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# Application of modified hat functions for solving nonlinear quadratic integral equations

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**Abstract**

A numerical method to solve nonlinear quadratic integral equations (QIE) is presented in this work. The method is based upon modification of hat functions (MHFs) and their operational matrices. By using this approach and the collocation points, solving the nonlinear QIE reduces to solve a nonlinear system of algebraic equations. The proposed method does not need any integration for obtaining the constant coefficients. Hence, it can be applied in a simple and fast technique. Convergence analysis and associated theorems are considered. Some numerical examples illustrate the accuracy and computational efficiency of the proposed method.

**Keywords:** Modification of hat functions; Nonlinear quadratic integral equation; Vector forms; Operational matrix; Error analysis.

## 1 Introduction

Over the last years, the integral equations have been used increasingly in different areas of applied science. This tendency could be explained by the deduction of knowledge models which describe real physical phenomena. For details, we refer to [1, 2, 4-9, 12-18, 23, 25, 26]. In particular, quadratic integral equations (QIEs) have many useful applications in the real world. For example, QIEs are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory. The QIEs can be very often encountered in many applications. The quadratic integral equations have been studied in

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several papers and monographs [1, 2, 4-26, 29, 30]. In this paper, we study the numerical solution of a QIE:

$$f(x) = g(x) + \left( \int_0^x k_1(x,y)U_1(y,f(y))dy \right) \left( \int_0^x k_2(x,y)U_2(y,f(y))dy \right), \tag{1}$$

where $x \in D = [0,1]$, $g(x) \in C^3(D)$, $U_1(x,f(x)), U_2(x,f(x)) \in C^3(D \times \mathbb{R})$ and $k_1(x,y), k_2(x,y) \in C^3(D \times D)$ are known functions, $f(x) \in C^3(D)$ is an unknown function and we will obtain the approximate solution in the truncated MHFs series form

$$f_m(x) = \sum_{i=0}^{m} f_i h_i(x),$$

so that $f_i$; $i = 0, 1, \ldots, m$, are the unknown MHFs coefficients and $h_i(x)$; $i = 0, 1, \ldots, m$, are the MHFs.

To mention some recent works on QIEs, see e.g., [1,8,12,14,15,25,26] and for some applications we refer readers to [18,23]. Existence, uniqueness and some other properties of the solution to these problems were established in [33]. It should be recalled that nonlinear QIEs have been treated extensively with the measure of noncompactness and a fixed point theorem of Darbo type. This approach seems to be too restrictive. Furthermore, in most of the above investigations, some additional assumptions in terms of the measure of noncompactness were imposed on $g(x)$.

The plan for this paper is as follows: In Section 2, we describe MHFs and their properties. In Section 3, we will apply these sets of MHFs for approximating the solution of QIEs. In Section 4, theorems are proved for convergence analysis. Numerical results are given in Section 5 to illustrate the efficiency and the accuracy of our algorithms. Finally, Section 6 concludes the paper.

## 2 Modification of hat functions

The purpose of this section is to collect a number of definitions and lemmas concerning MHFs. we first construct the set of MHFs.

An $(m+1)$-set of MHFs consists of $(m+1)$ functions which are defined over district $D$ as follows [3,29]

$$h_0(x) = \begin{cases} \frac{1}{2h^2}(x-h)(x-2h) & 0 \leqslant x \leqslant 2h, \\ 0 & \text{otherwise}, \end{cases}$$

if $i$ is odd and $1 \leqslant i \leqslant m-1$,

$$h_i(x)=\begin{cases}\frac{-1}{h^2}(x-(i-1)h)(x-(i+1)h) & (i-1)h\leqslant x\leqslant (i+1)h,\\ 0 & \text{otherwise,}\end{cases}$$

if $i$ is even and $2\leqslant i\leqslant m-2$,

$$h_i(x)=\begin{cases}\frac{1}{2h^2}(x-(i-1)h)(x-(i-2)h) & (i-2)h\leqslant x\leqslant ih,\\ \frac{1}{2h^2}(x-(i+1)h)(x-(i+2)h) & ih\leqslant x\leqslant (i+2)h,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$h_m(x)=\begin{cases}\frac{1}{2h^2}(x-(1-h))(x-(1-2h)) & 1-2h\leqslant x\leqslant 1,\\ 0 & \text{otherwise,}\end{cases}$$

where $m\geqslant 2$ is an even integer and $h=\frac{1}{m}$. It is obvious that

$$h_i(jh)=\begin{cases}1 & i=j,\\ 0 & i\neq j,\end{cases} \tag{2}$$

$$h_i(x)h_j(x)=\begin{cases}0 & i \text{ is even and } |i-j|\geqslant 3,\\ 0 & i \text{ is odd and } |i-j|\geqslant 2,\end{cases} \tag{3}$$

and

$$\sum_{i=0}^{m} h_i(x)=1.$$

Let us write the MHFs vector $H(x)$ as follows

$$H(x)=[h_0(x),h_1(x),\ldots,h_m(x)]^T;\quad x\in D. \tag{4}$$

An arbitrary function $f(x)$ defined over $D$ can be expanded by the MHFs as

$$f(x)\simeq F^TH(x)=H^T(x)F,$$

where

$$F=[f_0,f_1,\ldots,f_m]^T,$$

and

$$f_i=f(ih);\qquad i=0,1,\ldots,m.$$

Similarly an arbitrary function of two variables, $k(x,y)$ on district $D\times D$ may be approximated with respect to MHFs such as

$$k(x,y) \simeq H^T(x)KH(y),$$

where $H(x)$ and $H(y)$ are MHFs vector of dimension $(m+1)$ and $K$ is the $(m+1)\times(m+1)$ MHFs coefficients matrix.

