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Explicit and implicit schemes for fractional–order Hantavirus model

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Abstract

In this paper, the fractional–order form of a mouse population model is introduced. Some explicit and implicit schemes, which are Theta methods and nonstandard finite difference (NSFD) schemes, are implemented to give a numerical solution of nonlinear ordinary differential equation system named Hantavirus epidemic model. These methods are compared and discussed that the method preserves the positivity properties of the integer order system. The numerical solutions are illustrated by means of some graphs. Numerical results of explicit and implicit methods denote that these methods are easy and accurate when applied to fractional–order Hantavirus model.

Keywords: Explicit and implicit methods; Theta method; Nonstandard finite difference scheme; Fractional–order nonlinear differential equation systems; Mouse population model.

1 Introduction

Although fractional calculus has a long history, its applications to natural science are just a recent focus of interest. The description of some phenomena is more accurate when the fractional derivative is used. In many scientific area specially in physics, chemistry, and engineering, the fractional differential equations (FDEs) become a popular subject, which are increasingly used to model problems in a number of research areas including dynamical systems, mechanical systems, signal processing, electronic circuit theory, control theory, chaos synchronization, mechanics, seismology, and many others.

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Some of these studies may be found in [4] and [35]. Lots of books written by the authors Podlubny [30], Lubich [19], Miller and Ross [24], Oldham and Spainer [28], Diethelm et al. [8], Samko et al. [34] played a significant role to understand the subject of FDEs and gave some methods for solutions.

Several methods, which are the Adams–Moulton method [14], [12], the homotopy perturbation method [25], the Adomian decomposition method [25], [26], the variational iteration method [36], the differential transform method [3], [11], the operational method [20], the predictor corrector methods [7], [8], [13], the product integration rules [15], [38], nonstandard finite difference scheme [29] have been used to solve FDEs. Numerical solution methods are frequently referred to as being explicit or implicit. These situations are being encountered numerical solutions of FDEs and equation systems.

The first and important study, which is relevant with Hantavirus model, is given by [2]. The stability analysis of model, which is discretized with nonstandard finite difference scheme, is given by [5], [6]. Further, variational iteration method [16], exponential matrix method (EMM) [39], multistage differential transformation method [17] have been used to solve Hantavirus model and FDE solutions of this model studied by [31], [37].

The effect of the environmental parameter on the Hantavirus infection through a fractional–order SI model was studied by Rida et al.(2012). They presented a fractional–order model of the Hantavirus infection in terms of simple differential equations involving the mice population. They studied that the effect of changes in ecological conditions and diversity of habitats can be observed by varying the value of the environmental parameter k. They used a generalized Euler method (GEM) to obtain an analytic approximate solution of the model [37].

In this paper, new numerical methods determined for fractional–order nonlinear differential equation model known as Hantavirus epidemic model. These model based on a nonlinear differential equation of order p, where p is a constant in range $0 < p < 1$. Some explicit and implicit methods such as theta method and nonstandard finite difference schemes are studied for the numerical solution of the fractional–order model. Especially the purpose of the studying to these two methods that both methods are more general face of classical methods and both methods allow to some arbitrary choice.

The following model is given in [2] has been used in the study of Hantavirus epidemics:

$$\begin{aligned} \frac{dM_s}{dt} &= bM - cM_s - \frac{M_s M}{K} - aM_s M_I, \\ \frac{dM_I}{dt} &= -cM_I - \frac{M_I M}{K} + aM_s M_I, \end{aligned} \tag{1}$$

where

M_s : the population of susceptible mice ($M_s \geqslant 0$),

M_I : the population of infected mice ($M_I \geqslant 0$),

b : the birth rate,

c : the death rate,

K : the carrying capacity of the environment.

In this model, mice movement as a process of diffusion is ignored and whole population is composed of two cases of mice, susceptible and infected. For the systems (1), the total population $M = M_s + M_I$ satisfying the logistic differential equation

$$\frac{dM}{dt} = (b-c)M - \frac{M^2}{K}. \tag{2}$$

The carrying capacity for the total population is $M^* = (b-c)K$. The critical value of the carrying capacity is

$$K_c = \frac{b}{a(b-c)}.$$

Let us $M_s = x, M_I = y$ in equation (1),

$$\begin{aligned} \frac{dx}{dt} &= (b-c)x + by - \frac{x^2}{K} - \left(\frac{1+aK}{K}\right)xy, \\ \frac{dy}{dt} &= -cy - \frac{y^2}{K} - \left(\frac{1-aK}{K}\right)xy. \end{aligned} \tag{3}$$

The dynamics of the continuous system (3) have been given in [2, 5, 6] as the following:

i) If $b \leq c$, then the system has a unique equilibrium $(0,0)$, and it is globally asymptotically stable.

ii) If $b > c$ and $K \leq \frac{b}{a(b-c)}$, then the system has two equilibria: $(0,0)$, which is unstable, and $(K(b-c), 0)$, which is globally asymptotically stable.

iii) If $b > c$ and $K > \frac{b}{a(b-c)}$, then the system has three equilibria: $(0,0)$ and $(K(b-c), 0)$, which are unstable; and $\left(\frac{b}{a}, K(b-c) - \frac{b}{a}\right)$, which is globally asymptotically stable.

