

# Two numerical methods for nonlinear constrained quadratic optimal control problems using linear B-spline functions

Y. Edrisi-Tabriz, M. Lakestani\* and A. Heydari

## Abstract

This paper presents two numerical methods for solving the nonlinear constrained optimal control problems including quadratic performance index. The methods are based upon linear B-spline functions. The properties of B-spline functions are presented. Two operational matrices of integration are introduced for related procedures. These matrices are then utilized to reduce the solution of the nonlinear constrained optimal control to a nonlinear programming one to which existing well-developed algorithms may be applied. Illustrative examples are included to demonstrate the validity and applicability of the presented techniques.

**Keywords:** Optimal control problem; Linear B-spline function; Integration matrix; Collocation method.

## 1 Introduction

Solving an optimal control problem is not easy. Because of the complexity of most applications, optimal control problems are most often solved numerically. Numerical methods for solving optimal control problems date back nearly five decades to the 1950s with the work of Bellman [2–4]. Numerical methods for solving optimal control problems are divided into two major classes: direct methods and indirect methods.

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\*Corresponding author

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Y. Edrisi-Tabriz

Department of Mathematics, Payame Noor University, Tehran, Iran.

e-mail: yousef\_edrisi@pnu.ac.ir

M. Lakestani

Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.

e-mail: lakestani@tabrizu.ac.ir

A. Heydari

Department of Mathematics, Payame Noor University, Tehran, Iran.

e-mail: a\_heidari@pnu.ac.ir

In an indirect method, the calculus of variations [14, 24] is used to determine the first-order optimality conditions of the original optimal control problem. The indirect approach leads to a multiple-point boundary-value problem that is solved to determine candidate optimal trajectories called extremals. Each of the computed extremals is then examined to see if it is a local minimum, maximum, or a saddle point. Of the locally optimizing solutions, the particular extremal with the lowest cost is chosen.

One of the widely used methods to solve optimal control problems is the direct method. There is a large number of research papers that employ this method to solve optimal control problems (see for example [5, 6, 10, 15, 25, 26, 28] and the references therein). This method converts the optimal control problem into a mathematical programming problem by using either the discretization technique [5, 6] or the parameterization technique [10, 25, 26, 28].

The discretization technique converts the optimal control problem into a nonlinear programming problem with a large number of unknown parameters and a large number of constraints [6]. On the other hand, parameterizing the control variables [10, 28] requires the integration of the state equations. While the simultaneous parameterization of both the state variables and the control variables [28] results in a nonlinear programming problem with a large number of parameters and a large number of equality constraints.

In the last several years, various methods have been proposed to solve these problems. Yen and Nagurka [32] proposed a method based on the state parameterization, using Fourier series, to solve the linear-quadratic optimal control problem (with equal number of state variables and control variables) subject to state and control inequality constraints. Also Razzaghi and El-nagar [30] proposed a method to solve the unconstrained linear-quadratic optimal control problem with equal number of state and control variables. Their approach is based on using the shifted Legendre polynomials to parameterize the derivative of each of the state variables. In [16] Jaddu and Shimemura proposed a method to solve the linear-quadratic and the nonlinear optimal control problems by using Chebyshev polynomials to parameterize some of the state variables, then the remaining state variables and the control variables are determined from the state equations. The approach proposed in [28] is based on approximating the state variables and control variables with hybrid functions.

In this paper, we present two computational methods for solving nonlinear constrained quadratic optimal control problems by using linear B-spline functions. The methods are based on approximating the state variables and the control variables with a semiorthogonal linear B-spline functions [21]. Our methods consists of reducing the optimal control problem to a NLP one by first expanding the state rate  $\dot{x}(t)$  and the control  $u(t)$  as a linear B-spline functions with unknown coefficients. These functions are introduced. For the approximation of the integral, the operational matrix of integration  $\mathbf{I}_\phi$  is

given. Two operational matrices of integration are calculated using (i) dual basis functions and (ii) interpolation basis functions.

The paper is organized as follows: In Section 2 we describe the basic formulation of the linear B-spline functions required for our subsequent development. Section 3 is devoted to the formulation of optimal control problems. Section 4 summarizes the application of these methods to the optimal control problems, and in Section 5, we report our numerical findings and demonstrate the accuracy of the proposed methods. Sections 6 completes this paper with a brief conclusion.

