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A New Hybrid Method Based on Pseudo Differential Operators and Haar Wavelet to Solve ODEs

M.B. Ghaemi*, M. Jamalpour Birgani and E. Nabizadeh Morsalfard

Abstract

In this paper we present a new and efficient method by combining pseudo differential operators and Haar wavelet to solve the linear and nonlinear differential equations. The present method performs extremely well in terms of efficiency and simplicity.

Keywords: Haar wavelet; Fourier transform; Differential equations; Numerical solution; Pseudo differential operators.

1 Introduction

In this paper we have offered a method based on Haar wavelet in combination with pseudo differential operators to solve ODEs.

Recently, wavelet methods have proved useful as widely applied tools for the computation of numerical approximations. Different types of wavelets and approximating functions have been proposed for the numerical solution of differential equations using various methods [2,3,9,15–20], such as Galerkin and collocations methods, the filter bank methods [20], and the Kalman filters. In [8], the Haar wavelet method was applied for solving the boundary

*Corresponding author
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M.B. Ghaemi
Department of Mathematics, Iran University of Science and Technology, Tehran, Iran. e-mail: mghaemi@iust.ac.ir

M. Jamalpour Birgani
Department of Mathematics, Iran University of Science and Technology, Tehran, Iran. e-mail: m.j.birgani@gmail.com

E. Nabizadeh Morsalfard
Department of Mathematics, Iran University of Science and Technology, Tehran, Iran. e-mail: elmira_nabizadeh@yahoo.com

and initial value differential equations. Lepik in [12, 14] applied Haar wavelet to solve differential equations. Also, he used Haar wevelet for solving evolution and integral equations in [11, 13] and PDEs in [10].

pseudo differential operators are considered as natural extensions of linear partial differential operators. The study of pseudo differential operators grows out of research in 1960s on the singular integral operators. The pseudo differential operator is the generalization of linear partial differential operator, with roots entwined deep in solving differential equations in the classic and quantum mechanics [4, 5].

At first by applying Fourier inversion formula to the given differential equation we obtain its associated pseudo differential equation. Then in the obtained Pseudo differential operator we expand $\hat{f}$ (f is assumed to be the solution of ODE) into Haar series, and we employ the collocation method for the equation to get a numerical solution for ODE. By some numerical examples the advantages of our tool in comparison with Haar wavelet method are shown.

The paper is organized in 5 sections in the following way. In section 2 we recall some preliminaries. Some numerical solutions for some examples of linear and nonlinear differential equations by using our new method are presented in sections 3 and 4, respectively. Finally error analysis is given in section 5.

2 Preliminaries

2.1 Wavelets

Haar wavelets are the simplest wavelets among various types of wavelets. The Haar wavelet family on $[0, 1]$ is defined as follows:

$$h_i(x) = \begin{cases} 1 & for\ x \in [\xi_1, \xi_2), \\ -1 & for\ x \in [\xi_2, \xi_3], \\ 0 & elsewhere, \end{cases}$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k + \frac{1}{2}}{m}, \quad \xi_3 = \frac{k+1}{m},$$

and $m = 2^j$, $j = 0, 1, \ldots, J$, indicate the level of the wavelet, and $k = 0, 1, \ldots, m-1$ is the translation parameter. The index i is calculated according to the formula $i = m+k+1$. The maximal value of i is $2M$ where $M = 2^J$. It is assumed that the value $i = 1$ corresponds to the scaling function of Haar wavelet $h_1(x) = 1$ for all $x \in [0, 1]$.

2.2 Fourier Transform and Pseudo Differential Operators

One of the leading ideas in the theory of pseudo differential operators is to reduce the study of properties of a linear differential operator

$$P = \sum_{\alpha} c_{\alpha}(x)\partial_x^{\alpha}, \tag{1}$$

which is a polynomial in the derivatives ∂_x with constants c_α depending on x, to its *symbol*

$$p(x,\xi) = \sum_{\alpha} c_{\alpha}(x)(i\xi)^{\alpha},$$

which is a polynomial in the phase variable $\xi \in \mathbb{R}$ with constants depending on the space variable x.

