simeq C^T \mathbf{D}^{(\alpha_j)} H(x), \ j = 1, \ldots, k.$$

By substituting these equations in (1) and (2), we get

$$F(x, C^T H(x), C^T \mathbf{D}^{(\alpha_1)} H(x), \ldots, C^T \mathbf{D}^{(\alpha_k)} H(x)) = 0, \tag{21}$$

$$Z_i(C^T H(\xi_i), C^T \mathbf{D}^{(1)} H(\xi_i), \ldots, C^T \mathbf{D}^{(p)} H(\xi_i)) = d_i, i = 0, 1, \ldots, p. \tag{22}$$

To find the solution y, we first collocate (21) at $m(n+1) - (p+1)$ points. For suitable collocation points, we use

$$x_i = \frac{2i-1}{2m(n+1)}, i = 1, \ldots, m(n+1) - (p+1).$$

These equations together with (22) generate $m(n+1)$ algebraic equations, which can be solved to find $c_{ji}, j = 1, \ldots, m, i = 0, \ldots, n$. Consequently, the approximate solution of the unknown function y, given in (20), can be calculated.

7 Numerical examples

In this section, numerical results of some examples are presented. The absolute errors of this method are compared with those of the existing methods reported in [35, 14, 22]. The computations associated with these examples were performed using Mathematica 9.0.

Example 1. Consider the fractional equation

$$\sqrt{x} D^{\frac{1}{2}} y(x) + e^x y(x) = x(\frac{2}{\sqrt{\pi}} + e^x), \quad 0 < x \leq 1, \tag{23}$$

with the following initial conditions $y(0) = 0$. The exact solution of this problem is $y(x) = x$. The numerical method developed in section 6 is applied to (23) for $m = n = 2$, we approximate solution as

$$y(x) \simeq c_{10}h_{10} + c_{11}h_{11} + c_{12}h_{12} + c_{20}h_{20} + c_{21}h_{21} + c_{22}h_{22} = C^T H(x).$$

Here, we have

$$\tilde{\mathbf{\Gamma}} = \begin{pmatrix} 3.16228 & -12.6491 & 12.6491 & 0 & 0 & 0 \\ -2.44949 & 29.3939 & -48.9898 & 0 & 0 & 0 \\ 1.41421 & -22.6274 & 56.5685 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.6491 & -25.2982 & 12.6491 \\ 0 & 0 & 0 & -29.3939 & 78.3837 & -48.9898 \\ 0 & 0 & 0 & 26.8701 & -79.196 & 56.5685 \end{pmatrix},$$

$$\tilde{\mathbf{\Psi}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.12838 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.50451 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.12838 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.50451 \end{pmatrix},$$

$$\tilde{\mathbf{P}} = \begin{pmatrix} 2.38514 & 0. & 0.666667 & 0.672262 & 0.430222 & 0.236893 \\ 0.170367 & 0.263932 & 0.161905 & 0.415476 & 0.383286 & 0.235524 \\ 0.0283945 & 0.0879772 & 0.084127 & 0.262718 & 0.331263 & 0.235838 \\ 2.38514 & 0. & 0.666667 & 0.672262 & 0.430222 & 0.236893 \\ 0.170367 & 0.263932 & 0.161905 & 0.415476 & 0.383286 & 0.235524 \\ 0.0283945 & 0.0879772 & 0.084127 & 0.262718 & 0.331263 & 0.235838 \end{pmatrix},$$

$$\mathbf{D}^{\frac{1}{2}} = \begin{pmatrix} -1.89128 & -2.09283 & -0.709874 & -0.930387 & 0.8335 & 1.12652 \\ 3.55781 & 2.26954 & -0.830648 & -5.58347 & -11.7032 & -9.57079 \\ -1.93327 & 0.748753 & 3.02605 & 11.7513 & 18.4068 & 14.0581 \\ -4.32293 & -5.85992 & -3.02074 & -6.86047 & -4.63714 & -2.23512 \\ 12.9755 & 16.8594 & 8.1193 & 17.3836 & 9.48449 & 3.44875 \\ -12.8079 & -16.0982 & -7.30845 & -14.7689 & -6.05866 & -0.975538 \end{pmatrix}.$$

Finally, the obtained results are reported in Table 1. It shows that our method has better accuracy with the piecewise constant orthogonal function developed in [35].

Example 2. Consider the following fractional differential equation:

$$D^{\frac{1}{3}}y(x) + x^{\frac{1}{3}}y(x) = \frac{3}{2\Gamma(2.3)}x^{\frac{2}{3}} + x^{\frac{4}{3}}, \quad 0 < x \leq 1,$$

with the following initial conditions

$$y(0) = 0.$$

We know that the exact solution is $y(x) = x$. The maximum errors for different values of x are listed in Table 1. It is shown that, we obtain the error of order 10^{-16} for $m = 2, n = 2$. In the Haar wavelets method [14], the error is 0.0014 obtained for $m = 64$, where m is the order of Haar wavelets. Clearly, the results obtained by our method are better than those in [14] in terms of accuracy if the exact solution is sufficiently smooth. Therefore, the HBB method is a valid method in solving fractional differential equations.

