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On algebraic characterizations for finiteness of the dimension of $\underline{E}G$

Olympia Talelli*†

Department of Mathematics, University of Athens

Panepistemiopolis, 15784 Athens - Greece

Abstract

Certain algebraic invariants of the integral group ring $\mathbb{Z}G$ of a group G were introduced and investigated in relation to the problem of extending the Farrell-Tate cohomology, which is defined for the class of groups of finite virtual cohomological dimension. It turns out that the finiteness of these invariants of a group G implies the existence of a generalized Farrell-Tate cohomology for G which is computed via complete resolutions.

In this article we present these algebraic invariants and their basic properties and discuss their relationship to the generalized Farrell-Tate cohomology. In addition we present the status of conjecture which claims that the finiteness of these invariants of a group G is equivalent to the existence of a finite dimensional model for $\underline{E}G$, the classifying space for proper actions.

Keywords and phrases: Farrell-Tate cohomology, virtual cohomological dimension, complete resolution, finitistic dimension of the integral group ring, classifying space for proper action.

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*E-mail: otalelli@cc.uoa.gr or otalelli@math.uoa

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1 Introduction

In their efforts to generalize the Farrell-Tate cohomology, which was defined for the class of groups of finite virtual cohomological dimension, Ikenaga in [12] and Gedrich and Gruenberg in [10] considered certain algebraic invariants of a group and showed that if these were finite then generalized Tate cohomology is defined for the group.

In particular, Ikenaga defined the generalized cohomological dimension of a group G, $\underline{\mathrm{cd}}\, G$, to be

$$\underline{\mathrm{cd}}\, G = \sup\{k : \mathrm{Ext}^k_{\mathbb{Z}G}(M, F) \neq 0,\ M\ \mathbb{Z}\text{-free},\ F\ \mathbb{Z}G\text{-free}\}$$

and showed that if G admits a complete resolution and $\underline{\mathrm{cd}}\, G < \infty$ then generalized Tate cohomology is defined for G.

A complete resolution of G is an acyclic complex $\{P_k\}_{k\in\mathbb{Z}}$ of projective $\mathbb{Z}G$-modules which agree with an ordinary projective resolution of G in sufficiently high (positive) dimensions.

Gedrich and Gruenberg considered the supremum of the projective lengths of injective $\mathbb{Z}G$-modules, $\mathrm{spli}\,\mathbb{Z}G$, and the supremum of the injective lengths of projective $\mathbb{Z}G$-modules, $\mathrm{silp}\,\mathbb{Z}G$. Then showed that if $\mathrm{spli}\,\mathbb{Z}G < \infty$ then G admits a complete resolution and moreover $\mathrm{silp}\,\mathbb{Z}G < \infty$ which implies that any two complete resolutions are homotopy equivalent, so generalized Tate cohomology is defined for G.

Note that $\mathrm{silp}\,\mathbb{Z}G$ and $\underline{\mathrm{cd}}\, G$ are closely related, namely $\underline{\mathrm{cd}}\, G \leq \mathrm{silp}\,\mathbb{Z}G \leq 1 + \underline{\mathrm{cd}}\, G$.

Mislin in [19] generalized these ideas and defined generalized Tate cohomology, $\hat{H}^n(G, -)$, for any group G and any integer n as follows: $\hat{H}^n(G, -) = \varinjlim_{j\geq 0} S^{-j}H^{n+j}(G, -)$ where $S^{-j}H^{n+j}(G, -)$ denotes the jth left satellite of the functor $H^{n+j}(G, -)$. Alternative but equivalent definitions were also given by Benson and Carlson [1] and Vogel (see [11]).

Note that the generalized Tate cohomology can not always be calculated via complete resolutions as they do not always exist. It turns out that the generalized Tate cohomology can be calculated via complete resolutions if and only if $\operatorname{spli}\mathbb{Z}G < \infty$ [24].

This article is a survey on the algebraic invariants of G that appeared in the search for the definition of generalized Tate cohomology for G.

We first discuss their basic properties and interrelations.

We then discuss the state of a conjecture (Conj. A in [26]) which claims that the finiteness of the above algebraic invariants, which imply that the generalized Tate cohomology can be calculated via complete resolutions, is the algebraic characterization of those groups G which admit a finite dimensional model for $\underline{E}G$, the classifying space for proper actions of G.

2 $\operatorname{spli}\mathbb{Z}G$

First we will establish some notation.

Let G be a group, $H \leq G$ and $i : \mathbb{Z}H \to \mathbb{Z}G$ the ring homomorphism induced from $H \hookrightarrow G$. Then the ring homomorphism i gives rise to the following functors:

1. $r : {}_{\mathbb{Z}G}\mathrm{Mod} \to {}_{\mathbb{Z}H}\mathrm{Mod}$, where any (left) $\mathbb{Z}G$-module can be regarded as a $\mathbb{Z}H$-module via i. If $M \in {}_{\mathbb{Z}G}\mathrm{Mod}$, then we denote $r(M)$ by $M\mid_H$.

2. $e : {}_{\mathbb{Z}H}\mathrm{Mod} \to {}_{\mathbb{Z}G}\mathrm{Mod}$

$N \to \mathbb{Z}G \underset{\mathbb{Z}H}{\otimes} N$, where the left $\mathbb{Z}G$-action on $\mathbb{Z}G \underset{\mathbb{Z}H}{\otimes} N$ is inherited from the $(\mathbb{Z}G, \mathbb{Z}H)$-bimodule structure of $\mathbb{Z}G$.

The module $e(N) = \mathbb{Z}G \underset{\mathbb{Z}H}{\otimes} N$ is called induced and we denote it by $\overset{\searrow}{\mathbb{Z}}G \underset{\mathbb{Z}H}{\otimes} N$.

3. $c : {}_{\mathbb{Z}H}\mathrm{Mod} \to {}_{\mathbb{Z}G}\mathrm{Mod}$

$N \to \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, N)$, where the left $\mathbb{Z}G$-action on $\operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, N)$ is inherited from the $(\mathbb{Z}H, \mathbb{Z}G)$-bimodule structure of $\mathbb{Z}G$.

