

# An approximation method for numerical solution of multi-dimensional feedback delay fractional optimal control problems by Bernstein polynomials

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## Abstract

In this paper, we present a new method for solving fractional optimal control problems with delays in state and control. This method is based upon Bernstein polynomials basis and feedback control. The main advantage of feedback or closed-loop control is that one can monitor the effect of such control on the system and modify the output accordingly. In this work, we use Bernstein polynomials to transform the fractional time-varying multi-dimensional optimal control system with both state and control delays, into an algebraic system in terms of the Bernstein coefficients approximating state and control functions. We use Caputo derivative of degree  $0 < \alpha \leq 1$  as the fractional derivative in our work. Finally, some numerical examples are given to illustrate the effectiveness of this method.

**Keywords:** Delay fractional optimal control problem; Caputo fractional derivative; Bernstein polynomial.

## 1 Introduction

The general definition of an optimal control problem requires the minimization of a functional over an admissible set of control and state functions sub-

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ject to dynamic constraints on the states and controls. A Fractional Optimal Control Problem (FOCP) is an optimal control problem in which either the performance index or the differential equations governing the dynamic of the system or both contain at least one fractional order derivative term [1, 2, 17].

Fractional Differential Equations ( FDEs ) have been the focus of many studies due to their appearance in various applications in real-world physical systems. For example, it has been illustrated that materials with memory and hereditary effects and dynamical processes including gas diffusion and heat conduction can be more adequately modeled by FDEs than integer-order differential equations [13, 18, 20]. Some other applications of FDEs are in behaviors of viscoelastic materials, biomechanics and electrochemical processes ( see [3, 5] for more details ).

Most FOCPs do not have exact solutions, so in these cases approximation methods and numerical techniques must be used. Recently, several approximation methods to solve FOCPs have been introduced [4, 14, 18].

Real life phenomena have been described more precisely by Delay Differential Equations, so Delay Fractional Optimal Control Problem ( DFOCP ) has become the focus of many researchers in the last decade. Baleanu in [6] and Jarad in [11] analyzed the fractional variational principles for some kinds of DFOCPs within Riemann-Liouville and Caputo fractional derivatives respectively and made their corresponding Euler-Lagrange equations. In this paper, we present a novel strategy based on Bernstein polynomials (BPs) to solve DFOCPs. Consider the following DFOCP

$$\text{Min } J = \frac{1}{2} \int_0^1 [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt, \quad (1)$$

s.t

$${}^c_0D_t^\alpha x_i(t) = \sum_{j=1}^r a_{i,j}(t)x_j(t) + \sum_{k=1}^s b_{i,k}(t)u_k(t) + \sum_{j=1}^r (a_d)_{i,j}(t)x_j(t - \eta_1) + \sum_{k=1}^s (b_d)_{i,k}(t)u_k(t - \eta_2), \quad 1 \leq i \leq r, \quad (2)$$

$$\begin{aligned} x_j(t) &= x_{j,0}, \quad t \in [-\eta_1, 0], & 1 \leq j \leq r, \\ u_k(t) &= u_{k,0}, \quad t \in [-\eta_2, 0], & 1 \leq k \leq s, \end{aligned} \quad (3)$$

where  $x(t) = [x_1(t) \cdots x_r(t)]^T$  and  $u(t) = [u_1(t) \cdots u_s(t)]^T$  are respectively the state and control functions. Also,  $Q(t)$  and  $R(t)$  are respectively,  $r \times r$  and  $s \times s$  semi-positive and positive definite time-varying matrices of the state and control's coefficients in the cost function with continuous functions as their entries. Furthermore,  $a_{i,j}(t)$ ,  $(a_d)_{i,j}(t)$ ,  $b_{i,k}(t)$  and  $(b_d)_{i,k}(t)$  are continuous functions which are respectively the coefficients of  $x_j(t)$ ,  $x_j(t - \eta_1)$  for  $(1 \leq j \leq r)$  and  $u_k(t)$ ,  $u_k(t - \eta_2)$  for  $(1 \leq k \leq s)$  in the  $i$ -th fractional differential equation (2) and  $\eta_1, \eta_2 > 0$  are given constant delays. The fractional derivative is defined in Caputo sense, i.e.

$${}^c_0D_t^\alpha x_i(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} x_i(\tau) d\tau, & 0 < \alpha < 1, \\ \dot{x}_i, & \alpha = 1. \end{cases} \quad (4)$$

In the numerical solution of dynamical systems, polynomials or piecewise polynomial functions are often used to present the approximate solutions [9, 10, 21]. The effectiveness of using Bernstein polynomials for solving FOCPs have been demonstrated before [4, 14]. In the present paper, we seek an optimal feedback control function to find the approximate solution of DFOCP (1) - (3) by using Bernstein polynomials.

This paper is organized as follows. In Section 2 we give some preliminaries in fractional calculus. In Section 3 Bernstein polynomials are introduced and their properties are shown in several lemmas. In Section 4, a FOCP with time delay will be solved using BPs. Section 5 contains some numerical examples. Finally Section 6 consists of a brief conclusion.

## 2 Some preliminaries in fractional calculus

**Definition 2.1.** A real function  $f(t)$ ,  $t > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in \mathbb{R}$ , if there exists a real number  $p > \mu$  such that  $f(t) = t^p f_1(t)$ , where  $f_1(t) \in C[0, +\infty)$  and it is said to be in the space  $C_\mu^m$  iff  $f^{(m)} \in C_\mu$  for  $m \in \mathbb{N}$ .

