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Technological returns to scale: Identification and visualization

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Abstract

One of the most critical issues for using data envelopment analysis models is the identification of technological returns to scale (TRTS). Recently, the angles method based on data mining is introduced for the identification of TRTS. This objective method uses the angles to measure the gap between the constant and variable TRTS. The gap is calculated in both the increasing and decreasing sections of the frontier. The larger the gap in the increasing and/or decreasing sections of the frontier, the closer TRTS is to the increasing and/or decreasing form of TRTS. In this paper, we propose a heuristic method for visualizing TRTS that would give a better understanding of identification of TRTS in the dataset. To this end, we introduce the maximum angles method for measuring the maximum possible deviation from constant TRTS assumption in the increasing and decreasing sections of the frontier. By the angles and the maximum angles , we can display the dataset's TRTS in a two-dimensional space. To validate the proposed method, we consider six one input/one output cases. Also, we apply the angles method and the maximum angles method for the Maskan bank of Iran. Using the proposed method, we show that how TRTS of the bank dataset can be displayed in a two-dimensional space.

Keywords: Data envelopment analysis; Returns to scale; Technology; Bank.

1 Introduction

Data envelopment analysis (DEA) is a nonparametric method for evaluating the efficiency of decision making units (DMUs) while each DMU utilizes

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multiple inputs to produce multiple outputs. DEA is introduced by the CCR model [11], for which technological returns to scale (TRTS) is assumed to be constant. In fact, it is assumed that the technological behavior of the dataset is constant from returns to scale point of view, and we call it constant technological returns to scale (CTRS). In another words, by multiplying inputs with α ($\alpha \geq 0$), outputs are also multiplied by the same value. Removing this full proportionality assumption, results in the BCC model [5] with variable technological returns to scale (VTRS). Overall, different assumptions on the proportionality between the inputs and outputs lead to various TRTS. The determination of returns to scale (RTS) is done at two levels: Technology and DMU. When RTS is identified at the technological level, it can be determined at the DMU level. The relation between these two levels is as follows:

- If TRTS is identified to be constant, then RTS of every DMUs is constant.
- If TRTS is identified to be increasing, then RTS of a DMU might be constant or increasing.
- If TRTS is identified to be decreasing, then RTS of a DMU might be constant or decreasing.
- If TRTS is identified to be variable, then RTS of a DMU might be constant or increasing or decreasing.

For the determination of RTS at the DMU level, basic methods have developed in two main paths [6]: One of the paths, which was developed by Färe, Grosskopf, and Lovell [13, 14], determines RTS by using ratio of radial measures. Model pairs are used for the development of these ratios and they only differ in satisfaction of convexity and sub-convexity conditions. The second path developed by Banker [3], Banker and Thrall [9], includes radial measure models as well as the additive and multiplicative models. Furthermore, many papers have focused on the determination of DMUs' RTS in different DEA models such as nonradial measure DEA models [17, 26], weight restricted DEA models [16, 18, 25], and FDH models [15, 19, 21]. Moreover, the measurement of RTS is considered in the presence of undesirable outputs [22, 23] and also negative data [2].

One of the most important issues in evaluating the efficiency of DMUs is the proper identification of RTS at the technological level. There are two basic approaches for the identification of TRTS: The subjective and objective approaches. Most of the objective methods are based on the statistics. The fundamental papers in this area are [4, 7, 8] in which the identification of TRTS is addressed by using DEA based hypothesis tests. Also, Simar and Wilson [20] have investigated TRTS identification by using nonparametric statistical tests. Using a statistical method mostly needs some assumptions to be made. Obviously, the wrong assumptions can lead to incorrect results. Although, these methods are strong from theoretical point of view, they may be difficult to use. Due to these deficiencies, Alirezaee, Hajinezhad, and Paradi [1] recently proposed a novel nonstatistical method for the objective identification of TRTS, which is called the angles method.

The angles method has been developed based on data mining for measuring RTS at the technological level. To achieve this aim, the gap between two assumptions; that is, constant and variable TRTS, is measured by an angle. For each DMU, a hyperplane with VTRS assumption and a hyperplane with CTRS assumption are constructed; the angle between these two hyperplanes is calculated as the DMU's gap. Based on the layered RTS [1] of the DMU under study, the calculated angle is stored in the increasing or decreasing set. Eventually, the angles of the increasing set are aggregated into an angle, which shows the gap from the constant form of TRTS in the increasing section of the frontier. Similarly, the aggregation of the angles in the decreasing set represents the deviation from CTRS in the decreasing section of the frontier. The larger the angles in the increasing and/or decreasing sections of the frontier, the closer TRTS will be to the increasing and/or decreasing assumptions.

In this paper, we propose a heuristic method for visualizing TRTS in a two-dimensional space. To this end, we introduce the maximum angles method to show how far a frontier with VTRS assumption may deviate from CTRS assumption. The angle between the efficient frontiers with CTRS assumption and the weak efficient frontier with VTRS assumption, is considered as a candidate for the maximum angle. For creating the weak efficient frontiers, anchor points have been used. Finally, the maximum angles are calculated in the increasing and decreasing sections of the frontier. By the angles, maximum angles and the percentages of DMUs in each section of the frontier, the dataset's TRTS in one input/one output space is displayed. The proposed method is validated by 6 cases of one input/one output. Application of the angles method and the maximum angles method has been investigated on an Iranian bank dataset.