The (left) $\mathbb{Z}G$-module $c(N) = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, N)$ is called co-induced and we denote it by $\operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}}G, N)$.

Let now G be a group and $A, B \in {}_{\mathbb{Z}G}\mathrm{Mod}$.

We denote by $\operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{A}, \overset{\searrow}{B})$ (resp. $\overset{\searrow}{A} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{B}$) the (left) $\mathbb{Z}G$-module $\operatorname{Hom}_{\mathbb{Z}}(A, B)$

(resp. $A \underset{\mathbb{Z}}{\otimes} B$) with the diagonal action $(gf)(\alpha) = gf(g^{-1}\alpha)$, $g \in G$, $f \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$, $\alpha \in A$ (resp. $g(\alpha \otimes \beta) = g\alpha \otimes g\beta$, $g \in G$, $\alpha \in A$, $\beta \in B$).

The following Proposition states the well-known relation between the diagonal action and the induced and co-induced actions. The Corollary after it, states some of the Proposition's well-known consequences.

We state both without proofs.

Proposition 2.1. *Let G be a group, $H \leq G$ and $M \in {}_{\mathbb{Z}G}\mathrm{Mod}$. If $\mathbb{Z}(G/H)$ is the permutation module, where G/H is the set of cosets gH and G acts on G/H by left translations then*

(i) $\mathbb{Z}(G\overset{\searrow}{/}H) \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{M} \cong \overset{\searrow}{\mathbb{Z}}G \underset{\mathbb{Z}H}{\otimes} M/H$

(ii) $\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}\overset{\searrow}{(}G/H), \overset{\searrow}{M}\right) \cong \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}\overset{\swarrow}{G}, M|_H)$.

Corollary 2.2. *Let $A \in {}_{\mathbb{Z}G}\mathrm{Mod}$ with $\mathrm{proj.dim}_{\mathbb{Z}G}\, A \leq m$. Then*

(i) If $B \in {}_{\mathbb{Z}G}\mathrm{Mod}$ with B $\mathbb{Z}$-free then $\mathrm{proj.dim}_{\mathbb{Z}G}\, \overset{\searrow}{A} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{B} \leq m$;

(ii) If $B \in {}_{\mathbb{Z}G}\mathrm{Mod}$ with B $\mathbb{Z}$-injective then $\mathrm{inj.dim}\,\mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{A}, \overset{\searrow}{B}) \leq m$.

The following proposition and theorem state some basic properties of $\mathrm{spli}\,\mathbb{Z}G$ [10].

$\mathrm{Spli}\,\mathbb{Z}G$ is the supremum of the projective lengths of the injective $\mathbb{Z}G$-modules. It is not difficult to see that $\mathrm{spli}\,\mathbb{Z}G < \infty$ iff every injective $\mathbb{Z}G$-module has finite projective dimension.

Proposition 2.3.

(i) If G is a finite group then $\mathrm{spli}\,\mathbb{Z}G = 1$

(ii) If G is a group with $\mathrm{cd}_Z G = n$ *then* $\mathrm{spli}\,\mathbb{Z}G \leq n+1$

(iii) Let G be a group and $H \leq G$. If I is an injective $\mathbb{Z}G$-module then $I|_H$ is an injective $\mathbb{Z}H$-module. Moreover $\mathrm{spli}\,\mathbb{Z}H \leq \mathrm{spli}\,\mathbb{Z}G$

(iv) If $H \leq G$ *and* $\mid G : H \mid < \infty$*, then* $\operatorname{spli} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}H$*.*

Proof.

(i) If I is an injective $\mathbb{Z}G$-module, with G finite, then I is cohomologically trivial [e.g. [2]] and hence $\operatorname{proj.dim} I \leq 1$ and since I is not $\mathbb{Z}$-free it follows that $\operatorname{proj.dim} I = 1$.

(ii) Since $\operatorname{cd}_{\mathbb{Z}} G = n$ we have that $\operatorname{proj.dim}_{\mathbb{Z}G} \mathbb{Z} = n$ hence by Corollary 2.2 (i), for any $\mathbb{Z}G$-module A with A $\mathbb{Z}$-free we have that $\operatorname{proj.dim}_{\mathbb{Z}G} A \leq n$.

Now if M is any $\mathbb{Z}G$-module and one takes a projective presentation of M

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

then K, being a submodule of P, is $\mathbb{Z}$-free. Hence $\operatorname{proj.dim}_{\mathbb{Z}G} K \leq n$ and since P is projective, it follows that $\operatorname{proj.dim}_{\mathbb{Z}G} M \leq n+1$. In particular if I is an injective $\mathbb{Z}G$-module then $\operatorname{proj.dim}_{\mathbb{Z}G} I \leq n+1$.

(iii) If I is an injective $\mathbb{Z}G$-module, then $I|_H$ is an injective $\mathbb{Z}H$-module since

$$\operatorname{Hom}_{\mathbb{Z}G}\left(\overset{\searrow}{\mathbb{Z}G} \underset{\mathbb{Z}H}{\otimes} -, I\right) \cong \operatorname{Hom}_{\mathbb{Z}H}(-, I|_H)$$

and $\overset{\searrow}{\mathbb{Z}G} \underset{\mathbb{Z}H}{\otimes} -$ is an exact functor: ${}_{\mathbb{Z}H}\mathrm{Mod} \rightarrow {}_{\mathbb{Z}G}\mathrm{Mod}$.

Now if K is an injective $\mathbb{Z}H$-module, then K is a $\mathbb{Z}H$-direct summand of the injective $\mathbb{Z}G$-module $\operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, K)$. Hence

$$\operatorname{proj.dim}_{\mathbb{Z}H} K \leq \operatorname{proj.dim}_{\mathbb{Z}H} \operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, K)|_H \leq \operatorname{proj.dim}_{\mathbb{Z}G} \operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, K)$$

, which implies that $\operatorname{spli} \mathbb{Z}H \leq \operatorname{spli} \mathbb{Z}G$.

(iv) Let $\mid G : H \mid < \infty$ and let $\operatorname{spli}\mathbb{Z}H = m$. By (iii), to show that $\operatorname{spli} \mathbb{Z}G = m$, it is enough to prove that every injective $\mathbb{Z}G$-module has projective dimension $\leq m$.

