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Research Article

Highly accurate collocation methodology for solving the generalized Burgers–Fisher's equation

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Abstract

An improvised collocation scheme is applied for the numerical treatment of the nonlinear generalized Burgers–Fisher's (gBF) equation using splines of degree three. In the proposed methodology, some subsequent rectifications are done in the spline interpolant, which resulted in the magnification of the order of convergence along the space direction. A finite difference approach is followed to integrate the time direction. Von Neumann methodology is opted to discuss the stability of the method. The error bounds and convergence study show that the technique has $(s^4 + \Delta t^2)$ order of convergence. The correspondence between the approximate and analytical solutions is shown by graphs, plotted using MATLAB and by evaluating absolute error.

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1 Introduction

Lu et al. [17] found that the generalized Burgers–Fisher's (gBF) problem is an extension of generalized Fisher's equation, which is as mentioned hereunder:

$$v_t = \beta v_{xx} + f(v, v_x), \quad x \in (a,b), \quad t \in (t_0, T). \tag{1}$$

Equation (1) can be expressed in the operator form as mentioned below:

$$L \equiv \beta v_{xx} - v_t + f(v, v_x), \tag{2}$$

with the initial condition as:

$$v = v^0, \quad \text{in} \quad [a,b] \times \{t_0\}, \tag{3}$$

and the boundary conditions as:

$$\mathcal{B}v = \Omega, \quad \text{on} \quad \partial\Phi_x \times [t_0, T], \tag{4}$$

where $f(v, v_x) = -\alpha v^\sigma v_x + \gamma v(1 - v^\sigma)$, $\Phi_x = (a,b)$, $\mathcal{B}$ is the boundary operator defined as $\mathcal{B}v = a_1(x,t)v(x,t) + a_2(x,t)v_x(x,t)$. Here v represents the traveling wave phenomena with $\sigma > 0$ and $T > t_0$. Also α, β, and γ correspond to the convection, diffusion, and reaction coefficients, respectively. With $\sigma = 1$, (1) becomes the Burgers–Fisher's equation given below:

$$v_t + \alpha v v_x = \beta v_{xx} + \gamma v(1 - v), \quad x \in (a,b), \quad t \in (t_0, T). \tag{5}$$

This is known as the Burgers–Fisher's equation because it has convective phenomena from the Burgers' problem, diffusion transport along with reaction characteristics from the Fisher's equation. Thus, it is a blending of convection, diffusion, and reaction mechanisms. The proposed problem was used by Sachdev [26] in self-similarity. When $\gamma = 0$, (5) becomes the Burgers' problem, which was used by Lighthill [16] in the investigation of sound waves in a viscous medium. When $\alpha = 0$, (5) reduces to the modified Fisher's problem, which was used by Murray [23] in mathematical biology.

Since two nonlinear terms occur in (5), therefore analytical methods such as Laplace, Fourier, and other classical approaches to integrate the system become invalid. Due to this, the traveling wave solution of the gBF equation was found by Fan [7] using the extended tanh-function and the Riccati equation. Mickens and Gumel [19] studied the properties of the Burgers–Fisher's problem and worked on its numerical solution using the nonstandard finite difference technique. Kaya and Sayed [15] obtained an explicit series solution of the gBF equation without any transformation and compared it with the numerical solution obtained using the Adomian decomposition technique. This technique was extended by Ismail, Raslan, and Abd Rabboh [12] to analyze the Burgers–Fisher's and Burgers–Huxley's equation. Javidi [13] solved this equation using a combination of pseudospectral Chebyshev and Runge-Kutta fourth-order methods. A variational iteration scheme based on Lagrange multipliers to construct correction functions for the gBF problem was adapted by Moghimi and Hejazi [21]. Wazwaz [38] derived the sets of traveling wave solutions as well as kinks and periodic solutions of the gBF equation using the tanh-coth method. The spectral domain decomposition technique with Chebyshev polynomials for spatial derivatives and RK4 for time integration was used by Golbabai and Javidi [10]. Zhu and Kang [40] applied the B-spline quasi-interpolation technique and opted for forward difference for temporal discretization to solve the Burgers–Fisher's equation. A finite difference technique of sixth-order for space, and the third-order Runge–Kutta method for temporal domain was applied by Sari, Gürarslan, and Dağ [30] for the gBF equation.

Bratsos [2] implemented the finite difference technique of order four for space discretization and a predictor-corrector technique for solving the resulting nonlinear system. Sari [29] adapted the polynomial-based differential quadrature technique for space and the SSP-RK scheme of third-order for time to solve the gBF equation. Tatari, Sepehrian, and Alibakhshi [35] used the collocation method with the radial basis function to solve the system of nonlinear equations by the predictor-corrector method. Zhao et al. [39] used the Legendre–Galerkin formulation for space discretization with Chebyshev–Gauss–Lobatto node points and leapfrog scheme for temporal discretization. Mohammadi [22] used an explicit exponential spline difference scheme for the gBF equation and analyzed convergence, error, and stability properties with the energy method. The limitation of the work was the large computational time. A modified spline collocation technique with the SSPRK-54 scheme was applied by Mittal and Tripathi [20] to analyze the gBF problem. Malik et al. [18] adapted a heuristic genetic algorithm scheme for the gBF equation based on an exp-function hybridization technique.

Chandraker, Awasthi, and Jayaraj [4] applied two implicit finite difference schemes to solve the Burgers–Fisher problem; one was semi-implicit and the other was based on the modified Crank–Nicolson method. Al-Rozbayani and Al-Hayalie [1] applied three different finite difference schemes to solve the Burgers–Fisher's equation. One is an explicit method, the other is an exponential method and the third one is the Du Fort–Frankel method. Hepson [11] implemented an extended B-spline collocation technique to solve the gBF equation. Saeed and Gilani [27] proposed a combination of the CAS wavelet method with a quasi-linearization scheme to solve the gBF equation. Sangwan and Kaur [28] applied a piecewise uniform Shishkin mesh with exponentially fitted splines and for temporal discretization, the implicit Euler method was adopted. The quasilinearization was used to deal with the nonlinear terms. Bratsos and Khaliq [3] adapted an exponential time differencing technique in which a nonlinear system was solved by a second-order modified predictor-corrector scheme.

In this study, we employ an extrapolated collocation algorithm to investigate the gBF equation. This method, previously utilized by Shallu, Kumari, and Kukreja [34, 31, 32], has been successfully applied to solve second-order self-adjoint equations, modified Burgers' equations, as well as RLW and MRLW equations. We enhance the methodology by utilizing improved cubic B-splines for spatial discretization and employing a weighted finite difference method for temporal discretization. These adjustments lead to a notable enhancement in the convergence order in the spatial domain.