**Definition 2.2.** The Riemann-Liouville fractional integral operator of order  $\alpha > 0$  of a function  $f \in C_\mu$ ,  $\mu > 1$ , is defined as:

$$\begin{aligned} {}_0I_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \\ {}_0I_t^0 f(t) &= f(t). \end{aligned} \quad (5)$$

**Definition 2.3.** The fractional derivative of  $f(t)$  in the Caputo sense is defined as follows:

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d^n}{d\tau^n} f(\tau), \quad n-1 < \alpha < n, \quad n \in \mathbb{N}, \quad f \in C_{-1}^m. \quad (6)$$

In [15], the following properties for  $f \in C_\mu$  and  $\mu \geq -1$  have been proved

1.  ${}_0I_t^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{\alpha+k}, \quad k \in \mathbb{N} \cup \{0\}, \quad t > 0,$
2.  ${}^cD_t^\alpha {}_0I_t^\alpha f(t) = f(t),$
3.  ${}_0I_t^\alpha {}^cD_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f(0^+) \frac{t^k}{k!}, \quad t > 0,$
4.  ${}^cD_t^\beta f(t) = {}_0I_t^{\alpha-\beta} {}^cD_t^\alpha f(t), \quad \alpha, \beta > 0.$

### 3 Properties of Bernstein polynomials

The Bernstein polynomial of degree  $n$  over the interval  $[a, b]$  is defined as follows:

$$B_{i,n} \left( \frac{t-a}{b-a} \right) = \binom{n}{i} \left( \frac{t-a}{b-a} \right)^i \left( \frac{b-t}{b-a} \right)^{n-i},$$

so, within the interval  $[0, 1]$  we have

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

Define  $\Phi_m(t) = [B_{0,m}(t) B_{1,m}(t) \cdots B_{m,m}(t)]^T$ . To consider the vector  $\Phi_m(t - \eta)$  ( $\eta$  is the given delay) in terms of  $\Phi_m(t)$ , we state the following lemmas.

**Lemma 3.1.** *We can write  $\Phi_m(t) = \Lambda T_m(t)$ , where  $\Lambda = (\Upsilon_{i,j})_{i,j=1}^{m+1}$  is an upper triangular  $(m+1) \times (m+1)$  matrix with entry*

$$\Upsilon_{i+1,j+1} = \begin{cases} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i}, & i \leq j, \\ 0, & i > j, \end{cases} \quad i, j = 0, 1, \dots, m,$$

and  $T_m(t) = [1 \ t \ \cdots \ t^m]^T$ .

*Proof.* [4].

**Lemma 3.2.** *For each given constant delay  $\eta > 0$ ,  $\Phi_m(t - \eta) = \Omega \Phi_m(t)$ , where  $\Omega$  is an  $(m+1) \times (m+1)$  matrix in terms of  $\eta$ .*

*Proof.* According to Lemma 3.1 we have

$$\Phi_m(t - \eta) = \Lambda T_m(t - \eta).$$

But, the right hand side of the above equation can be written as

$$\Lambda T_m(t - \eta) = \Lambda \begin{bmatrix} 1 \\ t - \eta \\ (t - \eta)^2 \\ \vdots \\ (t - \eta)^m \end{bmatrix} = \Lambda \Psi \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} = \Lambda \Psi T_m(t),$$

where

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\eta & 1 & 0 & \cdots & 0 \\ \eta^2 & -2\eta & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (-\eta)^m & \binom{m}{m-1}(-\eta)^{m-1} & \cdots & \cdots & 1 \end{bmatrix}.$$

By Lemma 3.1,  $T_m(t) = \Lambda^{-1}\Phi_m(t)$ , thus

$$\Phi(t - \eta) = \Lambda\Psi\Lambda^{-1}\Phi_m(t) = \Omega\Phi_m(t). \quad \square \quad (7)$$

**Lemma 3.3.** *Let  $L^2[0, 1]$  be a Hilbert space with inner product  $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$  and  $y \in L^2[0, 1]$ . Then one can find the unique vector  $C = [c_0 \ c_1 \ \cdots \ c_m]^T$  such that*

$$y(t) \approx \sum_{i=0}^m c_i B_{i,m}(t) = C^T \Phi_m(t). \quad (8)$$

*Proof.* [12].

In Lemma 3.3 we have  $C^T = Q^{-1}\langle y, \Phi_m \rangle$  such that

$$\langle y, \Phi_m \rangle = \int_0^1 y(t)\Phi_m(t)dx = [\langle y, B_{0,m} \rangle \ \langle y, B_{1,m} \rangle \ \cdots \ \langle y, B_{m,m} \rangle]^T,$$

and each entry of the matrix  $Q = (Q_{i+1,j+1})_{i,j=0}^m$  is defined as follows:

$$Q_{i+1,j+1} = \int_0^1 B_{i,m}(t)B_{j,m}(t)dx = \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}.$$

Since the set  $\{B_{0,m}(t), B_{1,m}(t), \dots, B_{m,m}(t)\}$  forms a basis for the vector space of polynomials of real coefficients and degree no more than  $m$  [7, 16], a polynomial of degree  $m$  can be expanded in terms of a linear combination of  $B_{i,m}(t)$ , ( $i = 0, 1, \dots, m$ ) as follows

$$P(t) = \sum_{i=0}^m c_i B_{i,m}(t),$$

moreover we have

$$t^k = \sum_{i=k-1}^{m-1} \frac{\binom{i}{k}}{\binom{m}{k}} B_{i,m}(t).$$

**Lemma 3.4.** *Derivatives of  $P_n(f) = \sum_{j=0}^n f(\frac{j}{n})B_{j,n}(t)$  of any order converge*

to corresponding derivatives of  $f$ . So if  $f \in C^k[0, 1]$  then

$$\lim_{n \rightarrow \infty} (P_n(f))^{(k)} = f^{(k)},$$

uniformly on  $[0, 1]$ .

