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Research Article

Effective numerical methods for nonlinear singular two-point boundary value Fredholm integro-differential equations

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Abstract

We deal with some effective numerical methods for solving a class of nonlinear singular two-point boundary value Fredholm integro-differential equations. Using an appropriate interpolation and a q-order quadrature rule of integration, the original problem will be approximated by the nonlinear finite difference equations and so reduced to a nonlinear algebraic system that can be simply implemented. The convergence properties of the proposed method are discussed, and it is proved that its convergence order will be of $\mathcal{O}(h^{\min\{\frac{7}{2}, q-\frac{1}{2}\}})$. Ample numerical results are addressed to confirm the expected convergence order as well as the accuracy and efficiency of the proposed method.

AMS subject classifications (2020): 65R20, 65M06, 65L10, 65L20, 65L70.

Keywords: Nonlinear Fredholm integro-differential equations; Singular two-point boundary value; Numerical method.

1 Introduction

In this study, we consider the following nonlinear singular two-point boundary value Fredholm integro-differential equation (SFIDE):

$$(t^\alpha y'(t))' = f(t) + \int_0^1 v(t,s)u(y(s))ds, \quad t \in (0,1], \quad 0 < \alpha < 1, \tag{1}$$

$$y(0) = a, \quad y(1) = b, \tag{2}$$

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where $f(t), y(t)$ and kernels $v(s,t), u$ are known L_2 functions, and all of them are in $C^4((0,1])$. The nonlinear singular problems are extensively arisen in many applications in physics and astrophysics [11, 9, 12, 19, 20], chemical and mechanical engineering [18, 5, 15, 13], physiological process [14], population dynamics and epidemiology [23], fluid mechanics, electro hydrodynamics, nuclear physics, and chemical kinetics [2, 10].

Also, the nonlinear singular problems have many applications in stellar structure, thermal explosions, isothermal gas spheres, radiative cooling, thermionic currents, and the thermal behavior of a spherical cloud of gas [3, 17, 22].

The majority of engineering applications and various branches of science, such as financial mathematics, oceanography, population dynamics, fluid mechanics, plasma physics, electromagnetic theory, artificial neural networks, and biological processes, have been dominated by Fredholm integro-differential equations (FIDEs) [4, 6]. In general, the analytical solution of FIDEs is not available. As a result, various numerical techniques for determining approximate solutions of FIDEs have been created. The situation is significantly more complicated for FIDEs with singularities. A particular type of them called singularly perturbed Fredholm integro-differential equations (SPFIDEs), was discussed in [1, 6, 8, 7]. Numerical analysis of SPFIDEs has not yet been widely utilized. In this study, we focus on a specific case of nonlinear singular two-point boundary value Fredholm integro-differential equations of the form (1)–(2). Since solving problems of this type is very difficult, the main motivation of this study is to construct an efficient and useful numerical method with $\mathcal{O}(h^{\frac{7}{2}})$ accuracy in the L_2 norm for nonlinear singular problems of the form (1).

To formulate some accurate and effective methods for (1), we first apply a finite difference method to discretize the singular ordinary differential equation part and a suitable quadrature rule of integration for the singular two-point boundary value Fredholm integro-differential part of (1).

Then, the original problem is converted into a system of nonlinear algebraic equations. The numerical solution of the derived nonlinear system is computed by using some solver like the Newton method. Also, the convergence analysis of the present method is established.

The main features of the new method are as follows:

- It can be simply implemented by converting the singular problem into a system of nonlinear algebraic equations.
- The convergence rate of the proposed method is $\mathcal{O}(h^4)$ with respect to the L_∞ norm when applied to nonlinear singular problems.
- The proposed method is successful in solving some classes of singular problems, such as SFIDEs and SPFIDE.

- The provided comparative numerical simulations confirm that the proposed method is more accurate than the existing methods reported in the literature.

2 Formulation of the method

In this section, we formulate a novel numerical method for solving the two-point boundary value Fredholm integro-differential equation (1). At first if one takes $q(t) = t^\alpha y'(t)$, then (1) reduces to

$$q'(t) = f(t) + \int_0^1 v(t,s)u(y(s))ds. \tag{3}$$

Consider the partition $\{t_k = kh \,:\, k = 0, 1, \ldots, N\}$ of the interval $[0, 1]$, where $t_0 = 0$ and $t_N = 1$ and $h = \frac{1}{N}$ denotes the step size. For $k = 0, 1, \ldots, N$, let Y_k and $V_{k,n}$ denote the approximate values of $y_k := y(t_k)$ and $v_{k,n} := v(t_k, t_n)$, respectively. For (3), we can conclude that

$$q(t) - q_k = \int_{t_k}^t f(\xi)d\xi + \int_{t_k}^t \int_0^1 v(\xi,s)u(y(s))dsd\xi. \tag{4}$$

Dividing both sides of (4) by t^α and then integrating over $[t_k, t_{k+1}]$ and $[t_{k-1}, t_k]$, we have

$$\int_{t_k}^{t_{k\pm1}} \left(y'(t) - \frac{q_k}{t^\alpha}\right) dt = \int_{t_k}^{t_{k\pm1}} \int_{t_k}^t \frac{f(\xi)}{t^\alpha} d\xi dt + \int_{t_k}^{t_{k\pm1}} \int_{t_k}^t \int_0^1 \frac{v(\xi,s)}{t^\alpha} u(y(s))dsd\xi dt.$$