The structure of the paper is as follows: In Section 2, we detail the construction and implementation of the improved B-spline collocation methodology for addressing the given problem. Subsequently, we conduct a convergence analysis in the spatial domain. In Section 3, we proceed with the discretization of the temporal domain. Section 4 involves the utilization of the von Neumann method to assess the stability of our proposed approach. In Section 5, we present solved examples to demonstrate the effectiveness of our technique and its superiority over existing data. Finally, in Section 6, we provide a summary of our findings.

2 New cubic B-Spline collocation technique

Consider the uniform subdivision of the Π_x space domain with $s = (b-a)/M$ as the step length of the space domain and $M+1$ is the number of nodal points. The structure of cubic splines $C_{p,3}(x)$ is given in [24]. The numerical solution can be written as follows:

$$W(x,t) = \sum_{p=-1}^{M+1} d_p(t) C_p(x). \tag{6}$$

2.1 Corrections in the second-order derivative

Assume that the spline interpolant $W(x,t)$ fulfills the given constraints:

(I) the interpolatory constraints, for $p = 0, 1, \ldots, M$:

$$W(x_p, t) = v(x_p, t), \tag{7}$$

(II) at the end nodal points, for $p = 0$ and M:

$$W_{xx}(x_p, t) = v_{xx}(x_p, t) - \frac{s^2}{12} v_{xxxx}(x_p, t). \tag{8}$$

Theorem 1. The following relations hold among the cubic spline interpolant (CSI) $W(x,t)$ and the exact solution $v(x,t)$, where $v(x,t)$ satisfy (7) and (8) for $p = 0, 1, \ldots, M$:

$$W_{xx}(x_p,t) = v_{xx}(x_p,t) - \frac{s^2}{12} v_{xxxx}(x_p,t) + O(s^4),$$
$$W_x(x_p,t) = v_x(x_p,t) + O(s^4).$$

In addition,

$$\| W^{(j)} - v^{(j)} \|_\infty = O(s^{4-j}), \quad j = 0,1,2,$$

where $W^{(j)}$ and $v^{(j)}$ represent the jth derivative with respect to "x".

Proof. See [5]. □

Lemma 1. For $v(x,t) \in \mathbb{C}^6[a,b]$, the below mentioned relations hold:

For $p = 0:$

$$v_{xxxx}(x_0,t) = \frac{W_{xx}(x_0,t) - 5W_{xx}(x_1,t) + 4W_{xx}(x_2,t) - W_{xx}(x_3,t)}{x^2} + O(s^2).$$

For $p = 1,2,\ldots,M-1:$

$$v_{xxxx}(x_p,t) = \frac{W_{xx}(x_{p-1},t) - 2W_{xx}(x_p,t) + W_{xx}(x_{p+1},t)}{x^2} + O(s^2).$$

For $p = M:$

$$v_{xxxx}(x_M,t) = \frac{W_{xx}(x_M,t) - 5W_{xx}(x_{M-1},t) + 4W_{xx}(x_{M-2},t) - W_{xx}(x_{M-3},t)}{x^2} + O(s^2).$$

Proof. See [5]. □

Corollary 1. For $v(x,t) \in \mathbb{C}^6[a,b]$, the below given relations hold:

For $p = 0,1,\ldots,M:$

$$v_x(x_p,t) = W_x(x_p,t) + O(s^4),$$

For $p = 0:$

$$v_{xx}(x_0,t) = \frac{14W_{xx}(x_0,t) - 5W_{xx}(x_1,t) + 4W_{xx}(x_2,t) - W_{xx}(x_3,t)}{12} + O(s^4),$$

For $p = 1,2,\ldots,M-1:$

$$v_{xx}(x_p,t) = \frac{W_{xx}(x_{p-1},t) + 10W_{xx}(x_p,t) + W_{xx}(x_{p+1},t)}{12} + O(x^4),$$

For $p = M:$

$v_{xx}(x_M,t)$

$$= \frac{14W_{xx}(x_M,t) - 5W_{xx}(x_{M-1},t) + 4W_{xx}(x_{M-2},t) - W_{xx}(x_{M-3},t)}{12} + O(s^4).$$

Proof. See [5]. □

2.2 System of equations

At the nodal points, (1) can be expressed as follows:

$$\begin{aligned} v_t(x_p,t) &= \beta v_{xx}(x_p,t) + f(v(x_p,t), v_x(x_p,t)), \quad x_p \in [a,b], \\ \mathcal{B}(v(x_p,t)) &= \Omega(v(x_p,t)), \quad x_p \in \partial\Phi_x. \end{aligned}$$

Substituting the values of $v(x_p,t)$, $v_x(x_p,t)$, and $v_{xx}(x_p,t)$ in the above equations and using Corollary 1, we have

$$\begin{aligned} \frac{\partial}{\partial t} W(x_0,t) = \frac{\beta}{12} &[14W_{xx}(x_0,t) - 5W_{xx}(x_1,t) + 4W_{xx}(x_2,t) - W_{xx}(x_3,t)] \\ &+ \mathfrak{f}(W(x_0,t), W_x(x_0,t)) + O(s^4), \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial}{\partial t} W(x_p,t) = \frac{\beta}{12} &[W_{xx}(x_{p-1},t) + 10W_{xx}(x_p,t) + W_{xx}(x_{p+1},t)] \\ &+ \mathfrak{f}(W(x_p,t), W_x(x_p,t)) + O(s^4), \quad p = 1,2,\dots,M-1, \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial}{\partial t} W(x_M,t) = \frac{\beta}{12} &[14W_{xx}(x_M,t) - 5W_{xx}(x_{M-1},t) + 4W_{xx}(x_{M-2},t) \\ &- W_{xx}(x_{M-3},t)] + \mathfrak{f}(W(x_M,t), W_x(x_M,t)) + O(s^4). \end{aligned} \tag{11}$$

and the boundary constraints:

$$a_1(x_p,t)W(x_p,t) + a_2(x_p,t)W_x(x_p,t) = \Omega(x_p,t) + O(s^4), \quad p = 0, M. \tag{12}$$

The above relations form a nonlinear vector initial value problem of first-order with (3) as initial constraint.