*Proof.* [8].

#### 4 Fractional optimal control problem with delays in control and state

Consider fractional delay control system (2). For each  $0 \leq i \leq r$ , one can apply the Riemann-Liouville fractional integral  ${}_0I_t^\alpha$  to both sides of that equation

$$x_i(t) - x_i(0) = \sum_{j=1}^r {}_0I_t^\alpha \{a_{i,j}(t)x_j(t)\} + \sum_{k=1}^s {}_0I_t^\alpha \{b_{i,k}(t)u_k(t)\} + \sum_{j=1}^r {}_0I_t^\alpha \{(a_d)_{i,j}(t)x_j(t - \eta_1)\} + \sum_{k=1}^s {}_0I_t^\alpha \{(b_d)_{i,k}(t)u_k(t - \eta_2)\}. \quad (9)$$

Assume that  $x_i(t) \approx X_i^T \Phi_m(t)$  ( $1 \leq i \leq r$ ) and  $u_k(t) \approx U_k^T \Phi_m(t)$  ( $1 \leq k \leq s$ ) where the entries  $X_i = [X_i(0) \cdots X_i(m)]^T$  and  $U_k = [U_k(0) \cdots U_k(m)]^T$  are respectively the coefficients of  $x_i(t)$  and  $u_k(t)$  in approximating them by Bernstein polynomials of degree  $m$  just like (8). Moreover, the Bernstein approximated coefficients vectors of functions  $a_{i,j}(t)$ ,  $b_{i,k}(t)$ ,  $(a_d)_{i,j}(t)$  and  $(b_d)_{i,k}(t)$  can be achieved by using equation (8). We denote the approximated vector coefficients of these functions respectively by  $(A^{i,j})_{(m+1) \times 1}$ ,  $(B^{i,k})_{(m+1) \times 1}$ ,  $(A_d^{i,j})_{(m+1) \times 1}$  and  $(B_d^{i,k})_{(m+1) \times 1}$ .

By substituting the so called approximated vectors and matrices in (1), one can find the following equations:

$$X_i^T \Phi_m(t) - x_{i,0} = \sum_{j=1}^r {}_0I_t^\alpha \{((A^{i,j})^T \Phi_m(t))(X_j^T \Phi_m(t))^T\} + \sum_{k=1}^s {}_0I_t^\alpha \{((B^{i,k})^T \Phi_m(t))(U_k^T \Phi_m(t))^T\} + \sum_{j=1}^r {}_0I_t^\alpha \{((A_d^{i,j})^T \Phi_m(t))(X_j^T \Phi_m(t - \eta_1))^T\} + \sum_{k=1}^s {}_0I_t^\alpha \{((B_d^{i,k})^T \Phi_m(t))(U_k^T \Phi_m(t - \eta_2))^T\}. \quad (10)$$

Moreover, from Lemma 3.2 there exist  $(m+1) \times (m+1)$  matrices  $\Omega_1, \Omega_2$  where  $\Phi_m(t - \eta_1) = \Omega_1 \Phi_m(t)$  and  $\Phi_m(t - \eta_2) = \Omega_2 \Phi_m(t)$ , while

$$\Omega_1 = \Lambda \Psi \Lambda^{-1},$$

$$\Omega_2 = \Lambda \Psi' \Lambda^{-1},$$

and  $\Psi, \Psi'$  are obtained respectively in terms of  $\eta_1$  and  $\eta_2$ .

As it was shown in [4], for each  $1 \leq i, j \leq r$  and  $1 \leq k \leq s$ , the  $(m+1) \times (m+1)$

matrices  $A^{i,j}$ ,  $B^{i,k}$ ,  $A_d^{i,j}$  and  $B_d^{i,k}$  can be calculated such that:

$$\begin{aligned}(A^{i,j})^T \Phi_m(t) \Phi_m^T(t) &= \Phi_m^T(t) A^{i,j}, \\ (B^{i,k})^T \Phi_m(t) \Phi_m^T(t) &= \Phi_m^T(t) B^{i,k}, \\ (A_d^{i,j})^T \Phi_m(t) \Phi_m^T(t) &= \Phi_m^T(t) A_d^{i,j}, \\ (B_d^{i,k})^T \Phi_m(t) \Phi_m^T(t) &= \Phi_m^T(t) B_d^{i,k}.\end{aligned}$$

