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Research Article

A family of eight-order interval methods for computing rigorous bounds to the solution to nonlinear equations

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Abstract

One of the major problems in applied mathematics and engineering sciences is solving nonlinear equations. In this paper, a family of eight-order interval methods for computing rigorous bounds on the simple zeros of nonlinear equations is presented. We present the convergence and error analysis of the introduced methods. Also, the introduced methods are compared with the well-known interval Newton method and interval Ostrowski-type methods. Finally, we propose a technique based on the combination of the newly introduced approach with the extended interval arithmetic to find all of the roots of a nonlinear equation that are located in an initial interval.

AMS subject classifications (2020): 65G30; 34G20

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1 Introduction

The main motivation for this study is to enclose the simple root x^* of the nonlinear equation

$$f(x) = 0, \tag{1}$$

by a bounded interval, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a real-valued nonlinear function on the open interval D.

Nonlinear problems are of interest to engineers, physicists, and many other scientists because most systems are inherently nonlinear in nature.

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Up to now, many modified methods for solving nonlinear equations have been developed to improve the local order of convergence of some classical methods, such as Newton, Chebyshev, Potra-Ptak, and Ostrowski methods; see [19, 18, 7, 8, 6, 3, 4, 9, 10, 14, 2, 23, 5, 13].

An optimal eight-order method for solving nonlinear equation (1) proposed by Bi, Ren, and Wu [2] that is based on King family [14], is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - h(\mu_n)\frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{2}$$

where $\mu_n = \frac{f(z_n)}{f(x_n)}$ and h is a real-valued function with $h(0) = 1$, $h'(0) = 2$ and $|h''(0)| < \infty$. Iterative method (2) with eight-order of convergence is very fast compared with many other methods. Solving the problems in floating-point arithmetic is inevitably associated with round-off errors, and so the obtained solution to the problem is accompanied by some errors. Interval analysis is a tool for bounding the errors and providing rigorous bounds on the solution to the problems. The interval extension of the Newton method with quadratic convergence [24, 16], the interval extensions of the Ostrowski method and modified Ostrowski method, respectively, with fourth-order and sixth-order of convergence [1, 11], and the interval extension of the n-step Traub method with $(n + 1)$-order of convergence [21], are examples of the interval methods that give rigorous bounds on the solution to the nonlinear equations.

In this work, we present an interval extension of (2), which has an eight-order of convergence and gives rigorous and outstanding results, that is, interval enclosures with sharp bounds that contain the exact solution. Also, we introduce a technique based on combining the new method and the extended interval arithmetic for enclosing all simple roots that are located in an initial interval. In contrast, many root-finding methods can only find one root of the function in the given initial interval.

Here, we use boldface letters to denote intervals. The set of real intervals is denoted by $\mathbb{IR} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \leq \overline{\mathbf{x}}\}$. The midpoint and width of an interval number $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ are defined by $\mathrm{m}(\mathbf{x}) = \frac{\overline{\mathbf{x}}+\underline{\mathbf{x}}}{2}$ and $\mathrm{w}(\mathbf{x}) = \overline{\mathbf{x}} - \underline{\mathbf{x}}$, respectively. The absolute value of $\mathbf{x}$ is $|\mathbf{x}| = \max\{|x| : x \in \mathbf{x}\}$. The interval extension of real-valued function g is denoted by its corresponding uppercase and bold letter $\mathbf{G}$.

2 Description of the methods

Many modified methods for solving nonlinear equation (1) with a high-order of convergence are based on the well-known Newton method. So, we first give a brief description of the interval Newton method.

2.1 Interval Newton method

The idea of the interval Newton method for the first time was discussed in [24, 16]. Suppose that the real differentiable function f in (1) has the inclusion of monotonic interval extension $\mathbf{F}'(\mathbf{x})$ of its derivative $f'(x)$ and that $\mathbf{x}_0$ is an initial point. Then the interval Newton method is

$$\mathbf{x}_{n+1} = \big\{\mathrm{m}(\mathbf{x}_n) - \frac{f(\mathrm{m}(\mathbf{x}_n))}{\mathbf{F}'(\mathbf{x}_n)}\big\} \cap \mathbf{x}_n, \qquad n = 0, 1, \dots. \tag{3}$$

Recursive relation (3) produces a sequence $\{\mathbf{x}_n\}$ of interval numbers. If the initial interval $\mathbf{x}_0$ contains a zero x^* of $f(x)$ and $0 \notin \mathbf{F}'(\mathbf{x}_0)$, then all iterates contain x^* and the method converges to x^*.

Theorem 1. [17] Let f be a real rational function of a single real variable x with rational extensions $\mathbf{F}$ and $\mathbf{F}'$ of f and f', respectively, such that f has a simple zero y in an interval $[x_1, x_2]$ for which $F([x_1, x_2])$ is defined and $\mathbf{F}'([x_1, x_2])$ is defined and does not contain zero. Then there is an interval $\mathbf{x_0} \subseteq [x_1, x_2]$ containing y and a positive real number K such that

$$\mathrm{w}(\mathbf{x_{n+1}}) \leq K(\mathrm{w}(\mathbf{x_n}))^2,$$

therein $\{\mathbf{x}_n\}$ is the produced interval sequence by (3).

2.2 Main results and convergence analysis

In this subsection, a new interval method is introduced to obtain sharp enclosures for the simple zeros of nonlinear equations. First, for theoretical considerations, we present the following lemmas.

Lemma 1. [17, 1] For real numbers a and b and interval numbers $\mathbf{x}$ and $\mathbf{y}$, we have
(i) $\mathrm{w}(a\mathbf{x} + b\mathbf{y}) = |a|\mathrm{w}(\mathbf{x}) + |b|\mathrm{w}(\mathbf{y})$,
(ii) $\mathrm{w}(\mathbf{xy}) \leq |\mathbf{x}|\mathrm{w}(\mathbf{y}) + |\mathbf{y}|\mathrm{w}(\mathbf{x})$.