## 2 Properties of B-spline functions

### 2.1 Linear B-spline functions on $[0,1]$

The  $m$ th-order cardinal B-spline  $N_m(t)$  has the knot sequence  $\{\dots, -1, 0, 1, \dots\}$  and consists of polynomials of order  $m$  (degree  $m-1$ ) between the knots. Let  $N_1(t) = \chi_{[0,1]}(t)$  be the characteristic function of  $[0,1]$ . Then for each integer  $m \geq 2$ , the  $m$ th-order cardinal B-spline is defined, inductively by [8, 13]

$$N_m(t) = (N_{m-1} * N_1)(t) = \int_{-\infty}^{\infty} N_{m-1}(t-\tau)N_1(\tau)d\tau = \int_0^1 N_{m-1}(t-\tau)d\tau. \quad (1)$$

It can be shown [7] that  $N_m(t)$  for  $m \geq 2$  can be achieved using the following formula

$$N_m(t) = \frac{t}{m-1}N_{m-1}(t) + \frac{m-t}{m-1}N_{m-1}(t-1),$$

recursively, and  $\text{supp}[N_m(t)] = [0, m]$ .

The explicit expressions of  $N_2(t)$  (linear B-spline function) are [7, 8, 13]

$$N_2(t) = \begin{cases} t & t \in [0, 1], \\ 2-t & t \in [1, 2], \\ 0 & \text{elsewhere.} \end{cases} \quad (2)$$

Suppose  $N_{j,k}(t) = N_2(2^j t - k)$ ,  $j, k \in \mathbb{Z}$  and  $B_{j,k} = \text{supp}[N_{j,k}(t)] = \text{clos}\{t : N_{j,k}(t) \neq 0\}$ . It is easy to see that

$$B_{j,k} = [2^{-j}k, 2^{-j}(2+k)], \quad j, k \in \mathbb{Z}.$$

To use these functions on  $[0, 1]$ ,

$$S_j = \{k : B_{j,k} \cap [0, 1] \neq \emptyset\}, \quad j \in \mathbb{Z}.$$

It is easy to see that  $\min\{S_j\} = -1$  and  $\max\{S_j\} = 2^j - 1$ ,  $j \in \mathbb{Z}$ .

The support of  $N_{j,k}(t)$  may be out of interval  $[0,1]$ , we need that these functions intrinsically defined on  $[0,1]$  so we put

$$\phi_{j,k}(t) = N_{j,k}(t)\chi_{[0,1]}(t), \quad j \in \mathbb{Z}, \quad k \in S_j. \quad (3)$$

## 2.2 The function approximation

Suppose  $\Phi_j(t)$  is a  $(2^j + 1)$ -vector as

$$\Phi_j(t) = [\phi_{j,-1}(t), \phi_{j,0}(t), \dots, \phi_{j,2^j-1}(t)]^T, \quad j \in \mathbb{Z}. \quad (4)$$

For a fixed  $j = M$ , a function  $f(t) \in L^2[0,1]$  may be represented by the linear B-spline functions as

$$f(t) \simeq \sum_{k=-1}^{2^M-1} s_k \phi_{M,k}(t) = S^T \Phi_M(t), \quad (5)$$

where

$$S = [s_{-1}, s_0, \dots, s_{2^M-1}]^T \quad (6)$$

and

$$s_k = f\left(\frac{k+1}{2^M}\right), \quad k = -1, \dots, 2^M - 1. \quad (7)$$

Note that the functions  $\phi_{M,k}(t)$  satisfy in the relation

$$\phi_{M,k}\left(\frac{i+1}{2^M}\right) = \delta_{k,i} = \begin{cases} 1, & k = i, \\ 0, & k \neq i, \end{cases} \quad i = -1, \dots, 2^M - 1.$$

So we have

$$\Phi_M(t_i) = e_i, \quad t_i = \frac{i+1}{2^M}, \quad i = -1, \dots, 2^M - 1, \quad (8)$$

where  $e_i$  is the  $i$ th column of unit matrix of order  $2^M + 1$  [21].

## 2.3 Two operational matrices of integration

Suppose

$$\Phi_M^f(t) = \int_0^t \Phi_M(\tau) d\tau, \quad (9)$$

then the integration of vectors  $\Phi_M$  in (4) can be expressed as



where using Eq. (4) we have

$$L_i(t) = \int_0^t \phi_{M,i-2}(\tau) d\tau, \quad i = 1, \dots, 2^M + 1, \quad M \in \mathbb{Z}.$$

Finally from Eq. (7) we get

$$(\mathbf{I}_\phi)_{ij} = L_i\left(\frac{j-1}{2^M}\right), \quad i = 1, \dots, 2^M + 1, \quad j = 1, \dots, 2^M + 1, \quad (17)$$

where  $(\mathbf{I}_\phi)_{ij}$  denotes the  $ij$ -th element of matrix  $\mathbf{I}_\phi$ . Final form of this matrix is as follows:

$$\mathbf{I}_\phi = 2^{-(M+1)} \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ & 1 & 2 & \cdots & 2 \\ & & 1 & \ddots & \vdots \\ & & & \ddots & 2 \\ & & & & 1 \end{bmatrix}. \quad (18)$$