In this article,the fractional–order Hantavirus model examined, which given as below:

$$\begin{aligned} &{}^C D^p_{t_0} x(t) = (b-c)x(t) + by(t) - \frac{x(t)^2}{K} - \left(\frac{1+aK}{K}\right)x(t)y(t), \\ &{}^C D^p_{t_0} y(t) = -cy(t) - \frac{y(t)^2}{K} - \left(\frac{1-aK}{K}\right)x(t)y(t), \\ &x(0) = x_0, \quad y(0) = y_0, \end{aligned}$$

where noninteger derivative is defined as the Caputo derivative. The fractional–order equation system is introduced and denoted that this model has a unique solution with given initial conditions while $t \geq 0$. Some explicit and implicit schemes are implemented to study the dynamics behaviors of fractional order Hantavirus epidemic system. Some numerical solutions illustrated by means of some graphs are provided in next sections.

2 Fractional derivatives

On this section, some definitions and relationship of fractional order integration and fractional–order differentiation mentioned [29].

The Grünwald–Letnikov (GL) operator of order $p > 0$ is defined as

$$_{GL}D^p_{t_0} f(t) = \lim_{N\to\infty} h_N^{-p} \sum_{j=0}^{N} w_j^{(p)} f(t - jh_N)$$

with

$$w_j^{(p)} = (-1)^j \binom{p}{j} = \frac{\Gamma(j-p)}{\Gamma(-p)\Gamma(j+1)}.$$

Note that the weights $w_j^{(p)}$ are the coefficient in the power series expansion of $(1-\xi)^p$; that is,

$$(1-\xi)^p = \sum_{j=0}^{\infty} w_j^{(p)} \xi^j$$

and they can be evaluated recursively by means of the recurrence

$$w_0^{(p)} = 1, \quad w_j^{(p)} = \left(1 - \frac{1+p}{j}\right) w_{j-1}^{(p)}, \quad j = 1, 2, \ldots. \tag{4}$$

The relationship between Riemann–Liouville (RL) and Caputo definitions is

$$_{C}D^p_{t_0} f(t) =_{RL} D^p_{t_0} \left(f(t) - T_{m-1}[f; t_0]\right),$$

where $T_{m-1}[f; t_0]$ is the $(m-1)$th degree Taylor polynomial for f centered at t_0

$$T_{m-1}[f; t_0](t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(t_0).$$

While, $0 < p < 1$, it is $m = 1$ and $T_0[f; t_0](t) = f(t_0)$. Thus, we obtained

$$_{C}D^p_{t_0} f(t) =_{RL} D^p_{t_0} (f(t) - f(t_0)).$$

Under suitable assumptions of regularity, the RL and GL operators coincide; that is,

$$ {}_{RL}D^p_{t_0} f(t) ={}_{GL} D^p_{t_0} f(t). $$

A consequence of the above two points is that in general

$$ {}_{C}D^p_{t_0} f(t) ={}_{GL} D^p_{t_0} f(t) \left(f(t) - T_{m-1}[f;t_0] \right), $$

and while $0 < p < 1$ it is

$$ {}_{C}D^p_{t_0} f(t) ={}_{GL} D^p_{0,t} (f(t) - f(t_0)). \tag{5} $$

In case it used in differential equations of fractional order, the Caputo definition is preferable since it allows the couple the equation with initial conditions of classical type (i.e., initial condition of Cauchy type). In this way we can obtain

$$ \begin{cases} {}_{C}D^p_{t_0} f(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases} $$

In case we use the RL definition; we can couple different initial conditions, as

$$ \begin{cases} {}_{RL}D^p_{t_0} f(t) = f(t, y(t)), \\ \lim\limits_{t \to t_0^+} J^{1-p}_{t_0} y(t) = b_1, \end{cases} $$

which does not have a clear physical meaning. Therefore, they are not useful for practical applications.

Lemma 1. *Let $0 < p < 1$, and let $w^{(p)}_n$ be the coefficients in the power series expansion of $(1-\xi)^\alpha$ given in $w^{(p)}_j$. Then for any $n = 1, 2, \ldots,$*

a) $-1 < w^{(p)}_1 < 0$,

b) $0 < w^{(p-1)}_1 < 1$.

Proof. The proof is an immediate consequence of the recursive relationship stated in (4). □

3 Fractional order Hantavirus epidemics

In this section, the fractional order applied in the model of Hantavirus epidemics [2]. The new system is given as the set of fractional differential equations (FDEs):

$$
\begin{aligned}
{}^{C}D_{t_0}^{p}x(t) &= (b-c)x(t) + by(t) - \frac{x(t)^2}{K} - \left(\frac{1+aK}{K}\right)x(t)y(t),\\
{}^{C}D_{t_0}^{p}y(t) &= -cy(t) - \frac{y(t)^2}{K} - \left(\frac{1-aK}{K}\right)x(t)y(t),
\end{aligned} \tag{6}
$$

$$
x(0) = x_0, \quad y(0) = y_0, \tag{7}
$$

where ${}_{C}D_{0,t}^{p}$ denotes the fractional derivative operator, with respect to the origin, according to Caputo's definition [30] and $x(t)$ and $y(t)$ are activator and inhibitor variables. The fractional derivatives are used to describe nonhomogeneous character of the ecosystems, with respect to the presence of competitors. The parameter p denotes the density of competitor species in the system. The reason, to use fractional order differential equations (FODE), is to be naturally related with systems with memory, which exists in most biological systems. Also FODE are closely related to fractals, which are abundant in biological systems [17]. The results derived of the fractional system (3) and (4) are more general nature.