Let I be an injective $\mathbb{Z}G$-module, then by (iii) $I|_H$ is an injective $\mathbb{Z}H$-module and since $\operatorname{spli}\mathbb{Z}H = m$, there is a $\mathbb{Z}H$-projective resolution

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow I|_H \longrightarrow 0,$$

which implies that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{\mathbb{Z}G} \underset{\mathbb{Z}H}{\otimes} I|_H \leq m$.

Since $\mid G : H \mid < \infty$, it follows that

$$\overset{\searrow}{\mathbb{Z}G} \underset{\mathbb{Z}H}{\otimes} I|_H \cong \operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, I|_H).$$

But I is a $\mathbb{Z}G$-direct summand of $\operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, I|_H)$, hence $\operatorname{proj.dim}_{\mathbb{Z}G} I \leq m$.

□

We will show that $\operatorname{spli}\mathbb{Z}G < \infty$ is an extension closed property. For this we need the following lemma.

Lemma 2.4. *Let G be a group and $J = \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Z})$. Then*

(i) $\operatorname{inj.dim}_{\mathbb{Z}G} J \leq 1$*;*

(ii) if $\operatorname{spli}\mathbb{Z}G = m$ *then* $\operatorname{proj.dim}_{\mathbb{Z}G} J \leq m$*;*

(iii) $\operatorname{spli}\mathbb{Z}G < \infty$ *iff* $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$*.*

Proof. The exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ gives rise to the following exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q}/\mathbb{Z})$$

from which follows (i) and (ii), since $\operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q})$ and $\operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q}/\mathbb{Z})$ are injective $\mathbb{Z}G$-modules.

Now let $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$. We will show that every injective $\mathbb{Z}G$-module I has finite projective dimension.

From the $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to IG \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where ε is the augmentation map, we obtain the $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to J \to \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{I} G, \overset{\searrow}{\mathbb{Z}}) \to 0$, which gives rise to the $\mathbb{Z}G$-exact sequence $0 \to I \to \overset{\searrow}{I} \otimes \overset{\searrow}{J} \to \overset{\searrow}{I} \otimes C \to 0$, where $C = \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{I} G, \overset{\searrow}{\mathbb{Z}})$. Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{G}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{\mathbb{Z}} G, \overset{\searrow}{\mathbb{Z}})$. Since I is a $\mathbb{Z}G$-direct summand of $\overset{\searrow}{I} \otimes \overset{\searrow}{J}$ it is enough to show that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{I} \otimes \overset{\searrow}{J} < \infty$.

Let $0 \to K \to P \to I \to 0$ be a $\mathbb{Z}G$-projective presentation of I. Since J is $\mathbb{Z}$-torsion-free we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \longrightarrow \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \longrightarrow \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \longrightarrow 0.$$

Since $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ and P, K are $\mathbb{Z}$-free it follows from Corollary 2.2 (i) that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} < \infty$ and $\operatorname{proj.dim} \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} < \infty$, hence $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} < \infty$. $\square$

It is clear from the proof of (iii) of the above lemma that we have

Corollary 2.5. $\operatorname{spli} \mathbb{Z}G < \infty$ *iff there is a $\mathbb{Z}$-split, $\mathbb{Z}G$-monomorphism* $0 \to \mathbb{Z} \to M$ *with* $\operatorname{proj.dim} M < \infty$ *and M $\mathbb{Z}$-torsion free.*

Theorem 2.6. *[10] Let $1 \to N \to G \xrightarrow{\pi} K \to 1$ be an extension of groups. Then* $\operatorname{spli} \mathbb{Z}G \le \operatorname{spli} \mathbb{Z}N + \operatorname{spli} \mathbb{Z}K$.

Proof. Let $\operatorname{spli} \mathbb{Z}N = n$ and $\operatorname{spli} \mathbb{Z}K = m$ and let I be an injective $\mathbb{Z}G$-module. We will show that $\operatorname{proj.dim}_{\mathbb{Z}G} I \le n + m$.

We consider the $\mathbb{Z}$-split $\mathbb{Z}K$-exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(IK, \mathbb{Z}) \longrightarrow 0$$

as a $\mathbb{Z}G$-exact sequence via $\pi : G \to K$ and tensoring it with I, we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow I \longrightarrow I \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z}).$$

Since I is a $\mathbb{Z}G$-direct summand of $\overset{\searrow}{I} \otimes \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z})$ it is enough to show that $\operatorname{proj.dim}_{\mathbb{Z}G} I \underset{\mathbb{Z}}{\otimes} J \leq n + m$, where J is the $\mathbb{Z}G$-module $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z})$.

Now by Lemma 2.4 (ii), $\operatorname{proj.dim}_{\mathbb{Z}K} J \leq m$ and since $\operatorname{spli} \mathbb{Z}N = n$ it follows that $\operatorname{proj.dim}_{\mathbb{Z}N} I|_N \leq n$.

Hence there exists $Q : 0 \to Q_m \to \cdots \to Q_0 \to J \to 0$ a $\mathbb{Z}K$-projective resolution of J of length m and $P : 0 \to P_n \to \cdots \to P_0 \to I \to 0$ a $\mathbb{Z}G$-exact sequence with P_i $\mathbb{Z}G$-projective modules for all $0 \leq i \leq n-1$ and $P_n|_N$ a projective $\mathbb{Z}N$-module.

Consider the following $\mathbb{Z}G$-complexes $Q' : 0 \to Q_m \to \cdots \to Q_0 \to 0$, a $\mathbb{Z}G$-complex via $\pi : G \to K$ and
$P' : 0 \to P_n \to \cdots \to P_0 \to 0$ and let $Q' \underset{\mathbb{Z}}{\otimes} P'$ be their tensor product.