### 3 Problem statement

The problem we are treating is to find the optimal control  $\mathbf{u}^*(t)$  and the corresponding optimal state trajectory  $\mathbf{x}^*(t)$  that minimizes the performance index

$$J = \frac{1}{2} \mathbf{x}^T(t_f) \mathbf{Z} \mathbf{x}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (\mathbf{x}^T(t) \mathbf{Q}(t) \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R}(t) \mathbf{u}(t)) dt, \quad (19)$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad (20)$$

$$\Psi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) = 0, \quad (21)$$

$$\mathbf{g}_i(\mathbf{x}(t), \mathbf{u}(t), t) \leq 0, \quad i = 1, 2, \dots, w, \quad (22)$$

where  $\mathbf{Z}$  and  $\mathbf{Q}(t)$  are positive semidefinite matrices,  $\mathbf{R}(t)$  is a positive definite matrix,  $t_0$  and  $t_f$  are known initial and terminal time respectively,  $\mathbf{x}(t) = (x_i(t))_{i=1}^l$  is the state vector,  $\mathbf{u}(t) = (u_j(t))_{j=1}^q$  is the control vector and  $\mathbf{f}, \mathbf{g}_i$  ( $i = 1, 2, \dots, w$ ) are nonlinear functions. This problem is defined on the time interval  $t \in [t_0, t_f]$ . Certain numerical techniques (like B-spline functions) require a fixed time interval, such as  $[0, 1]$ . The independent variable can be mapped to the general interval  $\tau \in [0, 1]$  via the affine transformation

$$\tau = \frac{t - t_0}{t_f - t_0}. \quad (23)$$

Note that this mapping is still valid with free initial and final times. Using Eq. (23), this problem can be redefined as follows:

Minimize the performance index

$$J = \frac{1}{2} \mathbf{x}^T(1) \mathbf{Z} \mathbf{x}(1) + \frac{1}{2} (t_f - t_0) \int_0^1 (\mathbf{x}^T(\tau) \mathbf{Q}(\tau) \mathbf{x}(\tau) + \mathbf{u}^T(\tau) \mathbf{R}(\tau) \mathbf{u}(\tau)) d\tau, \quad (24)$$

subject to

$$\frac{d\mathbf{x}}{d\tau} = (t_f - t_0) \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f), \quad (25)$$

$$\Psi(\mathbf{x}(0), t_0, \mathbf{x}(1), t_f) = 0, \quad (26)$$

$$\mathbf{g}_i(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f) \leq 0, \quad i = 1, 2, \dots, w, \quad \tau \in [0, 1]. \quad (27)$$

## 4 The proposed method

Let

$$\hat{\Phi}_{M,l}(t) = I_l \otimes \Phi_M(t), \quad (28)$$

$$\hat{\Phi}_{M,q}(t) = I_q \otimes \Phi_M(t), \quad (29)$$

where  $I_l$  and  $I_q$  are  $l \times l$  and  $q \times q$  dimensional identity matrices,  $\Phi_M(t)$  is  $(2^M + 1)$ -vector,  $\otimes$  denotes Kronecker product [20] and  $\hat{\Phi}_{M,l}(t)$  and  $\hat{\Phi}_{M,q}(t)$  are matrices of order  $l(2^M + 1) \times l$  and  $q(2^M + 1) \times q$ . Assume that each of  $\dot{\mathbf{x}}_i(t)$ ,  $i = 1, 2, \dots, l$ , and each of  $\mathbf{u}_j(t)$ ,  $j = 1, 2, \dots, q$ , can be written in terms of linear B-spline functions as

$$\dot{\mathbf{x}}_i(t) \simeq \Phi_M^T(t) \mathbf{X}_i,$$

$$\mathbf{u}_j(t) \simeq \Phi_M^T(t) \mathbf{U}_j.$$

Then using Eqs. (28) and (29) we have

$$\dot{\mathbf{x}}(t) \simeq \hat{\Phi}_{M,l}^T(t) \mathbf{X}, \quad (30)$$

$$\mathbf{u}(t) \simeq \hat{\Phi}_{M,q}^T(t) \mathbf{U}, \quad (31)$$

where  $\mathbf{X}$  and  $\mathbf{U}$  are vectors of orders  $l(2^M + 1)$  and  $q(2^M + 1)$ , respectively, given by

$$\mathbf{X} = [\mathbf{X}_1^T, \mathbf{X}_2^T, \dots, \mathbf{X}_l^T]^T,$$

$$\mathbf{U} = [\mathbf{U}_1^T, \mathbf{U}_2^T, \dots, \mathbf{U}_q^T]^T.$$

Similarly we have

$$\mathbf{x}(0) \simeq \hat{\Phi}_{M,l}^T(t) \mathbf{A}_0, \quad (32)$$

where  $\mathbf{A}_0$  is a vector of order  $l(2^M + 1)$  given by

$$\mathbf{A}_0 = [a_1^T, a_2^T, \dots, a_l^T]^T.$$

By integrating Eq. (30) from 0 to  $t$  we get

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \hat{\Phi}_{M,l}^T(\tau) \mathbf{X} d\tau \simeq (I_l \otimes \Phi_M^T(t)) (I_l \otimes \mathbf{I}_\phi^T) \mathbf{X} = \hat{\Phi}_{M,l}^T(t) \hat{\mathbf{I}}_\phi^T \mathbf{X}, \quad (33)$$

where  $\mathbf{I}_\phi$  is an operational matrix of integration given in Eq. (14). From Eqs. (32) and (33) we obtain

$$\mathbf{x}(t) \simeq \hat{\Phi}_{M,l}^T(t) (\mathbf{A}_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}). \quad (34)$$