Lemma 2. [17] Every nested sequence $\{\mathbf{x}_k\}$ converges and has the limit $\cap_{k=1}^{\infty}\mathbf{x}_k$.

Lemma 3. [17] If $\mathbf{F}$ is a natural interval extension of the real-valued rational function f with $\mathbf{F}(\mathbf{x})$ defined for $\mathbf{x} \subseteq \mathbf{x}_0$, where $\mathbf{x}$ and $\mathbf{x}_0$ are intervals, then there exists a constant L such that

$$\mathrm{w}(\mathbf{F}(\mathbf{x})) \leq L\mathrm{w}(\mathbf{x}).$$

Now we introduce the interval extension of (2) as follows:

$$\begin{cases} \mathbf{y}_n = \mathbf{N}(\mathbf{x}_n) \cap \mathbf{x}_n, \\ \mathbf{z}_n = \mathbf{R}(\mathbf{x}_n, \mathbf{y}_n) \cap \mathbf{x}_n, \\ \mathbf{x}_{n+1} = \mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n) \cap \mathbf{x}_n, \end{cases} \tag{4}$$

where

$$\mathbf{N}(\mathbf{x}) = \mathrm{m}(\mathbf{x}) - \frac{f(\mathrm{m}(\mathbf{x}))}{\mathbf{F}'(\mathbf{x})}, \tag{5}$$

$$\mathbf{R}(\mathbf{x}, \mathbf{y}) = \mathrm{m}(\mathbf{y}) - \frac{2f(\mathrm{m}(\mathbf{x})) - f(\mathrm{m}(\mathbf{y}))}{2f(\mathrm{m}(\mathbf{x})) - 5f(\mathrm{m}(\mathbf{y}))} \frac{f(\mathrm{m}(\mathbf{y}))}{\mathbf{F}'(\mathbf{x})}, \tag{6}$$

$$\mathbf{S}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathrm{m}(\mathbf{z}) - \mathbf{H}(\tilde{\mu}) \frac{f(\mathrm{m}(\mathbf{z}))}{\mathbf{F}'(\mathbf{z})}, \qquad \tilde{\mu} = \frac{\mathbf{F}(\mathbf{z})}{f(\mathrm{m}(\mathbf{x}))}, \tag{7}$$

in which $\mathbf{H}$ is the interval extension of the continuous rational function h.

Now we are ready to present the theoretical analysis of the proposed method (4).

Theorem 2. Assume that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that $0 \notin \mathbf{F}'(\mathbf{x}_0)$ for a given $\mathbf{x}_0 \subseteq D$. If $\mathbf{x}_0$ contains a zero x^* of $f(x)$, then so do all $\mathbf{x}_k$ for $k = 1, 2, \ldots,$ defined by (4). Furthermore, the intervals $\mathbf{x}_k$ form a nested sequence converging to x^*.

Proof. Using the Taylor expansion around $x \in \mathbf{x}_0$, we have

$$0 = f(x^*) = f(x) + (x^* - x) f'(\xi_1),$$

for some ξ_1 between x and x^*. Because $f'(\xi_1) \neq 0$, we obtain

$$x^* = x - \frac{f(x)}{f'(\xi_1)}, \tag{8}$$

which $f'(\xi_1) \in \mathbf{F}'(\mathbf{x}_0)$ yields

$$x^* = x - \frac{f(x)}{f'(\xi_1)} \in x - \frac{f(x)}{\mathbf{F}'(\mathbf{x}_0)}.$$

Since $x \in \mathbf{x}_0$ is arbitrary, so in particular for $x = \mathrm{m}(\mathbf{x}_0)$, and taking into account that $x^* \in \mathbf{x}_0$, we obtain

$$x^* \in \left\{\mathrm{m}(\mathbf{x}_0) - \frac{f(\mathrm{m}(\mathbf{x}_0))}{\mathbf{F}'(\mathbf{x}_0)}\right\} \cap \mathbf{x}_0 = \mathbf{N}(\mathbf{x}_0) \cap \mathbf{x}_0 = \mathbf{y}_0.$$

Now again using the Taylor theorem, for $y \in \mathbf{y}_0$, we can write

$$f(y) = f(x^*) + (y - x^*) f'(\xi_2),$$

for some ξ_2 between y and x^*. Since $f'(\xi_2) \neq 0$ and taking into account that $f(x^*) = 0$, we get

$$x^* = y - \frac{f(y)}{f'(\xi_2)}. \tag{9}$$

As previously mentioned, method (2) is based on the King family. King [14] proposed the following formula for approximating $f'(y_n)$:

$$f'(y_n) \approx f'(x_n)\frac{f(x_n) + \gamma f(y_n)}{f(x_n) + \beta f(y_n)}, \tag{10}$$

with $\gamma = \beta - 2$ to achieve a fourth-order of convergence. Let ξ_1 and ξ_2 be sufficiently close to x and y, respectively. Whereas in method (2), $\beta = -\frac{1}{2}$, and using (10), we have

$$f'(\xi_2) \approx f'(\xi_1)\frac{2f(x) - 5f(y)}{2f(x) - f(y)}. \tag{11}$$

Substituting (11) into (9) yields

$$x^* = y - \frac{f(y)}{f'(\xi_2)} = y - \frac{2f(x) - f(y)}{2f(x) - 5f(y)}\frac{f(y)}{f'(\xi_1)}. \tag{12}$$

Indeed $f'(\xi_1) \in \mathbf{F}'(\mathbf{x}_0)$ and (12) holds for any $x \in \mathbf{x}_0$ and $y \in \mathbf{y}_0$, in particular for $x = \mathrm{m}(\mathbf{x}_0)$ and $y = \mathrm{m}(\mathbf{y}_0)$. So, we obtain

$$x^* \in \left\{\mathrm{m}(\mathbf{y}_0) - \frac{2f(\mathrm{m}(\mathbf{x}_0)) - f(\mathrm{m}(\mathbf{y}_0))}{2f(\mathrm{m}(\mathbf{x}_0)) - 5f(\mathrm{m}(\mathbf{y}_0))}\frac{f(\mathrm{m}(\mathbf{y}_0))}{\mathbf{F}'(\mathbf{x}_0)}\right\} \cap \mathbf{x}_0 = \mathbf{R}(\mathbf{x}_0, \mathbf{y}_0) \cap \mathbf{x}_0 = \mathbf{z}_0.$$