2.3 Spatial convergence analysis

Let $\hat{L}$ and $\hat{\mathcal{B}}$ be the perturbation operators of L and $\mathcal{B}$. Then, the below given connections hold:

For $p = 1, 2, \ldots, M-1$:

$$\begin{aligned}
\hat{L}W(x_p,t) &\equiv L[W(x_p,t), W_x(x_p,t), W_{xx}(x_p,t) \\
&\quad + \frac{1}{12}[W_{xx}(x_p,t) - 2W_{xx}(x_p,t) + W_{xx}(x_p,t)], \\
\hat{L}W(x_0,t) &\equiv L[W(x_0,t), W_x(x_0,t), W_{xx}(x_0,t) \\
&\quad + \frac{1}{12}[2W_{xx}(x_0,t) - 5W_{xx}(x_1,t) + 4W_{xx}(x_2,t) - W_{xx}(x_3,t)], \\
\hat{L}W(x_M,t) &\equiv L[W(x_M,t), W_x(x_M,t), W_{xx}(x_M,t) \\
&\quad + \frac{1}{12}[2W_{xx}(x_M,t) - 5W_{xx}(x_{M-1},t) + 4W_{xx}(x_{M-2},t) \\
&\quad - W_{xx}(x_{M-3},t)]. \qquad (13) \\
\hat{\mathcal{B}}W(x_p,t) &= \mathcal{B}W(x_p,t), \quad p = 0, M.
\end{aligned}$$

Thus it is deduced that, for the unique CSI that satisfies (7)–(8), the following mentioned connections hold at the nodal points:

$$\hat{L}W(x_p,t) = O(s^4), \quad p = 0, 1, \ldots, M; \quad \hat{\mathcal{B}}W(x_p,t) = O(h^4), \quad p = 0, M.$$

The purpose is to find a cubic spline solution $\hat{v}(x,t)$, such that

$$\hat{L}\hat{v}(x_p,t) = 0, \quad p = 0, 1, \ldots, M; \quad \hat{\mathcal{B}}\hat{v}(x_p,t) = 0, \quad p = 0, M. \qquad (14)$$

Next, Green's function is applied for the establishment of error bounds.

Lemma 2. The coefficient matrix of $v_{xx} = g(x,t)$ having homogeneous boundary constraints has inverse with finite norm.

Proof. Using the steps of formation, the coefficient matrix $\beth$ of the equation $v_{xx} = g(x,t)$ is as mentioned below by using (13):

$$\gimel = \frac{1}{12}\begin{bmatrix} 14 & -5 & 4 & -1 & 0 & \dots & 0 \\ 1 & 10 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 10 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 1 & 10 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 10 & 1 \\ 0 & \dots & 0 & -1 & 4 & -5 & 14 \end{bmatrix}$$

Due to the diagonal dominance behavior of matrix, it is invertible. Moreover,

$$\|\gimel^{-1}\|_\infty \leq \max_{0\leq p\leq M} \frac{1}{\Delta_p \gimel},$$

where

$$\Delta_p \gimel = \mid \gimel_{pp} \mid - \sum_{j\neq p} \mid \gimel_{pj} \mid > 0 \quad \text{for } p = 0, 1, \dots, M.$$

So,

$$\|\gimel^{-1}\|_\infty \leq \frac{1}{\min_{0\leq p\leq M} \Delta_p(\gimel)} = \frac{12}{14-(5+4+1)} = 3.$$

Now, onwards $W^{(j)}$, $v^{(j)}$, and $\hat{v}^{(j)}$ are the jth differentiation with respect to space variable. Let $\hat{\gimel}$ denote the coefficient matrix of $W^{(1)}(x,t)$ in (9)–(12), that is, $\hat{\gimel} = diag(-\frac{5}{h}, 0, \frac{5}{h})$, which is invertible with finite norm. Since the boundary value problem of the form (1) with the boundary constraints (4) can be transformed into the Fredholm integral equation of order two. Let $v^{(2)} = z$ and $\hat{v}^{(2)} = w$ such that z and w fulfill the boundary constraints (4). Then v and $\hat{v}$ can be rebuilt by Green's function as

$$v^{(j)}(x,t) = \int_a^b \frac{\partial^j \mathcal{G}(x,t,r)}{\partial x^j} z(r,t)dr, \quad j = 0, 1,$$
$$\hat{v}^{(j)}(x,t) = \int_a^b \frac{\partial^j \mathcal{G}(x,t,r)}{\partial x^j} w(r,t)dr, \quad j = 0, 1.$$

Let $\delta(x,t)$ be any continuously differentiable function. The operators that are necessary for the establishment of the convergence analysis are given below:

$$\mathcal{A} : \mathbb{C}[a,b] \longrightarrow \mathbb{C}[a,b] \quad \text{such that} \quad \mathcal{A}\delta = \frac{1}{\beta}(G_0\delta_t - f(x,t,G_0\delta, G_1\delta)),$$

where $G_j\delta = \int_a^b \frac{\partial^j \mathcal{G}(x,t,r)}{\partial x^j \delta(r,t)dr}$, $j = 0,1$ are the operators from $[a,b]$ to $[a,b]$. Let $\mathcal{D}$ represent the piecewise linear interpolation operator at the points $\{(x_p,t)\}_{p=0}^{M}$. Let $\mathcal{S}$ be the following projection operator:

$$\mathcal{S} : \mathbb{C}[a,b] \longrightarrow \mathcal{R}^{M+1} \quad \text{such that} \quad \mathcal{S}\delta = [\delta(x_0,t), \delta(x_1,t), \dots, \delta(x_M,t)]^T.$$
$$\mathcal{E} : \mathbb{C}[a,b] \longrightarrow \mathbb{C}[a,b], \quad \text{such that} \quad \mathcal{E}\delta = [\mathcal{E}_0\delta, \mathcal{E}_1\delta, \dots, \mathcal{E}_M\delta]^T,$$

where $\mathcal{E}_p\delta = \frac{1}{\beta}(G_0\delta_t - f(x,t,G_0\delta,\mathcal{E}_p\mathcal{S}G_1\delta))$, in which $\mathcal{E}_p$ denotes the pth row of the coefficient matrix of $v_x(x,t)$. Using above definitions, (1) and (14) can be written as follows:

$$(I-\mathcal{A})z = 0,$$
$$(\daleth\mathcal{S} - \mathcal{E})w = 0. \tag{15}$$

Since $\daleth$ is an invertible, so

$$(\mathcal{S} - \daleth^{-1}\mathcal{E})w = 0.$$

Since w is a linear polynomial, therefore $\mathcal{D}\mathcal{S}w = w$ and

$$(I - \mathcal{D}\daleth^{-1}\mathcal{E})w = 0. \tag{16}$$

□

Lemma 3. For the uniform partition of $[a,b]$, $\| \mathcal{D}\daleth^{-1}\mathcal{E}\delta - \mathcal{A}\delta \|_\infty \to 0$ as $s \to 0$.