Since J is $\mathbb{Z}$-torsion free it follows from the Künneth formula that we obtain a $\mathbb{Z}G$-exact sequence $0 \to B_{m+n} \to \cdots \to B_0 \to \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \to 0$, where

$$B_\lambda = \left(Q' \underset{\mathbb{Z}}{\otimes} P' \right)_\lambda = \underset{r+s=\lambda}{\oplus} \overset{\searrow}{Q_r} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{P_s}.$$

□

By Proposition 2.1 (i), B_λ is a projective $\mathbb{Z}G$-module for $0 \leq \lambda \leq m+1$. Since $P_s|_N$ is a projective $\mathbb{Z}N$-module for all s, we obtain a $\mathbb{Z}G$-projective resolution of $\overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J}$ of length $m + n$.

3 $\operatorname{spli} \mathbb{Z}G$, $\operatorname{silp} \mathbb{Z}G$, $\operatorname{fin.dim} \mathbb{Z}G$, $K(\mathbb{Z}G)$

Silp $\mathbb{Z}G = \sup\{\operatorname{inj.dim}_{\mathbb{Z}G} P | P \text{proj.}\mathbb{Z}G\text{-module}\}$ and it is not difficult to see that $\operatorname{silp} \mathbb{Z}G < \infty$ iff every projective $\mathbb{Z}G$-module has finite injective dimension.
Note that $\operatorname{silp} \mathbb{Z}G \leq m$ is equivalent to the following extension condition [12]:

For every exact sequence

$$0 \longrightarrow \ker \partial_m \longrightarrow P_m \xrightarrow{\partial_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_i projective $\mathbb{Z}G$-modules for $0 \leq i \leq m$, any map $\ker \partial_m \to P$, P a projective $\mathbb{Z}G$-module, extends to a map $P_m \to P$.

It is not difficult to see that if $\operatorname{silp}\mathbb{Z}G$ and $\operatorname{spli}\mathbb{Z}G$ are both finite then they are equal.

The following Proposition, which we state without proof, gives some basic properties of $\operatorname{silp}\mathbb{Z}G$.

Proposition 3.1.

(i) If G is a finite group, then $\operatorname{silp}\mathbb{Z}G = 1$.

(ii) If G is a group with $\operatorname{cd}_{\mathbb{Z}}G = n$ *then* $\operatorname{silp}\mathbb{Z}G \leq n+1$.

(iii) If G is a group and $H \leq G$ then $\operatorname{silp}\mathbb{Z}H \leq \operatorname{silp}\mathbb{Z}G$.

Moreover, if $|G : H| < \infty$ then $\operatorname{silp}\mathbb{Z}G = \operatorname{silp}\mathbb{Z}H$.

Theorem 3.2. *[10] For any group G,* $\operatorname{silp}\mathbb{Z}G \leq \operatorname{spli}\mathbb{Z}G$.

Proof. It is enough to show that if $\operatorname{spli}\mathbb{Z}G < \infty$ then $\operatorname{silp}\mathbb{Z}G < \infty$. By Lemma 2.4 (iii), it is enough to show that if $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ then $\operatorname{silp}\mathbb{Z}G < \infty$, where $J = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z})$.

Let $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ and consider a projective $\mathbb{Z}G$-module P. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives rise to the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow P \longrightarrow P \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow P \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Hence to show that $\operatorname{inj.dim}_{\mathbb{Z}G} P$ is finite, it is enough to show that $\operatorname{inj.dim} P \underset{\mathbb{Z}}{\otimes} D$ is finite, where D is a $\mathbb{Z}$-injective abelian group.

Let $\widetilde{P} = P \underset{\mathbb{Z}}{\otimes} D$, where D is a divisible abelian group, then $\widetilde{P}$ is a direct summand of an induced module hence it is relative projective i.e. if

$$0 \longrightarrow A \longrightarrow B \longrightarrow \widetilde{P} \longrightarrow 0 \qquad (*)$$

is an exact sequence of $\mathbb{Z}G$-modules which is $\mathbb{Z}$-split, then $(*)$ is $\mathbb{Z}G$-split.

Consider the $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to J \to C \to 0$ where $C = \operatorname{Hom}_{\mathbb{Z}}(JG, \mathbb{Z})$. This gives rise to the following $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(C, \widetilde{P}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(J, \widetilde{P}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \widetilde{P}) \longrightarrow 0.$$

But $\mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{\mathbb{Z}}, \overset{\searrow}{\widetilde{P}}) \cong \widetilde{P}$, hence $\widetilde{P}$ is a $\mathbb{Z}G$-direct summand of $\mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{J}, \overset{\searrow}{\widetilde{P}})$.

Since $\mathrm{proj.\,dim}_{\mathbb{Z}G}\, J < \infty$ it follows from Corollary 2.2 (ii) that

$\mathrm{inj.\,dim}_{\mathbb{Z}G}\, \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{J}, \overset{\searrow}{\widetilde{P}}) < \infty$. □

Open questions 3.3.

a) It is not known if $\mathrm{silp}\, \mathbb{Z}G < \infty$ is an extension closed property.

b) It is not known if there is a group G such that $\mathrm{silp}\, \mathbb{Z}G < \infty$ and $\mathrm{spli}\, \mathbb{Z}G$ infinite.

c) It is conjectured in [6] that for any group G, $\mathrm{silp}\, \mathbb{Z}G = \underline{\mathrm{cd}}\, G + 1 = \mathrm{spli}\, \mathbb{Z}G$. This is proved in [6] for certain classes of groups.

Two more algebraic invariant of G, the finiteness dimensions of $\mathbb{Z}G$, and $k(G)$ are related to $\mathrm{spli}\, \mathbb{Z}G$, and $\mathrm{spli}\, \mathbb{Z}G$. The finiteness dimension of $\mathbb{Z}G$, $\mathrm{fin.\,dim}\, \mathbb{Z}G$, which is the supremum of the projective dimensions of the $\mathbb{Z}G$-modules of finite projective dimension and

$k(G) = \sup\{\mathrm{proj.\,dim}_{\mathbb{Z}G}\, M | \mathrm{proj.\,dim}_{\mathbb{Z}H}\, M < \infty$ for every finite subgroup $H \leq G\}$.

Proposition 3.4. *[26] Let G be any group, then*

$$\mathrm{fin.\,dim}\, \mathbb{Z}G \leq \mathrm{silp}\, \mathbb{Z}G \leq \mathrm{spli}\, \mathbb{Z}G \leq k(\mathbb{Z}G).$$

Moreover, if any of the above invariants is finite then it is equal to the ones less than equal to it.