By changing the order of integration, we get

$$y_{k\pm1} - y_k \mp q_k T_{k\pm\lfloor\frac{k\mp1}{k}\rfloor} = f_k^\pm + \int_0^1 u(y(s))v_k^\pm(s)ds, \tag{5}$$

where

$$T_{k-1} = \frac{t_k^{1-\alpha} - t_{k-1}^{1-\alpha}}{1-\alpha}, \qquad f_k^\pm = \int_{t_k}^{t_{k\pm1}} \frac{t_{k\pm1}^{1-\alpha} - \xi^{1-\alpha}}{1-\alpha} f(\xi)d\xi,$$
$$v_k^\pm(s) = \int_{t_k}^{t_{k\pm1}} \frac{t_{k\pm1}^{1-\alpha} - \xi^{1-\alpha}}{1-\alpha} v(\xi,s)d\xi, \tag{6}$$

and $k = 1, \ldots, N-1$. Eliminating q_k in (5) concludes that

$$\frac{1}{T_{k-1}}(y_k - y_{k-1} + f_k^-) + \frac{1}{T_k}(y_k - y_{k+1} + f_k^+) + \int_0^1 u(y(s))\left(\frac{1}{T_{k-1}}v_k^-(s) + \frac{1}{T_k}v_k^+(s)\right) ds = 0. \tag{7}$$

To solve (7), it is sufficient to utilize some suitable numerical integration methods to approximate $f^{\pm}$, $v^{\pm}$, and its integral part. By using the interpolating polynomials of $f(\xi)$ and $v(\xi,\cdot)$ at nodes t_k and $t_{k\pm1}$, we can approximate the integrals given in (6) as follows:

$$\begin{cases} f_k^{\pm} = a_{0,k}^{\pm} f(t_k) + a_{1,k}^{\pm} f(t_{k\pm1}) + a_{2,k}^{\pm} f''(t_k) + a_{3,k}^{\pm} f'''(\xi_k^{\pm}), \\ v_k^{\pm}(s) = a_{0,k}^{\pm} v(t_k,s) + a_{1,k}^{\pm} v(t_{k\pm1},s) + a_{2,k}^{\pm} \frac{\partial^2 v}{\partial t^2}(t,s)\Big|_{t=t_k} + a_{3,k}^{\pm} \frac{\partial^3 v}{\partial t^3}(t,s)\Big|_{t=\zeta_k^{\pm}}, \end{cases}$$

in which $\xi_k^-, \zeta_k^- \in (t_{k-1}, t_k)$ and $\xi_k^+, \zeta_k^+ \in (t_k, t_{k+1})$ and

$$\begin{cases} a_{0,k}^{\pm} = \sum_{j=0}^{1} \frac{(-1)^j}{2-\alpha-j}\binom{1}{j}(t_{k\pm1}^{2-\alpha-j} - t_k^{2-\alpha-j})t_k^j \\ \qquad\quad \mp \frac{1}{2h}\sum_{j=0}^{2} \frac{(-1)^j}{3-\alpha-j}\binom{2}{j}(t_{k\pm1}^{3-\alpha-j} - t_k^{3-\alpha-j})t_k^j & \text{(8a)} \\ a_{1,k}^{\pm} = \pm\frac{1}{2h}\sum_{j=0}^{2} \frac{(-1)^j}{3-\alpha-j}\binom{2}{j}(t_{k\pm1}^{3-\alpha-j} - t_k^{3-\alpha-j})t_k^j, & \text{(8b)} \\ a_{2,k}^{\pm} = \frac{1}{6}\sum_{j=0}^{3} \frac{(-1)^j}{4-\alpha-j}\binom{3}{j}(t_{k\pm1}^{4-\alpha-j} - t_k^{4-\alpha-j})t_k^j \\ \qquad\quad \mp \frac{h}{4}\sum_{j=0}^{2} \frac{(-1)^j}{3-\alpha-j}\binom{2}{j}(t_{k\pm1}^{3-\alpha-j} - t_k^{3-\alpha-j})t_k^j, & \text{(8c)} \\ a_{3,k}^{\pm} = \pm\frac{h}{4}\sum_{j=0}^{2} \frac{(-1)^j}{3-\alpha-j}\binom{2}{j}(t_{k\pm1}^{3-\alpha-j} - t_k^{3-\alpha-j})t_k^j. & \text{(8d)} \end{cases}$$

Assume that the functions $f^{(4)}(t)$ and $\frac{\partial^4 v}{\partial t^4}(t,s)$ are continuous. Then there are the values $\varsigma_k, \tilde{\varsigma}_k \in (t_{k-1}, t_{k+1})$ and $\zeta_k, \tilde{\zeta}_k \in (\min(\xi_k^{\pm}), \max(\xi_k^{\pm}))$ such that

$$\begin{cases} \frac{1}{T_{k-1}} f_k^- + \frac{1}{T_k} f_k^+ = \varphi_k^0 f(t_k) + \varphi_k^- f(t_{k-1}) + \varphi_k^+ f(t_{k+1}) + e_k(f), & \text{(9a)} \\ \frac{1}{T_{k-1}} v_k^-(s) + \frac{1}{T_k} v_k^+(s) = \varphi_k^0 v(t_k,s) + \varphi_k^- v(t_{k-1},s) + \varphi_k^+ v(t_{k+1},s) + e_k(v(\cdot,s)), & \text{(9b)} \end{cases}$$

where $\varphi_k^0 = b_{0,k} - \frac{2}{h^2} b_{2,k}, \quad \varphi_k^{\pm} = \frac{1}{T_{k-1}} a_{1,k}^{\pm} + \frac{1}{h^2} b_{2,k}$,