When the power exponent is $p = 1$, this corresponds to equation (3) and varies competitor's populations when $0 < p < 1$. While $p > 1$, the density of competitor or alien species will increase in the populations [1]. The stability analysis given by the study of [29].

For non–negative solutions, denote $\mathbb{R}_+^2 = \left\{x \in \mathbb{R}^2 \mid x \geq 0\right\}$, and let $x(t) = (X, Y)^T$. To prove the main theorem, we need following lemma and corollary given by [9, 27].

Lemma 2 (Generalized Mean Value Theorem)**.** *Suppose that $f(x) \in C[a,b]$ and $D_a^p f(x) \in C[a,b]$ for $0 < p \leq 1$; then we have*

$$
f(x) = f(a) + \frac{1}{\Gamma(p)} \left(D_a^p f\right)(\xi)(x-a)^p
$$

with $a \leq \xi \leq x$ for all $x \in (a, b]$.

Corollary 1. *Suppose that $f(x) \in C[a,b]$ and $D_a^p f(x) \in C[a,b]$ for $0 < p \leq 1$. If $D_a^p f(x) \geq 0$, for all $x \in (a,b)$, then $f(x)$ is nondecreasing for each $x \in [a,b]$. If $D_a^p f(x) \leq 0$, for all $x \in (a,b)$, then $f(x)$ is nonincreasing for each $x \in [a,b]$.*

Theorem 1. *There is a unique solution $x(t) = (X, Y)^T$ to equation (6) with initial condition (7) on $t \geq 0$, and the solution will remain in $\mathbb{R}_+^2$. Furthermore, X and Y are all bounded by K_c.*

Proof. We know that the solution on $(0, \infty)$ solving the initial value problem (6)–(7) is not only existence but also unique [18].

From equation (6), we find

$$D^pX\mid_{X=0}=bY\geqslant 0, D^pY\mid_{Y=0}=0.$$

By Corollary 1, the solution will remain in $\mathbb{R}^2_+$. From [2], X and Y are bounded by K_c. □

To evaluate the equilibrium points, let $D^px=0$ and $D^py=0$. Then $E_0=(0,0)$, $E_1=(K(b-c),0)$, and$E_2=(\frac{b}{a},K(b-c)-\frac{b}{a})$ are the equilibrium points. To give more detail about the local behavior near the equilibria, we find the Jacobian matrix of equation (6) at each equilibrium point:

$$J(E_0)=\begin{pmatrix} b-c & b\\ 0 & -c\end{pmatrix}, J(E_1)=\begin{pmatrix} -(b-c) & c-aK(b-c)\\ 0 & -b+aK(b-c)\end{pmatrix}$$

and

$$J(E_2)=\begin{pmatrix} -aK(b-c)+b(1-\frac{1}{aK}) & -\frac{b}{aK}\\ aK(b-c)+\frac{b}{aK}-2b+c & -b(1-\frac{1}{aK})+c\end{pmatrix}.$$

The equilibrium point E_0 is locally asymptotically stable, if all of the eigenvalues λ_i $(i=1,2)$ of the Jacobian matrix $J(E_i)$ for $i=0,1,2$ satisfy the following condition [10, 21]:

$$|\arg(\lambda_i)|>\frac{p\pi}{2}. \tag{8}$$

The eigenvalues of $J(E_0)$ are $\lambda_1=(b-c)$ and $\lambda_2=-c$. If $b\geq c$, then $\arg(\lambda_1)=0$, $\arg(\lambda_2)=\pi$, and E_0 is unstable. If $b<c$, $\arg(\lambda_1)=\arg(\lambda_2)=\pi$, and equation (8) is correct, then E_0 is asymptotically stable.

Now consider the case $b>c$ and the equilibrium point E_1. The eigenvalues of $J(E_1)$ are $\lambda_1=-(b-c)$ and $\lambda_2=-b+aK(b-c)$. $\arg(\lambda_1)=\pi$ and for $K<K_c$, $\arg(\lambda_2)=\pi$, thus E_1 is asymptotically stable. If $K>K_c$, then E_1 is unstable.

Finally, consider the case $b>c$ for the equilibrium point E_2. If $K>K_c$, then $\lambda_1=-(b-c)$ and $\lambda_2=b-aK(b-c)$ are negative eigenvalues and $\arg(\lambda_1)=\arg(\lambda_2)=\pi$; consequently, E_2 is asymptotically stable.

4 Theta method for fractional order models

In this section, theta method ($\theta--method$) is described and studied for the numerical solution of fractional–order equation systems by the way of Hantavirus model. Theta method is known as the weighted method and general form given by

$$D^py(t_n)=\theta f(t_n,y_n)+(1-\theta)f(t_{n-1},y_{n-1}) \tag{9}$$

while $\theta \in [0,1]$. There are two specific values of θ: while $\theta = 0$, we obtain the forward explicit Euler method and while $\theta = 1$ yields forward implicit Euler method. Note that $\theta = \frac{1}{2}$ in (9) corresponds the so–called mid–point rule and trapezoidal rule, respectively.

For numerical computation the Grünwald–Letnikov (GL) operator is truncated; we fixed a step–size $h > 0$ and $t_n = t_0 + hn, N = (T - t_0)/h$.