#### 4.1 The performance index approximation

By substituting Eqs. (31)-(34) in Eq. (24) we get

$$\begin{aligned} J = & \frac{1}{2} (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X})^T (\hat{\Phi}_{M,l}(1) \mathbf{Z} \hat{\Phi}_{M,l}^T(1)) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X}) \\ & + \frac{1}{2} (t_f - t_0) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X})^T \left( \int_0^1 \hat{\Phi}_{M,l}(t) \mathbf{Q}(t) \hat{\Phi}_{M,l}^T(t) dt \right) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X}) \\ & + \frac{1}{2} (t_f - t_0) \mathbf{U}^T \left( \int_0^1 \hat{\Phi}_{M,q}(t) \mathbf{R}(t) \hat{\Phi}_{M,q}^T(t) dt \right) \mathbf{U}. \end{aligned} \quad (35)$$

Eq. (35) can be computed more efficiently by writing  $J$  as

$$\begin{aligned} J = & \frac{1}{2} (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X})^T (\mathbf{Z} \otimes \Phi_M(1) \Phi_M^T(1)) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X}) \\ & + \frac{1}{2} (t_f - t_0) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X})^T \left( \int_0^1 \mathbf{Q}(t) \otimes \Phi_M(t) \Phi_M^T(t) dt \right) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X}) \\ & + \frac{1}{2} (t_f - t_0) \mathbf{U}^T \left( \int_0^1 \mathbf{R}(t) \otimes \Phi_M(t) \Phi_M^T(t) dt \right) \mathbf{U}. \end{aligned} \quad (36)$$

For problems with time-varying performance index,  $\mathbf{Q}(t)$  and  $\mathbf{R}(t)$  are functions of time and

$$\int_0^1 \mathbf{Q}(t) \otimes \Phi_M(t) \Phi_M^T(t) dt, \quad \int_0^1 \mathbf{R}(t) \otimes \Phi_M(t) \Phi_M^T(t) dt$$

can be evaluated numerically. For time-invariant problems,  $\mathbf{Q}(t)$  and  $\mathbf{R}(t)$  are constant matrices and can be removed from the integrals. In this case,



Eq. (36) can be rewritten as

$$\begin{aligned} J(\mathbf{X}, \mathbf{U}) = & \frac{1}{2}(\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X})^T (\mathbf{Z} \otimes \Phi_M(1) \Phi_M^T(1)) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X}) \\ & + \frac{1}{2}(t_f - t_0)(\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X})^T (\mathbf{Q} \otimes \mathbf{P}) (\mathbf{A}_0 + \mathbf{I}_\phi^T \mathbf{X}) \\ & + \frac{1}{2}(t_f - t_0) \mathbf{U}^T (\mathbf{R} \otimes \mathbf{P}) \mathbf{U}. \end{aligned} \quad (37)$$

## 4.2 The system constraints approximation

We approximate the system constraints as follows:

Using Eqs. (30), (31) and (34) the system constraints (25), (26) and (27) became

$$\hat{\Phi}_{M,l}^T(t) \mathbf{X} = (t_f - t_0) \mathbf{f}(\hat{\Phi}_{M,l}^T(t)(A_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}), \hat{\Phi}_{M,q}^T(t) \mathbf{U}, t; t_0, t_f), \quad (38)$$

$$\Psi(\hat{\Phi}_{M,l}^T(0)(A_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}), t_0, \hat{\Phi}_{M,l}^T(1)(A_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}), t_f) = 0, \quad (39)$$

$$\mathbf{g}_i(\hat{\Phi}_{M,l}^T(t)(A_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}), \hat{\Phi}_{M,q}^T(t) \mathbf{U}, t; t_0, t_f) \leq 0, \quad i = 1, 2, \dots, w. \quad (40)$$

We collocate Eqs. (38) and (40) at Newton-cotes nodes  $t_k$ ,

$$t_k = \frac{k-1}{2^M}, \quad k = 1, 2, \dots, 2^M + 1. \quad (41)$$

The optimal control problem has now been reduced to a parameter optimization problem which can be stated as follows:

Find  $\mathbf{X}$  and  $\mathbf{U}$  so that  $J(\mathbf{X}, \mathbf{U})$  is minimized (or maximized) subject to Eq. (39) and

$$\hat{\Phi}_{M,l}^T(t_k) \mathbf{X} = (t_f - t_0) \mathbf{f}(\hat{\Phi}_{M,l}^T(t_k)(A_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}), \hat{\Phi}_{M,q}^T(t_k) \mathbf{U}, t_k), \quad (42)$$

$$\mathbf{g}_i(\hat{\Phi}_{M,l}^T(t_k)(A_0 + \hat{\mathbf{I}}_\phi^T \mathbf{X}), \hat{\Phi}_{M,q}^T(t_k) \mathbf{U}, t_k; t_0, t_f) \leq 0, \quad (43)$$

$$i = 1, 2, \dots, w, \quad k = 1, 2, \dots, 2^M + 1.$$

Many well-developed nonlinear programming techniques can be used to solve this extremum problem (see, e.g. [1, 9, 11]).

## 5 Illustrative examples

This section is devoted to numerical examples. All problems were programmed in MAPLE, running on a Pentium 4, 2.4-GHz PC with 4 GB of RAM. Also we solved the obtained nonlinear programming that is minimize

(or maximize)  $J(\mathbf{X}, \mathbf{U})$  subject to Eqs. (39), (42) and (43), using "NLPsolve" command in MAPLE program. To illustrate our techniques, we present five numerical examples and make a comparison with some of the results in the literatures.

**Example 1.** This example is adapted from [18]. Find the control vector  $u(t)$  which minimizes

$$J = \frac{1}{2} \int_0^1 (x_1^2(t) + u^2(t)) dt, \quad (44)$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad (45)$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad (46)$$

and subject to the following inequality control constraint

$$|u(t)| \leq 1. \quad (47)$$

In Table 1, the minimum of  $J$  using the rationalized Haar functions [29], hybrid of block-pulse and Legendre polynomials [25], hybrid of block-pulse and Bernoulli polynomials [28] and present two methods are listed. In Figure 1, the control and state variables with the absolute value of constraint's errors for  $M = 8$ , are reported.

Table 1: Estimated values of  $J$  for Example 1

Method	$J$	CPUTime
Rationalized Haar functions [29]		
$K = 4$	8.07473	0.389
$K = 8$	8.07065	0.546
Hybrid of block-pulse and Legendre [25]		
$N = 4, M_1 = 3$	8.07059	1.592
$N = 4, M_1 = 4$	8.07056	4.304
Hybrid of block-pulse and Bernoulli [28]		
$N = 4, M = 2$	8.07058	0.858
$N = 4, M = 3$	8.07055	1.155
Present method 1		
$M = 6$	8.07056208507474	1.075
$M = 7$	8.07056206359357	1.453
$M = 8$	8.07056204949560	1.722
Present method 2		
$M = 6$	8.07043910532066	0.665
$M = 7$	8.07053132323846	1.009
$M = 8$	8.07055438812380	1.341