2 The angles method

In this section, the main points of the angles method introduced in [1], is briefly described. The main concept of this method is the gap, which shows the deviation from CTRS assumption. The gap is defined to show moving towards or away from the constant form of TRTS assumption. In this method, the gap is measured by the angle between two frontiers with different TRTS. For each DMU, a frontier with VTRS assumption and another one with CTRS assumption are constructed. Then the angle between these two frontiers is calculated.

In the angles method, RTS of the technology is determined by using all the DMUs' angles , because the technology influences all the DMUs' behavior. To consider all the DMUs for determining TRTS, the layering technique [12, 24] is applied. Let us consider N DMUs that use m inputs to produce s outputs in set PPS^1. In the layering technique, the first efficient layer is constructed by the efficient DMUs obtained from the execution of DEA model for PPS^1. By deleting the efficient DMUs from set PPS^1, the new set of DMUs PPS^2 is created. Again, the DEA model is executed for the DMUs of PPS^2 and the obtained efficient DMUs make up the second layer. In the same way, PPS^t is constructed by removing the DMUs on the layer $(t-1)$ from the set PPS^{t-1}; the layer t is formed by running the DEA model for the set PPS^t. This process continues until the number of DMUs is meaningful regarding the number of inputs and outputs. So, the layering continues until the number of DMUs is more than or equal to 3×(number of inputs + number of outputs). The DEA model used for the layering technique is the additive model with VTRS assumption.

After measuring a DMU's angle, it is required to determine whether it shows the movement to the increasing or decreasing TRTS assumption. To this end, the angles method uses RTS at the DMU level measured by the method presented in [6] with a little change. At iteration t of the layering, RTS of DMUs constructed the layer t, are measured regarding the DMUs in the set PPS^t. Since a DMU's RTS is determined based on the layer located on it, this RTS is called the layered RTS (LRTS). Assume that in iteration t of the layering, DMU j_o is under study and relies on layer t. Let j_o consume $x_o = (x_{1o}, \dots, x_{mo})$ to produce $y_o = (y_{1o}, \dots, y_{so})$. Moreover, let PPS^t be the set containing all the observed DMUs at iteration t. DMU j_o is evaluated by the following multiplier form of input oriented BCC model (it is clear that since j_o is strong efficient, it is efficient by the input oriented model as well):

$$
\begin{aligned}
Max \quad & uy_o - u_0 && && (1a)\\
s.t. \quad & uy_j - vx_j - u_0 \le 0, && j \in PPS^t, && (1b)\\
& vx_o = 1, && && (1c)\\
& u \ge 0,\ v \ge 0,\ u_0 \ free\ in\ sign. && && (1d)
\end{aligned}
$$

Let us suppose that the optimum solution obtained from the above model is in the form of (u^*, v^*, u_0^*). If $u_0^* = 0$, the DMU's LRTS is constant. Otherwise if $u_0^* > 0$ ($u_0^* < 0$), the following minimization (maximization) model for the LRTS identification is applied:

$$\begin{aligned} Min(Max) \quad & u_0 && (2a) \\ s.t. \quad & uy_j - vx_j - u_0 \leq 0, \qquad j \in PPS^t, && (2b) \\ & vx_o = 1, && (2c) \\ & uy_o - u_0 = 1, && (2d) \\ & u \geq 0,\ v \geq 0, && (2e) \\ & u_0 \geq 0\ (u_0 \leq 0). && (2f) \end{aligned}$$

If the optimized u_0 from the above model equals zero, the LRTS of the DMU under evaluation is constant. If u_0 is strictly positive (negative), its LRTS is decreasing (increasing).

Similarly, DMU j_o can be evaluated by the following multiplier form of output oriented BCC model (it is clear that since j_o is strong efficient, it is efficient by the output oriented model as well).

$$\begin{aligned} \min \quad & vx_o + v_0 && (3a) \\ s.t. \quad & uy_j - vx_j - v_0 \leq 0, \qquad j \in PPS^t, && (3b) \\ & uy_o = 1, && (3c) \\ & u \geq 0,\ v \geq 0,\ v_0\ free\ in\ sign. && (3d) \end{aligned}$$

Let us suppose that the optimum solution obtained from the above model is in the form of (u^*, v^*, v_0^*). If $v_0^* = 0$, the DMU's LRTS is constant. Otherwise if $v_0^* > 0$ ($v_0^* < 0$), the following minimization (maximization) model for the LRTS identification is applied:

$$\begin{aligned} \min(\max) \quad & v_0 && (4a) \\ s.t. \quad & uy_j - vx_j - v_0 \leq 0, \qquad j \in PPS^t, && (4b) \\ & uy_o = 1, && (4c) \\ & vx_o + v_0 = 1, && (4d) \\ & u \geq 0,\ v \geq 0, && (4e) \\ & v_0 \geq 0\ (v_0 \leq 0). && (4f) \end{aligned}$$

If the optimized v_0 from the above model equals zero, the LRTS of the DMU under study is constant. If v_0 is strictly positive (negative), its LRTS is decreasing (increasing).