Proof. It is easy to see that $\mathrm{fin.\,dim}\, \mathbb{Z}G \leq \mathrm{silp}\, \mathbb{Z}G\mathbb{Z}G$ and by Theorem 3.2, $\mathrm{silp}\, \mathbb{Z}G \leq \mathrm{spli}\, \mathbb{Z}G$. Now by Proposition 2.3 (i) and (iii) it follows that $\mathrm{spli}\, \mathbb{Z}G \leq k(G)$.

Now if $k(G) < \infty$ then clearly $k(G) \leq \mathrm{fin.\,dim}\, \mathbb{Z}G$, hence

$$\mathrm{fin.\,dim}\, \mathbb{Z}G = \mathrm{silp}\, \mathbb{Z}G = \mathrm{spli}\, \mathbb{Z}G = k(\mathbb{Z}G).$$

In [4], it was shown that if G is an $H\mathcal{F}$-group then $\operatorname{silp}\mathbb{Z}G = \operatorname{spli}\mathbb{Z}G = \operatorname{fin.dim}\mathbb{Z}G = k(\mathbb{Z}G)$.

The class $H\mathcal{F}$ of groups was defined by Kropholler in [14] as follows. Let $H_0\mathcal{F}$ be the class of finite groups. Now define $H_\alpha\mathcal{F}$ for each ordinal α by transfinite recursion: if α is a successor ordinal then $H_\alpha\mathcal{F}$ is the class of groups G which admits a finite dimensional contractible G-CW-complex with cell stabilizers in $H_{\alpha-1}\mathcal{F}$, and if α is a limit ordinal then $H_\alpha\mathcal{F} = \bigcup_{\beta<\alpha} H_\beta\mathcal{F}$. A group belongs to $H\mathcal{F}$ if it belongs to $H_\alpha\mathcal{F}$, for some ordinal α. □

Note that a G-CW-complex is a CW-complex on which G acts by self-homeomorphisms in such a way that the set-wise stabilizer of each cell coincides with its point-wise stabilizer.

The class $H\mathcal{F}$ contains among others all groups of finite virtual cohomological dimension and all countable linear groups of arbitrary characteristic. Moreover, it is extension closed, subgroup closed, closed under directed unions and closed under amalgamated free products and HNN-extensions.

4 Another characterization of $\operatorname{spli}\mathbb{Z}G < \infty$

Definition. A complete resolution for a group , $(\mathcal{F}, \mathcal{P}, n)$, consists of an acyclic complex $\mathcal{F} = \{(F_i, \partial_i) | i \in \mathbb{Z}\}$ of projective modules and a projective resolution $\mathcal{P} = \{(P_i, d_i) | i \leq 0\}$ of G such that $\mathcal{F}$ and $\mathcal{P}$ coincide in sufficiently high dimensions

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow F_{n+1} \longrightarrow F_n & \xrightarrow{\partial_n} & F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots \\ \| \qquad\quad \| & & \\ \cdots \longrightarrow P_{n+1} \longrightarrow P_n & \xrightarrow{d_n} & P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \end{array}$$

The number n is called the coincidence index of the complete resolution.

Inekaga in [12] defined the notion of generalized cohomological dimension of a group G, $\underline{\operatorname{cd}}\, G = \sup\{k : \operatorname{Ext}^k_{\mathbb{Z}G}(M, F) \neq 0,\ M\ \mathbb{Z}\text{-free},\ F\ \mathbb{Z}G\text{-free}\}$.

Note that $\underline{\mathrm{cd}}\, G \le \mathrm{silp}\, \mathbb{Z}G \le \underline{\mathrm{cd}}\, G + 1$.

He showed in [12] that if G admits a complete resolution then G admits a complete resolution of coincidence index $\underline{\mathrm{cd}}\, G$. In particular a group G with $v\underline{\mathrm{cd}}\, G < \infty$ admits a complete resolution of coincidence index $v\underline{\mathrm{cd}}\, G$.

Moreover, it was shown in [12] that if a group G admits a complete resolution of coincidence index n, then

(i) $H^i(G, P) \neq 0$ for some $\mathbb{Z}G$-projective module P and some $i \le n$

(ii) $\mathrm{fin.\,dim}\, \mathbb{Z}G \le n + 1$.

Since admitting a complete resolution is a subgroup closed property, and since if $\mathcal{A}$ is a free abelian group of infinite rank, $H^i(\mathcal{A}, P) = 0$ for any projective $\mathbb{Z}\mathcal{A}$-module and any i, it follows from (i) that if a group G contains a free abelian subgroup of infinite rank then G does not admit a complete resolution.

Proposition 4.1. *[24] If* $\mathrm{spli}\, \mathbb{Z}G < \infty$ *then there is a* $\mathbb{Z}$*-split* $\mathbb{Z}G$*-exact sequence* $0 \to \mathbb{Z} \to A$ *with* A $\mathbb{Z}$*-free and* $\mathrm{proj.\,dim}_{\mathbb{Z}G}\, A < \infty$.

Proof. It was shown in [24] that if $\mathrm{spli}\, \mathbb{Z}G < \infty$ then G admits a complete resolution.

Now consider a complete resolution for G

$$\begin{array}{ccccccccccc}
\longrightarrow F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n & \xrightarrow{\partial_n} & F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots \\
\| & & \| & & \\
\longrightarrow P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0
\end{array}$$

Let $R_n = \mathrm{im}\partial_n$, $n \in \mathbb{Z}$. If $\lambda : R_n \to P$ is a $\mathbb{Z}G$-homomorphism with P a projective $\mathbb{Z}G$-module, then by Theorem 3.2 $\mathrm{silp}\, \mathbb{Z}G < \infty$ hence there is a positive integer m_0 and an integer m so that λ represents the zero element in $\mathrm{Ext}^{m_0}_{\mathbb{Z}G}(R_m, P)$. Hence we obtain the following commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & F_{n-2} & \longrightarrow \cdots \longrightarrow & F_1 & \longrightarrow & F_0 & \xrightarrow{\searrow R_0 \nearrow} & F_{-1} \longrightarrow \\
\| & & \| & & \downarrow f_{n-1} & & \downarrow f_{n-2} & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f
\end{array}$$

$$\cdots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}$$

where $R_0 = \operatorname{im} \partial_0$.

