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# New S-ROCK methods for stochastic differential equations with commutative noise

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**Abstract**

The class of strong stochastic Runge–Kutta ($SRK$) methods for stochastic differential equations with a commutative noise proposed by Rößler (2010) is considered. Motivated by Komori and Burrage (2013), we design a class of explicit stochastic orthogonal Runge–Kutta Chebyshev ($SROCKC2$) methods of strong order one for the approximation of the solution of Itô SDEs with an m-dimensional commutative noise.The mean-square and asymptotic stability analysis of the newly proposed methods applied to a scalar linear test equation with a multiplicative noise is presented. Finally, some numerical experiments for stochastic models arising in applications are given that confirm the theoretical discussion.

**Keywords:** Stochastic differential equations; Runge–Kutta methods; Stochastic mean square stability, Stiff equations; Commutative noise.

## 1 Introduction

Stochastic differential equations ($SDEs$) arise in a variety of applied mathematics and physics areas; see [4,22]. Such equations are the results of adding random fluctuations in the parameters of a system of ordinary differential equations ($ODEs$). In the literature, there are plenty of work devoted to the study of stochastic differential calculus; see [10,12]. However, in many cases, the analytical solutions of such SDEs are not known; this makes numerical methods as very important tools for solving SDEs, which have been considered by many researchers. For solving different forms of SDEs, many efficient

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numerical methods have been constructed, for example, [6, 12, 16, 23]. Especially, SRK methods are an important derivative free class of numerical methods, which are proposed for strong approximations of SDEs; see [5, 15, 17]. In 2010, Rößler [19, 20] introduced a new class of SRK methods in which the number of stages does not depend on the dimension, $m$, of the Wiener process. On the other hand, the stability of numerical methods for SDEs is an essential factor in order to avoid possible explosion. Generally, the implicit methods have a wider range of acceptable step sizes compared to explicit ones, which makes them suitable for the solution of stiff systems; see [7, 8, 12, 24]. However, the implementation of implicit methods demands to solve a nonlinear system of equations per step, which may increase the computational cost significantly in the case of high dimensional SDEs. Therefore, explicit methods with extended stability regions could greatly help to solve stiff SDEs. In this regard, Abdulle et al. proposed two classes of explicit stochastic orthogonal Runge–Kutta Chebyshev ($SROCK$) methods of strong order 1/2 for SDEs in Stratonovich [1] and Itô [2] sense. Recently, Komori et al. in [14], constructed explicit strong SRK methods ($SROCKD1$ and $SROCKD2$) of order one for SDEs in Itô and Stratonovich sense. These methods are designed for approximation of SDEs with noncommutative noises. As we know, double stochastic integrals must be used in methods of strong order one for the approximation of the solution of a SDE with a noncommutative noise; see [12, 13]. Such an expensive computational cost can be reduced when the SDE has a single Wiener process or has a commutative noise. It should be noted, such SDEs can be used for modeling a variety of nonlinear physical systems; see [12]. In this article, the SRK methods of [20] for the strong approximation of SDEs with a commutative noise is briefly reviewed. Then, motivated by [14], a new class of SROCK methods of strong order one for the approximation of the solution of SDEs with an $m$-dimensional commutative noise is proposed. The mean-square stability (MS-stability) functions for the new proposed methods, applied to a scalar linear test equations with a multiplicative noise, are determined, and in some special cases, the regions of MS-stability and T-stability are compared between our proposed methods and the scalar test equation.

Consider the autonomous SDE of the Itô type in the form of

$$dX(t) = a(X(t))dt + b(X(t))dW_t, \quad X(t_0) = X_0, \quad t \in [t_0, T]. \tag{1}$$

Here, $X(t) \in \mathbb{R}^d$, $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, is a drift vector, $b = (b_1, b_2, \ldots, b_m) : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a diffusion matrix, and $W(t)$ is an $m$-dimensional standard Wiener process defined on probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\in[t_0,T]}$, which satisfies the usual conditions. It is assumed, $X_0$ is $\mathcal{F}_{t_0}$ -measurable and independent of the Wiener process with $E|X_0|^2 < \infty$, where $|.|$ denotes the Euclidean norm. Suppose that the conditions of the existence and uniqueness theorem [12] are fulfilled for SDE (1). For simplicity in this paper, the equidistant discretization $p_h = \{t_0, t_1, \ldots, t_N = T\}$ of the time interval $[t_0, T]$ with $t_n = t_0 + nh$, where $h = \frac{T-t_0}{N}$ for $n = 0, 1, \ldots, N$, has been considered. Also, for an equidistant grid point $t_n \in p_h$, the notation $y_n$ denotes a discrete approximation to the solution $X(t_n)$ of (1).

In the following, the SDE (1) is considered with a commutative noise, in the other words, the diffusion matrix $b$ satisfies the commutativity condition

$$\sum_{i=1}^{d} b^{i,r} \frac{\partial b^{k,s}}{\partial x^i}(t,x) = \sum_{i=1}^{d} b^{i,s} \frac{\partial b^{k,r}}{\partial x^i}(t,x), \tag{2}$$

for all $r, s = 1, \ldots, m$, $k = 1, \ldots, d$, and $(t, x) \in [t_0, T] \times \mathbb{R}^d$.

**Definition 1** (see [12, 20]). *A discrete time approximation $y^h$ is said to be convergent with the strong order $\alpha$ to the solution $X$ of SDE (1) at time $t_n$, if there exist constants $C > 0$ and $\delta_0 > 0$ such that, for each $h \in (0, \delta_0)$,*

$$\Big(E|X(t_n) - y_n^h|^2\Big)^{\frac{1}{2}} \leq Ch^{\alpha}. \tag{3}$$

*The constants $C$ and $\delta_0$ are independent of $h$.*

The outline of the paper is as follows: In Section 2, we will introduce second order Chebyshev methods proposed by Abdulle and Medovikov in [3] for ODEs. In Section 3, we introduce the class of SRK methods proposed by Rößler in [20] for the strong approximation of SDEs with a commutative noise. Then, we introduce a new class of methods of strong order one for the approximation of the solution of Itô SDEs with an m-dimensional commutative noise. In Section 4, a brief overview of the stochastic stability concepts of SDEs is presented. We also analyze the MS-stability and T-stability properties of the newly proposed method on a linear scalar SDE. In Section 5, some numerical results for stochastic models arising in applications are provided to confirm theoretical discussions.

## 2 Chebyshev methods of order two for ODEs

Consider the autonomous d-dimensional ODE system $X'(t) = a(X(t))$ with the initial condition $X(t_0) = y_0$. The s-stage explicit deterministic Runge–Kutta ($RK$) method for solving this system corresponding to step-size $h$ is as follows:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} \alpha_i a(H_i^{(0)}), \tag{4}$$

where

$$H_i^{(0)} = y_n + h \sum_{j=1}^{i-1} a_{ij} a(H_j^{(0)}), \;\; 1 \leq i \leq s.$$