To determine the angle corresponding to DMU j_o, two hyperplanes with constant and variable TRTS are required. Since j_o is a strong efficient DMU, it is efficient by the BCC model. So, the BCC hyperplane passing through j_o is considered as the hyperplane with VTRS assumption. However, it is possible that j_o is not CCR efficient. Therefore, the CCR hyperplanes passing through one of the reference DMUs for j_o from the additive model with CTRS assumption, is considered as the hyperplane with CTRS assumption. The set of mentioned reference DMUs is represented by Ref_{jo}. An important point is that the BCC and CCR hyperplanes satisfying those conditions are almost infinite. Therefore, for having unique hyperplanes and avoiding overestimation of the angle between the hyperplanes, the hyperplanes with the smallest angle are selected. So, to determine the angle for a DMU, the smallest angle model is introduced [1]. Suppose that DMU j_o is on layer t. Since j_o is strong efficient regarding the set PPS^t, it is efficient by the input oriented BCC model, too. So, the angle corresponding to DMU j_o is evaluated by using the following input oriented smallest angle model:

$$\begin{aligned}
\max \quad & \frac{(v^c,u^c)^T(v^v,u^v)}{||(v^c,u^c)||\ ||(v^v,u^v)||} && && \text{(5a)}\\
s.t. \quad & v^c x_{j_R} = 1, && && \text{(5b)}\\
& u^c y_{j_R} = 1, && && \text{(5c)}\\
& -v^c x_j + u^c y_j \leq 0 && \forall j \in PPS^t, && \text{(5d)}\\
& v^c \geq 0,\ u^c \geq 0, && && \text{(5e)}\\
& v^v y_o - u_0^* = 1, && && \text{(5f)}\\
& u^v x_o = 1, && && \text{(5g)}\\
& -v^v x_j + u^v y_j - u_0^* \leq 0 && \forall j \in PPS^t, && \text{(5h)}\\
& v^v \geq 0,\ u^v \geq 0, && && \text{(5i)}
\end{aligned}$$

where $v^c = (v_1^c, \ldots, v_m^c)$ and $u^c = (u_1^c, \ldots, u_s^c)$ are the input and output weights corresponding to the CCR model, $v^v = (v_1^v, \ldots, v_m^v)$ and $u^v = (u_1^v, \ldots, u_s^v)$ are the input and output weights corresponding to the BCC model, and u_0^* is the optimum value of u_0 from model (2). Maximizing the objective function (5a) —which is the inner product of the hyperplanes— leads to the hyperplanes with the smallest angle. Satisfying conditions (5b)–(5e) results in passing the hyperplane (v^c, u^c) through j_R –which is a member of the set Ref_{jo}– and supporting PPS^t with CTRS assumption. Similarly, satisfying conditions (5f)–(5i) results in passing the hyperplane (v^v, u^v, u_0^*) through j_o and supporting PPS^t with VTRS assumption. For every $j_R \in Ref_{jo}$, model (5) is executed and the smallest angle is selected as the angle corresponding to DMU j_o. If the DMU's LRTS is increasing, then the angle is saved in the increasing set represented by $Minangles^{ITRS}$. If the DMU's LRTS is decreasing, then the angle is stored in the decreasing set represented by $Minangles^{DTRS}$. ITRS and DTRS are the abbreviations for the increasing and decreasing technological returns to scale, respectively.

Similarly, it is assumed that DMU j_o is on the layer t. DMU j_o is efficient by the output oriented BCC model regarding the set PPS^t, since it is efficient by the additive model. So, the angle corresponding to DMU j_o can be evaluated by using the following output oriented smallest angle model:

$$\max \quad \frac{(v^c,u^c)^T(v^v,u^v)}{||(v^c,u^c)||\ ||(v^v,u^v)||} \tag{6a}$$

$$s.t. \quad v^c x_{j_R} = 1, \tag{6b}$$

$$u^c y_{j_R} = 1, \tag{6c}$$

$$-v^c x_j + u^c y_j \leq 0 \qquad \forall j \in PPS^t, \tag{6d}$$

$$v^c \geq 0,\ u^c \geq 0, \tag{6e}$$

$$v^v x_o + v_0^* = 1, \tag{6f}$$

$$u^v y_o = 1, \tag{6g}$$

$$-v^v x_j + u^v y_j - v_0^* \leq 0 \qquad \forall j \in PPS^t, \tag{6h}$$

$$v^v \geq 0,\ u^v \geq 0, \tag{6i}$$

where v_0^* is the optimum value of v_0 from model (4).

After finishing the layering process, the mean and median of the angles in each set; that is, $Minangles^{ITRS}$ and $Minangles^{DTRS}$, are calculated for determining the general gaps from CTRS assumption in the increasing and decreasing sections of the frontier. The geometric mean of the mean and median in the increasing section of the frontier; that is, $Mean^{ITRS}$ and Med^{ITRS}, is calculated as follows:

$$GM^{ITRS} = \sqrt{Mean^{ITRS} \times\ Med^{ITRS}}. \tag{7}$$

Moreover, the geometric mean of the mean and median in the decreasing section of the frontier; that is, $Mean^{DTRS}$ and Med^{DTRS}, is calculated as follows:

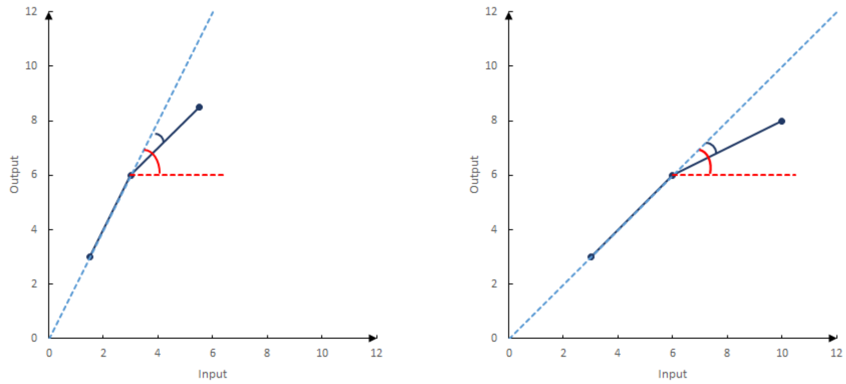
$$GM^{DTRS} = \sqrt{Mean^{DTRS} \times\ Med^{DTRS}} \tag{8}$$

GM^{ITRS} and GM^{DTRS} are also called the GM values in the increasing and decreasing sections of the frontier, respectively. GM^{ITRS} and GM^{DTRS} determine the tendency of the DMUs toward ITRS and DTRS assumptions.