$$\begin{cases} e_k(f) = -\frac{1}{12} h^2 b_{2,k} f^{(4)}(\varsigma_k) + b_{3,k} f^{(3)}(\zeta_k), \\ e_k(v(\cdot,s)) = -\frac{1}{12} h^2 b_{2,k} \frac{\partial^4 v}{\partial t^4}(t,s)\Big|_{t=\tilde{\varsigma}_k} + b_{3,k} \frac{\partial^3 v}{\partial t^3}(t,s)\Big|_{t=\tilde{\zeta}_k}, \end{cases}$$

and $b_{l,k} = \frac{1}{T_{k-1}} a_{l,k}^- + \frac{1}{T_k} a_{l,k}^+$, $l = 0, 2, 3$. Finally, by applying the relations (9) and utilizing a suitable numerical quadrature method of order q with weights $w = (w_0, w_1, \ldots, w_N)^\top$, (7) can be reformulated as

$$-\frac{1}{T_{k-1}}y_{k-1}+\left(\frac{1}{T_{k-1}}+\frac{1}{T_k}\right)y_k-\frac{1}{T_k}y_{k+1}+\hat{F}_k+h\sum_{n=0}^{N}w_nu_n\hat{V}_{k,n}=e_k(f)+\mathcal{O}(h^{q+1}), \tag{10}$$

where for $k=1,\dots,N-1$ and $n=0,1,\dots,N$, we set

$$\begin{cases} u_n := u(y_n), \\ \hat{F}_k := \varphi_k^0 f_k + \varphi_k^- f_{k-1} + \varphi_k^+ f_{k+1}, \\ \hat{V}_{k,n} := \varphi_k^0 v_{k,n} + \varphi_k^- v_{k-1,n} + \varphi_k^+ v_{k+1,n}. \end{cases}$$

Therefore, an approximate method to solve the problem (1) can be formulated as follows:

$$-\frac{1}{T_{k-1}}Y_{k-1}+\left(\frac{1}{T_{k-1}}+\frac{1}{T_k}\right)Y_k-\frac{1}{T_k}Y_{k+1}+\hat{F}_k+h\sum_{n=0}^{N}w_nU_n\hat{V}_{k,n}=0, \tag{11}$$

where $U_n=u(Y_n)$. Take note that the Newton method can be used to solve the derived nonlinear equations. Let us set $\mathbf{Y}=(Y_1,\dots,Y_{N-1})^\top$, $\mathbf{F}=(\hat{F}_1,\dots,\hat{F}_{N-1})^\top$, $\mathbf{V}=\left[\mathbf{V}^1,\dots,\mathbf{V}^{N-1}\right]$, and $\mathbf{V}^k=\left(\hat{V}_{1,k},\hat{V}_{2,k},\dots,\hat{V}_{N-1,k}\right)^\top$ for $k=1,\dots,N-1$. Then the matrix formulation of the proposed method (11) is also written in the following form:

$$\mathbf{T}\ \mathbf{Y}+h\ \mathbf{L}\ u(\mathbf{Y})=-\mathbf{F}-w_0U_0\mathbf{V}^0-w_NU_N\mathbf{V}^N+\tau_0Y_0\mathbf{I}_1+\tau_NY_N\mathbf{I}_{N-1}, \tag{12}$$

where $\mathbf{W}=\operatorname{diag}(w_1,\dots,w_{N-1})$, $\mathbf{L}=\mathbf{VW}$ and

$$\mathbf{T}=\operatorname{tridiag}(-\mathbf{T}^1_{N-2},\mathbf{T}^0_{N-2}+\mathbf{T}^1_{N-1},-\mathbf{T}^1_{N-2}), \tag{13}$$

is a tridiagonal matrix with $\mathbf{T}^k_n=[\tau_k,\tau_{k+1},\dots,\tau_n]^\top$ and $\tau_k=\frac{1}{T_k}$ for $k=0,1,\dots,N$. The symbol $\mathbf{I}_i$ signifies an $(N-1)$-column vector with entry 1 in the position i and 0 elsewhere, where $i=1,N-1$.

Remark 1. It is worth noting that according to the relation $t^\alpha y''=(t^\alpha y')'-\alpha t^{\alpha-1}y'$, the presented technique may be utilized for the singularly perturbed Fredholm integro-differential equations discussed in [1, 6, 8, 7], as well as the singularly perturbed boundary value problems considered in [10, 13].

2.1 Convergence analysis

In this section, the convergence analysis of the presented method (11) to solve the SFIDE (1) is performed. To this end, we set $\mathbf{y}=(y_1,y_2,\dots,y_{N-1})^\top$. Then a matrix formulation of (10) is also derived as

$$\begin{aligned}\mathbf{T}\ \mathbf{y}+h\ \mathbf{L}\ u(\mathbf{y}) =& \mathbf{E}(f)-\mathbf{F}-w_0U_0\mathbf{V}^0-w_NU_N\mathbf{V}^N+\tau_0Y_0\mathbf{I}_1\\ &+\tau_NY_N\mathbf{I}_{N-1}+\mathcal{O}(h^{q+1})\mathbf{1}_{N-1},\end{aligned}\tag{14}$$

where $\mathbf{E}(f) = (e_1(f), e_2(f), \ldots, e_{N-1}(f))^\top$ and $\mathbf{1}_{N-1} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^{N-1}$. By subtracting (12) from (14), the error equation can be derived as