We put the left–hand–side with the forward discrete derivative with

$$h^{-p}\sum_{j=0}^{n} w_j^p (y_{n-j} - y_0) = \theta f(t_n, y_n) + (1-\theta) f(t_{n-1}, y_{n-1}).$$

Case 1. While $\theta = 0$ on (9), in case we use it for the model equation of (6), we can get

$$\begin{cases} h^{-p}\sum\limits_{j=0}^{n} w_j^p (x_{n-j} - x_0) = (b-c)x_{n-1} + by_{n-1} - \frac{x_{n-1}^2}{K} - \left(\frac{1+aK}{K}\right) x_{n-1}y_{n-1}, \\ h^{-p}\sum\limits_{j=0}^{n} w_j^p (y_{n-j} - y_0) = -cy_{n-1} - \frac{y_{n-1}^2}{K} - \left(\frac{1-aK}{K}\right) x_{n-1}y_{n-1}. \end{cases}$$

Here, the left–hand derivatives are Grünwald–Letnikov derivatives.

$$\begin{cases} \sum\limits_{j=0}^{n} w_j^p x_{n-j} - x_0 \sum\limits_{j=0}^{n} w_j^p \\ \qquad\qquad = h^p \left[(b-c)x_{n-1} + by_{n-1} - \frac{x_{n-1}^2}{K} - \left(\frac{1+aK}{K}\right) x_{n-1}y_{n-1}\right], \\ \sum\limits_{j=0}^{n} w_j^p y_{n-j} - y_0 \sum\limits_{j=0}^{n} w_j^p = h^p \left[-cy_{n-1} - \frac{y_{n-1}^2}{K} - \left(\frac{1-aK}{K}\right) x_{n-1}y_{n-1}\right]. \end{cases}$$

In case we leave x_n and y_n alone on the left side,

$$\begin{cases} x_n = h^p \left[(b-c)x_{n-1} + by_{n-1} - \frac{x_{n-1}^2}{K} - \left(\frac{1+aK}{K}\right) x_{n-1}y_{n-1}\right] \\ \qquad\qquad - \sum\limits_{j=1}^{n} w_j^{(p)} x_{n-j} + w_n^{(p-1)} x_0, \\ y_n = h^p \left[-cy_{n-1} - \frac{y_{n-1}^2}{K} - \left(\frac{1-aK}{K}\right) x_{n-1}y_{n-1}\right] - \sum\limits_{j=1}^{n} w_j^{(p)} y_{n-j} - w_n^{(p-1)} y_0. \end{cases} \tag{10}$$

Since $w_1^{(\alpha)} = -p$, we obtain

$$\widetilde{x}_{n-1} = w_1^{(p-1)} x_0, \quad n = 1 \quad \textit{and} \quad w_1^{(p-1)} x_0 - \sum_{j=2}^{n} w_j^{(p)} x_{n-j}, \qquad n \geq 2,$$

and similarly we can say for $\widetilde{y}_{n-1}$. While

$$\begin{cases} w_n^{(p-1)}x_0 - X_n = \widetilde{x}_{n-1} + px_{n-1}, \\ w_n^{(p-1)}y_0 - Y_n = \widetilde{y}_{n-1} + py_{n-1}, \end{cases} \tag{11}$$

and

$$X_n = \sum_{j=1}^{n} w_j^{(p)} x_{n-j}, \qquad Y_n = \sum_{j=1}^{n} w_j^{(p)} y_{n-j} \ .$$

We get (11) in the equations system (10), and we obtain iteration equations as:

$$\begin{cases} x_n = \widetilde{x}_{n-1} + px_{n-1} \\ \qquad\qquad +h^p \left[(b-c)x_{n-1} + by_{n-1} - \frac{x_{n-1}^2}{K} - \left(\frac{1+aK}{K}\right) x_{n-1}y_{n-1}\right], \\ y_n = \widetilde{y}_{n-1} + py_{n-1} + h^p \left[-cy_{n-1} - \frac{y_{n-1}^2}{K} - \left(\frac{1-aK}{K}\right) x_{n-1}y_{n-1}\right]. \end{cases} \tag{12}$$

Case 2. While $\theta = 1$ on (9), use it for Equation (6), we obtain implicit form

$$\begin{cases} h^{-p} \sum\limits_{j=0}^{n} w_j^p (x_{n-j} - x_0) = (b-c)x_n + by_n - \frac{x_n^2}{K} - \left(\frac{1+aK}{K}\right) x_n y_n, \\ h^{-p} \sum\limits_{j=0}^{n} w_j^p (y_{n-j} - y_0) = -cy_n - \frac{y_n^2}{K} - \left(\frac{1-aK}{K}\right) x_n y_n. \end{cases}$$

In case we leave x_n and y_n alone on the left side,

$$\begin{cases} x_n - h^p \left[(b-c)x_n + by_n - \frac{x_n^2}{K} - \left(\frac{1+aK}{K}\right) x_n y_n\right] + X_n - w_n^{(p-1)}x_0 = 0, \\ y_n - h^p \left[-cy_n - \frac{y_n^2}{K} - \left(\frac{1-aK}{K}\right) x_n y_n\right] + Y_n - w_n^{(p-1)}y_0 = 0. \end{cases}$$