Therefore, by replacing the above equalities, (2) can be rewritten as follows:

$$\begin{aligned}X_i^T \Phi_m(t) - x_{i,0} &= \sum_{j=1}^r ({}_0I_t^\alpha \Phi_m^T(t)) (A^{i,j} X_j) + \sum_{k=1}^s ({}_0I_t^\alpha \Phi_m^T(t)) (B^{i,k} U_k) + \\ &\quad \sum_{j=1}^r ({}_0I_t^\alpha \Phi_m^T(t)) (A_d^{i,j} \Omega_1^T X_j) + \sum_{k=1}^s ({}_0I_t^\alpha \Phi_m^T(t)) (B_d^{i,j} \Omega_2^T U_k),\end{aligned}$$

or

$$\begin{aligned}X_i^T \Phi_m(t) - x_{i,0} &= \sum_{j=1}^r (A^{i,j} X_j)^T ({}_0I_t^\alpha \Phi_m(t)) + \sum_{k=1}^s (B^{i,k} U_k)^T ({}_0I_t^\alpha \Phi_m(t)) + \\ &\quad \sum_{j=1}^r (A_d^{i,j} \Omega_1^T X_j)^T ({}_0I_t^\alpha \Phi_m(t)) + \sum_{k=1}^s (B_d^{i,j} \Omega_2^T U_k)^T ({}_0I_t^\alpha \Phi_m(t)).\end{aligned}\tag{11}$$

where  $i = 1, \dots, r$ .

One can approximate  ${}_0I_t^\alpha \Phi_m(t)$  by  $I_\alpha \times \Phi_m(t)$ , where  $I_\alpha$  is an  $(m+1) \times (m+1)$  matrix called the operational matrix of Riemann-Liouville fractional integral.

Infact, from Lemma 3.1,  $\Phi_m(t) = \Lambda T_m(t)$ , so

$${}_0I_t^\alpha \Phi_m(t) = \Lambda {}_0I_t^\alpha T_m(t) = \Lambda [{}_0I_t^\alpha 1 \quad {}_0I_t^\alpha t \quad \dots \quad {}_0I_t^\alpha t^m]^T,$$

where  ${}_0I_t^\alpha t^j = \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} t^{j+\alpha}$ . Therefore,

$${}_0I_t^\alpha T_m(t) = \tilde{\Sigma} \tilde{T},\tag{12}$$

where  $\tilde{\Sigma} = (\tilde{\Sigma}_{i+1,j+1})$  and  $\tilde{T} = (\tilde{T}_{i+1})$  are respectively  $(m+1) \times (m+1)$  and  $(m+1) \times 1$  matrices, which are defined as follows:

$$\tilde{\Sigma}_{i+1,j+1} = \begin{cases} \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)}, & i = j, \\ 0, & o.w, \end{cases} \quad i, j = 0, \dots, m$$

and

$$(\tilde{T})_{i+1} = t^{i+\alpha}, \quad i = 0, \dots, m.$$

Also, from Lemma 3.3, since  $t^{i+\alpha} \in L^2([0, 1])$  for each integer  $i$  ( $0 \leq i \leq m$ ), one can find the  $(m+1) \times 1$  vector  $P_i$  such that

$$t^{i+\alpha} \approx P_i^T \Phi_m(t),\tag{13}$$

where  $P_i = Q^{-1} \langle t^{i+\alpha}, \Phi_m(t) \rangle$  and the entries of  $\bar{P}_i = \langle t^{i+\alpha}, \Phi_m(t) \rangle = [\bar{P}_{i,0} \bar{P}_{i,1} \cdots \bar{P}_{i,m}]^T$  can be attained as

$$\bar{P}_{i,j} = \int_0^1 t^{i+\alpha} B_{j,m}(t) dt = \frac{m! \Gamma(i+j+\alpha+1)}{j! \Gamma(i+m+\alpha+2)}, \quad i, j = 0, \dots, m.$$

Now if  $P$  is an  $(m+1) \times (m+1)$  matrix of the form  $[P_0 P_1 \cdots P_m]$ , then from (12) and (13) we have

$${}_0I_t^\alpha \Phi_m(t) \approx \Lambda \tilde{\Sigma} P^T \Phi_m(t), \quad (14)$$

therefore,  $I_\alpha = \Lambda \tilde{\Sigma} P^T$  is the aforementioned operational matrix of Riemann-Liouville fractional integral  ${}_0I_t^\alpha$ .

Hence, by replacing  ${}_0I_t^\alpha \Phi_m(t)$  from (14) into (4) and writing  $x_{i,0}$  in terms of BPs of degree  $m$ , equation (4) can be written as the following

$$\begin{aligned} X_i^T \Phi_m(t) - X_{i,0}^T \Phi_m(t) &= \sum_{j=1}^r (A^{i,j} X_j)^T I_\alpha \Phi_m(t) + \sum_{k=1}^s (B^{i,k} U_k)^T I_\alpha \Phi_m(t) \\ &\quad + \sum_{j=1}^r (A_d^{i,j} \Omega_1^T X_j)^T I_\alpha \Phi_m(t) + \sum_{k=1}^s (B_d^{i,j} \Omega_2^T U_k)^T I_\alpha \Phi_m(t), \end{aligned} \quad (15)$$

where

$$X_{i,0}^T = [X_{i,0}(0), \dots, X_{i,0}(m)]^T$$

is the known Bernstein approximated coefficients vector of  $x_{i,0}$  that can be computed using (8). By equalling the coefficients of  $\Phi_m(t)$  from both sides of (5), we found that

$$\begin{aligned} X_i^T &= X_{i,0}^T + \sum_{j=1}^r X_j^T (A^{i,j})^T I_\alpha + \sum_{k=1}^s U_k^T (B^{i,k})^T I_\alpha \\ &\quad + \sum_{j=1}^r X_j^T \Omega_1 (A_d^{i,j})^T I_\alpha + \sum_{k=1}^s U_k^T \Omega_2 (B_d^{i,j})^T I_\alpha, \end{aligned} \quad (16)$$

for  $i = 1, \dots, r$ . Equations (8) can be written in compact form as follows:

$$X^T = \Pi + U^T \Gamma, \quad (17)$$

where  $\Pi$  and  $\Gamma$  are respectively  $1 \times (m+1)$  and  $(m+1) \times (m+1)$  matrices that can be obtained by the following

$$\Pi = X_0^T (I_{m+1} - (\tilde{A} + \tilde{A}_d) I_\alpha)^{-1},$$

and

$$\Gamma = (\tilde{B} + \tilde{B}_d) I_\alpha (I_{m+1} - (\tilde{A} + \tilde{A}_d) I_\alpha)^{-1},$$

and  $I_{m+1}$  is the  $(m+1) \times (m+1)$  identity matrix.