Applying (4) to the scalar test equation yields

$$X'(t) = \lambda X(t), \quad t > 0, \quad y(0) = y_0, \tag{5}$$

where $\Re(\lambda) < 0$ and $y_0 \neq 0$, we have $y_{n+1} = R_s(\lambda h) y_n$. Here, $R_s(z)$ is called a stability function. Consequently, the stability region of (4) can be expressed as

$$R_S = \{z \in \mathbb{C} : |R_s(z)| \leq 1\}. \tag{6}$$

Riha [18] showed that the polynomial

$$R_s(z) = 1 + z + \frac{z^2}{2} + \sum_{j=3}^{s} c_{sj} z^j, \quad c_{sj} \in \mathbb{R},$$

exists such that $|R_s(z)| \leq 1$ for $z \in [-l_s, 0]$ with $l_s$ as large as possible. It is desirable in practice to replace $|R_s(z)| \leq 1$ by $|R_s(z)| \leq \eta \leq 1$ (damping). Abdulle et al. in [3] constructed approximations to these optimal stability polynomials. In this approach, at first for a given $\eta$ (damping factor) and $s$ (stage value), the values $a_s$ and $l_s$ and also the polynomial $\bar{w}(x)$ of the degree two with complex zeros $\alpha_s \pm \beta_s$ which satisfies $\bar{w}(a_s) = 1$, are calculated. Then, the orthogonal polynomials $\{\bar{Q}_j(x)\}_{j=0}^{s-2}$ with respect to the weight function $\frac{\bar{w}(x)}{\sqrt{1-x^2}}$ on $[-1, 1]$ are constructed in which $\bar{Q}_j(a_s) = 1$ for $j = 0, 1, \ldots, s-2$. Finally, the approximation of optimal stability polynomials is constructed as

$$R_s(x) := w(x) Q_{s-2}(x), \tag{7}$$

where $w(x) := \bar{w}(a_s + x/d_s)$ with $d_s = \frac{l_s}{1+a_s} > 0$ and $Q_j(x) := \bar{Q}_j(a_s + x/d_s)$ for $j = 0, 1, \ldots, s-2$. For more details, you can refer to [3]. The polynomials $Q_0(x), Q_1(x), \ldots, Q_{s-2}(x)$ satisfy in the below three-term recurrence relations as form

$$\begin{aligned} &Q_0(z) = 1, \quad Q_1(z) = 1 + \mu_1 z, \\ &Q_j(z) = (\mu_j z + \kappa_j + 1) Q_{j-1}(z) - \kappa_j Q_{j-2}(z), \end{aligned} \tag{8}$$

for $j = 1, \ldots, s-2$. The values of $\mu_j$ and $\kappa_j$ can be determined by inserting the different suitable values $r_1, r_2 \in \mathbb{R} \setminus \{0\}$ in (8) and solving the nonsingular linear system

$$(\mu_j r_i + \kappa_j + 1) Q_{j-1}(r_i) - \kappa_j Q_{j-2}(r_i) = Q_j(r_i), \quad i = 1, 2.$$

Abdulle et al. in [3], corresponding to stability polynomial (7), constructed the second order explicit RK method $\{y_k\}_{k \geq 0}$ for the approximation of the solution of the deterministic system $X'(t) = a(X(t))$. The mentioned method is implemented in a code called ROCK2 as below:

$$H_1^{(0)} = y_n,$$

$$
\begin{aligned}
H_2^{(0)} &= y_n + h\mu_1 a(H_1^{(0)}), \\
H_{j+1}^{(0)} &= h\mu_j a(H_j^{(0)}) + (\kappa_j + 1)H_j^{(0)} - \kappa_j H_{j-1}^{(0)}, \qquad 2 \le j \le s-2, \\
H_s^{(0)} &= H_{s-1}^{(0)} + h\theta_s a(H_{s-1}^{(0)}), \\
y_{n+1} &= H_s^{(0)} + h\theta_s a(H_s^{(0)}) - h\theta_s(1 - \frac{\tau_s}{\theta_s^2})(a(H_s^{(0)}) - a(H_{s-1}^{(0)})), \qquad (9)
\end{aligned}
$$

where the parameters $\theta_s$ and $\tau_s$ satisfy in the relation $w(x) = 1+2\theta_s x+\tau_s x^2$.

## 3 SRK methods for SDEs with a commutative noise

In this section, we will review the SRK method introduced by Rößler in [20] for the strong approximation of Itô SDEs with m-dimensional commutative noise. In this class, the SRK method is given by $y_0 = X_0$ and

$$
\begin{aligned}
y_{n+1} = y_n &+ \sum_{i=1}^{s} \alpha_i a(t_n + c_i^{(0)} h_n, H_i^{(0)}) h_n \\
&+ \sum_{i=1}^{s}\sum_{k=1}^{m} (\beta_i^{(1)} I_{(k),n} + \beta_i^{(2)} \sqrt{h_n}) b^k(t_n + c_i^{(1)} h_n, H_i^{(k)}), \qquad (10)
\end{aligned}
$$

for $n = 0, 1, 2, \ldots,$ with stage-values

$$
\begin{aligned}
H_i^{(0)} &= y_n + \sum_{j=1}^{s} A_{ij}^{(0)} a(t_n + c_j^{(0)} h_n, H_j^{(0)}) h_n + \sum_{j=1}^{s}\sum_{l=1}^{m} B_{ij}^{(0)} b^l(t_n + c_j^{(1)} h, H_j^{(l)}) I_{(l),n}, \\
H_i^{(k)} &= y_n + \sum_{j=1}^{s} A_{ij}^{(1)} a(t_n + c_j^{(0)} h_n, H_j^{(0)}) h_n - \sum_{j=1}^{s} B_{ij}^{(1)} b^k(t_n + c_j^{(1)} h_n, H_j^{(k)}) \frac{\sqrt{h_n}}{2} \\
&\quad + \sum_{j=1}^{s}\sum_{l=1}^{m} B_{ij}^{(1)} b^l(t_n + c_j^{(1)} h_n, H_j^{(l)}) \frac{I_{(k),n} I_{(l),n}}{2\sqrt{h_n}},
\end{aligned}
$$

for $i = 1, \ldots, s$ and $k = 1, \ldots, m$. The random variables of the method are defined by the stochastic Itô integrals

$$
I_{(k),n} = W^k(t_n + h) - W^k(t_n) = \int_{t_n}^{t_n+h} dW_s^k.
$$

Since the random variables of the method (10) are only increments of the Wiener process and the simulation of the multiple stochastic integrals is not required any more, the computational cost will be reduced significantly.

Table 1: Butcher tableau for the SRK methods $SRIC1$ (left) and $SRIC2$ (right)

$$
\begin{array}{c|ccc|ccc|}
0 & & & & & & \\
0 & 0 & & & 0 & & \\
0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & & & & & & \\
0 & 0 & & & 1 & & \\
0 & 0 & 0 & & -1 & & \\
\hline
 & 1 & 0 & 0 & 1 & 0 & 0 \\
\cline{2-7}
 & & & & 0 & \frac{1}{2} & -\frac{1}{2} \\
\cline{5-7}
\end{array}
\qquad
\begin{array}{c|ccc|ccc|}
0 & & & & & & \\
0 & 1 & & & 0 & & \\
0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & & & & & & \\
1 & 1 & & & 1 & & \\
1 & 1 & 0 & & -1 & & \\
\hline
 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\
\cline{2-7}
 & & & & 0 & \frac{1}{2} & -\frac{1}{2} \\
\cline{5-7}
\end{array}
$$

In the following, let $\mathcal{C}^{p,q}([t_0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ denote the space of all mappings from $[t_0,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$, which are $p$ and $q$ times continuously differentiable with respect to the time and the state variables, respectively.