$$\mathbf{T}(\mathbf{y}-\mathbf{Y})+h\ \mathbf{L}\big(u(\mathbf{y})-u(\mathbf{Y})\big)=\bar{\mathbf{E}},$$

where $\bar{\mathbf{E}} = \mathbf{E}(f) + \mathcal{O}(h^{q+1})\mathbf{1}_{N-1}$. Thus we have

$$(\mathbf{T}+h\ \mathbf{L}\mathbf{J}_U)(\mathbf{y}-\mathbf{Y})=\bar{\mathbf{E}},\tag{15}$$

where $\mathbf{J}_U$ is a diagonal matrix containing the Jacobian of kernel $u(y)$. That is, $\mathbf{J}_U = \mathrm{diag}(\big[\frac{\partial}{\partial y}u(y)\big|_{y=y_k}\big]_{k=1}^{N-1})$. To formulate an upper bound for the L_2-error $\|\mathbf{y}-\mathbf{Y}_2\|$ derived in (15), we first prove the following lemmas.

Lemma 1. Let $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$\|A\mathbf{x}\|_2 \geq \sigma_{\min}(A)\|\mathbf{x}\|_2,$$

in which $\sigma_{\min}(A)$ is the smallest singular value of matrix A.

Proof. If we consider the singular value decomposition of the form $A = S\Sigma Z^\top$, where S and Z are orthogonal and $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$ is the diagonal matrix with singular values $\sigma_k, k = 1, \ldots, n$. Then from the orthogonality of S and Z, we have

$$\begin{aligned}\|A\mathbf{x}\|_2 &= \left\|S\Sigma Z^\top\mathbf{x}\right\|_2 = \left\|S(\Sigma Z^\top\mathbf{x})\right\|_2 = \left\|\Sigma Z^\top\mathbf{x}\right\|_2\\ &= \left\|\begin{bmatrix}\sigma_1\sum_{i=1}^n z_{1,i}x_i\\ \vdots\\ \sigma_n\sum_{i=1}^n z_{n,i}x_i\end{bmatrix}\right\|_2 = \sqrt{\sum_{k=1}^n\sigma_k^2\left(\sum_{i=1}^n z_{k,i}x_i\right)^2},\end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_N)^\top$. Let $\sigma_{\min}(A) = \min\{\sigma_k; k = 1, \ldots, n\}$. This concludes that

$$\|A\mathbf{x}\|_2 \geq \sigma_{\min}(A)\sqrt{\sum_{k=1}^n\left(\sum_{i=1}^n z_{k,i}x_i\right)^2} = \sigma_{\min}(A)||Z^\top\mathbf{x}||_2 = \sigma_{\min}(A)||\mathbf{x}||_2.$$

□

In the following, we may try to construct lower and upper triangular matrices L_1 and U_1 such that the tridiagonal matrix A can be expressed as the product $A = L_1U_1$ of the form

$$\underbrace{\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & d_{n-1} & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & \ddots & \ddots & \\ & & l_{n-1} & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} p_1 & c_1 & & \\ & p_2 & \ddots & \\ & & \ddots & c_{n-1} \\ & & & p_n \end{bmatrix}}_{U_1}. \tag{16}$$

Indeed, multiplying L_1 by U_1 yields the following recursive relations:

$$p_1 = d_1, \quad l_k = \frac{a_k}{p_k}, \quad p_{k+1} = d_{k+1} - l_k c_k, \quad k = 1, 2, \ldots, n-1. \tag{17}$$

In the following lemma, we will exhibit that the decomposition (16)–(17) for the matrices with the property of strictly diagonally dominant is unique.

Lemma 2. If A is a strictly diagonally dominant matrix, then it has a unique LU-factorization of the form (16)–(17).

Proof. It is sufficient to show that the elements p_k in (17) are nonzero for $k = 1, \ldots, n$. It can be done by induction. So, we show that $|p_k| \geq \delta_k + |c_k|$, where $\delta_k = |d_k| - |a_{k-1}| - |c_k| > 0, k = 1, 2, \ldots, n$, and $a_0 = c_n = 0$. Since $|p_1| = |d_1| = \delta_1 + |c_1|$, assuming $|p_k| \geq \delta_k + |c_k|$ for some $k = 1, 2, \ldots, n-1$, concludes that $\frac{|c_k|}{|p_k|} < 1$. Therefore, according to (17) and the strictly diagonal dominant of the matrix A, we have

$$\begin{aligned} |p_{k+1}| = |d_{k+1} - l_k c_k| = \left| d_{k+1} - \frac{a_k c_k}{p_k} \right| &\geq |d_{k+1}| - \frac{|a_k||c_k|}{|p_k|} \\ &\geq |d_{k+1}| - |a_k| = \delta_{k+1} + |c_{k+1}|. \end{aligned}$$

□

In the next lemma, we can see that under some conditions there is an LU-factorization in the form (16)–(17) for every weak dominant tridiagonal matrix.

Lemma 3. Assume that A is a tridiagonal matrix with the property of weakly diagonally dominant. If in addition $|d_1| > |c_1|$ and $a_k \neq 0, k = 1, 2, \ldots, n-2$, then it has a unique LU-factorization of the form (16)–(17). Moreover, if $d_n \neq 0$, then A is nonsingular.