According to (2) and expanding $\int_0^x h_i(y)dy,\ i = 0,1,\dots,m$ by MHFs, integration of vector $H(x)$ defined in (4) can be expressed as

$$\int_0^x H(y)dy \simeq PH(x), \tag{5}$$

where $P$ is the $(m+1)\times(m+1)$ matrix as follows

$$P = \frac{h}{12}\begin{pmatrix} 0 & 5 & 4 & 4 & 4 & 4 & 4 & \dots & 4 & 4 \\ 0 & 8 & 16 & 16 & 16 & 16 & 16 & \dots & 16 & 16 \\ 0 & -1 & 4 & 9 & 8 & 8 & 8 & \dots & 8 & 8 \\ 0 & 0 & 8 & 16 & 16 & 16 & 16 & \dots & 16 & 16 \\ 0 & 0 & 0 & -1 & 4 & 9 & 8 & \dots & 8 & 8 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 8 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 4 \end{pmatrix}.$$

By considering (2), (3) and expanding entries of $H(x)H^T(x)$ by MHFs, we obtain

$$H(x)H^T(x) \simeq diag(H^T(x)),$$

so we have

$$H(x)H^T(x)F \simeq \widetilde{F}H(x), \tag{6}$$

where $F$ be an $(m+1)$-vector and $\widetilde{F}$ is an $(m+1)\times(m+1)$ diagonal matrix. Also, if $A$ is an $(m+1)\times(m+1)$-matrix, we have

$$H^T(x)AH(x) \simeq H^T(x)\widehat{A}, \tag{7}$$

where $\widehat{A}$ is an $(m+1)$-vector with elements equal to the diagonal entries of matrix $A$.

## 3 Basic idea

In this section, we will provide the basic idea. This idea includes of approximating the solution of nonlinear quadratic integral equations (1). To solve this equation, we first consider the approximations

$$
\begin{aligned}
w_1(x) &= U_1(x, f(x)) \\
&= U_1\left(x, g(x) + \left(\int_0^x k_1(x,y)w_1(y)dy\right)\left(\int_0^x k_2(x,y)w_2(y)dy\right)\right), \\
w_2(x) &= U_2(x, f(x)) \\
&= U_2\left(x, g(x) + \left(\int_0^x k_1(x,y)w_1(y)dy\right)\left(\int_0^x k_2(x,y)w_2(y)dy\right)\right).
\end{aligned} \tag{8}
$$

We approximate function $w_1(x), w_2(x), k_1(x,y)$ and $k_2(x,y)$ by MHFs,

$$
\begin{cases}
w_1(x) \simeq W_1^T H(x) = H^T(x) W_1, \\
w_2(x) \simeq W_2^T H(x) = H^T(x) W_2, \\
k_1(x,y) \simeq H^T(x) K1 H(y) = H^T(y) K1^T H(x), \\
k_2(x,y) \simeq H^T(x) K2 H(y) = H^T(y) K2^T H(x),
\end{cases} \tag{9}
$$

where $W_1, W_2,$ $K1$ and $K2$ are MHFs coefficients of $w_1(x), w_2(x), k_1(x,y)$ and $k_2(x,y)$, respectively. Substituting (9) in (8), we get

$$
\begin{aligned}
H^T(x)W_1 \simeq U_1\Big(x, g(x) + &\Big(\int_0^x H^T(x)K1H(y)H^T(y)W_1dy\Big) \\
&\times\Big(\int_0^x H^T(x)K2H(y)H^T(y)W_2dy\Big)\Big), \\
H^T(x)W_2 \simeq U_2\Big(x, g(x) + &\Big(\int_0^x H^T(x)K1H(y)H^T(y)W_1dy\Big) \\
&\times\Big(\int_0^x H^T(x)K2H(y)H^T(y)W_2dy\Big)\Big).
\end{aligned}
$$

Using (5) and (6), yields

$$
\begin{aligned}
H^T(x)W_1 &\simeq U_1\left(x, g(x) + \left(H^T(x)K1\widetilde{W_1}PH(x)\right)\left(H^T(x)K2\widetilde{W_2}PH(x)\right)\right), \\
H^T(x)W_2 &\simeq U_2\left(x, g(x) + \left(H^T(x)K1\widetilde{W_1}PH(x)\right)\left(H^T(x)K2\widetilde{W_2}PH(x)\right)\right),
\end{aligned}
$$

where $\widetilde{W_i} = diag(W_i),\ i = 1, 2,$ are an $(m+1)\times(m+1)$ diagonal matrices. From (7), we have