**Remark 1.** *Frequently, the coefficients of the SRK method (10) are shown by the following Butcher tableau :*

$$
\begin{array}{c|c|c|}
c^{(0)} & A^{(0)} & B^{(0)} \\
\hline
c^{(1)} & A^{(1)} & B^{(1)} \\
\hline
 & \alpha^T & {\beta^{(1)}}^T \\
\cline{2-3}
 & & {\beta^{(2)}}^T \\
\cline{3-3}
\end{array}
$$

**Theorem 1** (Order conditions)**.** *Let $a, b^j \in \mathcal{C}^{1,2}([t_0,T]\times\mathbb{R}^d,\mathbb{R})$ for $j = 1,\ldots,m$. Then, the SRK method (10) obtains the strong order* 0.5 *for approximation of the solution of (1), if its coefficients satisfy in the equations*

$$1.\quad \alpha^T e = 1, \qquad 2.\quad {\beta^{(1)}}^T e = 1, \qquad 3.\quad {\beta^{(2)}}^T e = 0.$$

*In addition, suppose $a, b^j \in \mathcal{C}^{1,3}([t_0,T]\times\mathbb{R}^d,\mathbb{R})$ for $j = 1,\ldots,m$. Then, the SRK method (10) obtains the strong order* 1.0*, if the equations*

$$
\begin{array}{llllll}
4. & {\beta^{(2)}}^T A^{(1)} e = 0, & 5. & {\beta^{(2)}}^T (B^{(1)} B^{(1)} e) = 0, & & \\
6. & {\beta^{(1)}}^T B^{(1)} e = 0 & 7. & {\beta^{(2)}}^T (B^{(1)} e)^2 = 0, & 8. & {\beta^{(2)}}^T B^{(1)} e = 1,
\end{array}
$$

*are fulfilled with $c^{(i)} = A^{(i)} e$ for $i = 1, 2$.*

*Proof.* see [20]. ☐

To date, suggestions have been presented for the coefficients of SRK method (10). As an example, Rößler in [20] purposed three stages of SRK methods $SRIC1$ and $SRIC2$ with orders $(p_D, p_S) = (1.0, 1.0)$ and $(p_D, p_S) = (2.0, 1.0)$, respectively, in which $p_D$ is the order of the method related to ODEs, that is, $b \equiv 0$ and $p_S$ is the order of the method related to SDEs; see Table 1.

**Remark 2.** *In the rest of the paper, we set* $B^{(0)} = 0$ *and*

$$B^{(1)} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & -1 & 0 & 0 \end{pmatrix}_{s\times s}, \ \beta^{(1)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{s\times 1}, \ \beta^{(2)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}_{s\times 1}, \quad (11)$$

*therefore, the equations 2. and 3. as well as the equations 5.–8. in Theorem 1 are satisfied.*

## 3.1 A new class of Chebyshev methods for SDEs

In this section, motivated by Komori et al. [14], we introduce a new class of $s$-stage explicit stochastic orthogonal Runge–Kutta Chebyshev method for Itô SDEs with a commutative noise. We denote this method by $SROCKC2$, which has the orders $(p_D, p_S) = (2.0, 1.0)$.

For the strong approximation of the solution of SDE (1) with a commutative noise, the s-stage ($s \geq 2$) SROCKC2 method is given by $y_0 = X_0$ and

$$y_{n+1} = H_{s+1}^{(0)} + \sum_{i=1}^{s}\sum_{k=1}^{m}(\beta_i^{(1)} I_{(k),n} + \beta_i^{(2)}\sqrt{h_n})b^k(t_n + c_i^{(1)}h_n, H_i^{(k)}), \quad (12)$$

with stage values

$$\begin{aligned}
H_1^{(0)} &= y_n, \qquad H_2^{(0)} = y_n + h\mu_1 a(H_1^{(0)}), \\
H_{j+1}^{(0)} &= h\mu_j a(H_j^{(0)}) + (\kappa_j + 1)H_j^{(0)} - \kappa_j H_{j-1}^{(0)}, \qquad 2 \leq j \leq s-2, \\
H_s^{(0)} &= H_{s-1}^{(0)} + h\theta_s a(H_{s-1}^{(0)}), \\
H_{s+1}^{(0)} &= H_s^{(0)} + h\theta_s a(H_s^{(0)}) - h\theta_s(1 - \frac{\tau_s}{\theta_s^2})(a(H_s^{(0)}) - a(H_{s-1}^{(0)})), \\
H_i^{(k)} &= Y_n + \sum_{j=1}^{s} A_{ij}^{(1)} a(H_j^{(0)})h_n - \sum_{j=1}^{s} B_{ij}^{(1)} b^k(H_j^{(k)})\frac{\sqrt{h_n}}{2} \\
&\quad + \sum_{j=1}^{s}\sum_{l=1}^{m} B_{ij}^{(1)} b^l(H_j^{(l)})\frac{I_{(k),n}I_{(l),n}}{2\sqrt{h_n}},
\end{aligned}$$

where $\mu_j$ and $\kappa_j$ ( $j = 1, \ldots, s-2$) as well as $\theta_s$ and $\tau_s$ are free parameters in $\mathbb{R}$, which are in accordance with the second method ROCK2 in (9); see [3] for more details. In addition, $B^{(0)}$, $B^{(1)}$, $\beta^{(1)}$ and $\beta^{(2)}$ would be defined as previously mentioned in relation (11).

**Remark 3.** *We note that in the deterministic case,* $b \equiv 0$*, the method (12) coincides with the second order* $ROCK2$ *method, introduced in (9). Therefore, assuming* $y_n + \sum_{i=1}^{s} \alpha_i a(H_i^{(0)}) = H_{s+1}^{(0)}$ *from second order properties of the ROCK2 method, we have* $\sum_{i=1}^{s} \alpha_i = 1$*, and the equation 1. in Theorem 1 is satisfied.*

## 4 Stability properties

In this section, we will first investigate the MS-stability of our proposed methods and compare them with the MS-stability regions of the methods SRIC1 and SRIC2 in [20]. Then, the T-stability properties of the proposed schemes on a scalar linear test equation with a multiplicative noise have been analyzed. In this regard, we first review the MS-stability analysis of the methods SRIC1 and SRIC2 on the linear test problem with the multiplicative noise

$$dX(t) = \lambda X(t)dt + \mu X(t)dW_t, \quad \lambda, \mu \in C, \tag{13}$$

with the deterministic initial condition $X(t_0) = y_0 \in \mathbb{R}\backslash\{0\}$. For $\lambda, \mu \in \mathbb{C}$, the solution of (13), $X(t) = y_0 exp\{(\lambda - \frac{1}{2}\mu^2)(t - t_0) + \mu(W_t - W_{t_0})\}$, is stochastically asymptotically stable in the large if and only if

$$\lim_{t\to\infty} |X(t)| = 0 \Leftrightarrow \Re(\lambda - \tfrac{1}{2}\mu^2) < 0. \tag{14}$$

Also, the equilibrium position of SDE (13) is asymptotically MS-stable if and only if

$$\lim_{t\to\infty} E(|X(t)|^2) = 0 \Leftrightarrow 2\Re(\lambda) + |\mu|^2 < 0, \tag{15}$$

for $\lambda, \mu \in \mathbb{C}$; see [12]. It worth mentioning, MS-stability always results in asymptotically stability in the large due to inequality $\Re(2\lambda - \mu^2) \le 2\Re(\lambda) + |\mu|^2$.

Here, we are going to find conditions in which the SRIC1 and SRIC2 methods, on the scalar test equation (13), have numerically stable solutions. For every $\lambda, \mu \in \mathbb{C}$ a numerical method applied on the test equation (13) is numerically asymptotically stable and MS-stable, respectively, if and only if $\lim_{n\to\infty} |y_n| = 0$ (with probability 1) and $\lim_{n\to\infty} E(|y_n|^2) = 0$. To determine the domain of asymptotic stability of a numerical method, we recall the following lemma from [9].