Now for $z \in \mathbf{z}_0$, by the Taylor theorem, we have

$$f(z) = f(x^*) + (z - x^*)f'(\xi_3), \tag{13}$$

for some ξ_3 between z and x^*. Using the Taylor expansion for $h(\mu)$ around zero with $\mu = \frac{f(z)}{f(x)}$, we get

$$h(\mu) \approx h(0) + \mu h'(0).$$

Since $h(0) = 1$ and $h'(0) = 2$, we obtain

$$h(\mu) \approx 1 + 2\frac{f(z)}{f(x)},$$

and so

$$f(z)h(\mu) = f(z) + 2\frac{f^2(z)}{f(x)}.$$

Because z is arbitrary, we can assume that z and x^* are sufficiently close together and so $f(z)h(\mu) \approx f(z)$. Now using (13), we obtain

$$x^* = z - h(\mu)\frac{f(z)}{f'(\xi_3)}. \tag{14}$$

Indeed $f'(\xi_3) \in \mathbf{F}'(\mathbf{z}_0)$ and (14) holds for any $x \in \mathbf{x}_0$ and $z \in \mathbf{z}_0$, in particular for $x = \mathrm{m}(\mathbf{x}_0)$ and $z = \mathrm{m}(\mathbf{z}_0)$. Therefore, since $x^* \in \mathbf{x}_0$, we obtain

$$x^* \in \left\{ \mathrm{m}(\mathbf{z}_0) - \mathbf{H}(\tilde{\mu}_0)\frac{f(\mathrm{m}(\mathbf{z}_0))}{\mathbf{F}'(\mathbf{z}_0)} \right\} \cap \mathbf{x}_0 = \mathbf{S}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \cap \mathbf{x}_0 = \mathbf{x}_1.$$

By continuing this process, we see that

$$x^* \in \mathbf{x}_k, \qquad k = 0, 1, \ldots. \tag{15}$$

Now by formula (4), it is obvious that $\mathbf{x}_{k+1} \subseteq \mathbf{x}_k$ for $k = 0, 1, \ldots$, which means that $\{\mathbf{x}_k\}$ is a nested sequence. By Lemma 2, this sequence is convergent to $\mathbf{a} = \cap_{k=1}^{\infty} \mathbf{x}_k$. Since $x^* \in \mathbf{x}_k$ for all k, then $x^* \in \mathbf{a}$. On the other hand, $\mathrm{m}(\mathbf{z}_n)$ is not contained in $\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n)$ unless $f(\mathrm{m}(\mathbf{z}_n)) = 0$. Since $\mathrm{m}(\mathbf{z}_n) \in \mathbf{z}_n \subseteq \mathbf{x}_n$, we conclude that $\mathrm{w}(\mathbf{x}_{n+1}) < \mathrm{w}(\mathbf{x}_n)$. Therefore $\mathbf{a} = x^*$. □

Note that procedure (4) stops when some stopping criteria are fulfilled, such as $\mathrm{w}(\mathbf{x}_n) < \epsilon$ for a tolerance ϵ or $\mathbf{x}_{n+1} = \mathbf{x}_n$. The computational scheme of the proposed interval method (4) for enclosing the simple roots of a given nonlinear equation $f(x) = 0$ is presented in Algorithm 1.

Algorithm 1 The new interval method (4) for enclosing roots of nonlinear equation $f(x) = 0$

procedure INTERVAL ROOT-FINDING($f, \mathbf{x}_0, tol$)
    $n = 0$;
    while $\mathrm{w}(\mathbf{x}_n) >= tol$ **do**
        Compute $\mathbf{N}(\mathbf{x}_n)$ from (5);
        $\mathbf{y}_n =$`intersect`$(\mathbf{N}(\mathbf{x}_n), \mathbf{x}_n)$;
        Compute $\mathbf{R}(\mathbf{x}_n, \mathbf{y}_n)$ from (6);
        $\mathbf{z}_n =$`intersect`$(\mathbf{R}(\mathbf{x}_n, \mathbf{y}_n), \mathbf{x}_n)$;
        Compute $\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n)$ from (7);
        $\mathbf{x}_{n+1} =$`intersect`$(\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n), \mathbf{x}_n)$;
        $n = n + 1$;
    end while
    return $\mathbf{x}_n$
end procedure

Theorem 3. Suppose that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that $0 \notin \mathbf{F}'(\mathbf{x}_0)$ for a given $\mathbf{x}_0 \subseteq D$. If $x^* \in \mathbf{x}_0$, then $\mathbf{x}_k$ contains a unique root of $f(x)$, for $k = 0, 1, \ldots$. Furthermore, if $\mathbf{S}(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k) \cap \mathbf{x}_k = \emptyset$ for some k, then $f(x) \neq 0$ for all $x \in \mathbf{x}_0$.

Proof. Let $x^* \in \mathbf{x}_0$. By Theorem 2, we conclude that $x^* \in \mathbf{x}_k$ for all k, which is unique because $0 \notin \mathbf{F}'(\mathbf{x}_k) \subseteq \mathbf{F}'(\mathbf{x}_0)$.
Now suppose $\mathbf{S}(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k) \cap \mathbf{x}_k = \emptyset$ for some k, but $x^* \in \mathbf{x}_0$ is a root of $f(x)$, so by Theorem 2, we conclude that $x^* \in \mathbf{x}_n$ for all n. Particularly $x^* \in \mathbf{x}_{k+1} = \mathbf{S}(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k) \cap \mathbf{x}_k$, which is a contradiction. □

Theorem 3 is in the category of verification methods. By verifying its assumptions with the aid of a computer, we can detect when a certain interval does not contain a root.