Fourier transform is the main tool in the theory of pseudo differential operators. Let f be integrable function; the Fourier transform and the inverse Fourier transform of f are defined as follows:

$$\hat{f}(\xi) := \mathfrak{F}[f](\xi) := \int_R e^{-ix\xi} f(x)dx$$

and

$$\mathfrak{F}^{-1}[\hat{f}](x) := \frac{1}{2\pi}\int_R e^{ix.\xi}\hat{f}(\xi)d\xi, \tag{2}$$

where $x, \xi \in R$. According to the inversion formula we have

$$\mathfrak{F}^{-1}[\mathfrak{F}(f)] \equiv \mathfrak{F}^{-1}[\hat{f}] = f. \tag{3}$$

Let P be the linear differential operator in (1) using the inversion formula,

$$\begin{aligned} Pf &= \sum_{\alpha} c_{\alpha}(x)\partial_x^{\alpha}\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix.\xi}\hat{f}(\xi)d\xi \\ &= \sum_{\alpha} c_{\alpha}(x)\frac{1}{2\pi}\int_{\mathbb{R}} (i\xi)^{\alpha} e^{ix.\xi}\hat{f}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix.\xi} p(x,\xi)\hat{f}(\xi)d\xi. \end{aligned}$$

Here p is considered as a symbol, and

$$(P(x,D_x)f)(x) := \int_{R^n} e^{ix.\xi} p(x,\xi)\hat{f}(\xi)d\xi$$

defines the associated *pseudo differential operator*. For more details one can refer to [1].

2.3 Combination of Haar Wavelet and Pseudo Differential Operators

Let $f \in L^2[0,1]$; then by (2) and (3), it can be approximated by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\hat{f}(\xi)d\xi, \tag{4}$$

since $f \in L^2$ then $\hat{f} \in L^2$, and for level J we can write

$$\hat{f}(\xi) = \sum_{p=1}^{2M} a_p h_p(\xi),$$

where $a_p = \int_0^1 h_p(\xi)\hat{f}(\xi)d\xi,\ p = 1,\dots,2M$. By replacing this value in (4) we obtain

$$f(x) = \sum_{p=1}^{2M} a_p \int_0^1 e^{ix\xi}h_p(\xi)d\xi. \tag{5}$$

With differentiation of (5) we have

$$f'(x) = \sum_{p=1}^{2M} a_p \int_0^1 i\xi e^{ix\xi}h_p(\xi)d\xi, \qquad f''(x) = \sum_{p=1}^{2M} a_p \int_0^1 (i\xi)^2 e^{ix\xi}h_p(\xi)d\xi,$$

in other words

$$f^{(n)}(x) = \sum_{p=1}^{2M} a_p \int_0^1 (i\xi)^n e^{ixy}h_p(\xi)d\xi, \qquad for\ all\ \ n \in N. \tag{6}$$

Now consider the linear differential equation $\sum_{\alpha=0}^{n} A_\alpha(x)f^{(n)}(x)$, by replacing the obtained value for $f, f', \dots, f^{(n)}$ in this equation we obtain

$$\sum_{\alpha=0}^{n} A_\alpha(x)f^{(\alpha)}(x) = \sum_{p=1}^{2M} a_p \int_0^1 p(x,\xi)e^{ix\xi}h_p(\xi)d\xi,$$

where $p(x,\xi) = \sum_{\alpha=0}^{n} A_\alpha(x)(i\xi)^\alpha$. This new form of pseudo differential operators is our main tool in order to obtain numerical solution for linear and nonlinear differential equations in the next sections.

2.4 Solving Linear Differential Equations

Consider the differential equation of order n

$$\sum_{m=0}^{n} A_m(x)y^{(m)}(x) = f(x), \tag{7}$$

where $x \in [0,1]$ and $y^{(n-1)}(0) = c_n, \ldots, y(0) = c_0$. The interval $[0,1]$ is divided into $2M$ subintervals with the length $\Delta x = \frac{1}{2M}$.
According to the equality (6) we have

$$y^{(n)}(x) = \sum_{p=1}^{2M} a_p \int_0^1 (i\xi)^n e^{ixy} h_p(\xi) d\xi, \tag{8}$$

where a_p, with $p = 1, \ldots, 2M$ are unknown. Performing integration on (8) from 0 to x, yields

$$y^{(n-1)}(x) = \sum_{p=1}^{2M} a_p [\int_0^1 (i\xi)^{n-1} e^{ix\xi} h_p(\xi) d\xi - \int_0^1 (i\xi)^{n-1} h_p(\xi) d\xi] + c_n \tag{9}$$

continuing the integration of (9), we get

$$\begin{aligned} y^{(m)}(x) = \sum_{p=1}^{2M} a_p [\int_0^1 (i\xi)^m e^{ixy} h_p(\xi) d\xi - x^{n-m-1} \int_0^1 (i\xi)^m h_p(\xi) d\xi] \\ + \sum_{\alpha=0}^{n-m-1} \frac{1}{\alpha!} x^\alpha y^{(\alpha+m)}(0) \end{aligned} \tag{10}$$

for $m = 0, 1, \ldots, n-1$. Putting the values obtained for $y^{(n)}$, $y^{(n-1)}, \ldots, y$ and the collocation points

$$x_l = \frac{(l - 0.5)}{2M}, \qquad l = 1, 2, \ldots, 2M, \tag{11}$$

in equation (7), the system below is obtained.