**Lemma 1.** *For a given sequence of real-valued, nonnegative, independent, and identically distributed random variables* $\{R_n(h, \lambda, \mu)y_n\}_{n=0}^{\infty}$*. Consider the sequence of random variable* $\{|y_n|\}_{n=0}^{\infty}$ *defined by* $y_{n+1} = R_n(h, \lambda, \mu)y_n$ *and* $|y_0| \neq 0$ *(w.p.1). If the random variables* $\log(|R_n(h, \lambda, \mu)|)$ *are square* *integrable, then*

$$\lim_{n\to\infty} |y_n| = 0 \Leftrightarrow E\Big(\log(|R_n(h,\lambda,\mu)|\Big) < 0. \tag{16}$$

So the domain of asymptotic stability of a numerical method can be expressed as

$$R_{AS} = \{\lambda, \mu \in \mathbb{C} : E\Big(\log(|R_n(h,\lambda,\mu)|\Big) < 0\}.$$

Set $a_1 := 1 + x - \frac{y^2}{2}$ and $a_2 := 1 + x + \frac{1}{2}x^2 - \frac{y^2}{2}$, where $x = h\lambda$ and $y = \mu\sqrt{h}$. Applying SRIC1 and SRIC2 methods on the scalar test equation (13), respectively, we have

$$y_{n+1} = \Big(a_j + \frac{I_{(1),n}}{\sqrt{h}} y + \frac{I^2_{(1),n}}{2h} y^2\Big) y_n, \quad j = 1, 2. \tag{17}$$

Here, we find the forms

$$y_{n+1} = R^j_n(x, y) y_n, \quad j = 1, 2,$$

for (17), and the MS-stability functions of the methods SRIC1 and SRIC2 will be given, respectively, by

$$E(|y_{n+1}|^2) = \hat{R}^j_n(x, y) E(|y_n|^2), \quad j = 1, 2.$$

Obviously, for $\lambda, \mu \in \mathbb{C}$ and $h > 0$, the methods SRIC1 and SRIC2 are MS-stable, if $\hat{R}^j_n(x, y) < 1$ for $j = 1, 2$. So, the domain of MS-stability of SRIC1 and SRIC2 methods can be expressed, respectively, as

$$R^j_{MS} = \{\lambda, \mu \in \mathbb{C} : \hat{R}^j_n(x, y) < 1\}, \quad j = 1, 2.$$

Because of better visualization of the regions $R^1_{MS}$ and $R^2_{MS}$ in the $x - y$ plane, the restriction $\lambda, \mu \in \mathbb{R}$ is considered. So, one can express the MS-stability function of the SRIC1 and SRIC2 methods, respectively, as

$$\hat{R}^j(x, y) = a_j^2 + y^2 + \frac{3}{4} y^4 + y^2 a_j, \quad j = 1, 2.$$

It should be noted that all the following figures, demonstrating regions of MS-stability for the numerical methods under consideration, are plotted, using the Mathematica software. Figure 1 gives the MS-stability regions of the methods SRIC1 and SRIC2. The light gray area shows the MS-stability region of the test equation (13) and the dark gray area shows the MS-stability regions of the methods. Comparison of the illustrated regions in Figure 1 shows that the SRIC1 and SRIC2 methods have small regions of MS-stability.

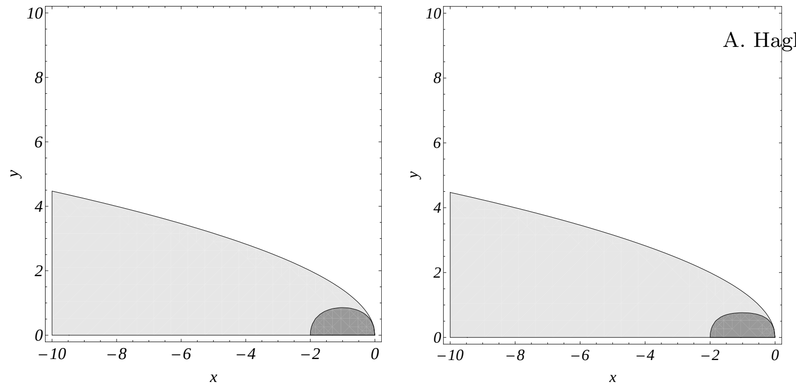


Figure 1: MS-stability region for the test equation (13) (light grey surface) and SRK methods (dark grey surface) (left) SRIC1, (right) SRIC2.

## 4.1 MS-stability analysis of SROCKC2

Consider the method (12), for simplification similar to [14], we set all components of $A^{(1)}$ as zero except

$$A^{(1)}_{s-2,j} = A^{(1)}_{s-1,j} = A^{(1)}_{s,j} = A^{(0)}_{s-1,j}, \quad 1 \leq j \leq s-2,$$

where $A^{(0)}$ is an $s \times s$ strictly lower triangular matrix which could be obtained from the following equation:

$$H_i^{(0)} = y_n + \sum_{j=1}^{i-1} A^{(0)}_{i,j} a(H^{(0)}_{j-1})$$

for $i = 1, \ldots, s$. Clearly, from the definition of $A^{(1)}$, the equation 4. in Theorem 1 is satisfied.

Now, suppose that $R_s(x) = w(x)Q_{s-2}(x)$ is the stability polynomial of the $s$-stage deterministic method ROCKD2 (9) with parameter $\eta$; see [3]. When the method (12) is applied to the test equation (13), we find the forms $y_{n+1} := R_s(h, \lambda, \mu, \Delta W_n)y_n$, where

$$R_s(h, \lambda, \mu, \Delta W_n) = \Big(w(\lambda h) + \Delta W_n \mu + \frac{1}{2}[\Delta W_n]^2 \mu^2 - \frac{1}{2} h \mu^2\Big) Q_{s-2}(\lambda h).$$

Applying the expectation operator and according to the properties of standard normal distribution in the terms of $x$ and $y$, we have

$$E\Big(|R_s(x,y)|^2\Big) = \Big(w(x)^2 + y^2 + \frac{y^4}{2}\Big)[Q_{s-2}(x)]^2.$$

Therefore, the mean square stability regions of the methods SROCKC2 is

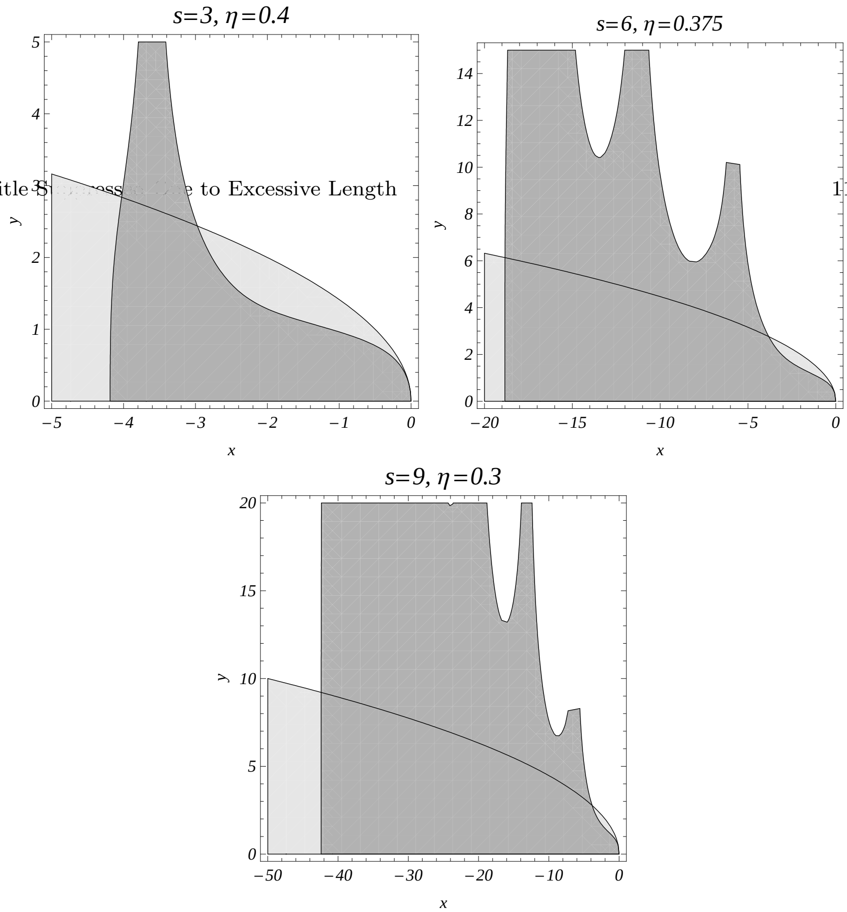


Figure 2: Profile of the MS-stability regions of the SROCKC2 schemes for some special values of $\eta$ and $s$.