Proof. Th proof holds as follows:

$$\begin{aligned}\| \mathcal{D}\daleth^{-1}\mathcal{E}\delta - \mathcal{A}\delta \|_\infty &\leq \| \mathcal{D}\daleth^{-1}\mathcal{E}\delta - \mathcal{D}\mathcal{S}\mathcal{A}\delta \|_\infty + \| \mathcal{D}\mathcal{S}\mathcal{A}\delta - \mathcal{A}\delta \|_\infty \\ &\leq \| \mathcal{D} \|_\infty \| \daleth^{-1} \|_\infty \| \mathcal{E}\delta - \daleth\mathcal{S}\mathcal{A}\delta \|_\infty + \| \mathcal{D}\mathcal{S}\mathcal{A}\delta - \mathcal{A}\delta \|_\infty \\ &\leq \| \mathcal{E}\delta - \daleth\mathcal{S}\mathcal{A}\delta \|_\infty + O(s^2).\end{aligned}$$

□

Theorem 2 (see [6])**.** Contemplate the curve $\mathrm{C} = (x,t,v,v_x) \in \mathbb{R}^4$, where $(x,t) \in [a,b] \times [t_0,T]$, and let $v(x,t) \in \mathbb{C}^6[a,b]$ represent the solution of the given equation (1) with the boundary constraint (4), let $f(u,y)$ be adequately smooth near v, and let the hereunder linear problem,

$$v_{xx} - \frac{\partial}{\partial y}\frac{1}{\beta}(u_t - f(u,y))v_x - \frac{\partial}{\partial u}(u_t - f(u,y))v = 0$$

with the boundary constraints (4) be distinctively solvable and acquire Green's function $\mathcal{G}(x,t,r)$. Then, there exist $\epsilon,\ \eta > 0$ (constants) such that

(I) there is no other solution $\hat{w}$ of equation (1) with boundary constraint (4) satisfying $\| v_{xx} - \hat{w}_{xx} \| < \eta$,

(II) for $s < \epsilon$, (16) has a unique spline approximate solution $W(x,.)$ in the same neighborhood of v.

(III) the Newton's method converges in the neighborhood of v for $s < \epsilon$ quadratically, which is used to solve (16).

Theorem 3. Let the presumption of Theorem 2 agree. Then the below given error bound exists:

$$\begin{aligned} &\| v^{(j)}(x,\cdot) - \hat{v}^{(j}(x,.) \|_\infty = O(s^{4-j}), \quad j = 0,1,2. \\ &| v^{(j)}(x,\cdot) - \hat{v}^{(j)}(x,\cdot) |_{x_p} = O(s^4), \quad j = 0,1. \\ &| v^{(2)}(x,\cdot) - \hat{v}^{(2)}(x,\cdot) |_{x_p} = O(s^2). \end{aligned}$$

Proof. Consider the equation $W^{(2)} = \hat{\mu}$, $\hat{\mathcal{B}}W = O(s^4)$. Then there exists a linear polynomial $\bar{w}$ by using Theorem 2, such that

$$\hat{\mathcal{B}}\bar{w} = \hat{\mathcal{B}}W = O(s^4), \quad \| \bar{w}^{(j)} \|_\infty = O(s^4), \quad j = 0,1.$$

Since $(W^{(2)} - \bar{w}^{(2)}) = \hat{\mu}$, $\hat{\mathcal{B}}(W - \bar{w}) = 0$ has a unique solution. Therefore using Theorem 2, we have

$$(I - \mathcal{D}\daleth^{-1}\mathcal{E})(W^{(2)} - \bar{w}^{(2)}) = O(s^4). \tag{17}$$

Deducting (16) from (17), we have

$$(I - \mathcal{D}\daleth^{-1}\mathcal{E})(W^{(2)} - \bar{w}^{(2)} - \hat{v}^{(2)}) = O(s^4).$$

Since $(I - \mathcal{D}\daleth^{-1}\mathcal{E})$ is bounded,

$$\| W^{(2)} - \bar{w}^{(2)} - \hat{v}^{(2)} \|_\infty = O(s^4).$$

The equation $(W - \bar{w} - \hat{v})^{(2)} = \bar{\eta}$, $\hat{\mathcal{B}}(\mathcal{W} - \trianglelefteq - \trianglerighteq) = 0$ has unique solution, hence it assures the existence of Green's function such that,

$$| (W - \bar{w} - \hat{v})^{(j)} | = \int_a^b \frac{\partial^j \mathcal{G}(x,t,r)}{\partial x^j} (W^{(2)} - \bar{w}^{(2)} - \hat{v}^{(2)}) dr, \quad j = 0, 1.$$

Thus,

$$\| (W - \bar{w} - \hat{v})^{(j)} \|_\infty = O(s^4), \quad j = 0, 1.$$

So,

$$\| (W - \hat{v})^{(j)} \|_\infty \leq \| (W - \bar{w} - \hat{v})^{(j)} \|_\infty + \| \bar{w}^{(j)} \|_\infty = O(s^4), \quad j = 0, 1, 2. \tag{18}$$

Using Theorem 1, (18), and the triangular inequality implies

$$\| (v - \hat{v})^{(j)} \|_\infty \leq \| (v - W)^{(j)} \|_\infty + \| (W - \hat{v})^{(j)} \|_\infty = O(s^{4-j}), \quad j = 0, 1, 2.$$

□

3 Time discretization

Substitute the approximate values of v, v_x, and v_{xx} in (1), which leads to an initial value problem system as follows:

$$\mathcal{Q}_1 \frac{d}{dt} \mathcal{C}(t) = \frac{1}{h^2} \mathcal{Q}_2 B \mathcal{C}(t) + \mathfrak{F}(t, \mathcal{C}(t)) \quad t \in (t_0, T), \tag{19}$$

with the initial constraint

$$\mathcal{Q}_1 \mathcal{C}(t_0) = v^0, \tag{20}$$

where $\mathcal{Q}_1 = tri[1, 4, 1]$ is a three diagonal matrix, $\mathcal{C}(t) = [d_0(t), d_1(t), \ldots, d_M(t)]^T$, $\mathfrak{F}(t, \mathcal{C}(t))$ is a column vector with the elements $f(t, \mathcal{Q}_{1j}\mathcal{C}(t), \mathcal{Q}_{3j}\mathcal{C}(t))$, $v^0 = [v(x_0, t_0), v(x_1, t_0), \ldots, v(x_M, t_0)]^T$, where $B = diag[\beta]$, $\mathcal{Q}_3 = \frac{1}{3h} tri[-1, 0, 1]$ and the matrix $\mathcal{Q}_2$ is as follows:

$$\mathcal{Q}_2 = \frac{1}{2}\begin{bmatrix} 14 & -33 & 28 & -14 & 6 & -1 & 0 & \dots & 0 \\ 1 & 8 & -18 & 8 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 8 & -18 & 8 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 1 & 8 & -18 & 8 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 8 & -18 & 8 & 1 \\ 0 & \dots & 0 & -1 & 6 & -14 & 28 & -33 & 14 \end{bmatrix}$$