Moreover, by applying the approximations  $x(t) \approx (X^T)_{1 \times r(m+1)} \Phi_m(t)$  and  $u(t) \approx (U^T)_{1 \times s(m+1)} \Phi_m(t)$  where  $X^T = [X_1^T, \dots, X_r^T]$  and  $U^T = [U_1^T, \dots, U_s^T]$ , the cost functional (1) can be approximated as bellow



$$\begin{aligned}
J &= \frac{1}{2} \int_0^1 \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\} dt \\
&\approx \frac{1}{2} \int_0^1 \{X^T \Phi_m(t)(Q^T \Phi_m(t) \Phi_m^T(t) X)^T + U^T \Phi_m(t)(R^T \Phi_m(t) \Phi_m^T(t) U)^T\} dt,
\end{aligned} \tag{18}$$

where  $Q = [Q_{i,j}]$  and  $R = [R_{i,j}]$  that  $Q_{i,j}, R_{i,j}$  are the  $(m+1) \times 1$  vectors of Bernstein coefficients in approximating  $Q_{i,j}(t)$  and  $R_{i,j}(t)$  respectively. Therefore,

$$J \approx \frac{1}{2} \int_0^1 \{X^T \Phi_m(t)(\Phi_m^T(t) \tilde{Q} X)^T + U^T \Phi_m(t)(\Phi_m^T(t) \tilde{R} U)^T\} dt,$$

or

$$J \approx \frac{1}{2} \int_0^1 \{(X^T \Phi_m(t))(X^T \tilde{Q}^T \Phi_m(t)) + (U^T \Phi_m(t))(U^T \tilde{R}^T \Phi_m(t))\} dt, \tag{19}$$

where  $\tilde{Q} = [\tilde{Q}_{i,j}]$  and  $\tilde{R} = [\tilde{R}_{i,j}]$ . Also  $\tilde{Q}_{i,j}$  and  $\tilde{R}_{i,j}$  are  $(m+1) \times (m+1)$  matrices that can be calculated from

$$\begin{aligned}
(Q^{i,j})^T \Phi_m(t) \Phi_m^T(t) &= \Phi_m^T(t) \tilde{Q}^{i,j}, \\
(R^{i,j})^T \Phi_m(t) \Phi_m^T(t) &= \Phi_m^T(t) \tilde{R}^{i,j}.
\end{aligned}$$

Let  $Z_{i,j} = H \otimes \tilde{Q}_{i,j}$  and  $W_{i,j} = H \otimes \tilde{R}_{i,j}$ , where  $\otimes$  is the Kronecker product and  $H = [H_{i,j}]_{(m+1) \times (m+1)}$  and each entry  $H_{i,j}$  is defined by

$$H_{i,j} = \int_0^1 B_{i,m}(t) B_{j,m}(t) dt,$$

then (19) can be rewritten in compact form as:

$$J \approx \frac{1}{2} \{(X^T Z X) + (U^T W U)\}, \tag{20}$$

where  $Z = [z_{i,j}]$  and  $W = [w_{i,j}]$ .

From (17) we know that  $X^T = \Pi + U^T \Gamma$ , so the necessary condition that  $U$  minimizes (20) and satisfy (17) is that

$$\frac{\partial J}{\partial U} = X^T Z \Gamma^T + U^T W = 0,$$

so

$$U^{*T} = X^T Z \Gamma^T W^{-1}. \tag{21}$$

The above equation gives the optimal feedback control and by replacing (21) in (17), we can easily find the optimal state as well.

We need to mention that, since the Bernstein coefficients of positive functions in  $L^2[0, 1]$  are positive [7] and it was assumed that  $R(t)$  is positive definite, then  $R_{i,j}$  is a positive vector. Also because  $B_{i,m}(t) > 0$  for  $t \in (0, 1)$ , it's clear that

$$\Phi_m^T(t)\tilde{R}_{i,j} = R_{i,j}^T\Phi_m(t)\Phi_m^T(t) > 0, \quad t \in [0, 1],$$

therefore  $\tilde{R}_{i,j}$  and as the result  $W_{i,j} = H \otimes \tilde{R}_{i,j}$  are positive definite and consequently invertible matrices.

## 5 Convergence of the method

In this section, we show the convergence of the presented method discussed in this article. First we prove the following lemma.