$$
\begin{aligned}
H^T(x)W_1 &\simeq U_1\left(x, g(x) + \left(H^T(x)\widehat{K1\widetilde{W_1}P}\right)\left(H^T(x)\widehat{K2\widetilde{W_2}P}\right)\right), \\
H^T(x)W_2 &\simeq U_2\left(x, g(x) + \left(H^T(x)\widehat{K1\widetilde{W_1}P}\right)\left(H^T(x)\widehat{K2\widetilde{W_2}P}\right)\right),
\end{aligned} \tag{10}
$$

where $\widehat{Ki\widetilde{W}_iP}$, $i = 1, 2$, be an $(m + 1)$-vector with elements equal to the diagonal entries of matrix $Ki\widetilde{W}_iP$. We can rewrite $\widehat{Ki\widetilde{W}_iP}$, $i = 1, 2$, as follows

$$\widehat{Ki\widetilde{W}_iP} = AiW_i, \quad i = 1, 2, \tag{11}$$

where

$$Ai_{pq} = Ki_{pq}P_{qp}, \quad p, q = 0, 1, \ldots, m.$$

Substituting (11) into (10) and replacing $\simeq$ by $=$, we obtain

$$H^T(x)W_1 = U_1\left(x, g(x) + H^T(x)A1W_1H^T(x)A2W_2\right),$$

$$H^T(x)W_2 = U_2\left(x, g(x) + H^T(x)A1W_1H^T(x)A2W_2\right).$$

Now, using Newton-Cotes nodes as

$$x_i = \frac{2i-1}{2(m+1)}, \qquad i = 1, 2, \ldots, m+1,$$

then

$$H^T(x_i)W_1 = U_1\left(x_i, g(x_i) + H^T(x_i)A1W_1H^T(x_i)A2W_2\right),$$

$$H^T(x_i)W_2 = U_2\left(x_i, g(x_i) + H^T(x_i)A1W_1H^T(x_i)A2W_2\right).$$

We have a system of $(m+1)^2$ nonlinear equations and $(m+1)^2$ unknowns. After solving the above nonlinear system, we can find $W_1$ and $W_2$ and then

$$f(x) \simeq f_m(x) = g(x) + H^T(x)A1W_1H^T(x)A2W_2.$$

## 4 Convergence analysis

In this section, for confirming the accuracy of the proposed scheme in the previous section analytically, we provide an upper bound for difference between the exact solution of (1) and our approximated solution. We show that the MHFs method, is convergent of order $O(h^3)$. We define

$$||g|| = \sup_{x\in D} |g(x)|.$$

**Theorem 1.** *Suppose $x_i = ih$, $i = 0, 1, \ldots, m$, $g(x) \in C^3(D)$ and $g_m(x)$ be the MHFs expansions of $g(x)$ that defined as $g_m(x) = \sum_{i=0}^{m} g(x_i)h_i(x)$. Also, assume that $e_m = \|g - g_m\|$ where $x \in D$, then*

$$e_m = O(h^3).$$

*Proof.* According to definition of MHFs, the $g_m(x)$ is the quadratic polynomial interpolation on $[x_{i-2}, x_i)$. Therefore, for the interpolation error on $[x_{i-2}, x_i)$, we have [31]

$$e_{i,m}(x) = g(x) - g_m(x) = \frac{1}{6}\frac{d^3 g(\xi_i)}{dx^3}\prod_{i'=i-2}^{i}(x - x_{i'}); \quad i = 2, 4, \dots, m,$$

where $x, \xi_i \in [x_{i-2}, x_i)$. Let $v(x) = \prod_{i'=i-2}^{i}(x - x_{i'})$, so

$$\|e_{i,m}\| = \frac{1}{6}\sup_{x\in[x_{i-2},x_i)}\left|\frac{d^3 g(\xi_i)}{dx^3}\, v(x)\right|, \quad i = 2, 4, \dots, m,$$

or

$$\|e_{i,m}\| \le \frac{1}{6}\sup_{x\in[x_{i-2},x_i)}\left|\frac{d^3 g(\xi_i)}{dx^3}\right|\, |v(x)|, \quad i = 2, 4, \dots, m.$$

On the other hand, we have

$$e_m = \sup_{x\in D}|g(x) - g_m(x)| = \max_{i=2,4,\dots,m}\ \sup_{x\in[x_{i-2},x_i)}|g(x) - g_m(x)|$$
$$= \max_{i=2,4,\dots,m}\|e_{i,m}\|.$$

So

$$e_m \le \frac{1}{6}\max_{i=2,4,\dots,m}\ \sup_{x\in[x_{i-2},x_i)}\left|\frac{d^3 g(\xi_i)}{dx^3}\right|\, |v(x)|\ .$$

Since $|v(x)| \le \sup_{x\in[x_{i-2},x_i)}\left|\prod_{i'=i-2}^{i}(x - x_{i'})\right|$ and the maximum value of $\left|\prod_{i'=i-2}^{i}(x - x_{i'})\right|$ is obtained at $x = (i - 1 - \frac{\sqrt{3}}{3})h$, we have

$$|v(x)| \le \frac{2\sqrt{3}h^3}{9}, \qquad \forall x \in [x_{i-2}, x_i).$$

Therefore, it is not difficult to verify that

$$e_m \le \frac{h^3}{9\sqrt{3}}\left\|\frac{d^3 g}{dx^3}\right\| = Ch^3. \tag{12}$$