$$\begin{aligned} \sum_{p=1}^{2M} a_p \sum_{m=0}^{n} S_m(x) = f(x_l) \\ + \sum_{\nu=0}^{n-1} \sum_{\alpha=0}^{n-\nu-1} \frac{1}{\alpha!} (x_l)^\alpha y^{(\alpha+\nu)}(0) A_\nu(x_l), \end{aligned} \tag{12}$$

for $l = 1, \ldots, 2M$, where

$$S_m(x) = A_m(x)[\int_0^1 (i\xi)^m e^{ix\xi} h_p(\xi) d\xi - x^{n-m-1} \int_0^1 (i\xi)^m h_p(\xi) d\xi].$$

By calculating the coefficients a_p, $p = 1, \dots, 2M$, and replacing them in (10), for $m = 0$, an approximation numerical solution is obtained. The matrix form of the system (12) is

$$a(\sum_{m=0}^{n} W_m) = F,$$

where W_m signifies $2M \times 2M$ matrices as follows:

$$W_m(p, l) = A_m(x_l) S_m(p, l), \qquad m = 0, 1, \dots, n,$$

such that

$$\begin{aligned} S_m(p, l) &= S_m(x_l), \\ S_n(p, l) &= \int_0^1 (i\xi)^n e^{ix_l\xi} h_p(\xi) d\xi, \end{aligned}$$

and F and a are $2M-$dimensional vectors

$$\begin{aligned} F(l) &= f(x_l) + \sum_{\nu=0}^{n-1} \sum_{\alpha=0}^{n-\nu-1} \frac{1}{\alpha!} (x_l)^\alpha y^{(\alpha+\nu)}(0) A_\nu(x_l), \\ a &= (a_1, \dots, a_{2M}). \end{aligned}$$

Also, the matrix representation of (1), for $m = 0$, is

$$y = aS_0 + T,$$

where the vector T is

$$T(l) = \sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} (x_l)^\alpha y^{(\alpha)}(0), \qquad l = 1, \dots, 2M.$$

Remark. For calculating the integrals, for example,

$$\int_0^1 (i\xi)^k e^{ix_l\xi} h_p(\xi) d\xi, \quad k = 0, 1, \dots, n,$$

we can use of the points $\tau_l = \frac{(l-1)}{2M}, \quad l = 1, \dots, 2M + 1$. In other words,

$$\int_0^1 (i\xi)^k e^{ix_l\xi} h_p(\xi) = \sum_{s=1}^{2M} \int_{\tau_s}^{\tau_{s+1}} (i\xi)^k e^{ix_l\xi} h_p(\xi) d\xi.$$

Since for $\xi \in (\tau_s, \tau_{s+1})$ we have $h_p(\xi) = h_p(x_s)$; then

$$\int_0^1 (i\xi)^k e^{ix_l\xi} h_p(\xi) = \sum_{s=1}^{2M} h_p(x_s) \int_{\tau_s}^{\tau_{s+1}} (i\xi)^k e^{ix_l\xi} d\xi.$$

3 The Numerical Solution of Linear Differential Equations

In this section, some examples of linear differential equations are solved by using the new method. To get the error estimates, problems with the known exact solution $y_{ex}(x)$ are considered.

Example 1. Consider the differential equation

$$(x-1)y'(x) + (x^2+3)y(x) = x^4 + 7x^2 - 2x + 6, \tag{13}$$

where $y(0) = 0$. This equation has the exact solution $y_{ex}(x) = 2 + x^2$.
Now, we seek solutions for the equation (13) based on our hybrid method. In this equation

$$W_0(p,l) = (x_l^2 + 3)S_0(p,l),$$

$$W_1(p,l) = (x_l - 1)S_1(p,l),$$

$$F(l) = x_l^4 + 7x_l^2 - 2x_l + 6.$$

The obtained results and absolute errors, for some values of $J = 2$, are shown in Table 1 and Figure 1, respectively.