Consider the uniform division of the time domain as $\Gamma_t \equiv \{t_i\}_{i=0}^{m}$ of $[t_0, T]$ with the temporal step size $\Delta t = t^{n+1} - t^n$. Use the weighted finite difference method to discretize (19) as used in [9], with Θ as a parameter and identity matrix I, we have

$$\mathcal{Q}_1 I \frac{\sigma_t}{\Delta t(1-\Theta\sigma_t)}\mathcal{C}^n = \frac{1}{h^2}\mathcal{Q}_2 B\mathcal{C}^n + \mathfrak{F}(t,\mathcal{C}^n), \quad n = 1,2,\dots,$$

$$\left[\mathcal{Q}_1 I - \frac{\Delta t}{h^2}(1-\Theta)\mathcal{Q}_2 BI\right]\mathcal{C}^n - \Delta t(1-\Theta)\mathfrak{F}^n$$

$$= \left[\mathcal{Q}_1 I + \frac{\Delta t}{h^2}\Theta\mathcal{Q}_2 BI\right]\mathcal{C}^{n-1} + \Delta t\Theta\mathfrak{F}^{n-1}, \quad n = 1,2,\dots, \tag{21}$$

with the following initial constraint:

$$\mathcal{Q}_1\mathcal{C}^0 = v^0.$$

Using the initial constraint, obtain the value of $\mathcal{C}^0$, and using (21), the value of $\mathcal{C}$ can be computed at every successive time level.

Lemma 4. Let $v(\cdot,t) \in \mathbb{C}^3[t_0,T]$ be the exact solution of (1). For $\Theta = \frac{1}{2}$, the time integration methodology has order two, and for $\Theta \in \left(\frac{1}{2}, 1\right]$, it has order one of convergence.

Proof. The proof is given in [14, Theorem 2]. □

Hence, the method is fourth-order convergent in the space and second-order convergent in the time direction for $\Theta = \frac{1}{2}$.

4 Stability analysis

Von Neumann technique is used to analyze the stability technique. Take v as a local constant $P = \max(v)$ and integrate the time domain using a weighted finite difference methodology with $\Theta = \frac{1}{2}$. We get

$$\frac{v_p^{n+1} - v_p^n}{\Delta t} + \alpha P^\sigma \left[\frac{(v_x)_p^{n+1} + (v_x)_p^n}{2} \right] = \beta \frac{(v_{xx})_p^{n+1} + (v_{xx})_p^n}{2} + \gamma(1 - P^\sigma) \left[\frac{v_p^{n+1} + v_p^n}{2} \right].$$

Expressing the $(n+1)$th level in terms of nth time level terms, we have

$$\left(\frac{1}{\Delta t} + \frac{\gamma(P^\sigma - 1)}{2} \right) v_p^{n+1} + \frac{\alpha P^\sigma}{2} (v_x)_p^{n+1} - \frac{\beta}{2} (v_{xx})_p^{n+1} = \left(\frac{1}{\Delta t} - \frac{\gamma(P^\sigma - 1)}{2} \right) v_p^n - \frac{\alpha p}{2} (v_x)_p^n + \frac{\beta}{2} (v_{xx})_p^n. \tag{22}$$

Let

$$e_1 = \frac{1}{\Delta t} + \frac{\gamma(P^\sigma - 1)}{2}; \quad q_1 = \frac{\alpha P^\sigma}{2}; \quad e_2 = \frac{1}{\Delta t} - \frac{\gamma(P^\sigma - 1)}{2}; \quad q_2 = -\frac{\alpha P^\sigma}{2}.$$

With the above substitution, (22) becomes

$$e_1 v_p^{n+1} + q_1 (v_x)_p^{n+1} - \frac{\beta}{2} (v_{xx})_p^{n+1} = e_2 v_p^n + q_2 (v_x)_p^n + \frac{\beta}{2} (v_{xx})_p^n.$$

Substituting the values v, v_x, and v_{xx} and using the improvised cubic B-splines imply

$$e_1(d_{p-1}^{n+1} + 4d_p^{n+1} + d_{p+1}^{n+1}) - \frac{3q_1}{h}(d_{p-1}^{n+1} - d_{p+1}^{n+1}) - \frac{\beta}{4h^2}(d_{p-2}^{n+1} + 8d_{p-1}^{n+1} - 18d_p^{n+1} + 8d_{p+1}^{n+1} + d_{p+2}^{n+1}) = e_2(d_{p-1}^n + 4d_p^n + d_{p+1}^n) - \frac{3q_2}{h}(d_{p-1}^n - d_{p+1}^n) \tag{23}$$

$$+ \frac{\beta}{4h^2}(d_{p-2}^n + 8d_{p-1}^n - 18d_p^n + 8d_{p+1}^n + d_{p+2}^n). \tag{24}$$

After simplifying, (24) becomes

$$-\frac{\beta}{4s^2} d_{p-2}^{n+1} + \left(e_1 - \frac{3q_1}{s} - \frac{2\beta}{s^2} \right) d_{p-1}^{n+1} + \left(4e_1 + \frac{9\beta}{2s^2} \right) d_p^{n+1}$$

$$
\begin{aligned}
&+\left(e_1+\frac{3q_1}{s}-\frac{2\beta}{s^2}\right)d_{p+1}^{n+1}-\frac{\beta}{4s^2}d_{p+2}^{n+1}\\
&\quad=\frac{\beta}{4s^2}d_{p-2}^{n}+\left(e_2-\frac{3q_2}{s}+\frac{2\beta}{s^2}\right)d_{p-1}^{n}+\left(4e_2-\frac{9\beta}{2s^2}\right)d_{p}^{n}\\
&\qquad+\left(e_2+\frac{3q_2}{s}+\frac{2\beta}{s^2}\right)d_{p+1}^{n}+\frac{\beta}{4s^2}d_{p+2}^{n}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&u_1d_{p-2}^{n+1}+u_2d_{p-1}^{n+1}+u_3d_{p}^{n+1}+u_4d_{p+1}^{n+1}+u_1d_{p+2}^{n+1}\\
&\quad=-u_1d_{p-2}^{n}+u_5d_{p-1}^{n}+u_6d_{p}^{n}+u_7d_{p+1}^{n}-u_1d_{p+2}^{n},
\end{aligned}
$$

where

$$
\begin{aligned}
&u_1=-\frac{\beta}{4s^2};\quad u_2=e_1-\frac{3q_1}{s}-\frac{2\beta}{s^2};\quad u_3=4e_1+\frac{9\beta}{2s^2};\quad u_4=e_1+\frac{3q_1}{s}-\frac{2\beta}{s^2};\\
&u_5=e_2-\frac{3q_2}{s}+\frac{2\beta}{s^2};\quad u_6=4e_2-\frac{9\beta}{2s^2};\quad u_7=e_2+\frac{3q_2}{s}+\frac{2\beta}{s^2}.
\end{aligned}
$$