**Lemma 5.1.** *Let  $X^T\Phi_m(t) = \sum_{j=0}^m X_j B_{j,m}(t)$  be the Bernstein polynomial of order  $m$  that approximates the function  $x(t) \in L^2[0, 1]$ . Then  ${}_0I_t^\alpha(X^T\Phi_m(t))$ , tends to  ${}_0I_t^\alpha x(t)$  as  $m$  tends to infinity.*

*Proof.* By Lemma 3.3 we have

$$\lim_{m \rightarrow \infty} \sum_{j=0}^m X_j B_{j,m}(t) = x(t). \quad (22)$$

Since  $B_{j,m}(t)$  is a continuous function, we have

$$\lim_{m \rightarrow \infty} \int_0^t \frac{\sum_{j=0}^m X_j B_{j,m}(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \lim_{m \rightarrow \infty} \sum_{j=0}^m X_j \int_0^t \frac{B_{j,m}(\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$

By (22) and from Definition 2.2, we obtain

$$\int_0^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \Gamma(\alpha) \lim_{m \rightarrow \infty} \sum_{j=0}^m X_j {}_0I_t^\alpha B_{j,m}(t),$$

or

$${}_0I_t^\alpha x(t) = \lim_{m \rightarrow \infty} \sum_{j=0}^m X_j {}_0I_t^\alpha B_{j,m}(t) = \lim_{m \rightarrow \infty} X^T {}_0I_t^\alpha \Phi_m(t). \quad (23)$$

In (14),  $I_\alpha = \Lambda \tilde{\Sigma} P^T$  where the  $i$ -th column of  $P^T$  is the Bernstein approximated coefficients of  $t^{i+\alpha}$  for  $i = 0, \dots, m$ . Now, regarding the convergence of the Bernstein approximation of every functions in  $L^2([0, 1])$ , one can write

$$\lim_{n \rightarrow \infty} P^T \Phi_n(t) = \lim_{n \rightarrow \infty} \begin{bmatrix} \sum_{j=0}^n P_{j,0} B_{j,n}(t) \\ \sum_{j=0}^n P_{j,1} B_{j,n}(t) \\ \vdots \\ \sum_{j=0}^n P_{j,m} B_{j,n}(t) \end{bmatrix} = \tilde{T} = \begin{bmatrix} t^\alpha \\ t^{1+\alpha} \\ t^{2+\alpha} \\ \vdots \\ t^{m+\alpha} \end{bmatrix},$$

therefore,  $\lim_{n \rightarrow \infty} \Lambda \tilde{\Sigma} P^T \Phi_n(t) = \Lambda \tilde{\Sigma} \lim_{n \rightarrow \infty} P^T \Phi_n(t) = \Lambda \tilde{\Sigma} \tilde{T}$ , or as explained in (12) and (13)

$$\lim_{n \rightarrow \infty} I_\alpha \Phi_n(t) = {}_0 I_t^\alpha \Phi_m(t). \quad (24)$$

From (23) and (24) we reach

$${}_0 I_t^\alpha x(t) = \lim_{m \rightarrow \infty} X^T \lim_{n \rightarrow \infty} I_\alpha \Phi_n(t).$$

Given  $n \geq m$  will complete the proof.  $\square$

**Theorem 5.1.** *The approximated solutions  $\bar{x}(t) = \bar{X}^T \Phi_m(t)$  and  $\bar{u}(t) = \bar{U}^T \Phi_m(t)$  in which  $(\bar{X}, \bar{U})$  is achieved from (17) and (21), converge to the optimal solutions  $x^*(t)$  and  $u^*(t)$  as the degree of the Bernstein polynomials tend to infinity.*

*Proof.* Suppose  $W_m$  is the set of all  $(U^T, X^T) \Phi_m(\cdot)$  where  $X, U \in \mathbb{R}^{m+1}$  and satisfy (17), also  $W$  is the set of all  $(u(\cdot), x(\cdot))$  satisfy (2) and (3). Let  $\bar{U}$  be the optimal solution of (20) where obtained from (21) and  $\bar{X}$  be the solution of (17) obtained by replacing  $\bar{U}$  in equation (17). Therefore  $(\bar{U}^T, \bar{X}^T) \Phi_m(\cdot) \in W_m$ . By the convergence property of Bernstein polynomials, for  $(\bar{U}^T, \bar{X}^T) \Phi_m(\cdot)$ , there exists a unique pair of functions  $(\bar{u}(\cdot), \bar{x}(\cdot))$  such that

$$(\bar{U}^T, \bar{X}^T) \Phi_m(\cdot) \longrightarrow (\bar{u}(\cdot), \bar{x}(\cdot)) \quad \text{as } m \rightarrow \infty.$$

Now according to Lemma 5.1 it is clear that  $(\bar{u}(\cdot), \bar{x}(\cdot)) \in W$ . Moreover as  $m \rightarrow \infty$ , then  $J(\bar{U}^T \Phi_m, \bar{X}^T \Phi_m) \longrightarrow \bar{J}$  where  $\bar{J}$  is the value of cost function (1) corresponding to the feasible solution  $(\bar{u}(\cdot), \bar{x}(\cdot))$ . Now, since

$$W_1 \subseteq \dots \subseteq W_m \subseteq W_{m+1} \subseteq \dots \subseteq W,$$

consequently

$$\text{Inf}_{W_1} J_1 \geq \dots \geq \text{Inf}_{W_m} J_m \geq \text{Inf}_{W_{m+1}} J_{m+1} \geq \dots \geq \text{Inf}_W J.$$

Let  $J_m^* = \text{Inf}_{W_m} J_m$ , so  $J_m^* = J(\bar{U}^T \Phi_m, \bar{X}^T \Phi_m)$ . Furthermore, the sequence  $\{J_m^*\}$  is nonincreasing and bounded below which converges to a number  $\bar{J} \geq \text{Inf}_W J$ . We want to show that  $\bar{J} = \lim_{m \rightarrow \infty} J_m^* = \text{Inf}_W J$ . Given  $\varepsilon > 0$ , let  $(u(\cdot), x(\cdot))$  be an element in  $W$  such that

$$J(u, x) < \text{Inf}_W J + \varepsilon, \quad (25)$$

by the definition of infimum, such  $(u(\cdot), x(\cdot)) \in W$  exists.