While system has implicit form, Newton–Raphson method needed to solve implicit system by converting to explicit form. Here, the implicit form, x_n and y_n connected to the explicit form of the first equation to be able to resort to Newton–Raphson method:

$$\begin{cases} f(z) = z + X_n - w_n^{(p-1)}x_0 - h^p \left[(b-c)z + bu - \frac{z^2}{K} - \left(\frac{1+aK}{K}\right) zu\right], \\ g(u) = u + Y_n - w_n^{(p-1)}y_0 - h^p \left[-cu - \frac{u^2}{K} - \left(\frac{1-aK}{K}\right) zu\right]. \end{cases}$$

Regarding to Newton –Raphson method, iterative equations as:

$$z_{i+1} = z_i - \frac{f(z_i)}{f'(z_i)}, \qquad i = 0, 1, \dots, n,$$
$$u_{i+1} = u_i - \frac{g(u_i)}{g'(u_i)}, \qquad i = 0, 1, \dots, n.$$

Here, we can write initial estimations as $z_0 = x_{n-1}$ and $u_0 = y_{n-1}$. Thus we obtain

$$\begin{cases} z_{i+1} = z_i - \dfrac{z_i - \widetilde{x}_{n-1} - pz_0 - h^p\left[(b-c)z_i + bu_i - \frac{z_i^2}{K} - \left(\frac{1+aK}{K}\right)z_i u_i\right]}{1 - h^p\left[(b-c) - \frac{2z_i}{K} - \left(\frac{1+aK}{K}\right)u_i\right]}, \\ u_{i+1} = u_i - \dfrac{u_i - \widetilde{y}_{n-1} - pu_0 - h^p\left[-cu_i - \frac{u_i^2}{K} - \left(\frac{1-aK}{K}\right)z_i u_i\right]}{1 - h^p\left[-c - \frac{2u_i}{K} - \left(\frac{1-aK}{K}\right)z_i\right]}. \end{cases} \quad (13)$$

5 Nonstandard difference schemes methods for fractional order Hanta model

In this section, we proposed and discussed some NSFD schemes to discretize the fractional order nonlinear system (6). NSFD schemes applied in combination with the truncation of the GL operator. NSFD schemes were firstly proposed by Mickens for Ordinary Differential Equations and successively studied, for instance, in [22, 23]. The stability analysis of model which is discretized with nonstandard finite difference scheme is given by [5, 6].

Case 1: As a first nonstandard scheme, for discretization we make the replacement of the nonlinear term in the right–hand side of (6) by means of $x(t) \to x(t_{n-1})$, $y(t) \to y(t_{n-1})$, $x^2(t) \to x(t_n)x(t_{n-1})$, $x(t)y(t) \to x(t_n)y(t_{n-1})$.

Since $w_1^p = 1$, the discretized model is

$$x_n + X_n - w_n^{p-1}x_0 = \phi(h)\overset{*}{X},$$
$$y_n + Y_n - w_n^{p-1}y_0 = \phi(h)\overset{*}{Y},$$

which seem as explicit form and $\phi(h)$ is a denominator function and could be selected in a different way. Here, $X_n, Y_n, \overset{*}{X}$ and $\overset{*}{Y}$ are defined as:

$$X_n = \sum_{j=1}^{n} w_j^{(p)} x_{n-j}, \qquad Y_n = \sum_{j=1}^{n} w_j^{(p)} y_{n-j},$$

$$\begin{cases} \overset{*}{X} = (b-c)x_{n-1} + by_{n-1} - \frac{(x_{n-1}+y_{n-1})x_n}{K} - ax_n y_{n-1}, \\ \overset{*}{Y} = -cy_{n-1} - \frac{(x_{n-1}+y_{n-1})y_n}{K} - ax_n y_{n-1}, \end{cases}$$

$$\begin{cases} x_n = \frac{K\left(\phi(h)(b-c)x_{n-1}+\phi(h)by_{n-1}-X_n+w_n^{p-1}x_0\right)}{K+\phi(h)(x_{n-1}+y_{n-1})+K\phi(h)ay_{n-1}}, \\ \\ y_n = \frac{K\left(-\phi(h)cy_{n-1}-a\phi(h)x_n y_{n-1}-Y_n+w_n^{p-1}y_0\right)}{K+\phi(h)(x_{n-1}+y_{n-1})}. \end{cases}$$

The result of explicit nonstandard finite differences scheme is

$$\begin{cases} x_n = \frac{K(\widetilde{x}_{n-1}+(p-c\phi(h))x_{n-1}+b\phi(h)(x_{n-1}+y_{n-1}))}{K+\phi(h)(x_{n-1}+y_{n-1})+aK\phi(h)y_{n-1}}, \\ \\ y_n = \frac{K(\widetilde{y}_{n-1}+(p-c\phi(h))y_{n-1}+a\phi(h)x_n y_{n-1})}{K+\phi(h)(x_{n-1}+y_{n-1})}, \end{cases} \tag{14}$$

where we selected $\phi(h) = \left(\frac{\exp(h(b-c))-1}{b-c}\right)^p$. In [32, 33], authors have chosen $(xy) \to x(t+h)y(t)$, $x^2 \to x(t)x(t+h)$ and $y^2 \to y(t)y(t+h)$ for discretization.