Put $d_p^n=E\eta^n\exp(ip\varphi s)$, where i is the iota, E is the amplitude, and φ is the mode number. We have

$$
\begin{aligned}
\eta&=\frac{-u_1\exp(-2i\varphi s)+u_5\exp(-i\varphi s)+u_6+u_7\exp(i\varphi s)-u_1\exp(2i\varphi s)}{u_1\exp(-2i\varphi s)+u_2\exp(-i\varphi s)+u_3+u_4\exp(i\varphi s)+u_1\exp(2i\varphi s)}\\
&=\frac{-2u_1\cos(2\varphi s)+u_6+(u_5+u_7)\cos(\varphi s)+i(u_7-u_5)\sin(\varphi s)}{2u_1\cos(2\varphi s)+u_3+(u_2+u_4)\cos(\varphi s)+i(u_4-u_2)\sin(\varphi s)}\\
&=\frac{A_1+iB_1}{A_2+iB_2},
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=\frac{\beta}{2s^2}\cos(2\varphi s)+\left(2e_2+\frac{4\beta}{s^2}\right)\cos(\varphi s)+4e_2-\frac{9\beta}{2s^2};\\
B_1&=\frac{6q_2}{s}\sin(\varphi s);\\
A_2&=-\frac{\beta}{2s^2}\cos(2\varphi s)+\left(2e_1-\frac{4\beta}{s^2}\right)\cos(\varphi s)+4e_1+\frac{9\beta}{2s^2};\\
B_2&=\frac{6q_1}{s}\sin(\varphi s).
\end{aligned}
$$

We prove $|\eta|\le 1$, that is, $A_1^2+B_1^2\le A_2^2+B_2^2$ for the stability of the technique. As $q_1=-q_2$ therefore $B_1^2=B_2^2$. Next, to prove that $A_2\ge A_1$, that is, $A_2-A_1\ge 0$, we have

$$A_2 - A_1 = -\beta\frac{\cos(2\varphi s)}{s^2} + \left(2(e_1-e_2) - \frac{8\beta}{s^2}\right)\cos(\varphi s) + 4(e_1-e_2) + \frac{9\beta}{s^2};$$
$$= -\frac{2\beta}{s^2}\cos^2(\varphi s) + \left(2(e_1-e_2) - \frac{8\beta}{s^2}\right)\cos(\varphi s) + 4(e_1-e_2) + \frac{10\beta}{s^2}. \tag{25}$$

For minimum possible value of $A_2 - A_1$, take $\cos(\varphi s) = 1$. So, $A_2 - A_1 = 6(e_1 - e_2) \geq 0$. Hence the technique is unconditionally stable.

5 Numerical examples

In this portion, the gBF problem is analyzed numerically for distinct values of $\alpha,\ \beta,\ \gamma$, and σ. The numerical results are represented using tabular form as well as figures and are contrasted with the outcomes in literature as well as with its solitary wave solution. The difference in the results is shown by calculating absolute error defined as

$$\epsilon = |(v_{exact})_p^m - (v_{num})_p^m|; \quad p = 0, 1, 2, \ldots, M,$$

where $(v_{exact})_p^m$ and $(v_{num})_p^m$ are the exact and improved B-spline solutions of degree three, respectively, at the node point x_p.

The solitary wave solution of (1) is given by Wazwaz [37] as follows:

$$v(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\alpha\sigma}{2\beta(\sigma+1)}\left(x - \left(\frac{\alpha}{\sigma+1} + \frac{\beta\gamma(\sigma+1)}{\alpha}\right)t\right)\right]\right]^{\frac{1}{\sigma}}, \tag{26}$$

with the below given initial constraints:

$$v(x,0) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\alpha\sigma x}{2\beta(\sigma+1)}\right]\right]^{\frac{1}{\sigma}},$$

and the boundary constraints,

$$v(0,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{\alpha\sigma}{2\beta(\sigma+1)}\left(\frac{\alpha}{\sigma+1} + \frac{\beta\gamma(\sigma+1)}{\alpha}\right)t\right]\right]^{\frac{1}{\sigma}},$$
$$v(1,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\alpha\sigma}{2\beta(\sigma+1)}\left(1 - \left(\frac{\alpha}{\sigma+1} + \frac{\beta\gamma(\sigma+1)}{\alpha}\right)t\right)\right]\right]^{\frac{1}{\sigma}}.$$

Example 1. Consider the gBF equation (1) in the domain $[0,1]$ with $\alpha = 0.001$, $\beta = 1$, and $\gamma = 0.001$ as follows:

$$v_t + 0.001v^\sigma v_x = v_{xx} + 0.001v(1 - v^\sigma).$$

The solitary wave solution is given in (26). Table 1 represents the contrast of absolute error with $s = 0.1$ and $\Delta t = 0.0001$ for $t = 0.001, 0.01, 100$, and $\sigma = 1, 4$. The contrast shows that results are superior to the Adomian decomposition scheme [12], compact finite difference method [30], and exponential time differencing method [3]. The CPU time required to compute the absolute error at $t = 0.001$ is 0.043872 sec, at $t = 0.01$, it is 0.053391 sec, and at $t = 100$, it is 6.489983 sec. Figure 1 shows the resemblance between a solitary wave and the approximate solution at distinct times, and Figure 2 represents the three-dimensional surface plot of the approximate solution.

Example 2. Consider the gBF equation (1) in the domain $[0,1]$ with $\alpha = 1$, $\beta = 1$, and $\gamma = 1$ as follows:

$$v_t + v^\sigma v_x = v_{xx} + v(1 - v^\sigma).$$

Table 2 gives the absolute error at distinct times with $s = 0.1$ and $\Delta t = 0.0001$ for $\sigma = 2, 8$. This table demonstrates that results are highly accurate as compared to many existing techniques [12, 30, 3]. The CPU time required to compute the absolute error at $t = 0.0005$ is 0.037888 sec, and at $t = 0.001$, it is 0.043125 sec. The solitary wave behavior and the numerical solution are also represented by graphs. Figure 3 gives the comparison between solitary wave and approximate solution at distinct times and depicts the similarity between them. Figure 4 represents the three-dimensional surface plot of the approximate solution.