Since  $J(u, x)$  is continuous, for this value of  $\varepsilon$ , there exists  $N(\varepsilon)$  so that if  $m > N(\varepsilon)$ ,

$$|J(u, x) - J(U^T \Phi_m, X^T \Phi_m)| < \varepsilon, \quad (26)$$

Now if  $m > N(\varepsilon)$ , then using (25) and (26) gives

$$J(U^T \Phi_m, X^T \Phi_m) < J(u, x) + \varepsilon < \text{Inf}_W J + 2\varepsilon,$$

on the other hand

$$\text{Inf}_W J \leq J_m^* = \text{Inf}_{W_m} J_m \leq J(U^T \Phi_m, X^T \Phi_m),$$

so

$$\text{Inf}_W J \leq J_m^* < \text{Inf}_W J + 2\varepsilon,$$

or

$$0 \leq J_m^* - \text{Inf}_W J < 2\varepsilon,$$

where  $\varepsilon$  is chosen arbitrary. Thus

$$\bar{J} = \lim_{m \rightarrow \infty} J_m^* = \text{Inf}_W J. \quad \square$$

## 6 Numerical examples

In this section we give some numerical examples and apply the method presented in Section 4 for solving them. Our examples are solved using *Matlab2011a* on an Intel Core i5-430M processor with 4 GB of DDR3 Memory. These test problems demonstrate the validity and efficiency of this technique.

**Example 6.1.** Consider the following delay fractional optimal control problem in which  $0 < \alpha \leq 1$ ,

$$\begin{aligned} \min J &= \frac{1}{2} \int_0^1 [x^2(t) + \frac{1}{2}u^2(t)]dt, \\ \text{s.t. } {}_0^c D_t^\alpha x(t) &= -x(t) + x(t - \frac{1}{3}) + u(t) - \frac{1}{2}u(t - \frac{2}{3}), & 0 \leq t \leq 1, \\ x(t) &= 1, & -\frac{1}{3} \leq t \leq 0, \\ u(t) &= 0, & -\frac{2}{3} \leq t \leq 0. \end{aligned}$$

For  $\alpha = 1$ , this problem has been numerically solved by applying hybrid functions based on Legendre polynomials in [19] and the objective value  $I = 0.3731$  has been achieved. Whilst, in the presented method the solution has the objective value  $J^* = 0.3956$  for  $\alpha = 1$  and  $m = 6$ . Thus, our results with  $m = 6$  are in good agreement with the results demonstrated in [19] for  $\alpha = 1$ . In addition, by varying the value of  $\alpha$  we can obtain the optimal control  $u(\cdot)$  and trajectory function  $x(\cdot)$  which are shown respectively

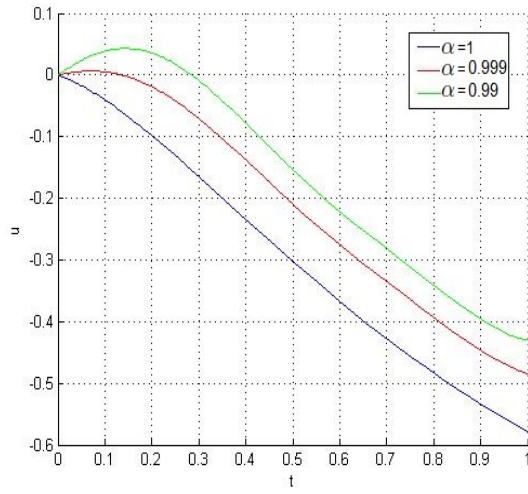


Figure 1: Approximate solution of  $u(\cdot)$  for  $\alpha = 1, 0.999, 0.99$  in Example 6.1

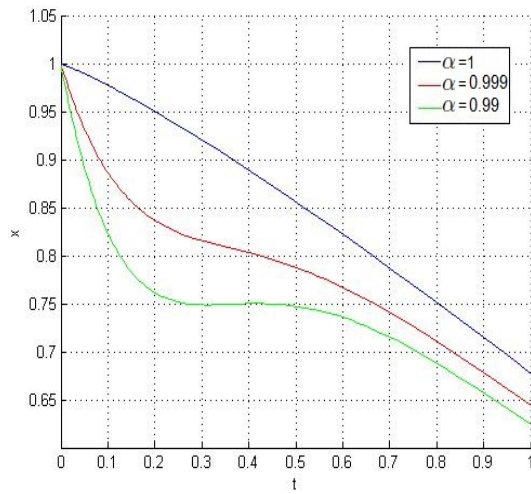


Figure 2: Approximate solution of  $x(\cdot)$  for  $\alpha = 1, 0.999, 0.99$  in Example 6.1

Table 1: The objective value and the end point of trajectory for  $\alpha = 1, 0.999, 0.99$  in Example 6.1

$\alpha$	objective value	end-point
1	0.3956	0.6775
0.999	0.3283	0.6443
0.99	0.2907	0.6249

Table 2: The objective value and the end points of trajectories for  $\alpha = 1, 0.9, 0.8$  in Example 6.2

$\alpha$	objective value	end points
1	0.7245	-0.4691 , -0.0113
0.9	1.0291	-0.6477 , 0.3202
0.8	0.7299	-0.4324 , 0.4674

for some values of  $\alpha$  in Fig.1 and Fig.2. Moreover, for these values of  $\alpha$  the objective values and the end points of optimal trajectory are shown in Table 1.