Example 2. Consider the differential equation

$$\begin{aligned} &(x-1)y''(x) + (x+1)y'(x) + y(x) = (2x^2 + 4x - 1)e^x, \\ &y(0) = 0, \qquad y'(0) = 1. \end{aligned} \tag{14}$$

This equation has the exact solution $y_{ex}(x) = xe^x$. In this equation

$$W_0(p,l) = S_0(p,l),$$

$$W_1(p,l) = (x_l + 1)S_1(p,l),$$

$$W_2(p,l) = (x_l - 1)S_2(p,l),$$

$$F(l) = (2x_l^2 + 4x_l - 1)e^{x_l} - (1 + 2x_l).$$

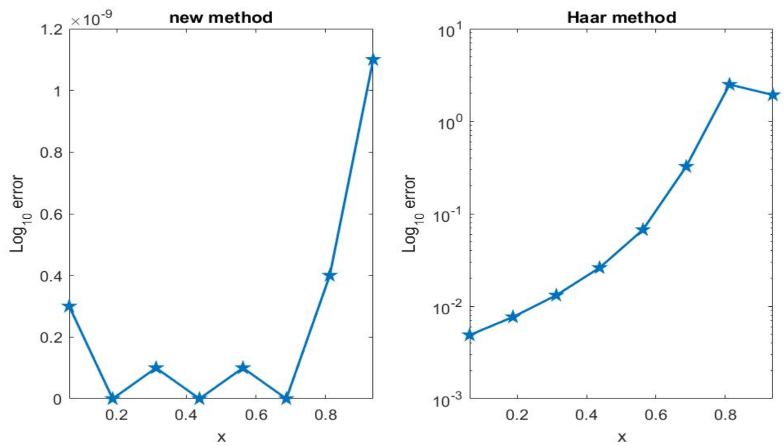

Figure 1: The absolute errors for example 1.

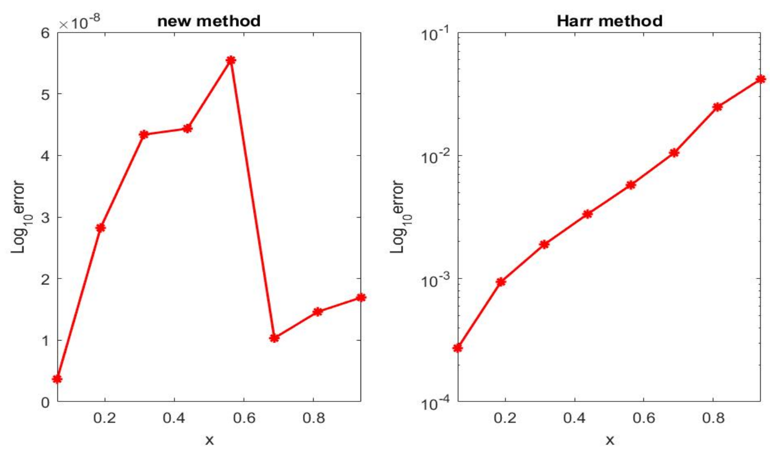

Figure 2: The absolute errors for example 2.

Table 1: Haar solution, Exact solution, The new method solution for equation (13)

$(x/16)$	Haar solution	Exact solution	The new method solution
1	2.0087906524	2.0039062500	2.0039062497
3	2.04280445967	2.0351562500	2.0351562500
5	2.11078811462	2.0976562500	2.0976562499
7	2.21748292022	2.1914062500	2.1914062500
9	2.38353209120	2.3164062500	2.3164062499
11	2.79651329536	2.4726562500	2.4726562500
13	0.16626536933	2.6601562500	2.6601562496
15	4.80205593409	2.8789062500	2.8789062511

The obtained results and absolute errors, for some values of J, are shown in Table 2 and Figure 2, respectively.

4 Nonlinear Differential Equations

In this section we adopt the new approach for nonlinear differential equations. A numerical solution of nonlinear differential equations can be obtained by using the new hybrid method. We consider a nonlinear form of differential equations as

$$K(x, u(x), u'(x), \ldots + u^{(n)}(x)) = f(x) \tag{15}$$

such that $x \in [0, 1]$ and $u^{(n-1)}(0) = c_n, \ldots, u(0) = c_0$. K is nonlinear respect to one of the functions u, u', … $u^{(n)}$ and the function f is given. Replacing x by x_l in the equation (15), for $l = 1, 2, \ldots, 2M$, gives

$$K(x_l, u(x_l), u'(x_l), \ldots + u^{(n)}(x_l)) = f(x_l),$$

where x_1, x_2, …, and x_{2M} are the collocation points introduced in (11). Now, by applying (8) and (9) for the functions $u, u', \ldots, u^{(n)}$, we get a system with $2M$ equations for calculating the coefficients a_p. In general, this system is nonlinear, and we use the following approach to solve it.