So

$$e_m = O(h^3).$$

This completes the proof. □

**Theorem 2.** *Let* $x_i = y_i = ih,\ i = 0, 1, \dots, m,\ k(x, y) \in C^3(D \times D)$ *and*

$$k_m(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{m} k(x_i,y_j)h_i(x)h_j(y),$$

*be the MHFs expansions of* $k(x,y)$*. Then*

$$e_m = O(h^3),$$

*where* $e_m = \|k - k_m\|$ *and* $(x,y) \in D \times D$*.*

*Proof.* $k_m(x,y)$ is the quadratic polynomial interpolation of $k(x,y)$ on $\Omega_{ij} = [x_{i-1},x_i) \times [y_{j-1},y_j)$. Therefore for the interpolation error on $\Omega_{ij}$, we have [27]

$$\begin{aligned} e_{ij,m}(x,y) &= k(x,y) - k_m(x,y) \\ &= \frac{1}{6}\frac{\partial^3 k(\xi_i,y)}{\partial x^3}\prod_{i'=i-2}^{i}(x - x_{i'}) + \frac{1}{6}\frac{\partial^3 k(x,\eta_j)}{\partial y^3}\prod_{j'=j-2}^{j}(y - y_{j'}) \\ &\quad - \frac{1}{36}\frac{\partial^6 k(\xi_i',\eta_j')}{\partial x^3 \partial y^3}\prod_{i'=i-2}^{i}(x - x_{i'})\prod_{j'=j-2}^{j}(y - y_{j'}), \end{aligned}$$

where $i,j = 2,4,\ldots,m,\ x,\xi_i,\xi_i' \in [x_{i-1},x_i)$ and $y,\eta_j,\eta_j' \in [y_{j-1},y_j)$. So we have

$$\|e_{ij,m}\| = \frac{1}{6}\sup_{(x,y)\in\Omega_{ij}}\left|\frac{\partial^3 k(\xi_i,y)}{\partial x^3}v(x) + \frac{\partial^3 k(x,\eta_j)}{\partial y^3}u(y) - \frac{1}{6}\frac{\partial^6 k(\xi_i',\eta_j')}{\partial x^3\partial y^3}v(x)u(y)\right|,$$

or

$$\begin{aligned} \|e_{ij,m}\| \le \frac{1}{6}\sup_{(x,y)\in\Omega_{ij}} &\left\{\left|\frac{\partial^3 k(\xi_i,y)}{\partial x^3}\right| |v(x)| + \left|\frac{\partial^3 k(x,\eta_j)}{\partial y^3}\right| |u(y)| \right. \\ &\left. + \frac{1}{6}\left|\frac{\partial^6 k(\xi_i',\eta_j')}{\partial x^3\partial y^3}\right| |v(x)|\,|u(y)|\right\}, \end{aligned}$$

where $i,j = 2,4,\ldots,m,\ v(x) = \prod_{i'=i-2}^{i}(x - x_{i'})$ and $u(y) = \prod_{j'=j-2}^{j}(y - y_{j'})$. On the other hand, we have

$$\begin{aligned} e_m &= \sup_{(x,y)\in D\times D}|k(x,y) - k_m(x,y)| \\ &= \max_{i,j=2,4,\ldots,m}\sup_{(x,y)\in\Omega_{ij}}|k(x,y) - k_m(x,y)| = \max_{i,j=2,4,\ldots,m}\|e_{ij,m}\|. \end{aligned}$$

So

$$e_m \le \frac{1}{6}\max_{i,j=2,4,\ldots,m}\sup_{(x,y)\in\Omega_{ij}}\left\{\left|\frac{\partial^3 k(\xi_i,y)}{\partial x^3}\right| |v(x)| + \left|\frac{\partial^3 k(x,\eta_j)}{\partial y^3}\right| |u(y)|\right.$$

$$+\frac{1}{6}\left|\frac{\partial^6 k(\xi_i', \eta_j')}{\partial x^3 \partial y^3}\right| \; |v(x)| \; |u(y)| \Big\} .$$

We know,

$$|v(x)| \leq \frac{2\sqrt{3}h^3}{9}, \qquad \forall x \in [x_{i-2}, x_i),$$

and

$$|u(y)| \leq \frac{2\sqrt{3}h^3}{9}, \qquad \forall y \in [y_{j-2}, y_j).$$

Therefore, it is not difficult to verify that

$$e_m \leq \frac{h^3}{9\sqrt{3}} \| \frac{\partial^3 k}{\partial x^3} \| + \frac{h^3}{9\sqrt{3}} \| \frac{\partial^3 k}{\partial y^3} \| + \frac{h^6}{243} \| \frac{\partial^6 k}{\partial x^3 \partial y^3} \| = Ch^3, \tag{13}$$

so

$$e_m = O(h^3).$$

This completes the proof. □

Let $f_m(x), g_m(x), k_{i,m}(x,y)$ and $U_{i,m}(x, f(x))$, for $i = 1, 2$, are the MHFs expansions of $f(x), g(x), k_i(x,y)$ and $U_i(x, f(x))$, respectively. According to Theorems 1, 2 and expression (1), we have