Theorem 4. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable and have a simple zero x^* in $\mathbf{x}_0$. If $0 \notin \mathbf{F}'(\mathbf{x}_0)$, then the interval method (4) has an eight-order of convergence, that is, there exists a positive real number C such that

$$\mathrm{w}(\mathbf{x_{n+1}}) \leq C(\mathrm{w}(\mathbf{x_n}))^8.$$

Proof. Since $\mathbf{x_{n+1}} \subseteq \mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n)$ so $\mathrm{w}(\mathbf{x}_{n+1}) \leq \mathrm{w}(\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n))$. By the mean value theorem, we can write

$$f(\mathrm{m}(\mathbf{z}_n)) = f'(\eta_1)(\mathrm{m}(\mathbf{z}_n) - x^*),$$

for some η_1 between $\mathrm{m}(\mathbf{z}_n)$ and x^*. Thus we get

$$\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n) = \mathrm{m}(\mathbf{z}_n) - \mathbf{H}(\tilde{\mu}_n)\frac{f'(\eta_1)(\mathrm{m}(\mathbf{z}_n) - x^*)}{\mathbf{F}'(\mathbf{z}_n)}. \tag{16}$$

Therefore, from (16) and Lemma 1, we obtain

$$\mathrm{w}(\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n)) \leq |\mathbf{H}(\tilde{\mu}_n)||\mathrm{m}(\mathbf{z}_n) - x^*||f'(\eta_1)|\mathrm{w}(\frac{1}{\mathbf{F}'(\mathbf{z}_n)}) + \mathrm{w}(\mathbf{H}(\tilde{\mu}_n))|\mathrm{m}(\mathbf{z}_n) - x^*||f'(\eta_1)||\tfrac{1}{\mathbf{F}'(\mathbf{z}_n)}|. \tag{17}$$

Because $x^* \in \mathbf{z}_n$, it is obvious that

$$|\mathrm{m}(\mathbf{z}_n) - x^*| \leq \mathrm{w}(\mathbf{z}_n). \tag{18}$$

On the other hand, we can write

$$\mathbf{z}_n \subseteq \mathrm{m}(\mathbf{y}_n) - \frac{2f(\mathrm{m}(\mathbf{x}_n)) - f(\mathrm{m}(\mathbf{y}_n))}{2f(\mathrm{m}(\mathbf{x}_n)) - 5f(\mathrm{m}(\mathbf{y}_n))}\frac{f(\mathrm{m}(\mathbf{y}_n))}{\mathbf{F}'(\mathbf{x}_n)}. \tag{19}$$

Using the mean value theorem, we have

$$f(\mathrm{m}(\mathbf{y}_n)) = f'(\eta_2)(\mathrm{m}(\mathbf{y}_n) - x^*), \qquad \text{and} \qquad f(\mathrm{m}(\mathbf{x}_n)) = f'(\eta_3)(\mathrm{m}(\mathbf{x}_n) - x^*), \tag{20}$$

for some η_2 between $\mathrm{m}(\mathbf{y}_n)$ and x^* and some η_3 between $\mathrm{m}(\mathbf{x}_n)$ and x^*. Because

$$|\mathrm{m}(\mathbf{y}_n) - x^*| \leq \mathrm{w}(\mathbf{y}_n) \leq \mathrm{w}(\mathbf{x}_n) \qquad \text{and} \qquad |\mathrm{m}(\mathbf{x}_n) - x^*| \leq \mathrm{w}(\mathbf{x}_n), \tag{21}$$

so using (20) and (21), we obtain

$$|f(\mathrm{m}(\mathbf{y}_n))| \leq |f'(\eta_2)|\mathrm{w}(\mathbf{y}_n), \tag{22}$$

$$\begin{aligned}|2f(\mathrm{m}(\mathbf{x}_n)) - f(\mathrm{m}(\mathbf{y}_n))| &\leq 2|f(\mathrm{m}(\mathbf{x}_n))| + |f(\mathrm{m}(\mathbf{y}_n))| \\ &= 2|f'(\eta_3)||\mathrm{m}(\mathbf{x}_n) - x^*| + |f'(\eta_2)||\mathrm{m}(\mathbf{y}_n) - x^*| \\ &\leq 2|f'(\eta_3)|\mathrm{w}(\mathbf{x}_n) + |f'(\eta_2)|\mathrm{w}(\mathbf{x}_n) \leq C_1\mathrm{w}(\mathbf{x}_n),\end{aligned} \tag{23}$$

where C_1 is an upper bound for $2|f'(\eta_3)| + |f'(\eta_2)|$. On the other hand, Theorem 1 yields

$$\mathrm{w}(\mathbf{y_n}) \leq C_2(\mathrm{w}(\mathbf{x_n}))^2, \tag{24}$$

for a positive constant C_2. So by (22) and (24), we obtain

$$|f(\mathrm{m}(\mathbf{y}_n))| \leq C_3(\mathrm{w},(\mathbf{x}_n))^2, \tag{25}$$

in which C_3 is an upper bound for $C_2|f'(\eta_2)|$. Using Lemma 3, we have

$$\mathrm{w}(\frac{1}{\mathbf{F}'(\mathbf{x}_n)}) \leq C_4\mathrm{w}(\mathbf{x}_n). \tag{26}$$

Now from (19) and Lemma 1, we can write

$$\mathrm{w}(\mathbf{z}_n) \leq \frac{|2f(\mathrm{m}(\mathbf{x}_n)) - f(\mathrm{m}(\mathbf{y}_n))|}{|2f(\mathrm{m}(\mathbf{x}_n)) - 5f(\mathrm{m}(\mathbf{y}_n))|}|f(\mathrm{m}(\mathbf{y}_n))|\mathrm{w}(\frac{1}{\mathbf{F}'(\mathbf{x}_n)}),$$