Table 2: Haar solution, Exact solution, The new method solution for equation (14)

$(x/16)$	Haar solution	Exact solution	The new method solution
1	0.0668034	0.0665309	0.0665309
3	0.2271089	0.2261681	0.2261682
5	0.4290282	0.4271368	0.4271369
7	0.6809506	0.6776132	0.6776133
9	0.9929719	0.9872182	0.9872183
11	1.3777507	1.3672570	1.3672570
13	1.8556966	1.8309970	1.8309970
15	2.4355610	2.3939901	2.3939901

We assume that (15) is previously solved for the level $J-1$, where the number of collocation points is $M = 2^J$. For the next level, the number of collocation points is doubled (the value J is increased by one). The new values for $u(x), u'(x), \ldots, u^{(n)}(x)$ and a_p are estimated as

$$\hat{u}^{(n)(J)}(x) = \sum_{p=1}^{2M} \hat{a}_p^{(J)} \int_0^1 (i\xi)^n e^{ix\xi} h_p(\xi) d\xi, \tag{16}$$

and for $m = 0, \ldots, n-1$

$$\begin{aligned} \hat{u}^{(m)(J)}(x) = & \sum_{p=1}^{2M} \hat{a}_p^{(J)} [\int_0^1 (i\xi)^m e^{ix\xi} h_p(\xi) d\xi - x^{n-m-1} \int_0^1 (i\xi)^m h_p(\xi) d\xi] \\ & + \sum_{\alpha=0}^{n-m-1} \frac{1}{\alpha!} x^\alpha u^{(\alpha+m)}(0), \end{aligned} \tag{17}$$

where

$$\hat{a}_p^{(J)} = \begin{cases} \hat{a}_p^{(J-1)} & \text{for } p = 1, \ldots, M, \\ 0 & \text{for } p = M+1, \ldots, 2M. \end{cases} \tag{18}$$

These estimates are corrected by Newton method; the application of which for $l = 1, \ldots, 2M$ leads to the equation

$$\sum_{p=1}^{2M}(\frac{\partial K}{\partial a_p})\Delta a_p = -K(x_l, \hat{u}^{(J)}(x_l), \hat{u}'^{(J)}(x_l), \ldots, \hat{u}^{(n)(J)}(x_l)) + f(x_l), \quad (19)$$

where

$$\frac{\partial K}{\partial a_p} = \frac{\partial K}{\partial u}\frac{\partial u}{\partial a_p} + \frac{\partial K}{\partial u'}\frac{\partial u'}{\partial a_p} + \cdots + \frac{\partial K}{\partial u^{(n)}}\frac{\partial u^{(n)}}{\partial a_p}.$$

By solving (19), Δa_p, $p = 1, \ldots, 2M$, are calculated and equations (16), (17), and (18) are corrected as follows:

$$a_p^{(J)} = \hat{a}_p^{(J)} + \Delta a_p,$$

$$u^{(n)(J)}(x) = \sum_{p=1}^{2M} a_p^{(J)} \int_0^1 (i\xi)^n e^{ix\xi} h_p(\xi) d\xi, \quad (20)$$

and

$$u^{(m)(J)}(x) = \sum_{p=1}^{2M} a_p^{(J)} [\int_0^1 (i\xi)^m e^{ixy} h_p(\xi) d\xi - x^{n-m-1} \int_0^1 (i\xi)^m h_p(\xi) d\xi] + \sum_{\alpha=0}^{n-m-1} \frac{1}{\alpha!} x^\alpha u^{(\alpha+m)}(0), \qquad m = 0, \ldots, n-1.$$

We start the procedure by taking an initial solution of $\hat{a}_1^{(0)} = 1$ and $\hat{a}_2^{(0)} = 0$. These estimates are corrected by solving (19) for $J = 0$.

Example 3. Consider the the differential equation

$$u(x) + x^2 u(x) u'(x)^{\frac{1}{2}} = x^2(1 + \sqrt{2}x^{\frac{5}{2}}), \quad (21)$$

which has the exact solution $u_{ex}(x) = x^2$.
The obtained results for $J = 2$ are shown in Table 3 and Figures 3 and 4.