Table 1: Numerical results for Examples 1 and 2 for $m = 2$ and $n = 2$.

x	0.1	0.3	0.5	0.7	0.9
Example 1	3.46945×10^{-17}	5.55112×10^{-17}	3.21965×10^{-15}	2.33147×10^{-15}	1.77636×10^{-15}
Example 2	1.38778×10^{-17}	5.55112×10^{-17}	1.5099×10^{-14}	1.55431×10^{-15}	1.27676×10^{-15}

Example 3. Consider the fractional equation

$$D^{\alpha}y(x) = -y(x) + x^2 + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}, \quad 0 < x \leq 1, \tag{24}$$

with the following initial conditions $y(0) = 0$. The accurate solution, in this case, is given by $y(x) = x^2$. For $\alpha = 0.25, 0.75$ and $m = n = 2$, the approximation solution for (24) is obtained by using the HBB method. The absolute error of HBB method and fast wavelet collocation method (FWCM) reported in [22] is shown in Table 2 when $\alpha = 0.25$, $\alpha = 0.75$, and $m = n = 2$.

The logarithm of absolute error for different values of m and n is shown in Figures 1 and 2. Table 2 and Figures 1 and 2 confirm that the HBB method approximates the solution of fractional differential equation uniformly.

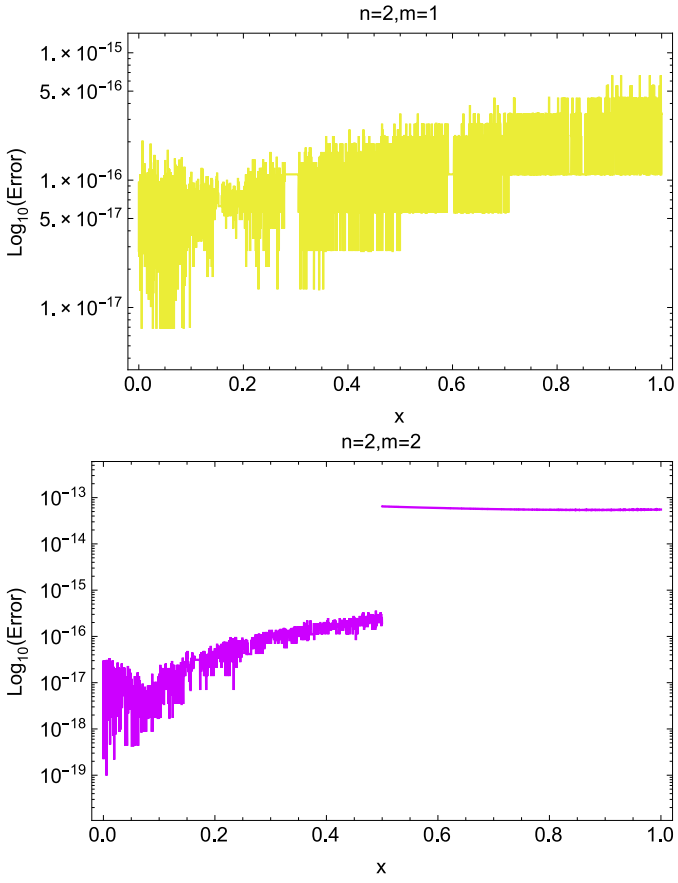

Figure 1: The curves of the logarithm of the absolute errors of Example 3 for $\alpha = 0.25$ and different value of m and n.