Proposition 1. *Let $a, b, c, K > 0$, $b \leq c$, and let step size $0 < h \leq (p/c)^{(1/p)}$ for the schemes have the iterations x_n, y_n given by the non–negative schemes for any $x_0 \geq 0$ and $y_0 \geq 0$.*

Proof. We proceed by induction on n. □

Case 2: In the second nonstandard scheme we use the replacement $x(t) \to x(t_n)$, $y(t) \to y(t_n)$, $x^2(t) \to x(t_n)x(t_n)$, $x(t)y(t) \to x(t_n)y(t_n)$. By operating in a similar way as in the previous case for system (6) , we can see that is implicit form:

$$x_n + X_n - w_n^{p-1}x_0 = \phi(h)\overset{**}{X},$$
$$y_n + Y_n - w_n^{p-1}y_0 = \phi(h)\overset{**}{Y},$$

$$\begin{cases} \overset{**}{X} = (b-c)x_n + by_n - \frac{(x_n+y_n)x_n}{K} - ax_n y_n, \\ \overset{**}{Y} = -cy_n - \frac{(x_n+y_n)y_n}{K} - ax_n y_n. \end{cases}$$

If we leave alone to x_n on the left side and y_n on the right side, we obtain implicit form:

$$\begin{cases} x_n - \phi(h)\overset{**}{X} + X_n - w_n^{(p-1)}x_0 = 0, \\ y_n - \phi(h)\overset{**}{Y} + Y_n - w_n^{(p-1)}y_0 = 0. \end{cases}$$

While Newton–Raphson method is needed, here, the implicit form, x_n and y_n connected to the explicit form of the first equation to be able to resort to the method of Newton–Raphson, we leave alone on the left side:

$$\begin{cases} f(z) = z + X_n - w_n^{(p-1)} y_0 - \phi(h) \left[(b-c)z + bu - \frac{z^2}{K} - \left(\frac{1+aK}{K} \right) zu \right], \\ g(u) = u + Y_n - w_n^{(p-1)} u_0 - \phi(h) \left[-cu - \frac{u^2}{K} - \left(\frac{1-aK}{K} \right) zu \right]. \end{cases}$$

Regarding to Newton–Raphson method, the initial estimations as $z_0 = x_{n-1}$ and $u_0 = y_{n-1}$. In this way, we can obtain

$$\begin{cases} z_{i+1} = z_i - \dfrac{z_i - \widetilde{x}_{n-1} - pz_0 - \phi(h) \left[(b-c)z_i + bu_i - \frac{z_i^2}{K} - \left(\frac{1+aK}{K} \right) z_i u_i \right]}{1 - \phi(h) \left[(b-c) - \frac{2z_i}{K} - \left(\frac{1+aK}{K} \right) u_i \right]}, \\ \\ u_{i+1} = u_i - \dfrac{u_i - \widetilde{y}_{n-1} - pu_0 - \phi(h) \left[-cu_i - \frac{u_i^2}{K} - \left(\frac{1-aK}{K} \right) z_i u_i \right]}{1 - \phi(h) \left[-c - \frac{2u_i}{K} - \left(\frac{1-aK}{K} \right) z_i \right]}. \end{cases} \tag{15}$$

6 Numerical simulations

In this section, numerical simulations of fractional–order Hantavirus model are presented by the solution of Theta method and NSFD schemes that are explicit and implicit form. We choose $a = 0.1$, $b = 1$, and $c = 0.5$ as used by study of [2]. The initial values is considered as $(x_0, y_0) = (25, 15)$; we fixed a step–size $h = 0.01$ and $N = 2000$. Various denominator functions listed in [29], we took into account the $\phi(h) = \left(\frac{\exp(h(b-c))-1}{b-c} \right)^p$. We choose $p = 1$ in Figure 1, $p = 0.6$ in Figure 2, $p = 0.3$ in Figure 3, $p = 1$ in Figure 4, and $(x_0, y_0) = (10, 10)$ and $K = 20$ used in Figure 5. In Figures 1–4, the top graphs in order of left–to–right are theta explicit and theta implicit methods, the bottom graphs in order of left–to–right are NSFD explicit and NSFD implicit methods. Figure 5 shows that for $p = 1$, $a = 0.1$, $b = 1$, $c = 0.5$ for x(t), relative errors between numerical schemes and exponential matrix method in [39]. All figures denoted the results of the simulation based on equation systems (12), (13), (14), and (15) the values are used as mentioned above. Equation systems (12), (13), (14), and (15) correspond to equations in (1).

We compared CPU times of numerical methods on the Table 1. Here, we used $N = 1000$ for all methods except Runge Kutta methods that we used $N = 50$. As we have seen in the Table 1, we can say, in case each explicit and implicit methods evaluated between oneself has not extremely difference. However we cannot say the same for the Runge Kutta. On the Table 2, we denoted qualitative results of the fixed point E_2 for different time step sizes for $N = 1000$, $p = 1$, $(x_0, y_0) = (25, 15)$, and $K = 40$.