Example 4. Consider the the differential equation

$$x \cos x u'(x) + (x+3)u(x) + x u(x)^2 = x(1 + \sin x) + 3 \sin x, u(0) = 0, \quad (22)$$

which has the exact solution $u_{ex}(x) = \sin x$. The obtained results for $J = 2$ are shown in Table 4 and Figures 5 and 6.

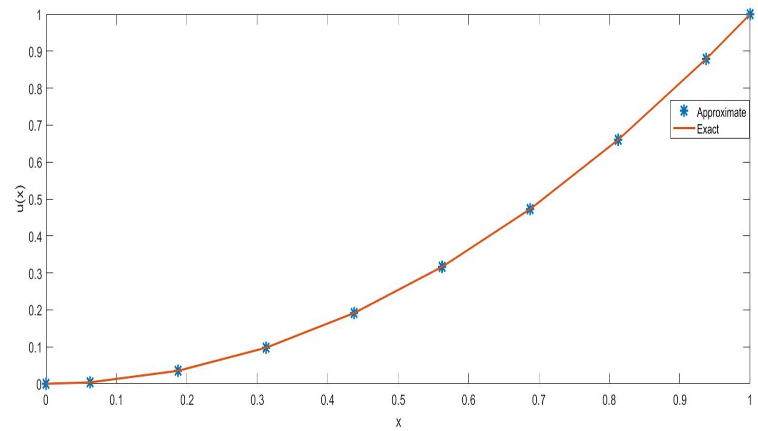

Figure 3: The comparison of the numerical solution and exact solution for example 3.

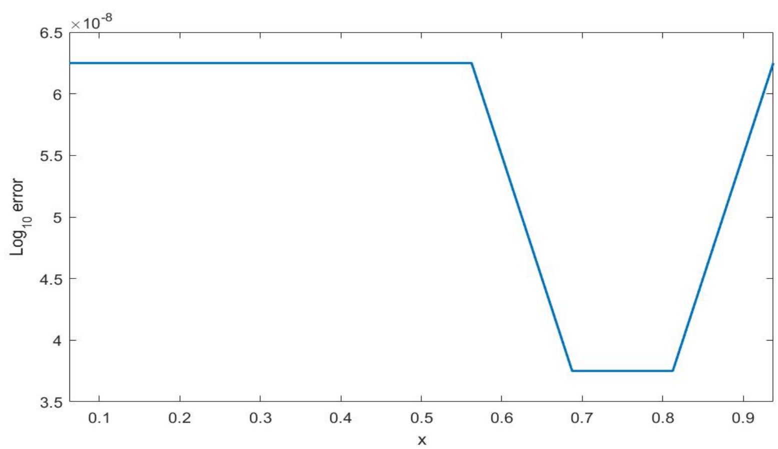

Figure 4: The absolute errors for example 3.

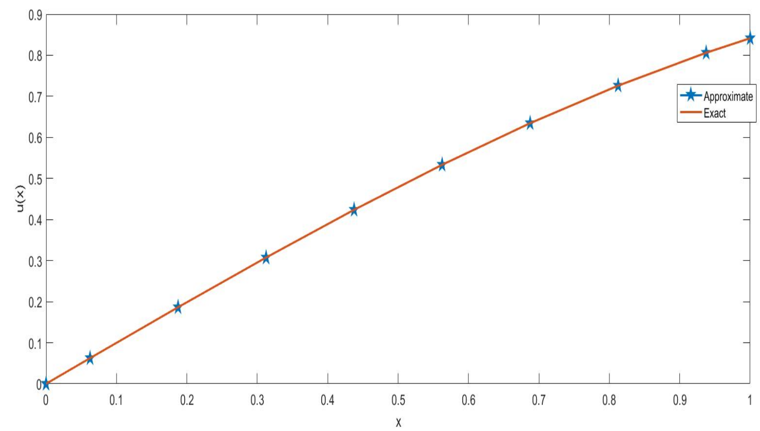

Figure 5: The comparison of the numerical solution and exact solution for Example 4.