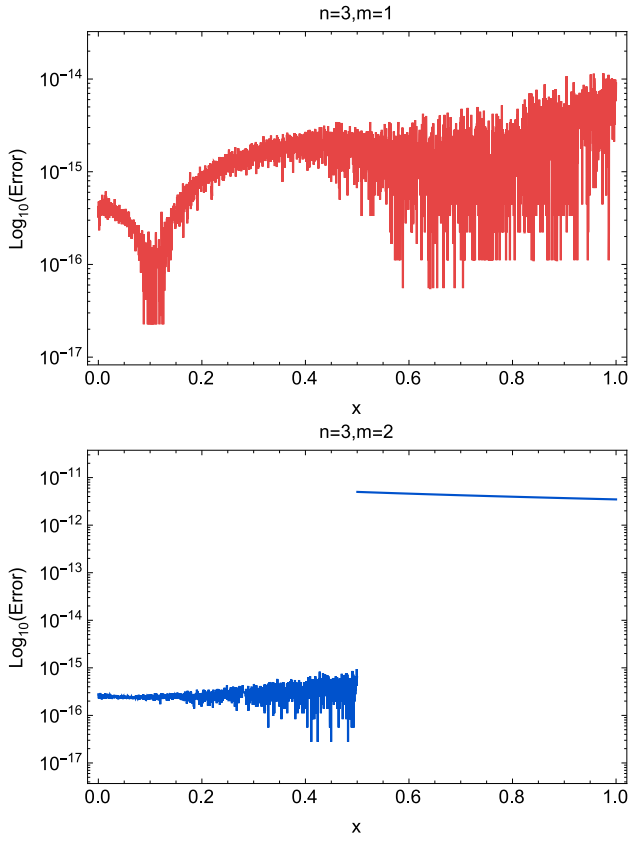

Figure 2: The curves of the logarithm of the absolute errors of Example 3 for $\alpha = 0.75$ and different value of m and n.

Table 2: The errors of Example 3 compared to [22]

x	$\alpha = 0.25$		$\alpha = 0.75$	
	Current Method	FWC method in [22]	Current Method	FWC method in [22]
0.03125	3.46945×10^{-18}	0.05958×10^{-12}	3.95517×10^{-16}	0.33012×10^{-13}
0.09375	2.77556×10^{-17}	0.04301×10^{-12}	4.02456×10^{-16}	0.34730×10^{-13}
0.18750	8.32667×10^{-17}	0.05390×10^{-12}	3.88578×10^{-16}	0.02296×10^{-13}
0.28125	1.38778×10^{-17}	0.02446×10^{-12}	3.88578×10^{-16}	0.19817×10^{-13}
0.37500	2.22045×10^{-16}	0.00394×10^{-12}	4.16334×10^{-16}	0.29393×10^{-13}
0.46875	6.15064×10^{-14}	0.06483×10^{-12}	5.55112×10^{-16}	0.22148×10^{-13}
0.56250	5.77316×10^{-14}	0.05401×10^{-12}	6.66134×10^{-15}	0.19428×10^{-13}
0.65625	5.54001×10^{-14}	0.00982×10^{-12}	8.88178×10^{-15}	0.17097×10^{-13}
0.75000	5.42899×10^{-14}	0.01798×10^{-12}	1.33227×10^{-15}	0.09103×10^{-13}
0.84375	5.45127×10^{-14}	0.03186×10^{-12}	1.77636×10^{-15}	0.11990×10^{-13}
0.93750	6.25614×10^{-14}	0.20128×10^{-12}	2.47315×10^{-15}	0.27533×10^{-13}
CPU time	$3.65s$		$2.57s$	

Example 4 (see [18]). Consider the following fractional Relaxation problem:

$$D^{\frac{1}{2}}y(x) = -y(x), \quad 0 < x \leq 1, \tag{25}$$

with initial condition $y(0) = 1$.

The exact solution of this equation is $y(x) = \mathbb{E}_\alpha(-x^\alpha)$. Here $\mathbb{E}_\alpha(x)$ is called the Mittag–Lefler function,

$$\mathbb{E}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

Table 3 shows a comparison of our method and the Taylor matrix method presented in [18]. Clearly, the results obtained by the HBB method are in agreement with other mentioned numerical method and in total this approach has high accuracy.

Table 3: Comparison of our method for $m = 3$ and $n = 8$, and the Taylor matrix method in [18] of Example 4

x	Current Method	Ref.[18]
0.0	3.4685×10^{-16}	0.0000
0.1	1.7745×10^{-10}	0.1000×10^{-9}
0.2	6.3421×10^{-10}	0.9000×10^{-9}
0.3	5.3461×10^{-9}	0.4000×10^{-9}
0.4	6.4326×10^{-10}	0.2000×10^{-9}
0.5	7.4902×10^{-9}	0.1000×10^{-8}
0.6	4.3721×10^{-8}	0.6000×10^{-8}
0.7	3.5684×10^{-7}	0.2400×10^{-7}
0.8	9.2341×10^{-7}	0.8400×10^{-7}
0.9	8.4563×10^{-7}	0.2480×10^{-6}
1.0	5.3475×10^{-7}	0.6740×10^{-6}

8 Conclusion

In this paper, we obtained operational matrices of the fractional derivatives of a combination of OBPs and BPFs. Then by using these matrices, we reduced the multi-order fractional differential equations to a system of algebraic equations that can be solved easily. The convergence analysis of the HBB bases is given in section 7. The numerical solutions obtained using the suggested method showed that numerical solutions are in very good coincidence with the exact solution. For future research, we will apply this numerical method for solving nonlinear fractional integro-differential equations, nonlinear Volterra–Fredholm–Hammerstein integral equations, and so on.

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