$$\begin{aligned} f_m(x) + O(h^3) &= g_m(x) + O(h^3) \\ &+ \left( \int_0^x \left(k_{1,m}(x,y) + O(h^3)\right) \left(U_{1,m}(y, f_m(y) + O(h^3)) + O(h^3)\right) dy \right) \\ &\times \left( \int_0^x \left(k_{2,m}(x,y) + O(h^3)\right) \left(U_{2,m}(y, f_m(y) + O(h^3)) + O(h^3)\right) dy \right), \end{aligned}$$

where $x \in D$. By ignoring the terms included $O(h^3)$, we have

$$\begin{aligned} f_m(x) = g_m(x) &+ \left( \int_0^x k_{1,m}(x,y) U_{1,m}(y, f_m(y)) dy \right) \\ &\times \left( \int_0^x k_{2,m}(x,y) U_{2,m}(y, f_m(y)) dy \right), \end{aligned} \tag{14}$$

where $x \in D$. Now, assume the following hypotheses:

(M1) Suppose that the error of MHFs is denoted by

$$E_m = \|f - f_m\|.$$

(M2) $\|f\| \leqslant M$.

(M3) The nonlinear term $U_1(x,y)$ and $U_2(x,y)$ satisfies in the Lipschitz and linear growth condition such that

$$|U_1(x, y_1) - U_1(x, y_2)| + |U_2(x, y_1) - U_2(x, y_2)| \leqslant L|y_1 - y_2| \; ,$$

where $(x, y_1), (x, y_2) \in D \times \mathbb{R}$ , and

$$|U_1(x,y)| + |U_2(x,y)| \leqslant L(1+|y|), \quad (x,y) \in D \times \mathbb{R} \ .$$

(M4) Let

$$\|k_1\| \leqslant M_1,$$

and

$$\|k_2\| \leqslant M_2.$$

(M5) Let

$$e_{1,m} = \|k_1 - k_{1,m}\| \leqslant C_1' h^3,$$

and

$$e_{2,m} = \|k_2 - k_{2,m}\| \leqslant C_2' h^3,$$

where $C_1'$ and $C_2'$ are constants that can be defined as coefficient $C$ in (13).

(M6) Let $2L(C_2'h^3 + M_2)(M_1 + C_1'h^3)(L(1+M) + C_1 h^3) < 1$, where

$$\|U_1 - U_{1,m}\| \leqslant C_1 h^3,$$

and $C_1$ is constant that can be defined as coefficient $C$ in (12).

**Theorem 3.** *Suppose $f(x)$ and $f_m(x)$ be the exact and approximate solutions of (1) respectively. Also, above assumptions of (M1)-(M6) are satisfied, then we have*

$$E_m = O(h^3).$$

*Proof.* Assume that $w_{i,m}(x)$ and $w_i(x)$ be the approximate and exact solution of (8). we define

$$w_{1,m}(x) = U_{1,m}(x, f_m(x)), \qquad w_{2,m}(x) = U_{2,m}(x, f_m(x)) \ ,$$

and

$$\hat{w}_{1,m}(x) = U_1(x, f_m(x)), \qquad \hat{w}_{2,m}(x) = U_2(x, f_m(x)) \ .$$

According to (1) and (14), we have

$$\begin{aligned} f(x) - f_m(x) &= g(x) - g_m(x) \\ &+ \left( \int_0^x k_1(x,y) w_1(y, f(y)) dy \right) \left( \int_0^x k_2(x,y) w_2(y, f(y)) dy \right) \\ &- \left( \int_0^x k_{1,m}(x,y) w_{1,m}(y, f_m(y)) dy \right) \left( \int_0^x k_{2,m}(x,y) w_{2,m}(y, f_m(y)) dy \right) \\ &+ O(h^3), \end{aligned}$$

Therefore,

$$f(x)-f_m(x) = g(x)-g_m(x)+\left(\int_0^x k_1(x,y)w_1(y,f(y))dy\right)$$
$$\times\left[\left(\int_0^x k_2(x,y)w_2(y,f(y))dy\right)-\left(\int_0^x k_{2,m}(x,y)w_{2,m}(y,f_m(y))dy\right)\right]$$
$$+\left(\int_0^x k_{2,m}(x,y)w_{2,m}(y,f_m(y))dy\right)$$
$$\times\left[\left(\int_0^x k_1(x,y)w_1(y,f(y))dy\right)-\left(\int_0^x k_{1,m}(x,y)w_{1,m}(y,f_m(y))dy\right)\right]$$
$$+\ O(h^3).$$

So

$$E_m \leqslant e_m + \|x\|^2\ \ \|k_1\|\ \ \|w_1\|\ \ \|k_2w_2 - k_{2,m}w_{2,m}\|$$
$$+\ \|x\|^2\ \ \|k_{2,m}\|\ \ \|w_{2,m}\|\ \ \|k_1w_1 - k_{1,m}w_{1,m}\| + O(h^3).$$

Since $x \in D$, then $\|x\|^2 \leqslant 1$. So

$$E_m \leqslant e_m + \|k_1\|\ \ \|w_1\|\ \ \|k_2w_2 - k_{2,m}w_{2,m}\|$$
$$+\ \|k_{2,m}\|\ \ \|w_{2,m}\|\ \ \|k_1w_1 - k_{1,m}w_{1,m}\| + O(h^3). \tag{15}$$