$$R^{MS} = \{(x, y) \in \mathbb{R}^2 : E\Big[|R_s(x, y)|^2\Big] < 1\}. \tag{18}$$

For more clear results concerning the MS-stability regions of SROCKC2 schemes, in the following, we confine our investigation to some optimal values of $\eta$ and $s$, which have been suggested by [14]. The regions of MS-stability for SROCKC2 schemes with different values of $\eta$ and $s$ (dark gray area) are illustrated in Figure 2. We can see in Figure 2 that the MS-stability region of methods SROCKC2 is larger than the methods SRIC1 and SRIC2. In what follows, SROCKC2 scheme with three stages ($\eta = 0.4$), six stages ($\eta = 0.375$), and nine stages ($\eta = 0.3$) are denoted by $SROCKC2 - 3$, $SROCKC2 - 6$, and $SROCKC2 - 9$, respectively.

## 4.2 T-stability analysis of SROCKC2

To measure the asymptotic stability of a numerical method with respect to the driving process, Saito and Mitsui established the definition of T-stability in [21].

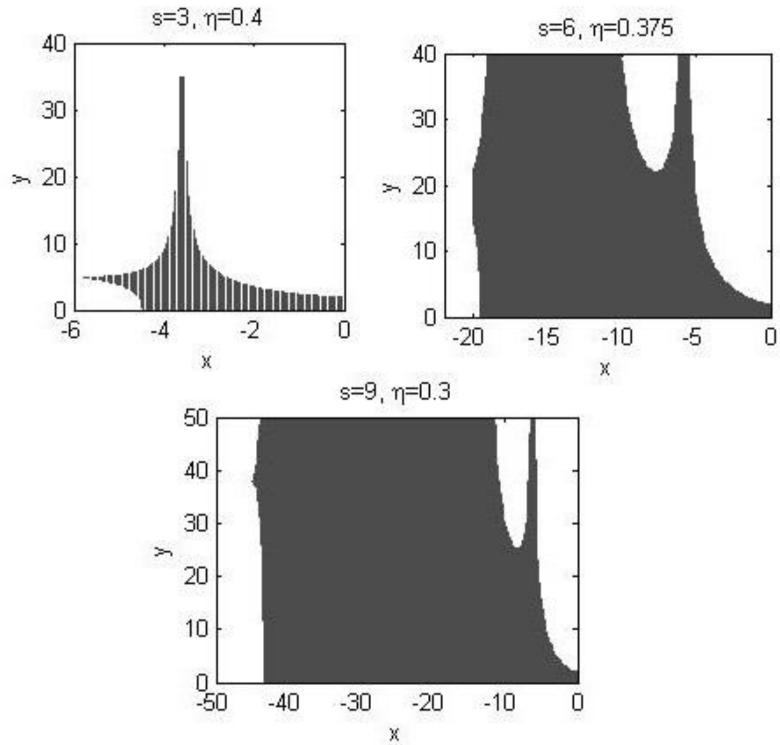


Figure 3: Profile of the T-stability regions of the SROCKC2 schemes for some optimal values of $\eta$ and $s$.

**Definition 2.** *Assume that the test equation* (13) *is stochastically asymptotically stable in the large. The numerical scheme equipped with a specified driving process said to be T-stable if* $\lim_{n\to\infty} |y_n| = 0$ *holds for the driving process.*

It can be deduced that the criterion of T-stability depends both on the scheme and the driving process. Accordingly, similar to the approach, which is used by Lemma 1, the criterion of T-stability of a numerical method with respect to normal random variables can express as

$$\log T(h, \lambda, \mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big(\log(|R_n(h, \lambda, \mu)| e^{-\frac{1}{2}z^2} dz\Big) < 0. \tag{19}$$

It seems it is not so simple to find the closed form of (19). Thus, we use the MATLAB software to approximate the integral in (19). The integral interval $(-\infty, +\infty)$ is approximated by $[-50, 50]$ because of the magnitude of the integrand in (19) becomes sufficiently small when $|z| > 50$. In Figure 3, the regions of T-stability for SROCKC2 schemes with different values of

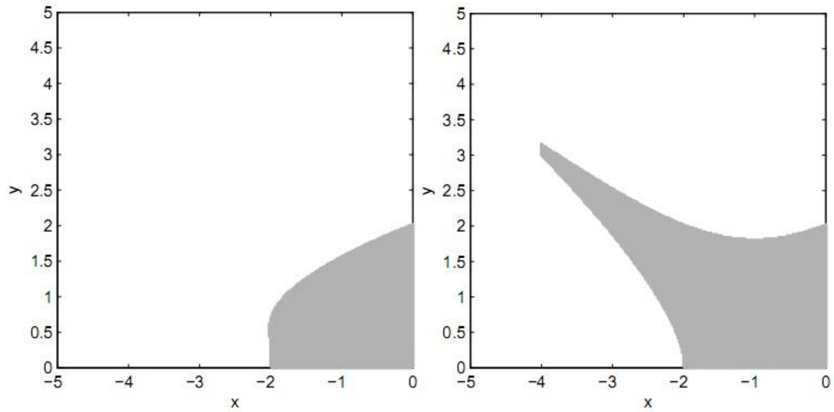


Figure 4: Profile of the T-stability regions of the methods SRIC1 (left) and SRIC2 (right).

$\eta$ and $s$ (dark gray area) have been illustrated. Similarly, the regions of T-stability for the methods SRIC1, SRIC2, Milstein and Platen (light gray area) are depicted in Figures 4 and 5. Comparison of the illustrated regions confirms that the proposed methods have reasonably larger regions of both MS-stability and T-stability.

## 5 Numerical experiments

In this section, we apply some optimal cases of SROCKC2 schemes (12) on some examples of Itô stochastic differential equations that arising in applications. These examples illustrate the efficiency of the new proposed methods. Denote $y_N^{(i)}$ and $X^{(i)}(t_N)$ as the numerical solutions and the exact solution at step point $t_N$ in $i$th simulation, respectively. The root mean square error (RMSE) is used by simulating 1000 independent trajectories for a given step-size $h$ as bellow:

$$RMSE := \left( \frac{1}{1000} \sum_{i=1}^{1000} \left| X^{(i)}(t_N) - y_N^{(i)} \right|^2 \right)^{\frac{1}{2}}.$$

We also investigate computational costs for a given $h$ to measure accuracy of proposed methods. In simulation results, the number of evaluations of the drift function $a$, of the diffusion function $b$, and the number of random numbers to be generated in every step is taken as measure of computational costs ($S_a$). In the following, we will compare the methods SROCKC2-3, SROCKC2-6, and SROCKC2-9 with some popular methods, including Euler-

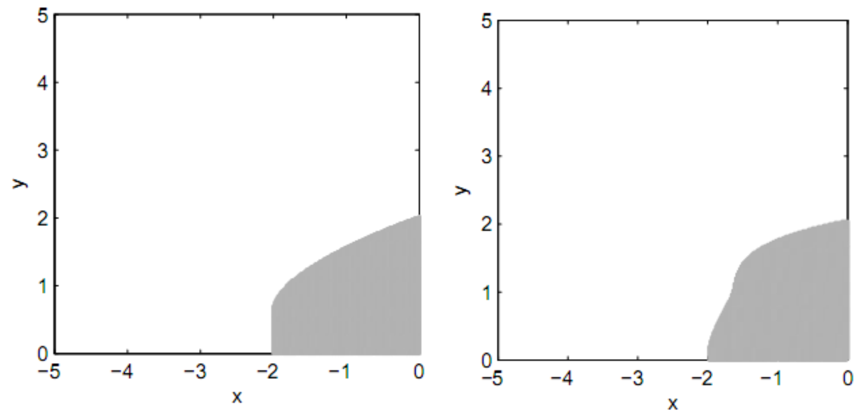


Figure 5: Profile of the T-stability regions of the methods Milstein (left) and Platen (right).