Example 3. Consider the gBF equation (1) in the domain $[0,1]$ with $\beta = 1$, $\alpha = 1$, and $\gamma = 0$ as follows:

$$v_t + v^\sigma v_x = v_{xx}.$$

The absolute error at distinct times and different spatial domain points with $h = 0.1$ and $\Delta t = 0.0001$ for $\sigma = 1, 2, 3$ is given in Table 3. Results are found to be more superior as compared to [12, 3]. The CPU time required to compute the absolute error at $t = 0.001$ is 0.042149 sec, and at $t = 2.0$, it is 6.187059 sec. Figure 5 gives the comparison between solitary wave and approximate solution at distinct times and depicts the similarity between them. Figure 6 represents the three-dimensional surface plot of the approximate solution.

Example 4. Consider the gBF equation (1) in the domain $[0, 1]$ with $\alpha = 0.1$, $\beta = 1$, and $\gamma = -0.0025$ as follows:

$$v_t + 0.1v^\sigma v_x = v_{xx} - 0.0025v(1 - v^\sigma).$$

Table 4 shows the absolute error with space step size $s = 0.1$ and time step size $\Delta t = 0.0001$ for $\sigma = 2, 4$, and 8. From the comparison it is clear that the results with the proposed methodology are superior to many other existing techniques used in [30], [3]. The CPU time required to compute the absolute error at $t = 0.1$ is 0.627901 sec, at $t = 0.5$, it is 0.237965 sec, and at $t = 2.0$, it is 6.489983 sec. Figure 7 demonstrates the resemblance between solitary wave and numerical solution at distinct times, and Figure 8 represents the three-dimensional surface plot of the approximate solution.

6 Conclusion

The gBF problem has been investigated using an innovative approach employing a three-degree spline collocation methodology. Through enhancements made to standard splines, we have achieved significantly improved accuracy in the approximate solution, accompanied by a notable reduction in absolute error. Our improvised methodology demonstrates a convergence order of four in the spatial domain and two in the temporal domain. These results surpass the performance of various established techniques, including the compact finite difference technique, exponential time differencing method, and Adomian decomposition scheme, among others. Furthermore, our method showcases computational efficiency across a range of relevant examples.

Table 1: Absolute error comparison of Example 1 with $\alpha = 0.001$, $s = 0.1$ and $\Delta t = 0.0001$, $\gamma = 0.001$

		$\sigma = 1$					$\sigma = 4$			
t	x	ICSCM	[12]	[30]	[3]	[36]	ICSCM	[30]	[3]	[36]
0.001	0.1	5.21E-15	1.940e-6	1.010e-7	1.150e-8	2.50e-8	1.77e-15	1.75e-8	7.71e-9	4.20e-8
	0.5	1.66E-16	1.940e-6	1.040e-7	3.070e-13	2.50e-8	3.33e-16	1.75e-8	2.07e-13	4.20e-8
	0.9	5.55E-17	1.940e-6	1.010e-7	1.150e-8	2.50e-8	1.11e-16	1.75e-8	7.71e-9	4.20e-8
0.010	0.1	3.25E-14	1.940e-5	7.530e-7	6.020e-8	2.50e-8	5.66e-15	1.27e-6	4.05e-8	4.20e-8
	0.5	1.11E-16	1.940e-5	1.040e-6	8.960e-13	2.50e-8	6.66e-16	1.75e-6	5.56e-13	4.20e-8
	0.9	5.55E-17	1.940e-5	7.530e-7	6.020e-8	2.50e-8	1.11e-16	1.27e-6	4.05e-8	4.20e-8
100	0.1	1.52E-14	-	7.530e-7	1.010e-7	2.50e-8	4.42e-14	-	5.73e-8	4.20e-8
	0.5	2.59E-14	-	1.040e-6	1.500e-11	2.50e-8	1.03e-14	-	3.51e-12	4.20e-8
	0.9	5.55E-15	-	7.530e-7	1.010e-7	2.50e-8	1.66e-15	-	5.73e-8	4.20e-8

Table 2: Absolute error comparison of Example 2 with $\alpha = 1$, $s = 0.1$, $\gamma = 1$, and $\Delta t = 0.0001$

		$\sigma = 2$					$\sigma = 8$			
t	x	ICSCM	[12]	[30]	[3]	[36]	ICSCM	[30]	[3]	[36]
0.0005	0.1	5.15e-11	1.40e-3	7.62e-5	5.67e-6	3.98e-5	2.4139e-9	1.02e-4	2.44e-6	5.16e-5
	0.5	1.35e-12	1.35e-3	9.14e-5	5.75e-9	4.15e-5	5.5742e-12	1.37e-4	1.82e-10	6.08e-5
	0.9	2.08e-11	1.28e-3	1.02e-4	5.95e-6	4.22e-5	1.6251e-9	1.69e-4	3.15e-6	6.91e-5
0.0010	0.1	1.03e-11	2.80e-3	1.50e-4	1.08e-5	3.97e-5	3.4743e-10	2.00e-4	4.65e-6	5.15e-5
	0.5	2.05e-12	2.69e-3	1.83e-4	1.15e-8	4.11e-5	2.8770e-11	2.74e-4	4.02e-10	6.09e-5
	0.9	2.40e-12	2.55e-3	2.00e-4	1.14e-5	4.16e-5	2.3789e-10	3.31e-4	6.00e-6	6.94e-5

Table 3: Absolute error comparison of Example 3 with $\alpha = 1$, $s = 0.1$, $\gamma = 0$, and $\Delta t = 0.0001$

σ	t	x	ICSCM	Ismail [12]	Bratsos [3]	t	ICSCM	Bratsos [3]
1	2	0.1	2.5046e-10	6.43e-5	9.82e-5	20	8.6084e-12	2.65e-6
		0.5	3.7073e-10	6.07e-5	1.45e-5		1.4161e-11	1.45e-7
		0.9	3.2483e-10	4.75e-5	9.29e-5		1.3391e-11	4.05e-6
2	2	0.1	1.1286e-10	1.19e-5	8.34e-5	20	1.1906e-12	2.96e-6
		0.5	6.8167e-10	1.50e-5	4.19e-6		5.1861e-11	7.14e-7
		0.9	1.2681e-10	1.44e-5	9.48e-5		1.2913e-11	5.66e-6
3	0.001	0.1	1.1582e-9	4.44e-4	9.10e-6	10	5.0684e-10	2.46e-5
		0.5	8.0534e-12	1.85e-3	6.75e-9		6.6355e-10	5.11e-6
		0.9	6.8473e-10	9.05e-4	1.09e-5		5.5408e-10	4.35e-5