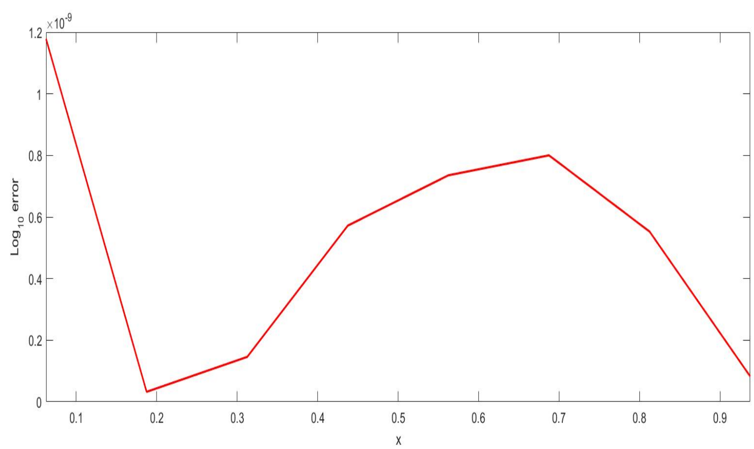

Figure 6: The absolute errors for example 4.

Table 3: Exact solution and The new method solution for equation (21)

$(x/16)$	Exact solution	The new method solution
1	0.00390	0.00390
3	0.03515	0.03515
5	0.09765	0.09765
7	0.19140	0.19140
9	0.31640	0.31640
11	0.47265	0.47266
13	0.66015	0.66016
15	0.87890	0.87890

5 Error analysis

In this section, the error analysis for the new method has been offered.

Lemma *Let $f \in L^2(R)$ be a continuous function defined in $(0,1)$. Then the error norm at Jth level satisfies the following inequalities*

$$E_J \leq \frac{3K^2}{16} 2^{-3J},$$

where $f(x) \leq K$, for all $x \in (0,1)$ and $K > 0$, and M is a positive number related to the Jth level resolution of the wavelet given by $M = 2^J$

Proof.

$$|E_J| = |u(x) - u_J(x)| = |\sum_{p=2M+1}^{\infty} a_p \int_0^1 e^{ix\xi} h_p(\xi) d\xi|,$$

where

$$u_J(x) = \sum_{p=1}^{2M} a_p \int_0^1 e^{ix\xi} h_p(\xi) d\xi$$

Table 4: Exact solution, The new method solution for equation (22)

$(x/16)$	Exact solution	The new method solution
1	0.0624593	0.0624592
3	0.1864032	0.1864033
5	0.3074385	0.3074385
7	0.4236762	0.4236762
9	0.5333026	0.5333026
11	0.6346070	0.6346070
13	0.7260086	0.7260086
15	0.8060811	0.8060811

$$\|E_J\|^2 = \int_0^1 (\sum_{p=2M+1}^{\infty} a_p \int_0^1 e^{ix\xi} h_p(\xi)d\xi)\overline{(\sum_{p=2M+1}^{\infty} a_p \int_0^1 e^{ix\xi} h_p(\xi)d\xi)}dx$$

$$= \sum_{p=2M+1}^{\infty} \sum_{k=2M+1}^{\infty} a_p \overline{a_k} \int_0^1 (\int_0^1 e^{ix\xi} h_p(\xi)d\xi)\overline{(\int_0^1 e^{ix\xi} h_k(\xi)d\xi)}dx$$

$$= \sum_{p=2M+1}^{\infty} \sum_{k=2M+1}^{\infty} a_p \overline{a_k} \int_0^1 (\sum_{s=1}^{2M} h_p(x_s) \int_{\tau_s}^{\tau_{s+1}} e^{ix\xi} d\xi)\overline{(\sum_{s=1}^{2M} h_k(x_s) \int_{\tau_s}^{\tau_{s+1}} e^{ix\xi} d\xi)}dx.$$

Applying mean value theorem

$$\sum_{p=2M+1}^{\infty} \sum_{k=2M+1}^{\infty} a_p \overline{a_k} \int_0^1 A_p(x) B_k(x) dx,$$

where

$$A_p(x) = \sum_{s=1}^{2M} \frac{1}{2M} h_p(x_s)(cos(x\eta_s) + isin(x\alpha_s))$$

and

$$B_k(x) = \sum_{k=1}^{2M} \frac{1}{2M} h_k(x_k)(cos(x\eta_k) - isin(x\alpha_k))$$

where $\eta_k, \alpha_k \in [\tau_k, \tau_{k+1}]$. Therefore

$$\begin{aligned}\|E_J\|^2 &\le \sum_{p=2M+1}^{\infty} \sum_{k=2M+1}^{\infty} \frac{a_p \overline{a_k}}{2M^2} \int_0^1 \sum_{s=1}^{2M} \sum_{s=1}^{2M} h_p(x_s) h_k(x_s) dx \\ &\le \sum_{p=2M+1}^{\infty} \frac{|a_p|^2}{M}.\end{aligned}$$