Table 2: A least squares fit for the parameters $C$ and $\alpha$, for problem 1 with $\lambda = 1, \mu = 1$ $X_{t_0} = 1$

| | SROCKC2-3 | SROCKC2-6 | SROCKC2-9 | Platen | Milstein | SRIC1 |
|---|---|---|---|---|---|---|
| $\alpha$ | 1.0514 | 1.0597 | 1.0615 | 1.0207 | 1.0109 | 1.0109 |
| $\log(C)$ | 1.8067 | 1.7033 | 1.6803 | 2.0750 | 2.2317 | 2.2317 |

Maruyama [12], Milstein [16], SRIC1 and SRIC2 [20], and also the method proposed by Platen [12].

**Problem 1:** The first problem is the scalar test equation (13) that is considered on $I = [0, 1]$ with the initial condition $X_{t_0} = 1$. For this setting, the simulation results are presented in Figure 6. In this figure, RSME versus step size of the methods are illustrated at $T = 1$. It is clearly seen in Figure 6 that the slopes of curves appear to match well and (3) for $\alpha = 1$ is valid. Further, with assumption $RSME \approx Ch^{\alpha}$ for some constants $C$ and $\alpha$, we can write

$$\log(RSME) \approx \log(C) + \alpha \log(h). \tag{20}$$

In Table 2, a least squares fit for the parameters $C$ and $\alpha$ are calculated, respectively. These results confirm that the proposed methods have a convergence with order 1 in mean square sense. Also, in Table 3, the RMSE for time steps $h = 2^{-9}, \ldots, 2^{-5}$ are computed.

As a further aspect of providing comparisons of the computational efficiency of the proposed methods, ratio CPU times of the methods SROCKC2, Milstein, Platen, and SRIC2 to the running time of the method SRIC1 are calculated in Table 4. It can be deduced from Table 4 that the CPU time of the

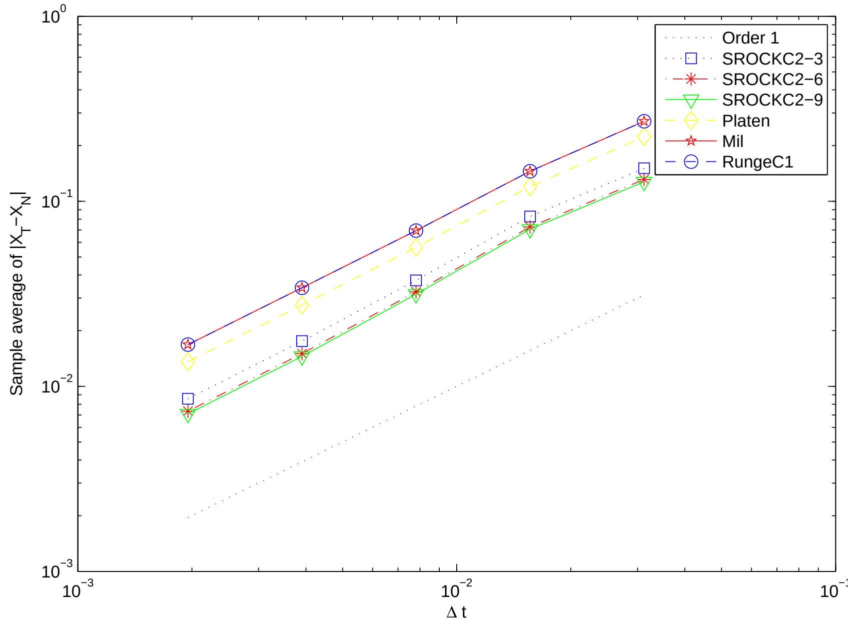


Figure 6: RMSE versus step size for test problem 1.

Table 3: Means of absolute errors, RMSE, for problem 1

| h | SROCKC2-3 | SROCKC2-3 | SROCKC2-3 | Platen | Milstein | SRIC2 |
|---|---|---|---|---|---|---|
| $2^{-5}$ | 0.150 | 0.131 | 0.127 | 0.223 | 0.271 | 0.271 |
| $2^{-6}$ | 0.082 | 0.072 | 0.070 | 0.121 | 0.145 | 0.145 |
| $2^{-7}$ | 0.037 | 0.032 | 0.031 | 0.056 | 0.069 | 0.068 |
| $2^{-8}$ | 0.017 | 0.015 | 0.014 | 0.027 | 0.035 | 0.034 |
| $2^{-9}$ | 0.008 | 0.007 | 0.007 | 0.013 | 0.017 | 0.016 |

method SROCKC2-3 and SRIC2 are approximately similar to the method SRIC1.

In following, because we do not know exact solutions to problems, reference solutions have been simulated with the Milstein scheme [16] based on the step size $h = 2^{-13}$.

**Problem 2:** The second problem is a chemical reaction model [12]. The mathematical model is

$$\begin{aligned} dx_1(t) &= \Big(c_1x_1(t) - c_2x_1(t)(x_1(t)-1) + 2c_3x_2(t)\Big)dt \\ &\quad + x_1(t)\Big(\alpha_1 dW_t^1 + \alpha_2 dW_t^2\Big), \\ dx_2(t) &= \Big(\frac{c_2}{2}x_1(t)(x_1(t)-1) - c_3x_2(t) - c_4x_2(t)\Big)\Big)dt \\ &\quad + x_2(t)\Big(\beta_1 dW_t^1 + \beta_2 dW_t^2\Big). \end{aligned}$$

The system parameters take the values $c_1 = c_2 = 200$, $c_3 = 100$, and $c_4 = 100$ while the stochastic coefficients are $\alpha_1 = \alpha_2 = 5$ and $\beta_1 = \beta_2 = 0.5$.

Table 4: Ratio CPU times of the proposed methods to the running time of the method SRIC1, for problem 1 with $\lambda = 1, \mu = 1$ $X_{t_0} = 1$ at $T = 1$

| h | SROCKC2-3 | SROCKC2-6 | SROCKC2-9 | Platen | Milstein | SRIC2 |
|---|---|---|---|---|---|---|
| $2^{0}$ | 0.842 | 1.736 | 2.157 | 0.115 | 0.105 | 0.473 |
| $2^{-1}$ | 1.346 | 1.846 | 3.891 | 0.269 | 0.262 | 1.000 |
| $2^{-2}$ | 0.957 | 2.015 | 3.723 | 0.414 | 0.106 | 1.063 |
| $2^{-3}$ | 0.970 | 2.000 | 3.303 | 0.284 | 0.176 | 0.961 |
| $2^{-4}$ | 0.984 | 2.035 | 3.533 | 0.329 | 0.203 | 1.011 |
| $2^{-5}$ | 1.032 | 2.060 | 3.486 | 0.319 | 0.151 | 1.001 |
| $2^{-6}$ | 0.995 | 1.949 | 3.212 | 0.386 | 0.239 | 1.022 |

Table 5: A least squares fit for the parameters $C$ and $\alpha$, for problem 2

| | SROCKC2-3 | SROCKC2-6 | SROCKC2-9 | Platen | Milstein | SRIC1 |
|---|---|---|---|---|---|---|
| $\alpha$ | 0.7247 | 0.7237 | 0.6993 | 1.5669 | 1.4207 | 0.8303 |
| $\log(C)$ | 8.5043 | 8.3181 | 7.9808 | 19.5330 | 17.8396 | 10.1329 |