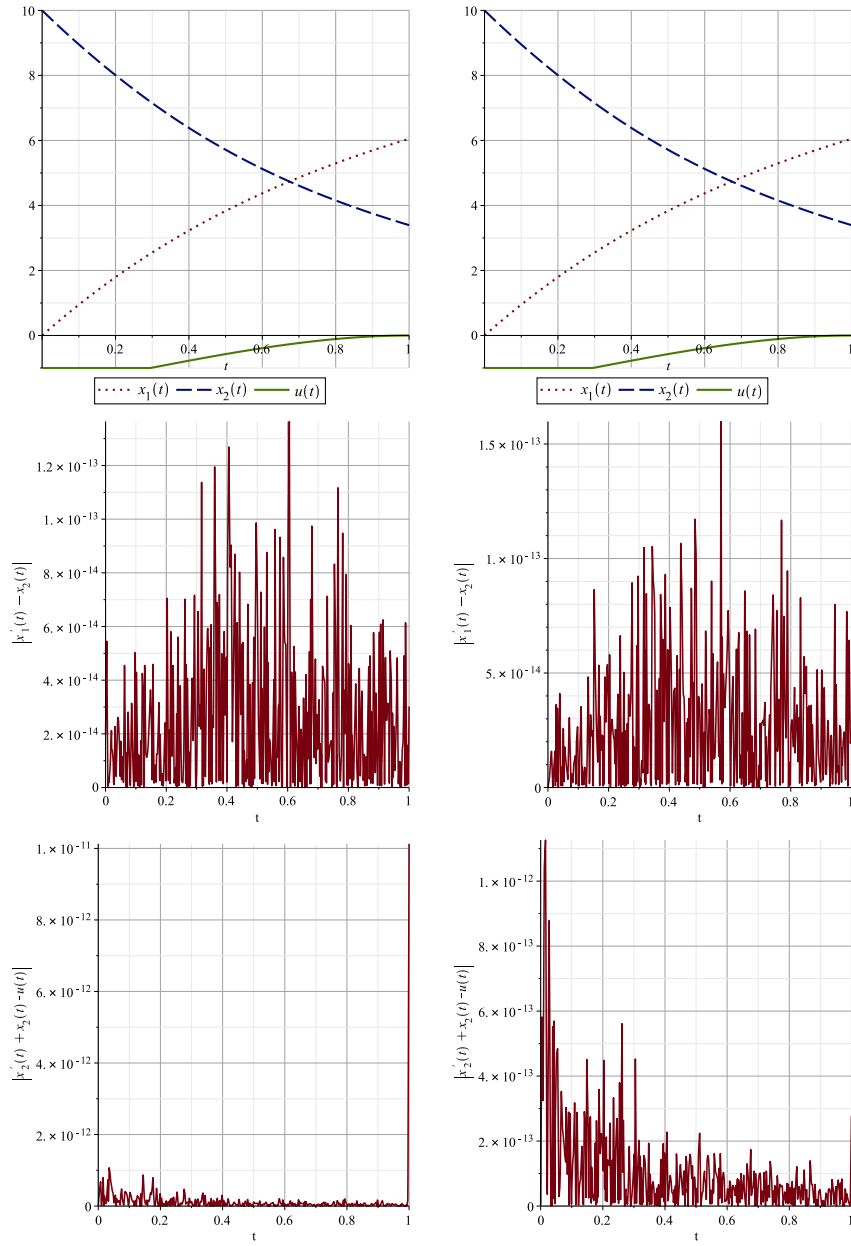


Figure 1: State and control variables and the constraint errors  $|\dot{x}_1(t) - x_2(t)|$  and  $|\dot{x}_2(t) + x_1(t) - u(t)|$  for Example 1 using Method 1 (left) and using Method 2 (right) with  $M = 8$

**Example 2.** Consider the Breakwell problem [12]. The performance index to be minimized is given by

$$J = \frac{1}{2} \int_0^1 u^2(t) dt, \quad (48)$$

subject to the state equations

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t), \quad (49)$$

with the endpoint conditions

$$x_1(0) = x_1(1) = 0, \quad x_2(0) = -x_2(1) = 1, \quad (50)$$

and the state constraint

$$x_1(t) \leq 0.1. \quad (51)$$

The exact solution to this problem is given by

$$u^*(t) = \begin{cases} \frac{200}{9}t - \frac{20}{3}, & t \in [0, 0.3], \\ 0, & t \in [0.3, 0.7], \\ -\frac{200}{9}t + \frac{140}{9}, & t \in [0.7, 1]. \end{cases} \quad (52)$$

This example was studied by using pseudospectral method [12] and ChFD scheme [27]. Here we applied the proposed method to solve this problem. Absolute errors between approximation and exact value of the performance index are reported in Table 2. The approximate solutions of  $x_1(t)$ ,  $x_2(t)$  and  $u(t)$ , obtained by linear B-spline functions using method 2 with  $M = 9$  and the exact solutions together error bounds  $|x_1^*(t) - x_1(t)|$ ,  $|x_2^*(t) - x_2(t)|$  and  $|u^*(t) - u(t)|$  are plotted in Figure 2. This results show that accuracy of our method in comparison with ChFD scheme [27] whose result are plotted in Figure 3.

Table 2: Errors of the estimated and exact values of the performance index,  $|J - J^*|$ , for Example 2

M	Method 1		Method 2	
	$ J - J^* $	CPUTime	$ J - J^* $	CPUTime
6	$3.67 \times 10^{-2}$	0.053	$5.93 \times 10^{-3}$	0.163
7	$1.86 \times 10^{-2}$	0.181	$1.48 \times 10^{-3}$	0.401
8	$9.34 \times 10^{-3}$	1.034	$3.74 \times 10^{-4}$	1.377
9	$4.68 \times 10^{-3}$	7.662	$9.38 \times 10^{-5}$	7.400

**Example 3.** This example is adapted from [19] and also studied by using rationalized Haar approach [26], hybrid of block-pulse and Legendre polynomials [25], hybrid of block-pulse and Bernoulli polynomials [28] and interpolating scaling functions [10]. Find the control vector  $u(t)$  which minimizes