Clearly $[f] \in \operatorname{Ext}_{\mathbb{Z}G}(R_{-1}, \mathbb{Z})$ and Yoneda product with $[f]$ induces an isomorphism: $\operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, -) \to \operatorname{Ext}^{i+1}_{\mathbb{Z}G}(R_{-1}, -)$. This implies (c.f. [27]) that $[f]$ is represented by an extension $0 \to \mathbb{Z} \to A \to R_{-1} \to 0$ with $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$. The result now follows since R_{-1} is $\mathbb{Z}$-free as a $\mathbb{Z}G$-submodule of a projective $\mathbb{Z}G$-module. □

Theorem 4.2. *[24] The following statements are equivalent for any group G.*

(i) $\operatorname{spli} \mathbb{Z}G < \infty$*;*

(ii) There is a $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$ with A $\mathbb{Z}$-free and $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$*.*

Proof. (i)⇒(ii) is Proposition 4.1.

For (ii)⇒(i). Let I be an injective $\mathbb{Z}G$-module and consider a $\mathbb{Z}G$-projective presentation of I

$$0 \longrightarrow K \longrightarrow P \longrightarrow I \longrightarrow 0.$$

Since A is $\mathbb{Z}$-free we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow 0.$$

By Corollary 2.2 (i), $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} < \infty$ and $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} < \infty$ hence $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} < \infty$, but tensoring $0 \to \mathbb{Z} \to A$ with I we obtain that I is a $\mathbb{Z}G$-direct summand of $\overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A}$, and the result follows. □

The following proposition states some of the properties of such a module A.

Proposition 4.3. *[26] Let G be a group and let $0 \to \mathbb{Z} \to A$ be a $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence with A $\mathbb{Z}$-free and* $\operatorname{proj.dim} A = n$*. Then*

(i) If $\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty$ *and M is $\mathbb{Z}$-free then* $\operatorname{proj.dim}_{\mathbb{Z}G} M \leq n$

(ii) $\operatorname{spli}\mathbb{Z}G \leq n+1$

(iii) For any finite subgroup H of G, $A|_H$ is a projective $\mathbb{Z}H$-module.

Proof. (i) Consider the $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A \to \bar{A} \to 0$. Clearly $\bar{A}$ is $\mathbb{Z}$-free.

Now let $\operatorname{proj.dim}_{\mathbb{Z}G} M = m$. By Corollary 2.2 (i) we have that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{M} \otimes \overset{\searrow}{A} \leq n, m$ and $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{M} \otimes \overset{\searrow}{\bar{A}} \leq m$. It now follows from the long exact Ext-sequence associated to

$$0 \longrightarrow M \longrightarrow \overset{\searrow}{M} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow \overset{\searrow}{M} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{\bar{A}} \longrightarrow 0$$

that if $\operatorname{proj.dim}_{\mathbb{Z}G} M > n$ then $\operatorname{proj.dim}_{\mathbb{Z}G} M \otimes \bar{A} \geq m+1$, which is a contradiction and hence $\operatorname{proj.dim}_{\mathbb{Z}G} M \leq n$.

(ii) Let I be an injective $\mathbb{Z}G$-module and $0 \to K \to P \to I \to 0$ a $\mathbb{Z}G$-projective presentation of I. By Theorem 4.2 $\operatorname{spli}\mathbb{Z}G < \infty$ hence $\operatorname{proj.dim}_{\mathbb{Z}G} K < \infty$ and by (i) $\operatorname{proj.dim}_{\mathbb{Z}G} K \leq n$ which implies that $\operatorname{proj.dim}_{\mathbb{Z}G} I \leq n+1$.

(iii) Since $A|_H$ is $\mathbb{Z}$-free and has $\operatorname{proj.dim}_{\mathbb{Z}H} A < \infty$ it follows that A is a projective $\mathbb{Z}H$-module (c.f. [2], Ch. VI). □

Theorem 4.4. $\operatorname{spli}\mathbb{Z}G < \infty$ *is a Weyl-group closed property i.e. if* $\operatorname{spli}\mathbb{Z}G < \infty$ *and H is a finite subgroup of G then* $\operatorname{spli}\mathbb{Z}(N_G(H)/H) < \infty$.

Proof. Assume that $\operatorname{spli}\mathbb{Z}G < \infty$ and let H be a finite subgroup of G. Let $N = N_G(A)$, then by Proposition 2.3 (iii) $\operatorname{spli}\mathbb{Z}N < \infty$ hence by Theorem 4.2 there is a $\mathbb{Z}$-split $\mathbb{Z}N$-exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \qquad (*)$$

with A $\mathbb{Z}$-free and $\operatorname{proj.dim}_{\mathbb{Z}N} A = n$.

Consider a $\mathbb{Z}N$-projective resolution of A of length n

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0. \qquad (*')$$

Since $A|_H$ is a projective $\mathbb{Z}H$-module, $(*')$ gives rise to the following $\mathbb{Z}(N/H)$-exact sequence

$$0 \longrightarrow P_n^H \longrightarrow P_{n-1}^H \longrightarrow \cdots \longrightarrow P_0^H \longrightarrow A^H \longrightarrow 0.$$

It is not difficult to see that P_i^H are projective $\mathbb{Z}(N/H)$-modules since $\mathbb{Z}N^H \cong \mathbb{Z}(N/H)$ as $\mathbb{Z}(N/H)$-modules, hence $\operatorname{proj.dim}_{\mathbb{Z}(N/H)} A^H \leq n$.

Moreover, $(*)$ gives rise to the $\mathbb{Z}$-split and $\mathbb{Z}(N/H)$-exact sequence $0 \to \mathbb{Z} \to A^H$. Hence by Theorem 4.2 $\operatorname{spli} \mathbb{Z}(N/H) < \infty$. □

5 The classes of groups $H_1\mathcal{F}$ and $\underline{E}G$

A group G belongs to $H_1\mathcal{F}$ if there is a finite dimensional contractible G-CW-complex with finite cell stabilizers.