Proof. The proof of this lemma is similar to Lemma 2. □

In the next step, we investigate some properties of the matrix $\mathbf{T}$ given by (13). Since $\mathbf{T}$ is a weak diagonal dominant symmetric matrix with positive diagonal elements, then it is a positive semidefinite matrix. It is easily seen that the elements of $\mathbf{T}_n^0$ and $\mathbf{T}_n^1$ are not vanished. Consequently, from Lemma 3, we conclude that $|\mathbf{T}| \neq 0$ and $p_k > 0$ for $k = 1, 2, \ldots, N-1$. This yields that all of the singular values σ_k as well as eigenvalues λ_k of $\mathbf{T}$ are positive. Now we establish that $\lambda_{\min}(\mathbf{T}) > h$. To this end, we first define $\mathbf{M} = \frac{1}{h}\mathbf{T} - \mathbf{I}$, in which $\mathbf{I}$ is the identity matrix of order $N-1$. So, if λ is an eigenvalue of $\mathbf{T}$, then $(\frac{1}{h}\lambda - 1)$ is the eigenvalue of $\mathbf{M}$. Hence, it is sufficient to prove that $\lambda_{\min}(\mathbf{M})$ (the smallest eigenvalue of $\mathbf{M}$) is positive. Since $\mathbf{M}$ is a strictly diagonally dominant matrix, then from Lemma 2, there exists a unique LU-factorization of the form (16)–(17) for this matrix with the following coefficients:

$$\begin{cases} d_k = \frac{1}{h}\left(\tau_{k-1} + \tau_k\right) - 1, & k = 1, 2, \ldots, N-1, \\ a_k = c_k = -\frac{1}{h}\tau_k, & k = 1, 2, \ldots, N-2. \end{cases}$$

Remark 2. Since $h = \frac{1}{N}$, the coefficients given by (8)–(9) can be approximated as

$$\tau_k \sim \frac{t_k^\alpha}{h}, \quad a_{1,k}^{\pm} \sim \frac{h^2}{6} t_k^{-\alpha}, \quad b_{0,k} \sim h, \quad b_{2,k} \sim -\frac{h^3}{12}, \quad b_{3,k} \sim -\frac{h^5}{24}, \tag{18}$$

as $h \to 0$. Therefore, the elements $\varphi_k^{\pm}, \varphi_k^0$, and V^n of the matrix $\mathbf{V}$ will be of order $\mathcal{O}(h)$. Finally, we conclude that the elements of the matrix $\mathbf{L}$ will be of order $\mathcal{O}(h)$.

Lemma 4. If there is an LU-factorization for the matrix $\mathbf{M}$ in the form (16)–(17), then $|\mathbf{M}| > 0$ and $p_k > 0$ for $k = 1, \ldots, N-1$.

Proof. Putting $\bar{p}_k = hp_k, \bar{l}_k = -l_k$ and utilizing (17) yield $\bar{p}_{k+1} = \tau_{k+1} + \tau_k - h - \tau_k \bar{l}_k$. Therefore, we get,

$$\bar{l}_{k+1} = \frac{\gamma_{k+1}}{\gamma_{k+1} + 1 - \bar{l}_k - h/\tau_k},$$

in which $\gamma_{k+1} = \frac{\tau_{k+1}}{\tau_k}, k = 1, \ldots, N-1$. It should be mentioned that $\bar{l}_1 = \frac{\tau_1}{\tau_0 + \tau_1} < 1$ and $\lim_{k\to\infty} \bar{l}_k = 1$. So, for sufficiently small h, assuming $\bar{l}_k < 1 - \frac{h}{\tau_{k-1}} < 1$ concludes $\bar{l}_{k+1} < 1 - \frac{h}{\tau_k} < 1$. Totally, we have $0 < \bar{l}_k < 1, k = 1, \ldots, N-1$. Therefore, with the help of (18), we get

$$\bar{p}_{k+1} > \tau_{k+1} - h > k.$$

□

From Lemma 4, we can reach that the determinant of all upper-left submatrices of $\mathbf{M}$ is positive. Thanks [24, Theorem 7.2], this implies that $\mathbf{M}$ is a positive definite matrix and all its eigenvalues are positive. It means that eigenvalues of $\mathbf{T}$ must be fulfilled $\lambda > h$. Now, We set $A = (\mathbf{T} + h\mathbf{L}\mathbf{J}_U)$ and $\mathbf{x} = \mathbf{y} - \mathbf{Y}$. Then, using the Lemma 1 for (15), we conclude that

$$\|\mathbf{y}-\mathbf{Y}\|_2 \leq \frac{\|\bar{\mathbf{E}}\|_2}{\sigma_{\min}\left(\mathbf{T}+h\mathbf{L}\mathbf{J}_U\right)}. \tag{19}$$

Since $\mathbf{T}$ is a nonsingular positive semidefinite tridiagonal matrix and using Remark 2, we can easily deduce that $\sigma_{\min}\left(\mathbf{T}+h\mathbf{L}\mathbf{J}_U\right) \sim \sigma_{\min}\left(\mathbf{T}\right)$.

Theorem 1. Let the fourth-order derivatives of functions $f(t), v(t,s), u(y(t))$ are continuous. Then for all $\alpha \in (0,1)$, we have $\|\mathbf{y}-\mathbf{Y}\|_2 = \mathcal{O}(h^{\min\{\frac{7}{2}, q-\frac{1}{2}\}})$.