**Example 6.2.** Consider the following two-dimensional DFOCP in which  $0 < \alpha \leq 1$ ,

$$\begin{aligned} \min J &= \frac{1}{2} \int_0^1 \{ [x_1(t) \ x_2(t)] \begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix} [x_1(t) \ x_2(t)]^T + (t^2 + 1)u^2(t) \} dt, \\ \text{s.t. } {}_0D_t^\alpha \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} t^2 + 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t - \frac{1}{2}) \\ x_2(t - \frac{1}{2}) \end{bmatrix} + \begin{bmatrix} 1 \\ t + 1 \end{bmatrix} u(t) + \begin{bmatrix} t + 1 \\ t^2 + 1 \end{bmatrix} u(t - \frac{1}{4}), \quad 0 \leq t \leq 1, \\ [x_1(t) \ x_2(t)] &= [1, 1], \quad -\frac{1}{2} \leq t \leq 0, \\ u(t) &= 1, \quad -\frac{1}{4} \leq t \leq 0. \end{aligned}$$

This problem for  $\alpha = 1$  has been studied in [19], where the obtained approximated cost function is  $I = 1.5622$ . Using the presented method for  $\alpha = 1$  and  $m = 6$ , gives the approximated cost function as  $J^* = 0.7245$ . So we achieved satisfactory numerical results in comparison with what have been obtained in [19] for  $\alpha = 1$ . Also by varying the value of  $\alpha$  the obtained control and trajectories functions are shown respectively in Fig.3, Fig.4 and Fig.5. Moreover, for these values of  $\alpha$  the objective values and the end points of optimal trajectories are shown in Table 2.

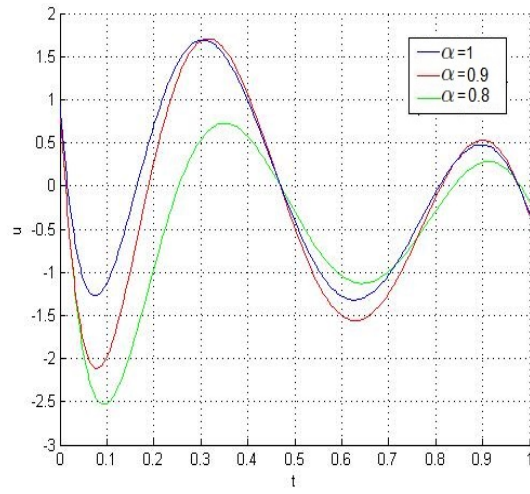


Figure 3: Approximate solution of  $u(\cdot)$  for  $\alpha = 1, 0.9, 0.8$  in Example 6.2

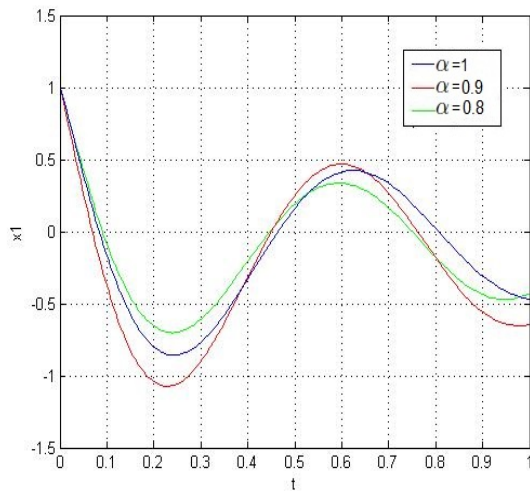


Figure 4: Approximate solution of  $x_1(\cdot)$  for  $\alpha = 1, 0.9, 0.8$  in Example 6.2

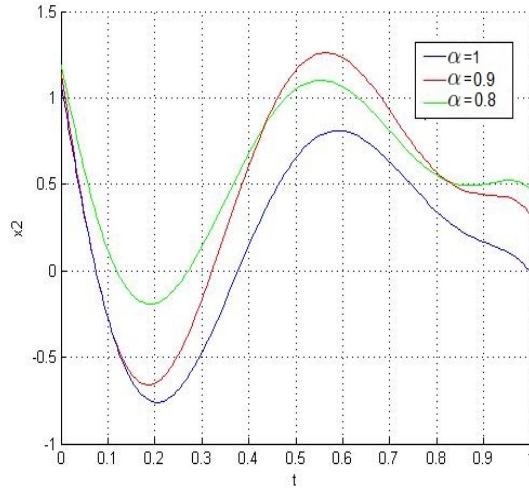


Figure 5: Approximate solution of  $x_2(\cdot)$  for  $\alpha = 1, 0.9, 0.8$  in Example 6.2

## 7 Conclusion

In this paper, we present a new method of using Bernstein polynomials for solving DFOCP's. We approximate the objective function and find a feedback control which minimizes the cost function. Then by replacing the optimal control in the constraints, we get an algebraic system which can be solved in terms of the approximate coefficients of trajectory. The convergence of the method is extensively discussed and some test problems are included to show the efficiency of this very easy to use and accurate method.

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