By a theorem of Serre (see also Exercise in [2], p. 191) it follows that $H_1\mathcal{F}$ contains all groups of finite virtual cohomological dimension.

It also contains infinite torsion groups, for example a countable locally finite group G is in $H_1\mathcal{F}$, since G acts on a tree with finite vertex stabilizers. It was proved in [5] that if G is a locally finite group of cardinality less than N_w then G is in $H_1\mathcal{F}$.

For sufficiently large e it is known [13] that the free Burnside groups of exponent e admit actions on contractible 2-dimensional complexes with cyclic stabilizers, hence these groups are in $H_1\mathcal{F}$.

If G is in $H_1\mathcal{F}$ and X is a finite dimensional contractible G-CW-complex with finite cell stabilizers, then the argumented cellular chain complex of X gives rise to the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \underset{i_n \in I_n}{\otimes} \mathbb{Z}(G/G_{i_n}) \longrightarrow \cdots \longrightarrow \underset{i_0 \in I_0}{\otimes} \mathbb{Z}(G/G_{i_0}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

with G_{i_j} finite for all i_j.

So if G is in $H_1\mathcal{F}$ and G is torsion free then $\operatorname{cd}_{\mathbb{Z}} G < \infty$.

In particular a free abelian group of infinite rank is not in $H_1\mathcal{F}$.

Proposition 5.1. *If G is in $H_1\mathcal{F}$ then*

$$\text{fin.}\dim \mathbb{Z}G = \operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G = k(\mathbb{Z}G) < \infty.$$

Proof. Since G is in $H_1\mathcal{F}$, there is a $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \bigotimes_{i_n \in I_n} \mathbb{Z}(G/G_{i_n}) \longrightarrow \cdots \longrightarrow \bigotimes_{i_0 \in I_0} \mathbb{Z}(G/G_{i_0}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

with G_{i_j} finite subgroups of G for all i_j.

If M is a $\mathbb{Z}G$-module such that $\operatorname{proj.dim}_{\mathbb{Z}H} M|_H < \infty$ for every finite subgroup H of G, and $0 \to K \to P \to M \to 0$ is a $\mathbb{Z}G$-projective presentation of M then $K|_H$ is a projective $\mathbb{Z}H$-module for every finite subgroup H of G.

Hence if we tensor $(*)$ with K we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \bigotimes_{i_n \in I_n} \mathbb{Z}(\overset{\searrow}{G}/G_{i_n}) \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{K} \longrightarrow \cdots \longrightarrow \bigotimes_{i_0 \in I_0} \mathbb{Z}(\overset{\searrow}{G}/G_{i_0}) \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{K} \longrightarrow K \longrightarrow 0$$

which by Proposition 2.1 (i), is a $\mathbb{Z}G$-projective resolution of K, since $K|_H$ is a projective $\mathbb{Z}H$-module for every finite subgroup H of G.

Hence $\operatorname{proj.dim}_{\mathbb{Z}G} K \leq n$ which implies that $k(\mathbb{Z}G) \leq n$. The result now follows from Proposition 3.4. □

In [15] Kropholler and Mislin proved

Theorem A. Every $H\mathcal{F}$-group of type FP_∞ is in $H_1\mathcal{F}$.

A group G is said to be of type FP_∞ if there is a $\mathbb{Z}G$-projective resolution of G

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with P_i finitely generated $\mathbb{Z}G$-modules for all $i \geq 0$.

Notation. If $\mathcal{X}$ is a class of groups, we denote by $\mathcal{X}_b$ the subclass of $\mathcal{X}$ consisting of those groups in $\mathcal{X}$, for which there is a bound on the orders of the finite subgroups.

To prove Theorem A, they first considered the following two properties of $H\mathcal{F}$-groups of type FP_∞, which were both shown using complete cohomology.

• If G is an $H\mathcal{F}$-group of type FP_∞ then G is in $H\mathcal{F}_b$.

In particular, if $|\Lambda(G)|$ is the G-simplicial complex determined by the poset of the non-trivial finite subgroups of G, then $\dim|\Lambda(G)| < \infty$.

• If G is an $H\mathcal{F}$-group of type FP_∞, then $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$, where $B(G,\mathbb{Z})$ is the $\mathbb{Z}G$-module of bounded functions from G to $\mathbb{Z}$.

They then proved, by induction on $\dim|\Lambda(G)|$

Theorem B. If G is an $H\mathcal{F}$-group such that $\dim|\Lambda(G)| < \infty$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ then G is in $H_1\mathcal{F}$.

Clearly Theorem A follows from Theorem B.

Generalizations of this Theorem were obtained in [17], [20], [26].

Note that $B(G,\mathbb{Z})$, the $\mathbb{Z}G$-module of bounded functions from G to $\mathbb{Z}$, is a $\mathbb{Z}$-free $\mathbb{Z}G$-module and there is a $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \xrightarrow{i} B(G,\mathbb{Z})$ where $i(n) : G \to \mathbb{Z}$ is the constant function c_n [16].

By Theorem 4.2 $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ implies that $\operatorname{spli}\mathbb{Z}G < \infty$. Now if G is in $H\mathcal{F}$ then it is known [4] that $\operatorname{spli}\mathbb{Z}G = k(\mathbb{Z}G)$.

So if G is in $H\mathcal{F}$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ then $k(\mathbb{Z}G) < \infty$. It is easy to see that $k(\mathbb{Z}G) < \infty$ is a subgroup closed property and by Theorem 4.4 a Weyl-group closed property [26].

These properties, which are implications of the finiteness of the proj. dim of $B(G,\mathbb{Z})$, for G in $H\mathcal{F}$, are crucial for the proof of Theorem B.

The following Conjecture, (Conj. A in [26]), claims that the finiteness of the algebraic invariants we've studied here, give an algebraic characterization for the class $H_1\mathcal{F}$.