3 A heuristic method for displaying TRTS

By using the angles method on a dataset, the angles in the increasing and decreasing sections of the frontier is calculated. However, these angles do not give any information about the maximum deviations from CTRS assumption. For clarification of this problem, consider two datasets in Figure 1. For both the datasets, the angle corresponding to the increasing section of the frontier is 18°. However, the maximum angles of case Figure 1a and Figure 1b are 63° and 45°, respectively.

Thus, for having a better picture of TRTS, the maximum angles in the increasing and decreasing sections of the frontier is required to be calculated. By having the maximum angles, we can represent a dataset's TRTS with



(a) (b)

Figure 1: The angles from CTRS assumption obtained by the angles method give no information about the maximum deviations

multiple inputs and multiple outputs in a two-dimensional space. In the following, we introduce a method that we call it the maximum angles method and it is used for calculating the maximum possible angles from CTRS assumption.

3.1 The maximum angles method

The weak efficient frontiers with VTRS assumption possess the maximum deviations from CTRS assumption. So, the maximum angle from CTRS assumption corresponds to a DMU lying on a weak efficient frontier. It means that the angles corresponding to the DMUs locating on the weak efficient frontiers need to be examined.

In order to calculate the maximum deviations for each layer in the increasing and decreasing sections of the frontier, it is necessary to examine all the weak efficient frontiers corresponding to that layer. On the other hand, it is possible that the weak efficient DMUs do not generate all the weak efficient frontiers. Therefore, we consider some artificial DMUs, which are weak efficient and produce all the weak efficient frontiers. To construct these artificial DMUs, the anchor points are used.

The intersections of the weak and strong efficient frontiers are called the anchor points [10]. In other words, a strong efficient DMU is an anchor point if and only if it is located on an unbounded hyperplane or a weak efficient frontier. By using anchor points, we create new DMUs on the weak efficient frontier and their corresponding angles are studied as the candidates for the maximum angles in both sections of the frontier; that is, increasing and decreasing.

Consider the layer t. It can be easily shown that $\{e_1, e_2, \ldots, e_m\}$ and $\{-e_1, -e_2, \ldots, -e_s\}$ are the extreme directions of the production possibility set. e_i is a zero vector, where component i equals one. Suppose that j_o with (x_o, y_o) is a strong efficient DMU on the layer t. We construct $m+s$ artificial DMUs: Artificial DMU ${j_o}^i$ $(i = 1, \ldots, m)$ with $(x_o + e_i, y_o)$ and artificial DMU ${j_o}^r$ $(r = 1, \ldots, s)$ with $(x_o, y_o + 0.1y_o e_r)$. To construct the artificial DMU ${j_o}^r$, the output of DMU j_o decreases proportionally to prevent negative outputs. If at least one of $m+s$ artificial DMUs is weak efficient, then DMU j_o is an anchor point on the layer t. However, the efficiency of all artificial DMUs are measured. Then, the angles corresponding to the weak efficient DMUs are calculated.

In a loop, artificial DMU ${j_o}^i$ $(i = 1, \ldots, m)$ is inserted to the set PPS^t (PPS^t is the set containing all the observed DMUs at iteration t of the layering) and evaluated by using the output oriented BCC model. Let artificial DMU ${j_o}^\iota$ with $(x_o + e_\iota, y_o)$ be a weak efficient DMU. Since, DMU j_o^ι is output oriented BCC efficient, its corresponding angle is calculated by using the output oriented smallest angle (6). If the DMU's LRTS is increasing (decreasing), then the smallest angle is stored in the set $MaxSet_t^{ITRS}$ ($MaxSet_t^{DTRS}$). Eventually, DMU j_o^ι is removed from the set PPS^t. Similarly, for every output component r $(r = 1, \ldots, s)$, artificial DMU j_o^r is inserted with $(x_o, y_o + 0.1y_o e_r)$ to the set PPS^t and evaluated by using the input oriented BCC model. Suppose that DMU j_o^ρ is founded to be efficient. Since, DMU j_o^ρ is input oriented BCC efficient, its corresponding angle is calculated by using the input oriented smallest angle (5). If the DMU's LRTS is increasing (decreasing), then the smallest angle is stored in the set $MaxSet_t^{ITRS}$ ($MaxSet_t^{DTRS}$). Finally, the artificial DMU is removed from the set PPS^t. After the termination of both loops, the maximum members of the sets $MaxSet_t^{ITRS}$ and $MaxSet_t^{DTRS}$ are stored in Max_t^{ITRS} and Max_t^{DTRS}. Now, we can calculate the maximum angles in the two sections of the frontier by using equations (9)–(10).