Table 6: Means of absolute errors, RMSE, for problem 2

| h | SROCKC2-3 | SROCKC2-3 | SROCKC2-3 | SRIC2 | Milstein | Euler | Platen |
|---|---|---|---|---|---|---|---|
| $0.01 \times 2^{-9}$ | unst. | 1.17 | 1.21 | unst. | unst. | unst. | unst. |
| $0.01 \times 2^{-10}$ | 0.86 | 0.91 | 0.92 | unst. | unst. | unst. | unst. |
| $0.01 \times 2^{-11}$ | 0.71 | 0.60 | 0.58 | 0.98 | 1.72 | unst. | 1.58 |
| $0.01 \times 2^{-12}$ | 0.41 | 0.34 | 0.33 | 0.55 | 0.51 | 0.51 | 0.41 |
| $0.01 \times 2^{-13}$ | 0.26 | 0.22 | 0.22 | 0.31 | 0.24 | 0.24 | 0.18 |

Table 7: Ratio CPU times of the proposed methods to the running time of the method SRIC1, for problem 2

| h | SROCKC2-3 | SROCKC2-6 | SROCKC2-9 | Platen | Milstein | SRIC2 |
|---|---|---|---|---|---|---|
| $0.01 \times 2^{-6}$ | 1.066 | 1.333 | 1.600 | 0.733 | 0.667 | 0.933 |
| $0.01 \times 2^{-7}$ | 0.935 | 1.225 | 1.581 | 0.709 | 0.645 | 0.967 |
| $0.01 \times 2^{-8}$ | 1.034 | 1.310 | 1.672 | 0.706 | 0.689 | 1.034 |
| $0.01 \times 2^{-9}$ | 1.008 | 1.288 | 1.661 | 0.711 | 0.678 | 1.016 |
| $0.01 \times 2^{-10}$ | 0.991 | 1.287 | 1.665 | 0.707 | 0.671 | 1.042 |
| $0.01 \times 2^{-11}$ | 1.006 | 1.319 | 1.696 | 0.724 | 0.698 | 1.002 |
| $0.01 \times 2^{-12}$ | 1.004 | 1.304 | 1.685 | 0.719 | 0.691 | 1.000 |
| $0.01 \times 2^{-13}$ | 1.007 | 1.314 | 1.684 | 0.730 | 0.699 | 1.009 |

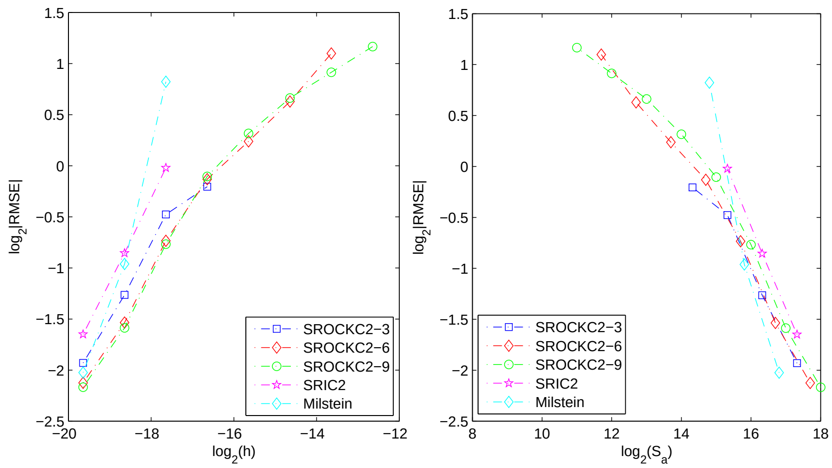


Figure 7: RMSE versus step size (left) and RMSE versus computational cost (right) with double logarithmic (ld) measures for test problem 2.

Table 8: A least squares fit for the parameters $C$ and $\alpha$, for problem 3

| | SROCKC2-3 | SROCKC2-6 | SROCKC2-9 | Platen | Milstein | SRIC1 |
|---|---|---|---|---|---|---|
| $\alpha$ | 0.5454 | 0.5715 | 0.5244 | 1.4623 | 1.3736 | 1.4243 |
| $\log(C)$ | 0.6772 | 0.8068 | 0.5372 | 6.7938 | 6.2749 | 6.5723 |

The initial conditions are $x_1(0) = 1000$ and $x_2(0) = 100$, and the integration is performed on the interval $[0, 0.01]$. The results are indicated in Figure 7 and Tables 5–7. Table 5 shows that although all mentioned methods have same slope values, $\alpha$, but the values of $C$, for the methods SROCKC2-3, SROCKC2-6, and SROCKC2-9 are much less that other methods. Also, from Table 6, it can be deduced that the methods SROCKC2-6 and SROCKC2-9 are MS-stable for step sizes $h \leq 7.81 \times 10^{-5}$ and $h \leq 1.56 \times 10^{-4}$. Clearly, the simulation results suggest proposed methods as the most efficient methods with respect to the root mean-square errors and computational costs. Also, the evolution of the $x_1(t)$ and $x_2(t)$ is given in Figure 8.

**Problem 3:** The third problem is Marine bacteriophage infection model, which is a stiff SDEs; see [11]. The mathematical model is

$$
\begin{aligned}
ds(t) &= \Big(as(t)(1-(i(t)+s(t))) - s(t)p(t)\Big)dt + \sigma_1\Big(s(t)-s^*\Big)dW_t^1,\\
di(t) &= \Big(s(t)p(t) - \ell i(t)\Big)dt + \sigma_2\Big(i(t)-i^*\Big)dW_t^2, \qquad (21)\\
dp(t) &= \Big(-s(t)p(t) - mp(t) + b\ell i(t)\Big)dt - \sigma_3\Big(p(t)-p^*\Big)dW_t^3,
\end{aligned}
$$

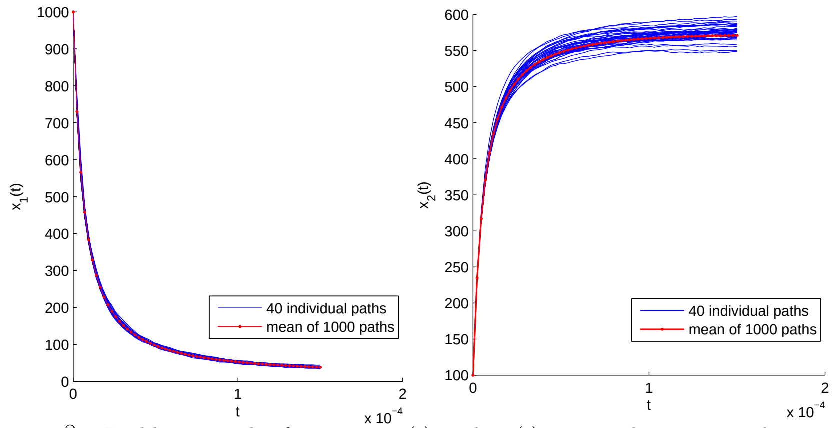


Figure 8: Problem 2: The functions $x_1(t)$ and $x_2(t)$ averaged over 1000 discretized Brownian paths and along 40 individual paths with the SROCKC2-6 method with $h = 0.01 \times 2^{-8}$.