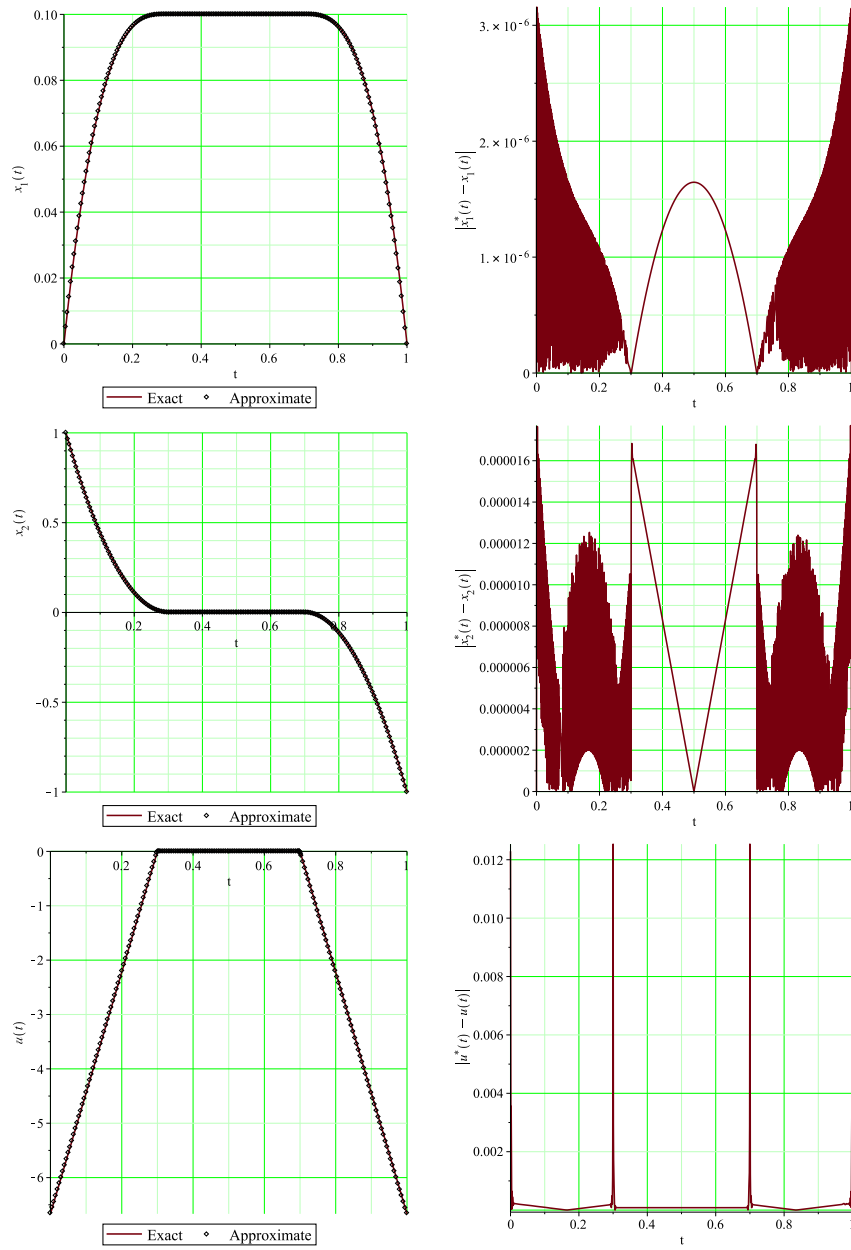


Figure 2: Exact value, approximation of optimal control, state variables and error bounds using method 2 for Example 2 with  $M = 9$

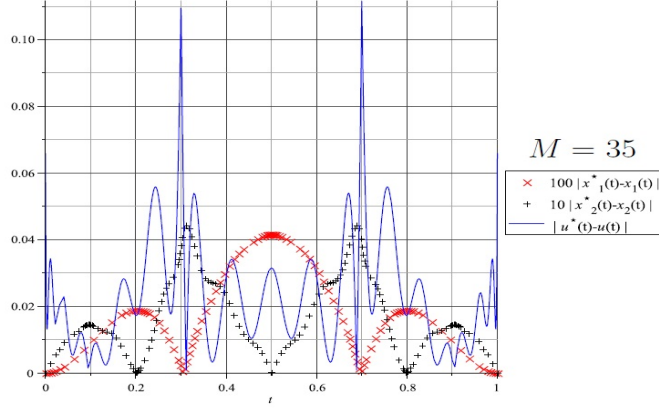


Figure 3: Exact value and approximation errors of  $|x_1^*(t) - x_1(t)|$ ,  $|x_2^*(t) - x_2(t)|$  and  $|u^*(t) - u(t)|$  using ChFD scheme [27] for Example 2 with  $M = 35$

$$J = \int_0^1 (x_1^2(t) + x_2^2(t) + 0.005u^2(t)) dt, \quad (53)$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad (54)$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad (55)$$