Table 4: Absolute error comparison of Example 4 with $\alpha = 0.1$, $s = 0.1$, $\gamma = -0.0025$, and $\Delta t = 0.0001$

		$\sigma = 2$			$\sigma = 4$			$\sigma = 8$		
t	x	ICSCM	[30]	[3]	ICSCM	[30]	[3]	ICSCM	[30]	[3]
0.1	0.1	5.040e-14	1.210e-5	9.470e-6	4.787e-13	1.340e-5	6.760e-6	3.140e-12	1.470e-5	4.090e-6
	0.5	1.221e-15	2.900e-5	2.740e-8	6.994e-15	3.490e-5	1.030e-8	6.362e-14	3.830e-5	1.840e-8
	0.9	3.197e-14	1.540e-5	9.570e-6	4.969e-13	1.390e-5	6.920e-8	3.041e-12	1.530e-5	4.240e-6
0.5	0.1	4.252e-14	1.670e-5	9.580e-6	5.746e-13	2.000e-5	6.830e-6	3.148e-12	2.200e-5	4.140e-6
	0.5	2.109e-15	4.690e-5	5.180e-8	2.420e-14	5.640e-5	1.930e-8	2.331e-14	6.220e-5	3.470e-8
	0.9	3.442e-14	1.710e-5	9.660e-6	5.044e-13	2.070e-5	7.010e-6	3.068e-12	2.280e-5	4.300e-6
2.0	0.1	6.051e-14	-	9.590e-6	5.369e-13	-	6.860e-6	3.116e-12	-	4.200e-6
	0.5	3.553e-15	-	5.260e-8	5.329e-15	-	1.890e-8	3.919e-14	-	3.450e-8
	0.9	3.186e-14	-	9.670e-6	4.998e-13	-	7.040e-6	3.069e-12	-	4.350e-6

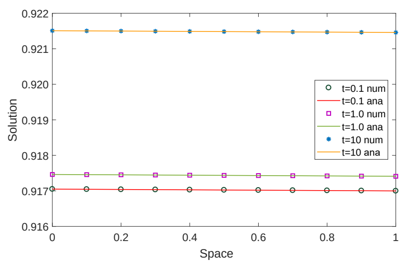

Figure 1: Solution of Example 1 at distinct times with $s = 0.1$, $\Delta t = 0.0001$, and $\sigma = 8$.

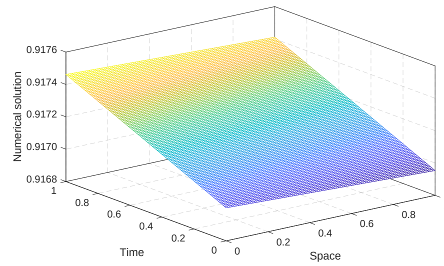

Figure 2: Three-dimensional representation of numerical solution of Example 1 with $s = 0.01$, $\Delta t = 0.001$, and $\sigma = 8$.

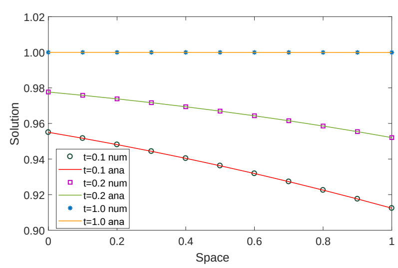

Figure 3: Solution of Example 2 at distinct times with $s = 0.1$, $\Delta t = 0.0001$, and $\sigma = 8$.

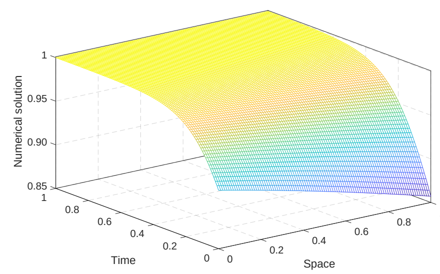

Figure 4: Three-dimensional representation of numerical solution of Example 2 with $s = 0.01$, $\Delta t = 0.001$, $\sigma = 8$.

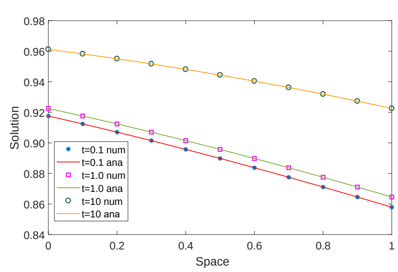

Figure 5: Solution of Example 3 at distinct times with $s = 0.1$, $\Delta t = 0.0001$, and $\sigma = 8$.

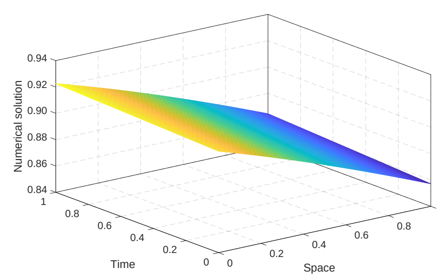

Figure 6: Three-dimensional representation of numerical solution of Example 3 with $s = 0.01$, $\Delta t = 0.001$, and $\sigma = 8$.

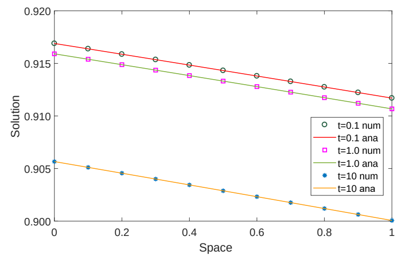

Figure 7: Solution of Example 4 at distinct times with $s = 0.1$, $\Delta t = 0.0001$, and $\sigma = 8$.

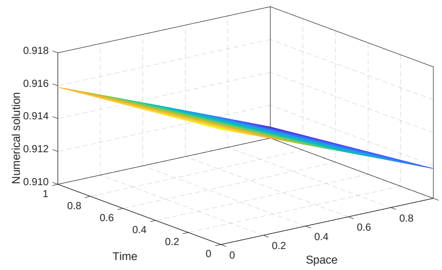

Figure 8: Three-dimensional representation of numerical solution of Example 4 with $s = 0.01$, $\Delta t = 0.001$, and $\sigma = 8$.

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