We have

$$\|w_i - w_{i,m}\| \leqslant \|w_i - \hat{w}_{i,m}\| + \|\hat{w}_{i,m} - w_{i,m}\| \leqslant LE_m + C_ih^3, \quad i=1,2, \tag{16}$$

where $C_1$ and $C_2$ are constants that can be defined as coefficient $C$ in (12). Also, we have

$$\|w_{i,m}\| \leqslant \|w_i - w_{i,m}\| + \|w_i\| \leqslant LE_m + C_ih^3 + L(1+M)\ , \tag{17}$$

and

$$\|k_{2,m}\| \leqslant \|k_2 - k_{2,m}\| + \|k_2\| \leqslant C_2'h^3 + M_2\ . \tag{18}$$

Now, according to Theorem 2 and inequalities (14), (15) and assumptions (M4)-(M5), we have

$$\|k_iw_i - k_{i,m}w_{i,m}\| \leqslant \|k_i\|\ \ \|w_i - w_{i,m}\| + \|w_{i,m}\|\ \ \|k_i - k_{i,m}\|$$
$$\leqslant M_i(LE_m + C_ih^3) + C_i'h^3(LE_m + C_ih^3 + L(1+M))\ . \tag{19}$$

From Theorem 1 and inequalities (15)-(17) and assumptions (M3)-(M4), we can rewrite (15), as follows

$$E_m \leqslant Ch^3 + M_1L(1+M)\Big(L(M_2 + C_2'h^3)E_m + C_2h^3(M_2 + C_2'h^3)$$
$$+LC_2'h^3(1+M)\Big) + \left(C_2'h^3 + M_2\right)\left(LE_m + C_1h^3 + L(1+M)\right)$$

$$\times\Big(L(M_1+C_1'h^3)E_m + C_1h^3(M_1+C_1'h^3) + LC_1'h^3(1+M)\Big) + O(h^3),$$

where $C$ is defined in (12). Without loss of generality, ignoring the term included $E_m^2$ and $h^6$, we have

$$E_m \leqslant \frac{\Big(C + \Big((M_1LC_2' + M_2LC_1')(1+M) + M_1M_2(C_1+C_2)\Big)L(1+M)\Big)h^3}{1-2L(M_1+C_1'h^3)(M_2+C_2'h^3)(L(1+M)+C_1h^3)}$$

$$+ O(h^3).$$

This completes the proof. □

## 5 Numerical examples

To illustrate the accuracy and efficiency of proposed method, some examples are provided. The algorithms associated with the numerical method were performed using Matlab. We have checked that when more points are used the accuracy improves significantly.

**Example 1.** Consider the following nonlinear QIE [33]

$$f(x) = \left(x^2 + \frac{x^{15}}{1350}\right) + \left(\int_0^x yf^2(y)dy\right)\left(\int_0^x \frac{y^2}{25}f^3(y)dy\right), \quad x \in [0,1], \tag{20}$$

with the exact solution $f(x) = x^2$.

Table 1 and Figure 1 illustrate the error results for this example. Also, we compare the maximum absolute error computed by the present method, repeated trapezoidal (RT) method [33] and Adomian decomposition (AD) method [33] in Table 2.

**Example 2.** Consider the following nonlinear QIE [33]

$$f(x) = x-(e^x-1)\left(\frac{x^3}{30}+\frac{x^5}{50}\right)+\left(\int_0^x \frac{y^2+1}{10}f^2(y)dy\right)\left(\int_0^x e^{f(y)}dy\right), \tag{21}$$

where $x \in [0,1]$ with the exact solution $f(x) = x$.

Table 3 and Figure 2 illustrate the error results for this example. Also, we compare the maximum absolute error computed by the present method, RT method [33] and AD method [33] in Table 4.

**Example 3.** Consider the following nonlinear QIE

Table 1: Absolute error for $m = 8, 16, 32$ of $f(x)$ of Equation (18)

| Nodes x | Present method | | |
|---|---|---|---|
| | m=8 | m=16 | m=32 |
| x = 0.0 | 0 | 0 | 0 |
| x = 0.1 | 8.8569950e-13 | 1.9081958e-17 | 0 |
| x = 0.2 | 2.9361930e-13 | 1.0984269e-14 | 6.3490879e-15 |
| x = 0.3 | 2.7433902e-11 | 4.7872678e-12 | 3.7166104e-13 |
| x = 0.4 | 1.4692486e-10 | 2.1893068e-10 | 1.1033702e-11 |
| x = 0.5 | 1.4339294e-09 | 1.2865681e-10 | 1.0435374e-11 |
| x = 0.6 | 1.4664141e-07 | 1.0643045e-08 | 6.9546968e-10 |
| x = 0.7 | 4.2146640e-07 | 2.6710975e-08 | 1.5046137e-08 |
| x = 0.8 | 6.8445266e-06 | 3.6807176e-07 | 5.1551159e-08 |
| x = 0.9 | 3.1263755e-06 | 2.7156875e-06 | 1.6706609e-07 |
| x = 1.0 | 4.2731139e-06 | 6.9580496e-07 | 1.1520629e-07 |