Moreover, using (23), (25), and (26) yields

$$\mathrm{w}(\mathbf{z}_n) \leq C_5(\mathrm{w}(\mathbf{x}_n))^4, \tag{27}$$

where C_5 is an upper bound for $\frac{C_1C_3C_4}{|2f(\mathrm{m}(\mathbf{x}_n))-5f(\mathrm{m}(\mathbf{y}_n))|}$. By Lemma 3 and (27), we get

$$\mathrm{w}(\frac{1}{\mathbf{F}'(\mathbf{z}_n)}) \leq \mathrm{w}(\mathbf{z}_n) \leq C_5(\mathrm{w}(\mathbf{x}_n))^4. \tag{28}$$

Therefore, from (18), (27) and (28), we have

$$|\mathbf{H}(\tilde{\mu}_n)||\mathrm{m}(\mathbf{z}_n) - x^*||f'(\eta_1)|\mathrm{w}(\frac{1}{\mathbf{F}'(\mathbf{z}_n)}) \leq C_6(\mathrm{w}(\mathbf{x}_n))^8, \tag{29}$$

in which C_6 is an upper bound for $C_5^2|f'(\eta_1)||\mathbf{H}(\tilde{\mu}_n)|$. Now by Lemma 3, there exists a positive constant C_7 such that

$$\mathrm{w}(\mathbf{H}(\tilde{\mu}_n)) \leq C_7\mathrm{w}(\tilde{\mu}_n). \tag{30}$$

Using Lemmas 1 and 3 and (27), we obtain

$$\mathrm{w}(\tilde{\mu}_n) = \mathrm{w}(\frac{\mathbf{F}(\mathbf{z}_n)}{f(\mathrm{m}(\mathbf{x}_n))}) = \frac{\mathrm{w}(\mathbf{F}(\mathbf{z}_n))}{|f(\mathrm{m}(\mathbf{x}_n))|} \leq \frac{C_8 \mathrm{w}(\mathbf{z}_n)}{|f(\mathrm{m}(\mathbf{x}_n))|} \leq C_9(\mathrm{w}(\mathbf{x}_n))^4, \quad (31)$$

where C_8 is a positive constant and C_9 is an upper bound for $\frac{C_5 C_8}{|f(\mathrm{m}(\mathbf{x}_n))|}$. From (30) and (31), we can write

$$\mathrm{w}(\mathbf{H}(\tilde{\mu}_n)) \leq C_{10}(\mathrm{w}(\mathbf{x}_n))^4, \quad (32)$$

in which $C_{10} = C_7 C_9$. Using (18), (27), and (32), we obtain

$$\mathrm{w}(\mathbf{H}(\tilde{\mu}_n))|\mathrm{m}(\mathbf{z}_n) - x^*||f'(\eta_1)||\frac{1}{\mathbf{F}'(\mathbf{z}_n)}| \leq C_{11}(\mathrm{w}(\mathbf{x}_n))^8, \quad (33)$$

where C_{11} is an upper bound for $C_5 C_{10}|f'(\eta_1)||\frac{1}{\mathbf{F}'(\mathbf{z}_n)}|$. Finally, since $\mathrm{w}(\mathbf{x}_{n+1}) \leq \mathrm{w}(\mathbf{S}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n))$, by (17), (29), and (33), we conclude that $\mathrm{w}(\mathbf{x}_{n+1}) \leq C(\mathrm{w}(\mathbf{x}_n))^8$, where $C = C_6 + C_{11}$. □

As one can see, the new interval method (4) with three-step has an eight-order of convergence, while some other interval methods with the same number of steps have a lower order of convergence; for some of them, see [1, 21].

3 Test problems

In this section, we give some numerical examples to illustrate the performance of the new approach proposed in Section 2. The new method is compared with the interval Newton method, interval Ostrowski method, and interval modified Ostrowski method. In all examples, the procedures are stopped when $\mathrm{w}(\mathbf{x}_k) < 10^{-16}$. We utilize INTLAB [22] to compute the verified results on the computer. We study the following examples:

$$\begin{aligned} f_1(x) &= \arcsin(x^2 - 1) - \frac{1}{2}x + 1, & x_1^* &\approx 0.5948109683983692, \\ f_2(x) &= \ln(x^2 + x + 2) - x + 1, & x_2^* &\approx 4.1525907367571583, \\ f_3(x) &= x^2 - e^x - 3x + 2, & x_3^* &\approx 0.25753028543986079, \\ f_4(x) &= \arctan(x) + x - 8, & x_4^* &\approx 6.58002470991429699, \\ f_5(x) &= x - 1/x, & x_5^* &= 1. \end{aligned}$$

The first two examples are taken from [2] and the latest is taken from [15]. For all examples, we use rational function h as follows:

$$h(t) = 1 + \frac{2t}{1+t}.$$

In Figure 1, one can see the graphs of five functions f_1, f_2, f_3, f_4, and f_5, respectively, over the initial intervals $\mathbf{x}_0^1 = [0.4, 1]$, $\mathbf{x}_0^2 = [3.5, 5]$, $\mathbf{x}_0^3 = [0.1, 2]$,