The maximum GM values are calculated similar to the GM values. The mean and median of Max_t^{ITRS} for all layers, are saved in $Mean^{ITRS-M}$ and Med^{ITRS-M}. The maximum GM value in the increasing section of the frontier is calculated as follows:

$$GM^{ITRS-M} = \sqrt{Mean^{ITRS-M} \times Med^{ITRS-M}} \tag{9}$$

Moreover, The mean and median of Max_t^{DTRS} for all layers, are saved in $Mean^{DTRS-M}$ and Med^{DTRS-M}. The maximum GM value in the decreasing section of the frontier is calculated as follows:

$$GM^{DTRS-M} = \sqrt{Mean^{DTRS-M} \times Med^{DTRS-M}} \tag{10}$$

3.2 Displaying TRTS in the two sections of the frontier: increasing and decreasing

In this section, we want to introduce a heuristic method for displaying a dataset's TRTS in a two-dimensional space by using the results of the angles method as well as the maximum angles method. It should be noted that plotting the production possibility set (PPS) even in ($m + s =$) three-dimensional space is difficult; so it is not a proper method for representing the dataset's TRTS. On the other hand, the scattering of data makes the PPS inappropriate for giving a clear image of TRTS.

By using the angles method, two angles and the percentage of DMUs in the increasing and decreasing sections of the frontier are obtained. By using the maximum angles method described in section 3.1, the maximum deviations from CTRS are calculated in the two sections of the frontier. These results are independent of the number of inputs and outputs; so it is assumed that these results are obtained for a one input/one output dataset. The angles are displayed by using a CCR and a BCC frontier. We explain the method of displaying TRTS on a case study, which is completely described in section 4.2.

Consider the bank dataset (management code 256) with 22 DMUs that use two inputs to produce three outputs. The results of the angles method and the maximum angles method are represented in Table 1 . The values of GM and GM^{-M} represent the angle and the maximum possible angle from CTRS assumption, respectively. Also, Num represents the percentage of DMUs in both sections of the frontier.

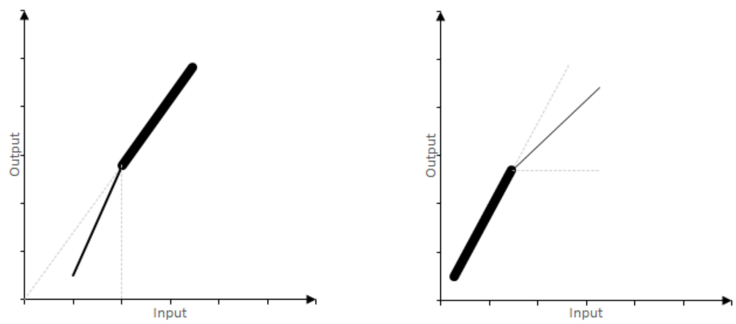
Table 1: The results of the angles method and the maximum angles method for a bank dataset with 22 DMUs

Increasing section of the frontier			Decreasing section of the frontier		
$Num(\%)$	$GM(°)$	$GM^{-M}(°)$	$Num(\%)$	$GM(°)$	$GM^{-M}(°)$
22.2	11.7	35.7	11.1	18.0	61.6

By considering the results represented in Table 1, TRTS is displayed in Figure 2 according to the following rules:

- The CCR and BCC frontiers are represented by two lines.

- The CCR frontier is drawn in such a way that the angle between the CCR frontier and the weak efficient frontier in the increasing/decreasing section is equal to the maximum possible angle in those sections. This weak efficient frontier in the increasing and decreasing sections of the frontier are respectively, the vertical and horizontal lines represented by the dotted line.



(a) (b)

Figure 2: Displaying TRTS in the increasing (a) and decreasing (b) sections of the frontier for a bank dataset with 22 DMUs

- The percentage of DMUs in the constant section of the frontier is equal to 100 minus the total percentage of DMUs in the increasing and decreasing sections of the frontier.
- The higher the percentage of DMUs in the constant section of the frontier is the thicker CCR line .
- The BCC frontier in the increasing/decreasing section is drawn in such a way that the angle between the BCC frontier and the CCR frontier is equal to GM value in the increasing/decreasing section of the frontier.
- The higher the percentage of DMUs in the increasing/decreasing section of the frontier is the thicker BCC line .

The angles in the increasing and decreasing sections are represented separately because the maximum angles in both sections of the frontier may not be complementary; therefore the corresponding CCR frontiers do not coincide.

4 Experimental results

To study the results of the angles method and the maximum angles method, 6 cases one input/one output and a real dataset are considered. For each case, the GM values (by using equations (7)–(8)), the percentages of the DMUs and the maximum GM values (by using equations (9)–(10)) in the increasing and decreasing sections of the frontier are calculated.

To analyze the results of the angles method and the maximum angles method, the followings should be considered:

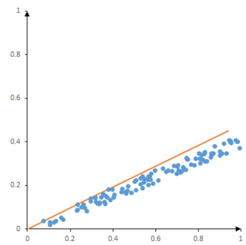
- The existence of small angles in the increasing and decreasing sections of the frontier is normal. Therefore, the angles smaller than $10°$ are insignificant and do not conflict with CTRS assumption.
- Large GM value in the increasing section of the frontier and small GM value in the decreasing section of the frontier, support ITRS assumption to be accepted.
- Small GM value in the increasing section of the frontier and large GM value in the decreasing section of the frontier, support DTRS assumption to be accepted.
- Large GM value in both sections of the frontier, supports VTRS assumption to be accepted.
- Large GM value produced by a small number of DMUs in the increasing and/or decreasing sections of the frontier, cannot reject CTRS assumption. So, if the number of DMUs in a section of the frontier is less than 10% of its total then the angle of that section is neglected.
- The experiments show clearly that the angles method does not merely accept or reject ITRS, CTRS or DTRS technology, but it determines the rate of increasing/decreasing if ITRS/DTRS is revealed. A larger GM value resulted in a larger rate of increasing/decreasing TRTS.
- It is obvious that a larger sample size would lead to more reliable results.