Conjecture A. The following statements are equivalent for a group G:

(1) G is in $H_1\mathcal{F}$;

(2) G is of type Φ;

(3) $\operatorname{spli}\mathbb{Z}G < \infty$;

(4) $\operatorname{silp}\mathbb{Z}G < \infty$;

(5) $\operatorname{fin.dim} \mathbb{Z}G < \infty$,

where, a group G is said to be of type Φ if it has the property that for every $\mathbb{Z}G$-module M, $\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $\operatorname{proj.dim}_{\mathbb{Z}H} M|_H < \infty$ for every finite subgroup H of G.

Note that G is of type Φ it if has the property that for every $\mathbb{Z}G$-module M, $\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $M|_H$ is a cohomologically trivial $\mathbb{Z}H$-module, for every finite subgroup H of G.

Proposition 5.1 shows that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) in Conjecture A.

Kropholler and Mislin's Theorem show essentially that (5) $\Rightarrow$ (1) if G is in $(H\mathcal{F})_b$.

In [26] it was shown that (5) $\Rightarrow$ (1) if G is a torsion-free locally soluble group.

In support of Conj. A is also a result obtained in [5] which says that a group G is finite if and only if $\operatorname{spli} \mathbb{Z}G = 1$. It is worth mentioning that its proof uses the theory of groups acting on trees.

If G is in $H_1\mathcal{F}$ then there is a $\mathbb{Z}G$-resolution of G by direct sums of permutation modules of finite subgroups of G, i.e.

$$0 \longrightarrow \underset{i_n \in I_n}{\otimes} \mathbb{Z}(G/G_{i_n}) \longrightarrow \cdots \longrightarrow \underset{i_0 \in I_0}{\otimes} \mathbb{Z}(G/G_{i_0}) \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad (*)$$

with G_{i_j} finite subgroups of G for all i_j.

It follows from $(*)$ that if G is in $H_1\mathcal{F}$ then $\operatorname{cd}_{\mathbb{Q}} G < \infty$.

It is likely that the existence of $(*)$ is another algebraic characterization for the $H_1\mathcal{F}$-class of groups.

As we mentioned before if G is a group of finite virtual cohomological dimension, $v\underline{\operatorname{cd}}\, G < \infty$ then G is in $H_1\mathcal{F}$, actually G is in $H_1\mathcal{F}_b$.

We consider the class $H_1\mathcal{F}$ or rather the class $H_1\mathcal{F}_b$ as a more "natural class" than the class of groups of finite $v\underline{\operatorname{cd}}$.

The class $H_1\mathcal{F}_b$ is closed under extensions and taking fundamental groups of finite graphs of groups [21] unlike the class of groups of finite $v\underline{\operatorname{cd}}$.

The following example of a group G, which was constructed by Dyer in [8] as a counter example to a conjecture related to residual finiteness, has the following properties

• G is a free product with amalgamation of groups of finite $v\underline{\mathrm{cd}}$,

• G is an extension of a finite group by a group of finite cohomological dimension and yet G is not of finite $v\underline{\mathrm{cd}}$.

$G = A \underset{H,\varphi}{*} B$ where

$$A = < a_1, a_2, a_3, a, d \mid [a_i, a_j] = [a_i, d] = [a, d] = d^p = 1, \\ a_1^a = a_2, a_2^a = a_3, a_3^a = a_1 a_2^{-3} a_3^2 >$$

$$B = < b_1, b_2, b_3, b, e \mid [b_i, b_j] = [b_i, e] = [b, e] = e^p = 1, \\ b_1^b = b_2, b_2^b = b_3, b_3^b = b_1 b_2^{-3} b_3^2 >$$

and

$$H = < a_1, a_2^p, a_3, d > \qquad \varphi(H) = < b_1^p, b_2, b_3^p, e >$$

and

$$\varphi(a_1) = b_1^p e \quad \varphi(a_2^p) = b_2 \quad \varphi(a_2) = b_3^p \quad \varphi(d) = e.$$

Note that $A \cong B \cong (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times C_p) \lhd \mathbb{Z}$, hence $v\mathrm{cd}A = v\mathrm{cd}B = 4$. It follows that $< d > \subseteq \cap\{N | G : N| < \infty\}$ hence G does not have a torsion-free subgroup of finite index.

Moreover we have the group extension

$$1 \longrightarrow < d > \longrightarrow G \longrightarrow K \longrightarrow 1 \qquad (**)$$

where K is a group with $\mathrm{cd}_{\mathbb{Z}} K < \infty$ which implies that the class of groups of finite $v\mathrm{cd}$ is not extension closed.

Now since K is in $H_1\mathcal{F}$ and $< d >$ is finite it follows from $(**)$ that G is in $H_1\mathcal{F}$.

It is worth mentioning that, it is not known whether $H_1\mathcal{F}$ is extension closed.

The class $H_1\mathcal{F}$ is closely related to the class of groups which admit a finite dimensional model for $\underline{E}G$, the classifying space for proper actions.

For every group G, there exists up to G-homotopy a unique G-CW-complex $\underline{E}G$ such that the fixed point space $\underline{E}G^H$ is contractible for every finite subgroup H of G and empty for infinite H. A G-CW-complex is called proper if all point stabilizers are finite (equivalently, if all its G-cells are of the form $G/H \times \sigma$ with H a finite subgroup of G). The space $\underline{E}G$ is an example of a proper G-CW-complex and it is referred to as the classifying space for proper actions, because it has the universal property, "for any proper G-CW-complex X there is a unique G-homotopy class of G-maps $X \to \underline{E}G$".

For a survey on classifying spaces see [18]. It is clear that the class of groups that admit a finite dimensional model for $\underline{E}G$ is a subclass of $H_1\mathcal{F}$.

Kropholler and Mislin in [15] actually proved that if G is an $H\mathcal{F}$-group such that $\dim|\Lambda(G)| < \infty$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ then G admits a finite dimensional model for $\underline{E}G$.

Moreover, it was shown in [26] that the condition (5) of Conjecture A implies that G admits a finite dimensional $\underline{E}G$, if G is a torsion-free locally soluble group. However it is an open question whether the class of groups which admit a finite dimensional $\underline{E}G$ is indeed a proper subclass of $H_1\mathcal{F}$.

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