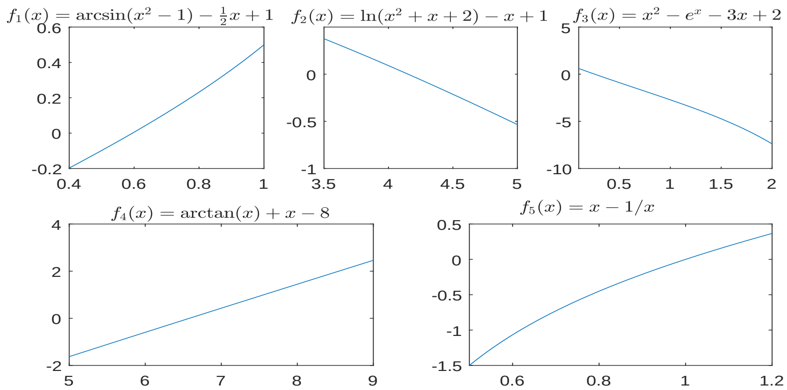

Figure 1: Graphs of functions f_1, f_2, f_3, f_4, f_5

$\mathbf{x}_0^4 = [5,9]$, and $\mathbf{x}_0^5 = [0.5, 1.2]$. By this figure, in addition to obtaining an intuitive view of the functions, we can understand the behavior of the functions for computing the following parameter:

$$\rho_k = \max_{x \in \mathbf{x}_k} |f(x)|. \tag{34}$$

In the tables below, one can see the results obtained by implementing the interval Newton method, the interval Ostrowski method, the interval modified Ostrowski method, and the new method (4) introduced in this paper. The third and fourth columns of the tables show, respectively, the tolerance parameters $\delta_k = \frac{\mathrm{w}(\mathbf{x}_k)}{\max\{|\mathbf{x}_k|, 1\}}$ and ρ_k introduced by (34). Note that in some tables, mark "—" in the last step shows that the method fails in solving the problem.

As the first example for $f_1(x) = \arcsin(x^2-1) - \frac{1}{2}x + 1$, we present the obtained results by the mentioned methods in Tables 1–4. The presented results in these tables show that the new method (4) achieves the desired result with less number of iterations and higher accuracy. Also, the interval Ostrowski method fails in solving the problem.

Tables 5–8 show the results obtained by executing different methods for enclosing the root of $f_2(x) = \ln(x^2+x+2) - x + 1$. It can be seen that the new method (4) successes in the least number of iterations. Also, the interval Ostrowski method fails in solving the problem.

For the third function $f_3(x) = x^2 - e^x - 3x + 2$, the reported values in Tables 9–12 show that only the new method (4) successes in getting the result and the other methods fail.

Tables 13–16 display the results obtained by executing four methods for enclosing the root of $f_4(x) = \arctan(x) + x - 8$. As one can see, the interval Ostrowski method has failed to obtain a result, and the new approach gives better results than the other methods.

The results of different methods for obtaining appropriate enclosures for the positive root of $f_5(x) = x - 1/x$ have been displayed in Tables 17–20. The interval Newton method does not yield any result. Whereas the new approach yields the exact root of the function.

Table 1: Results of the interval Newton method for $f_1(x) = \arcsin(x^2 - 1) - \frac{1}{2}x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.40000000000000002, 0.66396313641487115]	2.64×10^{-1}	7.47×10^{-2}
2	[0.56560254826011236, 0.66396313641487115]	9.84×10^{-2}	7.47×10^{-2}
3	[0.59018815218397114, 0.59856980551945871]	8.38×10^{-3}	3.98×10^{-3}
4	[0.59480310218157917, 0.59481912020532601]	1.60×10^{-5}	8.63×10^{-6}
5	[0.59481096839332148, 0.59481096840342751]	1.01×10^{-11}	5.36×10^{-12}
6	[0.59481096839836900, 0.59481096839836911]	2.22×10^{-16}	0
7	[0.59481096839836911, 0.59481096839836911]	0	0

Table 2: Results of the interval Ostrowski method for $f_1(x) = \arcsin(x^2 - 1) - \frac{1}{2}x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.54158214865149934, 0.63394129754193074]	9.24×10^{-2}	4.19×10^{-2}
2	[0.59477478728793232, 0.59485799844400755]	8.32×10^{-5}	4.98×10^{-5}
3	[0.59481096839836756, 0.59481096839837055]	3.11×10^{-15}	1.33×10^{-15}
4	—	—	—

Table 3: Results of the interval modified Ostrowski method for $f_1(x) = \arcsin(x^2 - 1) - \frac{1}{2}x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.58885410911304559, 0.59936304066316770]	1.05×10^{-2}	4.83×10^{-3}
2	[0.59481096839549719, 0.59481096840132608]	5.83×10^{-12}	3.13×10^{-12}
3	[0.59481096839836922, 0.59481096839836933]	2.22×10^{-16}	1.11×10^{-16}
4	[0.59481096839836933, 0.59481096839836933]	0	1.11×10^{-16}

Table 4: Results of the new method (4) for $f_1(x) = \arcsin(x^2-1) - \frac{1}{2}x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.58015286826057066, 0.60890961953980971]	2.88×10^{-2}	1.50×10^{-2}
2	[0.59481096839720404, 0.59481096839958292]	2.38×10^{-12}	1.29×10^{-12}
3	[0.59481096839836911, 0.59481096839836911]	0	0

Table 5: Results of the interval Newton method for $f_2(x) = \ln(x^2+x+2) - x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[4.09482718955130400, 4.17132082850488750]	1.83×10^{-2}	3.47×10^{-2}
2	[4.15231696283340760, 4.15292802943720400]	1.47×10^{-4}	2.03×10^{-4}
3	[4.15259073289156170, 4.15259074074274000]	1.90×10^{-9}	2.40×10^{-9}
4	[4.15259073675715750, 4.15259073675715840]	4.28×10^{-16}	4.44×10^{-16}
5	[4.15259073675715840, 4.15259073675715840]	0	0

Table 6: Results of the interval Ostrowski method for $f_2(x) = \ln(x^2+x+2) - x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[4.14427225093898070, 4.15515943057456380]	2.62×10^{-3}	5.01×10^{-3}
2	[4.15259073560489430, 4.15259073791874480]	5.57×10^{-10}	6.10×10^{-10}
3	[4.15259073675715840, 4.15259073675715840]	0	0

Table 7: Results of the interval modified Ostrowski method for $f_2(x) = \ln(x^2+x+2) - x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[4.15136705154255560, 4.15297239536206850]	3.87×10^{-4}	7.37×10^{-4}
2	[4.15259073675715750, 4.15259073675715840]	4.28×10^{-16}	4.44×10^{-16}
3	—	—	—