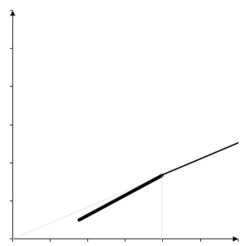
4.1 One input/one output samples

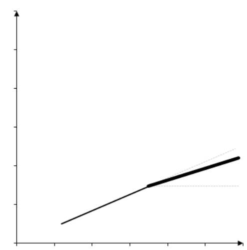
To study the results of the angles method and the maximum angles method, 6 one input/one output cases are considered. The cases are from [1] and plotted in the first column of Figure 3. TRTS of each case can easily be identified from Figure 3. TRTS of cases 3a–3c are constant. Also, TRTS of cases 3d, 3(e), and 3(f) are decreasing, increasing, and variable, respectively. For each cases, the angles method and the maximum angles method are applied. The results shown in Table 2, represent the percentage of the DMUs (Num), the GM values (GM), and the maximum GM values (GM^{-M}) in both sections of the frontier.

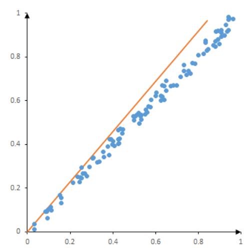
Regarding Table 2, we would make some points:

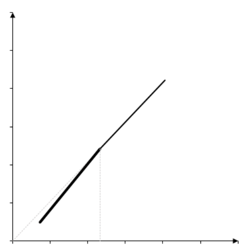
- For all the cases, the percentage of DMUs in both sections of the frontier is more than 17%. So, no angles are negligible.
- From Figure 3, it can be seen that the value of GM^{-M} in the decreasing section of the frontier is approximately equal to the angle between the CCR frontier and the horizontal line. Also, the value of GM^{-M} in the

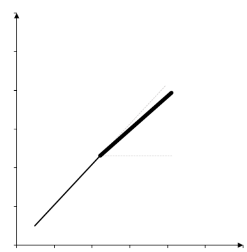

(a)

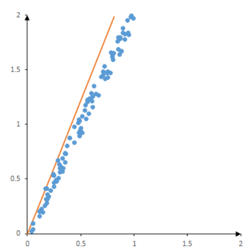

a - Increasing section

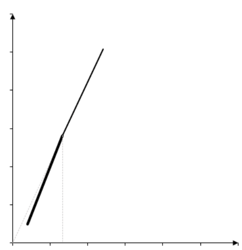

a - Decreasing section

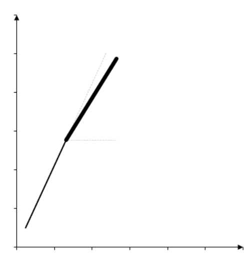

(b)

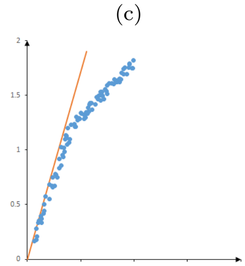

b - Increasing section

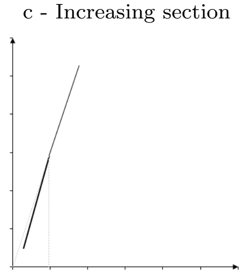

b - Decreasing section

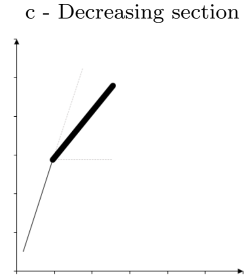

(c)


c - Increasing section


c - Decreasing section


(d)


d - Increasing section


d - Decreasing section

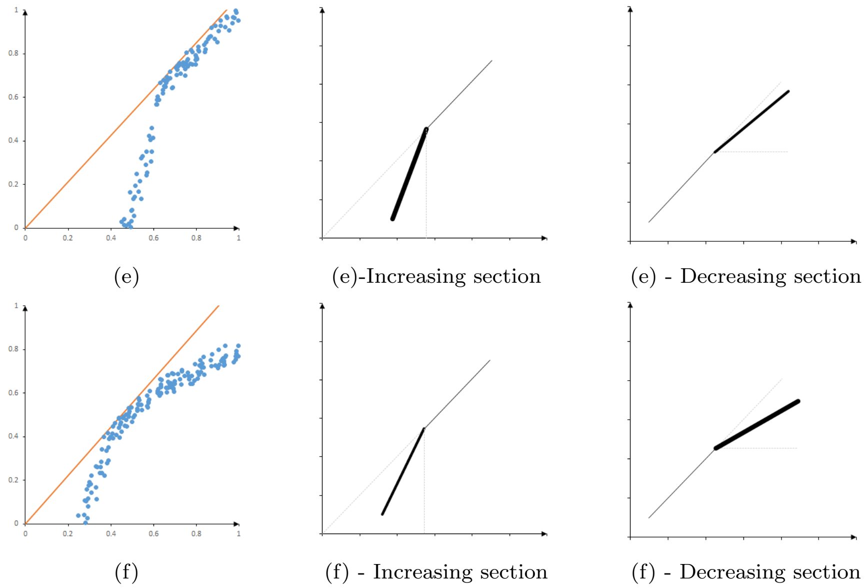


Figure 3: Displaying the original one input/one output cases and their TRTS in the increasing and the decreasing sections of the frontier. In each row, the first cell represents the original one input/one output case [1], the second and the third cells display TRTS in the increasing and the decreasing sections of the frontier for the case in the first cell. The red line (the first column) shows the CCR frontier for the first layer. The horizontal and the vertical coordinates are the input and output, respectively