Table 9: Means of absolute errors, RMSE, for problem 3

| h | SROCKC2-3 | SROCKC2-3 | SROCKC2-3 | SRIC1 | Milstein | Euler | Platen |
|---|---|---|---|---|---|---|---|
| $5 \times 2^{-5}$ | unst. | 0.353 | 0.351 | unst. | unst. | unst. | unst. |
| $5 \times 2^{-6}$ | 0.354 | 0.350 | 0.344 | unst. | unst. | unst. | unst. |
| $5 \times 2^{-7}$ | 0.312 | 0.311 | 0.309 | unst. | unst. | unst. | unst. |
| $5 \times 2^{-8}$ | 0.238 | 0.237 | 0.235 | unst. | unst. | unst. | unst. |
| $5 \times 2^{-9}$ | 0.162 | 0.161 | 0.159 | unst. | unst. | unst. | unst. |
| $5 \times 2^{-10}$ | 0.108 | 0.107 | 0.105 | 0.365 | 0.355 | 0.362 | 0.372 |
| $5 \times 2^{-11}$ | 0.074 | 0.072 | 0.073 | 0.136 | 0.137 | 0.133 | 0.135 |
| $5 \times 2^{-12}$ | 0.055 | 0.054 | 0.055 | 0.039 | 0.039 | 0.038 | 0.039 |

with the initial condition $(s_0, i_0, p_0) = (0.3, 0.2, 5)$. Here $s$ represents the susceptible bacteria, $i$ is the infected bacteria, and $p$ is the phage (viruses). The drift coefficients have the values $a = 8.65$, $\ell = 24.628$, $m = 14.925$, and $b = 60$ while the noise coefficients have the values $\sigma_1 = \sigma_2 = \sigma_3 = 0.4$ and

$$s^* = \frac{m}{b-1}, i^* = \frac{as^*(1-s*)}{\ell + as^*}, p^* = \frac{a\ell(1-s^*)}{\ell + as^*}.$$

In Figure 9, we report the errors versus step size of the methods and errors versus computational effort with double logarithmic (ld) measures on the interval $[0, 5]$. Also, Figure 9 illustrates the reduction in computational effort of methods SROCKC2-3, SROCKC2-6, and SROCKC2-9 in the same level of accuracy. Clearly, the simulation results in Tables 8–10 suggest the proposed methods as the most efficient methods with respect to root mean-square errors and computational costs. The evolution of the three interacting species

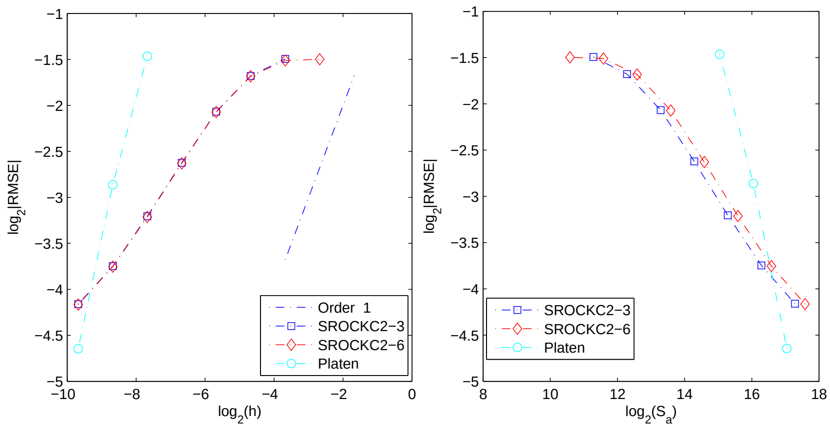


Figure 9: RMSE versus step size (left) and RMSEs versus computational cost (right) with double logarithmic (ld) measures for test problem 3.

Table 10: Ratio CPU times of the proposed methods to the running time of the method SRIC1, for problem 3

| h | SROCKC2-3 | SROCKC2-6 | SROCKC2-9 | Platen | Milstein | SRIC2 |
|---|---|---|---|---|---|---|
| $5 \times 2^{-5}$ | 1.200 | 1.600 | 3.000 | 0.400 | 0.401 | 1.200 |
| $5 \times 2^{-6}$ | 0.909 | 1.545 | 2.454 | 0.363 | 0.181 | 1.000 |
| $5 \times 2^{-7}$ | 1.001 | 1.476 | 2.428 | 0.381 | 0.285 | 1.095 |
| $5 \times 2^{-8}$ | 1.024 | 1.561 | 2.243 | 0.390 | 0.268 | 1.048 |
| $5 \times 2^{-9}$ | 0.964 | 1.511 | 2.178 | 0.392 | 0.272 | 1.023 |
| $5 \times 2^{-10}$ | 1.189 | 1.561 | 2.298 | 0.445 | 0.274 | 1.158 |
| $5 \times 2^{-11}$ | 0.972 | 1.413 | 2.011 | 0.356 | 0.263 | 1.087 |
| $5 \times 2^{-12}$ | 0.952 | 1.384 | 2.158 | 0.371 | 0.267 | 1.069 |

is given in Figure 10.

## 6 Conclusions

In this paper, the class of strong stochastic Runge–Kutta methods for stochastic differential equations with commutative noise due to Rößler (2010) is considered. For stiff SDEs, a family of explicit stochastic orthogonal Runge–Kutta Chebyshev methods of strong order one for the approximation of the solution of Itô SDEs with $m$-dimensional commutative noise are designed. The mean-square and asymptotic stability of newly proposed methods applied to a scalar linear test equation with multiplicative noise have been analyzed. Some numerical results for stochastic models arising in applications are provided to confirm theoretical discussions. In future works, we will con-

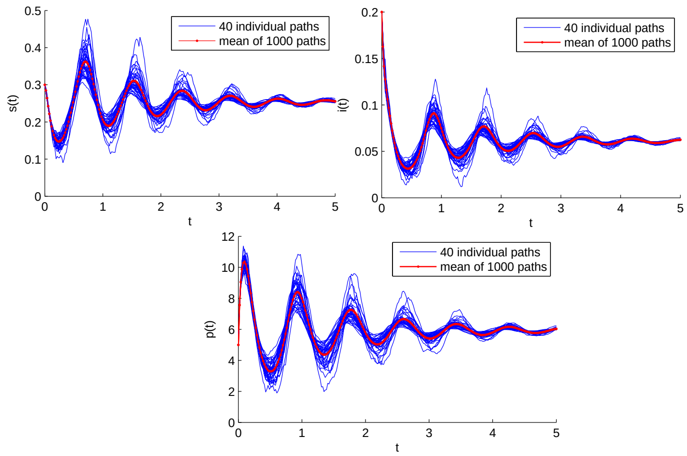


Figure 10: Problem 3: The functions $s(t)$, $i(t)$, and $p(t)$ averaged over 1000 discretized Brownian paths and along 40 individual paths with the SROCKC2-6 method with $h = 5 \times 2^{-8}$.

sider constructing methods with higher strong global convergence orders and better stability properties.

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# روشهای S-ROCK جدید برای حل معادلات دیفرانسیل تصادفی با نویز جابجایی

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**چکیده :** در این مقاله، ابتدا کلاس روشهای رونگ–کوتای تصادفی بیان شده توسط روسلر (۲۰۱۰) برای حل معادلات دیفرانسیل تصادفی با نویز جابجایی در نظر گرفته میشود. سپس، با استفاده از ایده بیان شده توسط بُرج و کُموری (۲۰۱۳)، یک کلاس از روشهای رونگ–کوتای تصادفی متعامد صریح (SROCKC۲) از مرتبهی قوی یک برای تقریب جواب معادلات دیفرانسیل تصادفی ایتو با نویز جابجایی طراحی میگردد. در ادامه، پایداری میانگین مربعی و مجانبی روشهای پیشنهادی روی یک معادلهی اسکالر خطی آزمون با نویز ضربی مورد تحلیل قرار میگیرد. در انتها، با ارائه نتایج عددی حاصل از شبیهسازی برخی مدلهای تصادفی کاربردی مطالب بیان شده مورد تایید قرار خواهد گرفت.

**کلمات کلیدی :** معادلات دیفرانسیل تصادفی؛ روشهای رونگ–کوتا؛ پایداری میانگین مربعی؛ معادلات سخت؛ نویز جابجایی.