Table 8: Results of the new method (4) for $f_2(x) = \ln(x^2+x+2) - x + 1$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[4.15167922809522590, 4.15321948581378480]	3.71×10^{-4}	5.49×10^{-4}
2	[4.15259073675715840, 4.15259073675715840]	0	0

Table 9: Results of the interval Newton method for $f_3(x) = x^2 - e^x - 3x + 2$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.10000000000000001, 0.76487534371627797]	6.65×10^{-1}	1.86
2	[0.17953909948997981, 0.30082399330312792]	1.21×10^{-1}	2.97×10^{-1}
3	[0.25663052647850410, 0.25844642836458781]	1.82×10^{-3}	3.46×10^{-3}
4	[0.25753027894621072, 0.25753029191301735]	1.30×10^{-8}	2.45×10^{-8}
5	[0.25753028543986067, 0.25753028543986073]	1.11×10^{-16}	4.44×10^{-16}
6	—	—	—

Table 10: Results of the interval Ostrowski method for $f_3(x) = x^2 - e^x - 3x + 2$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.10000000000000001, 0.31655239623745746]	2.17×10^{-1}	6.05×10^{-1}
2	[0.25752321108442017, 0.25753842849505237]	1.52×10^{-5}	3.08×10^{-5}
3	[0.25753028543986073, 0.25753028543986078]	1.11×10^{-16}	0
4	—	—	—

Table 11: Results of the interval modified Ostrowski method for $f_3(x) = x^2 - e^x - 3x + 2$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.24154741311026207, 2.00000000000000000]	8.79×10^{-1}	7.39
2	[0.25749104640972659, 0.39675078835778121]	1.39×10^{-1}	5.20×10^{-1}
3	[0.25753043640242368, 0.25753384076872499]	3.40×10^{-6}	1.34×10^{-5}
4	—	—	—

Table 12: Results of the new method (4) for $f_3(x) = x^2 - e^x - 3x + 2$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.22110828457567316, 0.27623770073133980]	5.51×10^{-2}	1.38×10^{-1}
2	[0.25753028543982470, 0.25753028543989787]	7.32×10^{-14}	1.40×10^{-13}
3	[0.25753028543986078, 0.25753028543986078]	0	0

Table 13: Results of the interval Newton method for $f_4(x) = \arctan(x) + x - 8$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[6.5762681889199976482, 6.5869858860385530619]	1.62×10^{-3}	7.11×10^{-3}
2	[6.5800246452848929479, 6.5800247578416417582]	1.71×10^{-8}	6.60×10^{-8}
3	[6.5800247099142961105, 6.5800247099142978868]	2.69×10^{-16}	1.77×10^{-15}
4	[6.5800247099142969986, 6.5800247099142969986]	0	0

Table 14: Results of the interval Ostrowski method for $f_4(x) = \arctan(x) + x - 8$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[6.5799958235806119689, 6.5800370828300822623]	6.27×10^{-6}	2.95×10^{-5}
2	[6.5800247099142961105, 6.5800247099142969986]	1.34×10^{-16}	8.88×10^{-16}
3	—	—	—

Table 15: Results of the interval modified Ostrowski method for $f_4(x) = \arctan(x) + x - 8$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[6.5800246462005800296, 6.5800248588084278012]	3.23×10^{-8}	1.52×10^{-7}
2	[6.5800247099142969986, 6.5800247099142969986]	0	0

Table 16: Results of the new method (4) for $f_4(x) = \arctan(x) + x - 8$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[6.5800247087713694683, 6.5800247104028359857]	2.47×10^{-10}	1.16×10^{-9}
2	[6.5800247099142969986, 6.5800247099142969986]	0	0

Table 17: Results of the interval Newton method for $f_5(x) = x - 1/x$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	—	—	—

Table 18: Results of the interval Ostrowski method for $f_5(x) = x - 1/x$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.99046958119024919309, 1.0128785276723828446]	2.21×10^{-2}	2.55×10^{-2}
2	[0.9999999856310709534, 1.000000014532778092]	2.89×10^{-8}	2.90×10^{-8}
3	[1.000000000000000000, 1.000000000000000222]	2.22×10^{-16}	3.33×10^{-16}
4	[1.000000000000000222, 1.000000000000000222]	0	3.33×10^{-16}

Table 19: Results of the interval modified Ostrowski method for $f_5(x) = x - 1/x$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.99900511706023975567, 1.0007695812111181421]	1.76×10^{-3}	1.99×10^{-3}
2	[1.000000000000000000, 1.000000000000000222]	2.22×10^{-16}	3.33×10^{-16}
3	[1.000000000000000222, 1.000000000000000222]	0	3.33×10^{-16}

Table 20: Results of the new method (4) for $f_5(x) = x - 1/x$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	[0.99968995513425429333, 1.0004281041560696419]	7.37×10^{-4}	8.56×10^{-4}
2	[1.0000000000000000000, 1.000000000000000222]	2.22×10^{-16}	3.33×10^{-16}
3	1	0	0

4 Enclosing the roots using extended interval arithmetic

In this section, we want to introduce a technique that can find all the roots of a nonlinear equation $f(x) = 0$ located in a wide initial interval. Many root-finding methods in floating-point arithmetic can only find one root of the function in a given interval. Our technique is based on combining the new method (4) introduced in this paper and the extended interval arithmetic [12, 17].