Proof. From the continuity of third and fourth order derivative of the corresponding functions and according to the relations (9) and (18), we conclude that there exists constant $\bar{c} \in \mathbb{R}$ such that

$$\left\|\bar{\mathbf{E}}\right\|_2 \leq \bar{c}h^{\frac{9}{2}} + ch^{q+\frac{1}{2}},$$

and so, from (19), we get

$$\|\mathbf{y}-\mathbf{Y}\|_2 \leq \frac{\bar{c}h^{\frac{9}{2}} + ch^{q+\frac{1}{2}}}{\lambda_{\min}\left(\mathbf{T}\right)} \leq \frac{\bar{c}h^{\frac{9}{2}} + ch^{q+\frac{1}{2}}}{h} = \bar{c}h^{\frac{7}{2}} + ch^{q-\frac{1}{2}}.$$

□

According to Theorem 1, the maximum order of convergence is achieved when $q \geq 4$. Therefore, we use a Simpson quadrature rule to discretize the integral parts of (6).

Remark 3. As is well known, all norms are equivalent for every $\mathbf{z} \in \mathbb{R}^n$; that is, $\|\mathbf{z}\|_1 \leq \sqrt{n}\|\mathbf{z}\|_2 \leq n\|\mathbf{z}\|_\infty$. As a result, if $q \geq 4$, then $\|\mathbf{y}-\mathbf{Y}\|_1 = \mathcal{O}(h^3)$ and $\|\mathbf{y}-\mathbf{Y}\|_\infty = \mathcal{O}(h^4)$ for the proposed method (11).

3 Numerical examples

The performance of the proposed method to solve the SFIDE (1) is demonstrated in this section. In the following numerical simulations, the step size is selected as $h = 2^{-k}, k = 2, \ldots, 8$, and then the error $\|\mathbf{y}-\mathbf{Y}\|_2$ is computed.

Example 1. As a first example for SFIDE (1), we consider $u(y) = \exp(-y)$ and

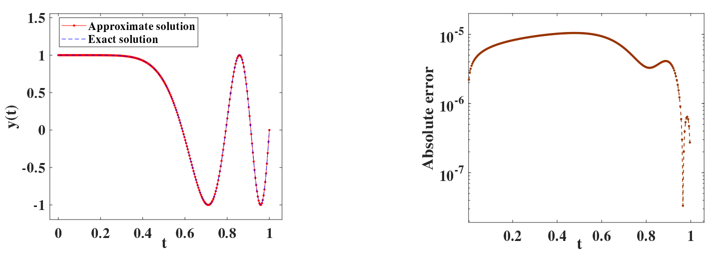
$$\begin{aligned} v(t,s) =& v_0\ t^{1+2\alpha}s^{2+\alpha}\sin\left(\mu\pi t^{3+\alpha}\right)\sin\left(\mu\pi s^{3+\alpha}\right), \\ f(t) =& t^{1+2\alpha}\left(3+\alpha\right)\left(-\mu^2\pi^2 t^{3+\alpha}\left(3+\alpha\right)\cos\left(\mu\pi t^{3+\alpha}\right)\right. \\ & \left. -2\left(\mu\pi+1\right)\left(1+\alpha\right)\sin\left(\mu\pi t^{3+\alpha}\right)\right). \end{aligned}$$

Then the exact solution is $y(t) = \cos\left(\mu\pi t^{\alpha+3}\right)$, where $v_0 = \frac{2\mu\pi(\alpha+1)(\alpha+3)^2}{\exp(-\cos(\mu\pi))-\exp(-1)}$ and $\alpha \in (0,1)$.

Table 1: L_2 error and order of the method (11) for Example 1.

N	$\alpha=\mu=\frac{1}{2}$	Order	$\alpha=\frac{1}{2},\mu=1$	Order	$\alpha=\frac{1}{8},\mu=\frac{5}{2}$	Order
4	8.9150e-02	—	6.1400e-01	—	3.0755e+00	—
8	5.9426e-03	3.90	7.0005e-02	3.13	7.0548e-01	2.12
16	4.6722e-04	3.66	6.2258e-03	3.49	6.8774e-02	3.35
32	3.2741e-05	3.83	4.3233e-04	3.84	8.1641e-03	3.07
64	1.1031e-04	3.60	3.5398e-05	3.61	6.9052e-04	3.56
128	2.6914e-06	3.52	3.0591e-06	3.53	6.1420e-05	3.49
256	2.0531e-08	3.50	2.6884e-07	3.50	5.4073e-06	3.50

It should be mentioned that, in this example, the Jacobian of the kernel $u(y)$ is not positive and that increasing μ causes more oscillations of the solution y. We computed the numerical solution of this singular problem by the proposed method (11). The numerical results of this test problem in the form of the L_2 error and the order of the method are reported in Table 1. From this table, we can see that the desired order of convergence of the presented method is obtained. In Figure 1(a), the exact solution of the problem given by Example 1 is compared with the approximate solution derived by the proposed method (11) when $h=2^{-8}$, $\alpha=\frac{2}{3}$, and $\mu=\frac{7}{2}$. The absolute error of the present method to solve this example is plotted in Figure 1(b).



(a) Exact and approximate solutions for $h=2^{-8}$. (b) Absolute error.

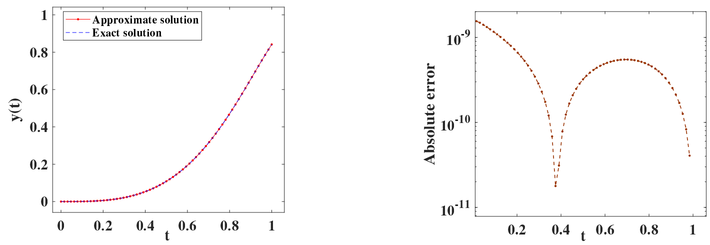
Figure 1: Numerical results of the proposed method to solve Example 1 with $\alpha=\frac{2}{3},\mu=\frac{7}{2}$.