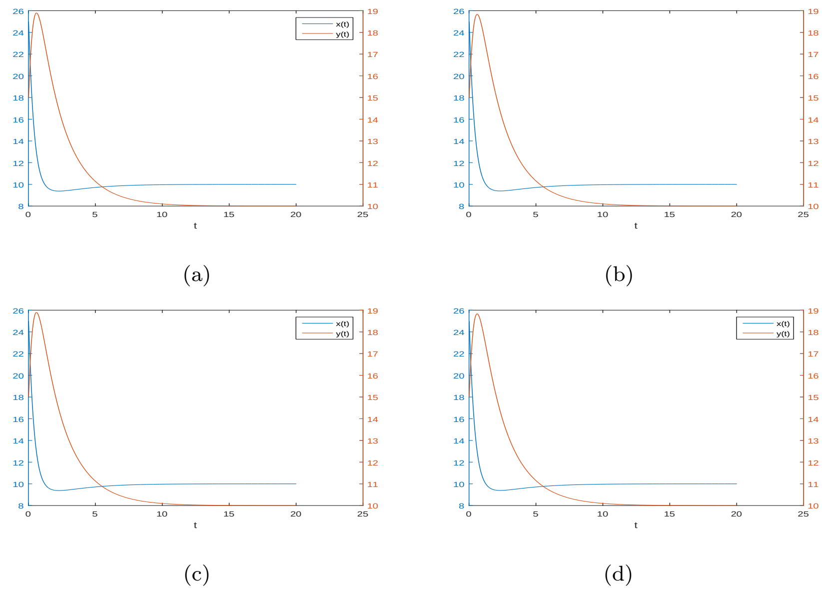


Figure 1: For p=1, Theta and NSFD Schemes in explicit and implicit form:
(a) Theta–explicit (b) Theta–implicit (c) NSFD–explicit (d) NSFD–implicit

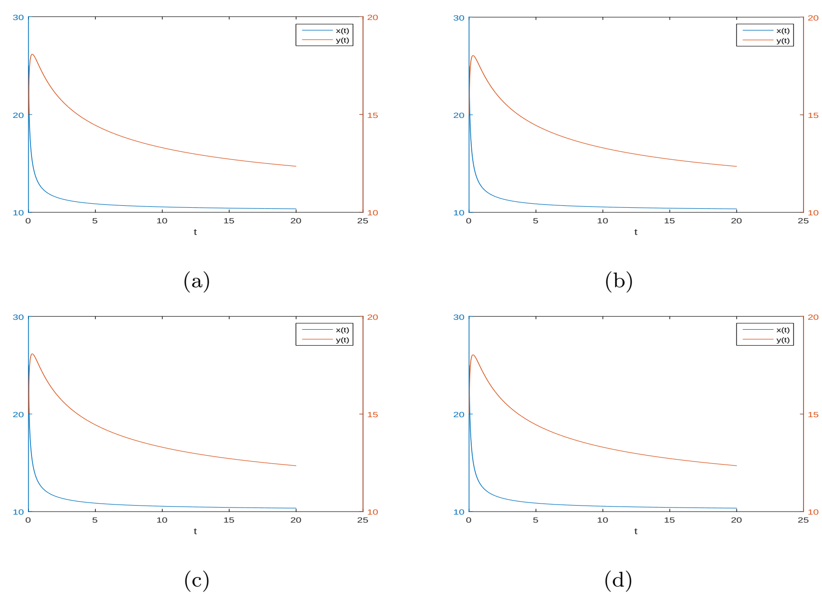
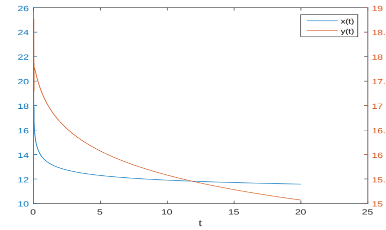
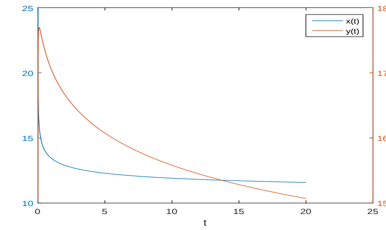


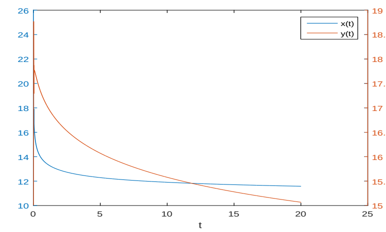
Figure 2: For p=0.6, Theta and NSFD Schemes in explicit and implicit form:
(a) Theta–explicit (b) Theta–implicit (c) NSFD–explicit (d) NSFD–implicit



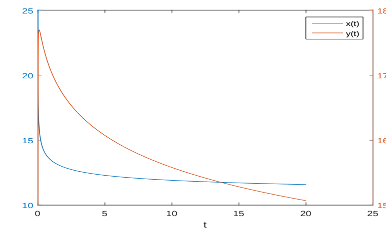
(a)



(b)

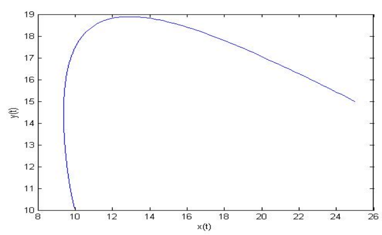


(c)

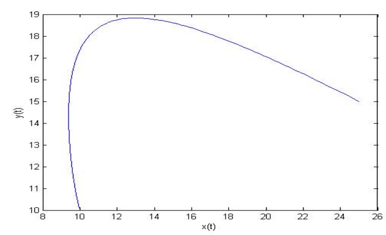


(d)

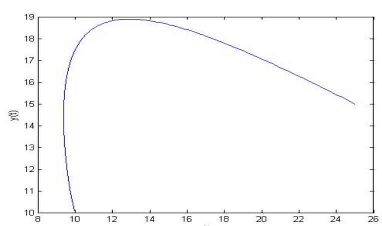
Figure 3: For p=0.3, Theta and NSFD Schemes in explicit and implicit form:
(a) Theta–explicit (b) Theta–implicit (c) NSFD–explicit (d) NSFD–implicit