Table 2: The results of the angles method and the maximum angles method for one input/one output cases

Case	DMUs	Increasing section			Decreasing section		
		Num (%)	GM (°)	GM^{-M} (°)	Num (%)	GM (°)	GM^{-M} (°)
(a)	104	40.4	5.1	67.2	44.4	5.7	22.8
(b)	104	34.6	4.1	43.8	50.0	5.7	46.2
(c)	103	36.4	3.4	25.3	48.5	7.0	64.7
(d)	97	17.2	2.6	18.5	72.0	21.2	71.5
(e)	104	55.0	23.4	44.4	32.0	7.0	45.6
(f)	155	31.3	18.2	44.8	58.0	16.4	45.2

increasing section of the frontier is approximately equal to the angle between the CCR frontier and the vertical line.

- For cases 3a–3c, the GM values are less than 7° in both sections of the frontier, which shows that TRTS of these cases is constant.
- For case 3d, the GM value represents a small deviation from CTRS assumption in the increasing section of the frontier. The GM values for cases 3(e) and 3(f) show clearly that they are ITRS and VTRS, respectively.
- By using the GM, maximum GM values and percentage of DMUs, TRTS of all cases are displayed in the second and the third columns of Figure 3. The first column shows the original data, the second and third columns display the cases' TRTS in the increasing and decreasing sections of the frontier, respectively.

4.2 The real dataset; Maskan bank

To study the results of the proposed method, the real dataset of Maskan bank incorporated in 1979 is used. Its basic mission as a specialized bank is to support the development of housing and construction activities of the government and private sectors in Iran. Its total employees is about 10,860. The dataset includes the information of 1213 branches, which are subdivided into 37 management codes. The number of branches in each management code ranges from 9 to 85. For each branch, there are two inputs: personnel costs and location index and, three outputs: resources, expenses, and services. The personnel costs are normalized to 1,000,000. TRTS of this dataset is subjectively identified to be increasing.

Applying the angles method and the maximum angles method for each dataset requires that the number of DMUs to be at least equal to 3×(number of inputs+number of outputs). So, we applied the angles method and the maximum angles method for the management codes with at least 15 branches. For each management code and all the branches –which is represented by "All" in the first row–, the GM values (GM), the percentage of DMUs (Num), and the maximum GM values (GM^{-M}) are obtained in both sections of the frontier represented in Table 3.

Regarding Table 3, we mention some points:

- For all the subdivisions except codes 228 and 256, the percentage of DMUs in the decreasing section of the frontier is less than 10. Therefore, TRTS of all of the divisions except these two codes, are nondecreasing.
- The percentage of DMUs in the increasing section of the frontier for all the divisions ranges from 16.7 to 69. Therefore, the angles in the increasing section of the frontier are not negligible.

- For management codes 225, 242, and 245, the number of DMUs in the decreasing section is negligible. Moreover, their angles in the increasing section of the frontier are absolutely less than 10°. Thus, their TRTS are constant.

- The angles in the increasing section of the frontier for management codes 219, 224, and 249 are a bit less than 10°. So, the increasing feature of their TRTS is weak.

- Regarding the percentage of DMUs and the angles in the increasing and decreasing sections of the frontier for management codes 228 and 256, these two codes are VTRS.

- For all the divisions except the management codes 219, 224, 225, 228, 242, 245, 249, and 256, the angles in the increasing section of the frontier, range from 10.0° to 22.05°, which represent that their TRTS are increasing.

- TRTS of the entire branches; that is, "All", and most of the management codes are increasing.

- By using the GM and maximum GM values, we can display TRTS of each management code. Here, TRTS of "All" and management code 211 are displayed in Figure 4.

- From Figure 4, it can be seen that for both cases, there is a large angle from CTRS frontier in the decreasing section of the frontier. However, the narrow line shows that this large angle is inconsiderable.

5 Conclusion

Recently, the angles method, which is a heuristic method based on data mining, has been proposed for identifying a dataset's TRTS. In this paper, we extended the angles method by proposing the maximum angles method to measure the maximum deviation from CTRS assumption. We use the angles and the maximum angles for heuristically visualizing TRTS in a two-dimensional space. The methods are validated by 6 one input/one output cases. Also, these methods are applied for the dataset of Maskan bank of Iran. The results showed that all bank branches and the most of the management codes are ITRS. Moreover, the dataset of two cases are depicted in a two-dimensional space by the obtained results.

Table 3: The results of the angles method and the maximum angles method for Maskan bank dataset