Example 2. As a second example for SFIDE (1), we consider $u(y)=\exp(y)$ and

$$v(t,s)=\mu_0\; t^{1+2\alpha}s^{2+\alpha}\cos\left(t^{3+\alpha}\right)\cos\left(s^{3+\alpha}\right), \quad f(t)=-(\alpha+3)^2\; t^{4+3\alpha}\sin\left(t^{3+\alpha}\right).$$

Then the exact solution will be $y(t)=\sin\left(t^{\alpha+3}\right)$, where $\mu_0=\frac{2(1+\alpha)(3+\alpha)^2}{\exp(\sin(1))-1}$ for $\alpha\in(0,1)$.

This example and Example 1 are similar, but the sign of its Jacobian J_U of the kernel $u(y)$ is unlike that of the ones in Example 1. We report the



(a) Exact and approximate solutions for $h = 2^{-8}$. (b) Absolute error.

Figure 2: Numerical results of the proposed method to solve Example 2 with $\alpha = \frac{1}{5}$.

Table 2: L_2 error and order of the method (11) for Example 2.

N	$\alpha = \frac{1}{3}$	Order	$\alpha = \frac{1}{2}$	Order	$\alpha = \frac{2}{3}$	Order
4	2.2839e-02	—	3.1452e-02	—	4.3256e-02	—
8	2.6238e-03	3.12	3.8926e-03	3.01	5.7418e-03	2.91
16	2.1676e-04	3.59	3.3580e-04	3.53	5.1593e-04	3.47
32	1.7916e-05	3.59	2.8937e-05	3.53	4.5736e-05	3.49
64	1.4922e-06	3.58	2.5547e-06	3.50	4.1658e-06	3.45
128	1.2372e-07	3.59	2.1784e-07	3.55	3.7527e-07	3.47
256	1.0570e-08	3.54	1.8648e-08	3.54	3.3067e-08	3.50

numerical results of this example with various values of α and step size h. Again, from Table 2, we can observe that the expected order of convergence $\frac{7}{2}$ is achieved. For $\alpha = \frac{1}{5}$ and $h = 2^{-8}$, the exact and approximate solutions are depicted in Figure 2(a). The absolute error plotting in Figure 2(b) shows that the present method is accurate and successful.

Example 3. As a third example for SFIDE (1), we consider $u(y) = -y^5$ and

$$v(t,s) = v_0\ t^{2\alpha} s^{1+\alpha} \left(1 + t^{2+\alpha}\right)^{\beta-1}, \quad f(t) = f_0\ t^{2+3\alpha} \left(1 + t^{2+\alpha}\right)^{-2+\beta}.$$

Then the exact solution will be $y(t) = \left(1 + t^{\alpha+2}\right)^{\beta}$, where $v_0 = \frac{\beta(5\beta+1)(1+2\alpha)(2+\alpha)^2}{1-2^{5\beta+1}}$ and $f_0 = \beta(\beta-1)(2+\alpha)^2$ for $\alpha \in (0,1)$ and $\beta > 0$.

We solved this singular boundary value problem for some selected values of α and β. The L_2 norm of the errors is computed for the presented method (11) and then is exhibited in Table 3. It can be seen that the numerical results verify that the claimed order of the convergence of the method is achieved.

Example 4. Consider the nonlinear SPFIDE from [6] as

$$\begin{cases} -\varepsilon y''(t) + (2 - \exp(-t))y(t) + \frac{1}{2}\int_0^1 \left(\exp(t\cos(\pi s)) - 1\right) y(s) ds = \frac{1}{1+t}, \\ y(0) = 1 \quad y(1) = 0, \end{cases}$$

Table 3: L_2 error and order of the method (11) for Example 3.

N	$\alpha=0.25,\beta=3$	Order	$\alpha=0.5,\beta=3$	Order	$\alpha=0.75,\beta=4$	Order
4	9.9948e-01	—	1.5924e+00	—	1.0037e+01	—
8	4.9934e-01	1.00	2.5881e-01	2.62	2.1306e+00	2.23
16	7.3623e-02	2.76	3.0132e-02	3.10	2.9483e-01	2.85
32	5.6781e-03	3.69	2.9081e-03	3.37	3.1017e-02	3.24
64	5.0210e-04	3.49	2.6321e-04	3.46	2.8995e-03	3.41
128	4.0819e-05	3.62	2.3407e-05	3.49	2.6351e-04	3.45
256	3.4354e-06	3.57	2.0688e-06	3.50	2.3132e-05	3.50

where $\varepsilon \in (0,1]$ is a perturbation parameter. Since the exact solution to this problem is unknown, we can use the double-mesh principle to estimate the errors and compute numerical solutions [6]. The errors obtained so are denoted by

$$E_\varepsilon^h = \max_k \left| Y_k^{\varepsilon,h} - Y_k^{\varepsilon,h/2} \right|,$$

in which $Y_k^{\varepsilon,h}$ is the corresponding approximate solution with respect to ε and step size h.