For a continuously differentiable function $f(x)$, if $\mathbf{x}_0$ contains more than one zero of $f(x)$, then $o \in \mathbf{F}'(\mathbf{x}_0)$, and the discussed theorems in Section 2 will not be applicable. Using the extended interval arithmetic, this problem can be handled. As said in [17], the definition of interval division can be extended as follows:

$$[a, b]/[c, d] = [a, b](1/[c, d]),$$

where

$$1/[c, d] = \{1/y : y \in [c, d]\}.$$

If $0 \notin [c, d]$, the we are using the ordinary interval arithmetic. If $0 \in [c, d]$, leaving aside the case $c = d = 0$, then the extended interval arithmetic

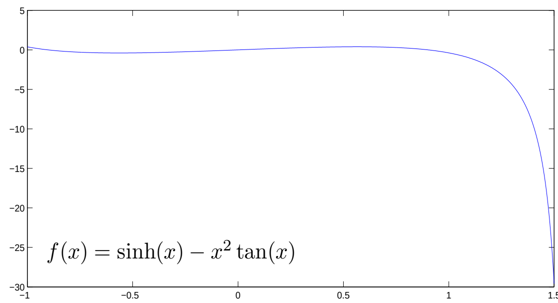

Figure 2: Graph of function $f(x) = \sinh(x) - x^2 \tan(x)$

specifies the following cases:

$$1/[c,d] = \begin{cases} [1/d, +\infty) & \text{if } c = 0 < d, \\ (-\infty, 1/c] \cup [1/d, +\infty) & \text{if } c < 0 < d, \\ (-\infty, 1/c] & \text{if } c < d = 0. \end{cases}$$

Now if the initial interval $\mathbf{x}_0$ is such that $0 \in \mathbf{F}'(\mathbf{x}_0)$, then the quotient $\frac{f(\text{mid}(\mathbf{x}_0))}{\mathbf{F}'(\mathbf{x}_0)}$ in (5) splits into two unbounded intervals. Thereafter intersecting $\mathbf{N}(\mathbf{x}_0)$ with the finite interval $\mathbf{x}_0$ yields two disjoint intervals $\mathbf{y}_{11}$ and $\mathbf{y}_{12}$. First, for $\mathbf{y}_{11}$, if $0 \notin \mathbf{F}'(\mathbf{y}_{11})$, then we take $\mathbf{y}_{11}$ as the initial point for the new method (4), otherwise again by computing $\mathbf{N}(\mathbf{y}_{11})$ and then intersecting it with $\mathbf{y}_{11}$, we obtain two other intervals. By repeating this process, we find some intervals that contain a simple zero of $f(x)$ and $\mathbf{F}'$ over them does not contain zero. The process for $\mathbf{y}_{12}$ is similar. Considering these intervals as initial points for the new method (4), we find all roots of $f(x)$ on the initial interval $\mathbf{x}_0$. A similar idea previously has been used for the interval Newton method; see [17].

For an example, we consider $f(x) = \sinh(x) - x^2 \tan(x)$ on the initial interval $\mathbf{x}_0 = [-1, 1.5]$. The graph of this function on $\mathbf{x}_0 = [-1, 1.5]$ is shown in Figure 2. We have $0 \in \mathbf{F}'(\mathbf{x}_0) = 10^2[-4.9097, 0.3056]$. Using the extended interval arithmetic, we obtain

$$\begin{aligned} \mathbf{N}(\mathbf{x}_0) &= \text{m}(\mathbf{x}_0) - \frac{f(\text{mid}(\mathbf{x}_0))}{\mathbf{F}'(\mathbf{x}_0)} \\ &= (-\infty, 0.24225490053166] \cup [0.25048201511344, +\infty). \end{aligned}$$

Intersecting $\mathbf{N}(\mathbf{x}_0)$ with $\mathbf{x}_0$, we get

$$\mathbf{y}_1 = [-1, 0.24225490053166] \cup [0.25048201511344, 1.5].$$

Indeed $\mathbf{F}'([-1, 0.24225490053166])$ and $\mathbf{F}'([0.25048201511344, 1.5])$ contain zero, too. We repeat the above process by putting $\mathbf{x}_0 = [-1, 0.24225490053166]$ and $\mathbf{x}_0 = [0.25048201511344, 1.5]$, separately. Doing this work several times, we obtain three appropriate intervals, and then we apply the new method (4) on these intervals. The obtained results are shown in Table 21.

As one can see, in a few iterations, all three roots of $f(x) = \sinh(x) - x^2\tan(x)$ in $\mathbf{x}_0 = [-1, 1.5]$ have been enclosed with sharp bounds and high accuracy.

Table 21: Results of the new technique in Section 4 for $f(x) = \sinh(x) - x^2\tan(x)$

k	$\mathbf{x}_k$	δ_k	ρ_k
1	$\mathbf{x}_{11}$=[0.87539095698495750, 0.93264721412667118]	5.73×10^{-2}	9.89×10^{-2}
2	$\mathbf{x}_{12}$=[$\underline{0.90196}$399818943263, $\underline{0.90196}$401144268923]	1.33×10^{-8}	2.08×10^{-8}
3	$\mathbf{x}_{13}$=[$\underline{0.901964005208589}$55, $\underline{0.901964005208589}$66]	2.22×10^{-16}	4.44×10^{-16}
1	$\mathbf{x}_{21}$=[-0.00000014204496225, 0.00000008340020063]	2.25×10^{-7}	1.42×10^{-7}
2	$\mathbf{x}_{22}$=10^{-50}[-0.20045735325692, 0.46773382426614]	6.68×10^{-51}	4.68×10^{-51}
1	$\mathbf{x}_{31}$=[$\underline{-0.90}$414914681585001, $\underline{-0.90}$027520645356984]	3.87×10^{-3}	6.51×10^{-3}
2	$\mathbf{x}_{32}$=[$\underline{-0.90196400520858}$988, $\underline{-0.90196400520858}$899]	8.88×10^{-16}	1.33×10^{-15}

5 Concluding remarks

In this work, a new family of numerical methods for enclosing the simple roots of the nonlinear equations was proposed. We showed that the new methods have an eight-order of convergence and also that the convergence analysis of the methods was studied. Some numerical examples were presented to show the feasibility and effectiveness of the new method proposed in Section 2. Also, we proposed a technique based on combining the new method (4) with the extended interval arithmetic to find all the roots of a nonlinear equation located in an initial interval. Finally, a numerical example for testing this technique was presented.

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