Table 2: Table 2: Comparison of the absolute errors of Example 1

| Methods | Maximum error |
|---|---|
| RT Method | |
| m = 10 | 6.38458E-5 |
| m = 100 | 6.30669E-7 |
| m = 1000 | 6.30590E-9 |
| AD Method | |
| q = 5 | 3.62460E-5 |
| q = 10 | 9.23545E-7 |
| q = 15 | 2.35318E-8 |
| Present method | |
| m = 10 | 1.15435E-5 |
| m = 100 | 5.72511E-9 |
| m = 1000 | 1.05104E-11 |

Table 3: Absolute error for $m = 8, 16, 32$ of $f(x)$ of Equation (19)

| Nodes x | Present method | | |
|---|---|---|---|
| | m=8 | m=16 | m=32 |
| x = 0.0 | 0 | 0 | 0 |
| x = 0.1 | 1.7137234e-6 | 3.3597747e-7 | 2.0934872e-8 |
| x = 0.2 | 6.0344397e-6 | 3.7285951e-7 | 1.1336895e-7 |
| x = 0.3 | 1.7380694e-5 | 1.0109144e-6 | 2.0167842e-7 |
| x = 0.4 | 8.0824578e-6 | 3.4573577e-6 | 2.0948964e-7 |
| x = 0.5 | 1.1099971e-6 | 7.1694685e-8 | 5.0373049e-9 |
| x = 0.6 | 4.5992070e-5 | 7.5092211e-6 | 4.7997155e-7 |
| x = 0.7 | 8.4403496e-5 | 5.4418147e-6 | 1.7002589e-6 |
| x = 0.8 | 1.9036412e-4 | 1.1238576e-5 | 2.3184972e-6 |
| x = 0.9 | 7.8122715e-5 | 3.0755442e-5 | 1.8469227e-6 |
| x = 1.0 | 1.9011556e-5 | 2.0920818e-6 | 2.8234706e-6 |

Table 4: Comparison of the absolute errors of Example 2

| Methods | Maximum error |
|---|---|
| RT Method | |
| m = 10 | 1.07275E-3 |
| m = 100 | 8.44338E-7 |
| m = 1000 | 8.44337E-9 |
| AD Method | |
| q = 5 | 8.44492E-5 |
| q = 10 | 8.44338E-7 |
| q = 15 | 8.44337E-8 |
| Present method | |
| m = 10 | 1.25539E-4 |
| m = 100 | 1.27663E-8 |
| m = 1000 | 2.35536E-11 |

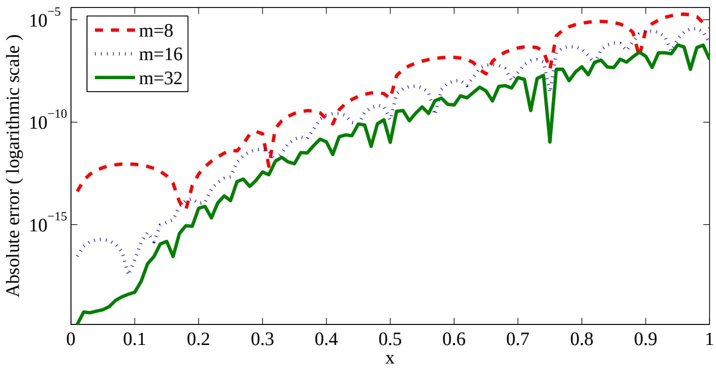


Figure 1: Absolute errors (on logarithmic scale) for Example 1, with $m = 8,\ 16,\ 32$

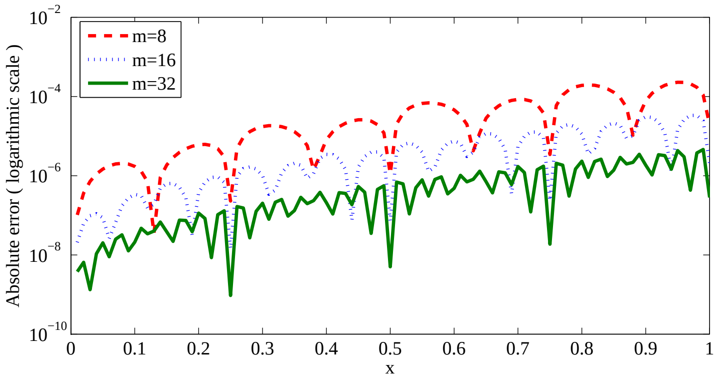


Figure 2: Absolute errors (on logarithmic scale) for Example 2, with $m = 8,\ 16,\ 32$

$$f(x) = g(x) + \left(\int_0^x (y^2+1)f^2(y)dy\right)\left(\int_0^x cos(y)e^{f(y)}dy\right), \quad x \in [0,1], \tag{22}$$

where

$$g(x) = \sin(x) + \left(\frac{x^3}{6} + \frac{x}{2} - \frac{\sin(2x)(1+2x^2)}{8} - \frac{x\cos(2x)}{4}\right)\left(1 - e^{\sin(x)}\right),$$

and the exact solution is $f(x) = \sin(x)$.