Now

$$a_p = \int_0^1 2^{j/2} h_p(\xi) \hat{f}(\xi) d\xi = \int_0^1 2^{j/2} h_p(\xi) Re(\hat{f}(\xi)) d\xi + i \int_0^1 2^{j/2} h_p(\xi) Im(\hat{f}(\xi)) d\xi.$$

By applying the mean value theorem for $\beta_1 \in (\xi_1, \xi_2)$ and $\beta_2 \in (\xi_2, \xi_3)$

$$\begin{aligned}\int_0^1 2^{j/2} h_p(\xi) Re(\hat{f}(\xi)) d\xi &= 2^{j/2} [\int_{\xi_1}^{\xi_2} Re(\hat{f}(\xi)) d\xi - \int_{\xi_2}^{\xi_3} Re(\hat{f}(\xi)) d\xi] \\ &= 2^{j/2} [(\xi_2 - \xi_1) Re(\hat{f}(\beta_1)) - (\xi_3 - \xi_2) Re(\hat{f}(\beta_2))].\end{aligned}$$

Applying the mean value theorem

$$\int_0^1 2^{j/2} h_p(\xi) Re(\hat{f}(\xi)) d\xi = 2^{-j/2-1} (\beta_1 - \beta_2)(Re\hat{f}(\lambda_1))',$$

where $\lambda_1 \in (\beta_1, \beta_2)$ and also

$$\int_0^1 2^{j/2} h_p(\xi) Im(\hat{f}(\xi)) d\xi = 2^{-j/2-1} (\beta_3 - \beta_4)(Im\hat{f}(\lambda_2))',$$

where $\lambda_2 \in (\beta_3, \beta_4)$, $\beta_3 \in (\xi_1, \xi_2)$, and $\beta_4 \in (\xi_2, \xi_3)$.
Therefore

$$a_p = \int_0^1 2^{j/2} h_p(\xi) \hat{f}(\xi) d\xi = 2^{-j/2-1} ((\beta_1 - \beta_2) Re\hat{f}(\lambda_1) + i(\beta_3 - \beta_4) Im\hat{f}(\lambda_2))'.$$

This implies that

$$|a_p|^2 \le K^2 2^{-3j}.$$

Therefore,

$$\|E_J\|^2 \le \sum_{i=2M+1}^{\infty} \frac{|a_i|^2}{M} \le \sum_{i=2M+1}^{\infty} \frac{K^2 2^{-3j}}{M}$$

$$
\begin{aligned}
&\leq \frac{K^2}{M} \sum_{j=J+1}^{\infty} \sum_{i=2^j+1}^{2^{j+1}} 2^{-3j} \leq \frac{K^2}{M} \sum_{j=J+1}^{\infty} 2^{-3j}(2^{j+1}-1-2^j+1) \\
&\leq \frac{K^2}{M} \sum_{j=J+1}^{\infty} 2^{-2j} \leq \frac{K^2}{M}\left(\frac{2^{-2(J+1)}}{1-\frac{1}{4}}\right) \\
&\leq \frac{3K^2}{16} 2^{-3J}.
\end{aligned}
$$

□

It is obvious that the error bound is inversely proportional to the level of resolution J of Haar wavelet. Hence, the accuracy in our new method improves as we increase the level of resolution J.

Conclusions In this paper a new method for numerical solution of ordinary differential equations based on pseudo differential operators and Haar wavelets is proposed. Its applicability and efficiency are checked on some problems. It is shown that with a small number of collocation points, more accurate solutions in comparison with Haar wavelet method can be obtained.

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یک روش ترکیبی جدید بر اساس عملگرهای شبه دیفرانسیل و موجک هار برای حل معادلات دیفرانسیل معمولی

محمد باقر قائمی، مجید جمال‌پور بیرگانی و المیرا نبی زاده مرصل فرد

دانشکده ریاضی، دانشگاه علم و صنعت ایران

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چکیده : در این مقاله ما با ترکیب عملگرهای شبه دیفرانسیل و موجک هار یک روش جدیدی برای حل معادلات دیفرانسیل خطی و غیر خطی ارائه می‌دهیم. روش ارائه شده از لحاظ کارایی و سادگی بسیار عالی عمل می کند.

کلمات کلیدی : موجک هار؛ تبدیل فوریه؛ معادلات دیفرانسیل؛ جواب عددی؛ عملگرهای شبه دیفرانسیل.