In the reported numerical results, we try to compute the estimated convergence rates

$$p_\varepsilon^h = \log_2\left(\frac{E_\varepsilon^h}{E_\varepsilon^{h/2}}\right), p^h = \log_2\left(\frac{E^h}{E^{h/2}}\right),$$

where $E^h = \max_\varepsilon E_\varepsilon^h$. The maximum pointwise errors and the rates of convergence $p_\varepsilon^h, p_\varepsilon$ are obtained for the values $\varepsilon = 4^{-j}, j = 0,\ldots,4$, and $N = 2^l, l = 5,\ldots,10$, by the proposed method. In this example, the numerical results of the presented method will be compared with those of the numerical reports in [6]. According to the computational results in Table 4, we observe that the presented method is more accurate than that of the method presented in [6]. It can be seen that the numerical results confirm that the methods have achieved the declared order of convergence. Based on Table 4, we can conclude that the order of convergence of the present method is 4 in the L_∞ norm, while the method of [6] is of order 2 with respect to the L_∞ norm.

Example 5. As a final example, we consider the following linear singularly perturbed boundary value problem [21, 16, 13]

$$\begin{cases} \varepsilon y''(t) + y'(t) = 1 + 2t, & 0 \le t \le 1, \\ y(0) = 0, \qquad y(1) = 1. \end{cases}$$

The analytical solution of this problem is

Table 4: L_∞ error and order of the methods for Example 4.

	Present method with $\alpha = 1e-6$					
ε	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
1	7.024e-07	4.390e-08	2.751e-09	1.719e-10	1.074e-11	6.528e-13
	3.999	3.997	4.000	4.001	4.040	
4^{-1}	2.591e-06	1.594e-07	9.958e-09	6.218e-10	3.888e-11	2.323e-12
	4.023	4.001	4.001	3.999	4.065	
4^{-2}	3.537e-05	2.638e-06	1.727e-07	1.090e-08	6.827e-10	4.264e-11
	3.745	3.933	3.986	3.997	4.001	
4^{-3}	2.305e-04	1.931e-05	1.283e-06	8.137e-08	5.104e-09	3.197e-10
	3.577	3.912	3.979	3.995	3.999	
4^{-4}	4.185e-04	3.902e-05	2.625e-06	1.669e-07	1.048e-08	6.563e-10
	3.423	3.894	3.975	3.993	3.997	
E^h	4.185e-04	3.902e-05	2.625e-06	1.669e-07	1.048e-08	6.563e-10
p^h	3.423	3.894	3.975	3.993	3.997	
	Method of [6]					
ε	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
1	2.882e-02	7.291e-03	1.839e-03	4.624e-04	1.161e-04	2.906e-05
	1.983	1.987	1.992	1.994	1.998	
4^{-1}	2.861e-02	7.251e-03	1.832e-03	4.614e-04	1.507e-04	3.785e-05
	1.98	1.985	1.989	1.991	1.993	
4^{-2}	4.001e-02	1.016e-02	2.575e-03	6.508e-04	1.643e-04	4.139e-05
	1.978	1.98	1.984	1.986	1.989	
4^{-3}	4.331e-02	1.100e-02	2.791e-03	7.070e-04	1.790e-04	4.527e-05
	1.977	1.979	1.981	1.982	1.983	
4^{-4}	4.343e-02	1.105e-02	2.804e-03	7.123e-04	1.809e-04	4.593e-05
	1.975	1.978	1.977	1.977	1.978	
E^h	4.343e-02	1.105e-02	2.804e-03	7.123e-04	1.809e-04	4.593e-05
p^h	1.975	1.978	1.977	1.977	1.978	

$$y(t) = t(t + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - \exp(-t/\varepsilon))}{1 - \exp(-1/\varepsilon)}.$$

In this example, we consider the methods [21, 16, 13] to compare the obtained numerical results with the present method. Table 5 contains the computed numerical solution achieved by our method and other methods. From this table, it can be seen that the present method is more accurate than methods [21, 16, 13].

Table 5: Comparison of the approximate solutions of Example 5 for $\varepsilon = h = 10^{-3}$.

t	Method [21]	Method [16]	Method [13]	Present method ($\alpha = 1e-6$)	Analytical solution
0.001	-1.0009970	-0.6311195	-0.6293169	-0.6298615715	-0.6298573177
0.010	-0.9918800	-0.9898546	-0.9878740	-0.9878746043	-0.9878746909
0.020	-0.9815600	-0.9796000	-0.9776400	-0.9776399939	-0.9776399980
0.030	-0.9710400	-0.9691000	-0.9671600	-0.9671599962	-0.9671599999
0.040	-0.9603199	-0.9584000	-0.9564800	-0.9564800000	-0.9564800000
0.050	-0.9493999	-0.9475000	-0.9456000	-0.9456000079	-0.9456000000
0.100	-0.8918000	-0.8900000	-0.8882000	-0.8882000032	-0.8882000000
0.300	-0.6114000	-0.6100000	-0.6086000	-0.6086000008	-0.6086000000
0.500	-0.2510000	-0.2500000	-0.2490000	-0.2490000008	-0.2490000000
0.700	0.1894000	0.1900000	0.1906000	0.1906000004	0.1906000000
0.900	0.7098000	0.7099999	0.7102000	0.7102000001	0.7102000000
1.000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

4 Conclusions

In this work, an effective and high-order numerical method for solving a class of nonlinear singular two-point boundary value Fredholm integro-differential equations was presented. After formulation of the method, as well as utilizing an appropriate numerical integration, the original problem was converted to a nonlinear algebraic system. The error analysis was performed to demonstrate the robustness of the method. It was observed that the present methods achieved the order of convergence $O(h^{\min\{\frac{7}{2},q-\frac{1}{2}\}})$ in the L_2 norm, where q is the order of the quadrature method. Here, some test problems of type SFIDEs and SPFIDEs are solved numerically. Numerical simulations confirmed the theoretical analysis and efficiency of the new method.

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