Table 5 and Figure 3 illustrate the error results for this example. Also, we compare the maximum absolute error computed by the present method, block-pulse functions (BPFs) method [30] and hat functions (HFs) method [28] in Table 6.

Table 5: Absolute error for $m = 8, 16, 32$ of $f(x)$ of Equation (20)

| Nodes x | Present method | | |
|---|---|---|---|
| | m=8 | m=16 | m=32 |
| x = 0.0 | 0 | 0 | 0 |
| x = 0.1 | 1.6976811e-5 | 3.1978554e-6 | 1.9245960e-7 |
| x = 0.2 | 5.3369972e-5 | 3.0671554e-6 | 1.0777490e-6 |
| x = 0.3 | 1.4115880e-4 | 9.0002882e-6 | 1.5689730e-6 |
| x = 0.4 | 5.0416870e-5 | 2.5604305e-5 | 1.6783596e-6 |
| x = 0.5 | 1.5197939e-5 | 2.3033416e-6 | 2.9173156e-7 |
| x = 0.6 | 2.5006880e-4 | 3.6295409e-5 | 2.0906182e-6 |
| x = 0.7 | 3.2816702e-4 | 1.8134154e-5 | 6.9241732e-6 |
| x = 0.8 | 5.4658908e-4 | 3.4864263e-5 | 6.6704623e-6 |
| x = 0.9 | 7.9549634e-5 | 5.0605385e-5 | 2.3233806e-6 |
| x = 1.0 | 1.9488394e-4 | 4.0158163e-5 | 6.8004887e-6 |

Table 6: Comparison of the absolute errors of Example 3

| Methods | Maximum error |
|---|---|
| BPFs Method | |
| m = 8 | 1.15609E-1 |
| m = 16 | 6.36185E-2 |
| m = 32 | 3.32452E-2 |
| HFs Method | |
| m = 8 | 2.22432E-2 |
| m = 16 | 5.79464E-3 |
| m = 32 | 1.51873E-3 |
| Present method | |
| m = 8 | 5.57591E-4 |
| m = 16 | 7.12826E-5 |
| m = 32 | 1.10246E-5 |

## 6 Conclusion

The MHFs method have been proposed for solving nonlinear quadratic integral equations. One of the advantages of this method is that the numerical solution of the nonlinear QIEs can be converted into a system of algebraic equations using the operational matrices. Furthermore, it is proved that MHFs method is convergence and the order of convergence is $O(h^3)$. The

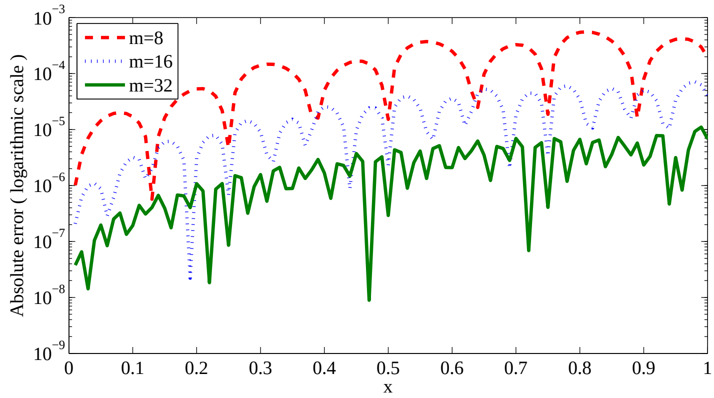


Figure 3: Absolute errors (on logarithmic scale) for Example 3, with $m = 8,\ 16,\ 32$

proposed method does not need any integration for obtaining the constant coefficients hence, it can be applied in a simple and fast technique. The comparison of the obtained results with those based on other methods shows that the present method is a powerful mathematical tool for finding the numerical solutions of such equations.

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# استفاده از توابع کلاهی اصلاح شده برای حل معادلات انتگرال کوادراتور غیر خطی

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دریافت مقاله ۲۵ اردیبهشت ۱۳۹۴، دریافت مقاله اصلاح شده ۲ آذر ۱۳۹۴، پذیرش مقاله ۵ اسفند ۱۳۹۴

**چکیده :** این مقاله یک روش عددی برای حل معادلات انتگرال کوادراتور غیرخطی ارائه می دهد. این روش بر مبنای توابع کلاهی اصلاح شده و ماتریس عملیاتی آنها می باشد. با استفاده از این روش و نقاط هم محلی حل معادلات انتگرال کوادراتور غیرخطی به حل یک دستگاه معادلات جبری غیر خطی کاهش می یابد. روش ارائه شده برای بدست آوردن ضرایب ثابت به انتگرال گیری نیاز ندارد. از اینرو، می توان به عنوان یک تکنیک ساده و سریع مورد استفاده قرار بگیرد. تجزیه و تحلیل همگرایی و قضایای مربوط به آن مورد بررسی قرار گرفته است. با چند مثال عددی کارایی و دقت روش ارائه شده، نشان داده شده است.

**کلمات کلیدی :** توابع کلاهی اصلاح شده؛ معادلات انتگرال کوادراتور غیرخطی؛ فرم برداری؛ ماتریس عملیاتی؛ تجزیه و تحلیل خطا.