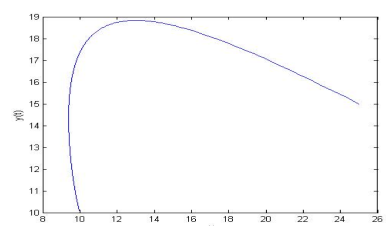
(a)



(b)



(c)



(d)

Figure 4: Phase–portraits for $p = 1$, $N = 2000$, $a = 0.1$, $b = 1$, $c = 0.5$, $h = 0.01$, $(x_0, y_0) = (25, 15)$, $K = 40$
(a) Theta–explicit (b) Theta–implicit (c) NSFD–explicit (d) NSFD–implicit

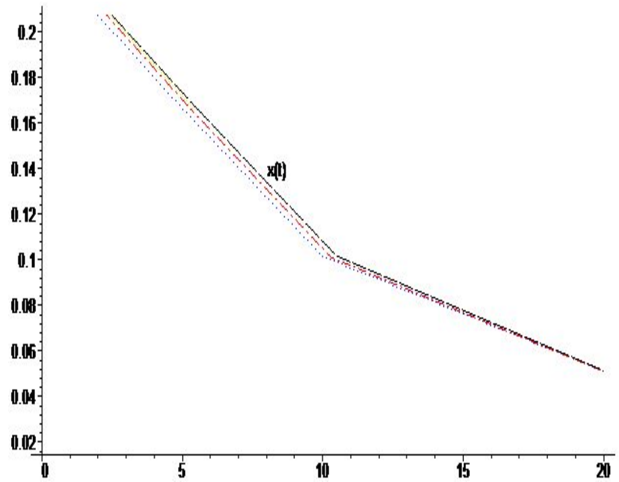


Figure 5: For $p = 1$, $a = 0.1$, $b = 1$, $c = 0.5$, $(x_0, y_0) = (10, 10)$, $K = 20$, for x(t), relative errors between numerical schemes and exponential matrix method in [39]
$- - --$ EMM and Theta-explicit
. . . . . . EMM and NSFD implicit
.... EMM and Theta-implicit
-.-.-.-.-.-. EMM and NSFD-explicit

Table 1: CPU times (seconds) for $(x_0, y_0) = (25, 15), h = 0.01$

p	Theta Explicit	Theta Implicit	NSFD Explicit	NSFD Implicit	Runge Kutta
0.3	1.912602	2.216342	2.259337	1.813677	3.597516
0.6	1.909790	2.522203	2.331288	1.77084	3.6738
1	1.86862	2.273820	4.053474	1.767188	4.222919

Table 2: Qualitative results of the fixed point E_2 for different time step sizes for $N = 1000, P = 1, (x_0, y_0) = (25, 15), K = 40$(Con. and Div., resp., Convergence and Divergence)

h	Theta Explicit	Theta Implicit	NSFD Explicit	NSFD Implicit	Runge Kutta
0.01	$Con.$	$Con.$	$Con.$	$Con.$	$Con.$
0.1	$Con.$	$Con.$	$Con.$	$Con.$	$Con.$
1	$Div.$	$Con.$	$Div.$	$Con.$	$Div.$
10	$Div.$	$Con.$	$Div.$	$Con.$	$Div.$
100	$Div.$	$Con.$	$Div.$	$Con.$	$Div.$

7 Conclusion

In this paper, fractional–order form of the Hantavirus model is introduced. The stability of the equilibrium points is studied. Two numerical methods have been presented in the explicit and implicit form for solving fractional order Hantavirus epidemic model. When we use implicit solutions, we need to use the Newton–Raphson method to solve implicit system by converting to explicit form. For NSFD schemes, one denominator function and different nonlocal terms have been proposed, and the results have been compared with each other. As it seen clearly that explicit and implicit NSFD methods should be used ultimately depending on the choices of the nonlocal terms. On the other hand, explicit methods produce the same accuracy, but with less computational effort and time than implicit methods.

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طرح های صریح و ضمنی برای مدل ویروس هانته از مرتبه کسری

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چکیده :
در این مقاله، یک مدل از مرتبه کسری برای جمعیت موش ها معرفی شده است. برخی از طرح های صریح و ضمنی مانند روش های تتا و تفاضلات متناهی غیر استاندارد (NSFD) برای حل عددی دستگاه معادلات دیفرانسیل معمولی غیر خطی به نام مدل ویروس اپیدمی هانته اجرا شده است. این روش ها با هم مقایسه شده و بحث شده است که این روش خواص مثبت بودن جواب را در حالت سیستم های از مرتبه صحیح را حفظ می کنند. جوا بهای عددی با استفاده از چند نمودار تشریح شده است. نتایج عددی روش های صریح و ضمنی نشان می دهد که این روش ها زمانی که برای مدل ویروس هانته از مرتبه کسری به کار برده می شوند آسان و دارای دقت خوبی هستند.

کلمات کلیدی : روش های صریح و ضمنی؛ روش تتا؛ روش تفاصلات متناهی غیر استاندارد؛ دستگاه معادلات دیفرانسیل غیر خطی از مرتبه کسری؛ مدل جمعیت موش ها.