		Increasing section			Decreasing section		
Code	DMUs	Num (%)	GM (°)	GM^{-M} (°)	Num (%)	GM (°)	GM^{-M} (°)
All	1213	66.2	15.6	44.1	3.4	12.3	57.1
211	85	53.8	13.3	40.4	5.1	17.6	51.8
212	63	63.5	14.4	39.4	1.9	4.5	49.7
214	68	51.5	12.4	42	1.5	19.0	43.1
215	38	69.0	19.2	47.9	0.0	–	48.5
216	69	35.5	15.4	34.8	6.5	8.0	48.7
217	48	58.1	22.0	46.5	0.0	–	41.4
218	83	46.2	19.0	45.7	0.0	–	34.6
219	51	23.4	9.6	30.8	4.3	7.6	48.9
221	47	52.4	12.8	33.9	2.4	0.9	55.7
222	47	43.2	15.1	39.2	0.0	–	9.3
224	49	40.9	9.6	34.1	0.0	–	6.5
225	28	53.3	7.5	30	6.7	19.4	59.4
226	45	45.0	16.1	48.1	0.0	–	39.9
227	19	44.4	17.4	48.7	0.0	–	27.8
228	34	20.0	17.6	35.3	20.0	9.9	35.4
229	25	61.1	17.7	40.2	0.0	–	37.9
241	25	16.7	11.7	46.2	0.0	–	38.6
242	24	46.7	7.4	24.4	0.0	–	33.6
244	24	47.1	12.3	26.2	0.0	–	17.1
245	22	50.0	6.2	21.7	0.0	–	34.4
246	20	44.4	10.0	37	0.0	–	31.0
247	19	50.0	15.0	39.8	0.0	–	18.7
248	29	50.0	11.5	25.9	0.0	–	26.2
249	21	20.0	9.6	31.7	0.0	–	27.7
252	50	40.0	12.7	43.5	6.7	31.0	39.1
254	21	50.0	18.6	53.5	0.0	–	38.3
255	17	28.6	11.4	23.9	0.0	–	17.1
256	22	22.2	11.7	35.7	11.1	18.0	61.6
264	49	42.9	14.9	45.1	0.0	–	19.5

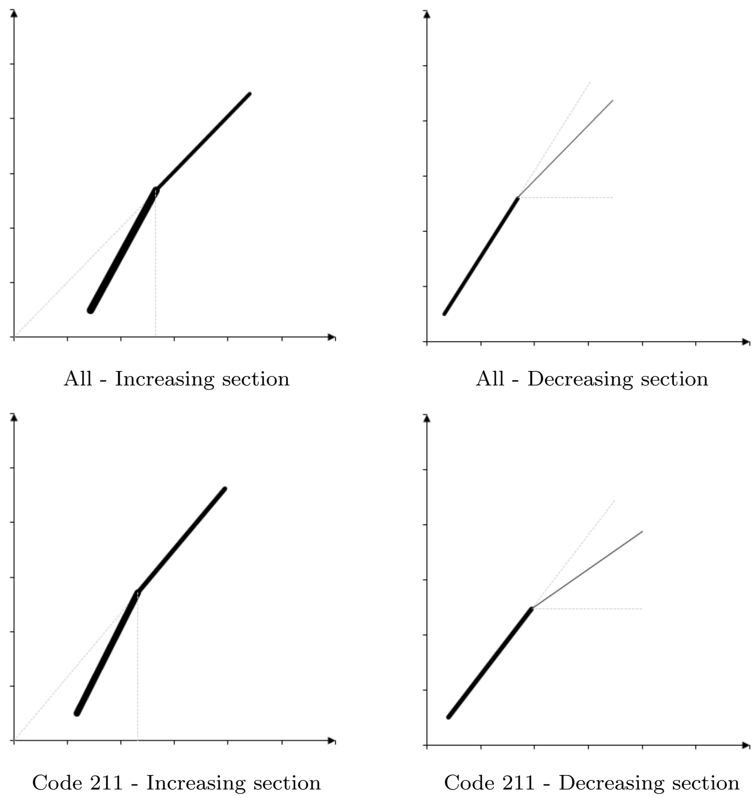


Figure 4: Displaying TRTS in the increasing and the decreasing sections of the frontier for all branches ("All") and management code 211. The horizontal and the vertical coordinates are the input and output, respectively.

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بازده به مقیاس تکنولوژیک: شناسایی و ترسیم

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دریافت مقاله ۲۷ آذر ۱۳۹۴، دریافت مقاله اصلاح شده ۱۱ بهمن ۱۳۹۶، پذیرش مقاله ۵ اردیبهشت ۱۳۹۷

چکیده : یکی از مهمترین مسایل برای به کارگیری مدل‌های تحلیل پوششی داده‌ها، شناسایی بازده به مقیاس تکنولوژیک (TRTS) است. اخیرا روش زاویه‌ها مبتنی بر داده‌کاوی برای شناسایی TRTS معرفی شده است. این روش عینی از زاویه‌ها برای اندازه‌گیری شکاف میان TRTS ثابت و متغیر استفاده می‌کند. شکاف در دو بخش افزایشی و کاهشی مرز محاسبه می‌گردد. هر چه شکاف در بخش افزایشی و/یا کاهشی مرز بیشتر باشد، TRTS به شکل افزایشی و/یا کاهشی نزدیک‌تر می‌باشد. در این مقاله، ما روشی ابتکاری برای ترسیم TRTS پیشنهاد می‌دهیم که درک بهتری از TRTS مجموعه داده ارایه می‌دهد. برای این منظور، روش بیشترین زاویه‌ها را برای محاسبه حداکثر انحراف ممکن از فرض TRTS ثابت در بخش‌های افزایشی و کاهشی مرز معرفی نموده‌ایم. با استفاده از زاویه‌ها و بیشترین زاویه‌ها، می‌توان TRTS مجموعه داده را در یک فضای دو بعدی رسم نمود. برای درستیابی روش پیشنهادی، از ۶ نمونه یک ورودی/یک خروجی استفاده نموده‌ایم. همچنین روش زاویه‌ها و روش بیشترین زاویه‌ها را بر روی داده‌های بانک مسکن ایران مورد استفاده قرار داده و TRTS مجموعه داده بانک را در یک فضای دو بعدی رسم کردیم.

کلمات کلیدی : تحلیل پوششی داده‌ها؛ بازده به مقیاس؛ تکنولوژی؛ بانک.