and the following state variable inequality constraint

$$x_2(t) \leq r(t), \quad (56)$$

where

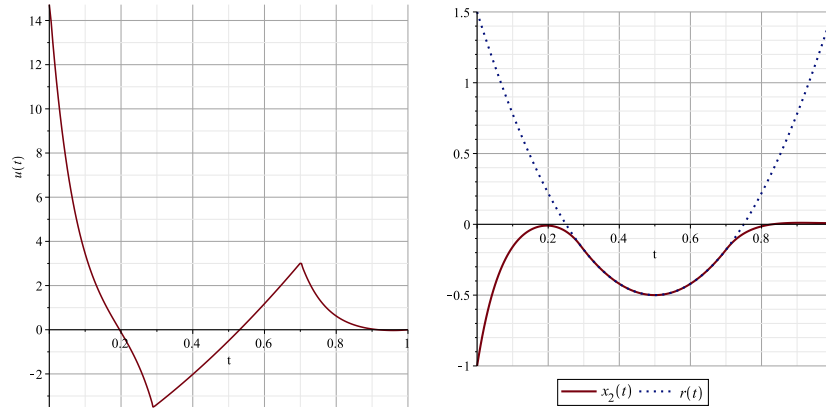
$$r(t) = 8(t - 0.5)^2 - 0.5, \quad 0 \leq t \leq 1.$$

The computational result for  $x_2(t)$  using method 2 for  $M = 8$  together with  $r(t)$  are given in Fig. 4. In Table 3, we compare the minimum of  $J$  using the proposed two methods with other solutions in the literature.

**Example 4.** We consider the optimal maneuvers of a rigid asymmetric spacecraft [17]. This example is studied by using quasilinearization and Chebyshev polynomials [15] and hybrid of block-pulse and Bernoulli polynomials [28]. The system state equations are

Table 3: Results for Example 3

Method	$J$	CPUTime
Rationalized Haar functions [26]		
$K = 64, w = 100$	0.170115	1.877
$K = 128, w = 100$	0.170103	1.983
Hybrid of block-pulse and Legendre [25]		
$N = 4, M_1 = 3$	0.17013645	0.951
$N = 4, M_1 = 4$	0.17013640	1.545
Hybrid of block-pulse and Bernoulli [28]		
$N = 4, M = 3$	0.1700305	0.756
$N = 4, M = 4$	0.1700301	0.921
Interpolating scaling functions [10]		
$n = 4, r = 5$	0.16982646	2.251
$n = 5, r = 5$	0.16982636	3.175
Present method 1		
$M = 6$	0.169672402102247	1.512
$M = 7$	0.169782602033829	1.607
$M = 8$	0.169811048165412	1.985
Present method 2		
$M = 6$	0.170071967582200	0.599
$M = 7$	0.169885295276034	1.003
$M = 8$	0.169837051920398	1.141

Figure 4: Control and state variables and constraint boundary for Example 3 using method 2 with  $M = 8$ 

$$\begin{aligned}
\dot{x}_1(\tau) &= -\frac{I_3 - I_2}{I_1}x_2(\tau)x_3(\tau) + \frac{u_1(\tau)}{I_1}, \\
\dot{x}_2(\tau) &= -\frac{I_1 - I_3}{I_2}x_1(\tau)x_3(\tau) + \frac{u_2(\tau)}{I_2}, \\
\dot{x}_3(\tau) &= -\frac{I_2 - I_1}{I_3}x_1(\tau)x_2(\tau) + \frac{u_3(\tau)}{I_3}, \\
x_1(\tau) - (5 \times 10^{-6}\tau^2 - 5 \times 10^{-4}\tau + 0.016) &\leq 0, \\
x_1(100) = x_2(100) = x_3(100) &= 0,
\end{aligned}$$

where  $I_1 = 86.24$ ,  $I_2 = 85.07$ ,  $I_3 = 113.59$ . The performance index to be minimized, starting from the initial states  $x_1(0) = 0.01$ ,  $x_2(0) = 0.005$  and  $x_3(0) = 0.001$  is

$$J = \frac{1}{2} \int_0^{100} (u_1^2(\tau) + u_2^2(\tau) + u_3^2(\tau)) d\tau.$$

We use transformation  $\tau = 100t, 0 \leq t \leq 1$ , in order to use our proposed method. In Table 4, the results for  $J$  using linear B-spline functions, hybrid of block-pulse and Bernoulli polynomials [28] and quasilinearization and Chebyshev polynomials [15] are listed. Optimal control and state variables and constraint boundary using method 2, for  $M = 7$ , are shown in Figure 5.

Table 4: Results for Example 4

Method	$J$	CPUTime
Quasilinearization and Chebyshev polynomials [15]		
$N = 6$	0.00536584	0.07
$N = 8$	0.00534427	0.10
$N = 10$	0.00534063	0.12
Quasilinearization and Chebyshev polynomials [15] with using 2 subintervals		
$M2 = 10$	0.00530902	0.36
Hybrid of block-pulse and Bernoulli [28]		
$N = 6, M = 3$	0.00531097	1.89
$N = 6, M = 4$	0.00530263	2.12
$N = 6, M = 5$	0.00530213	2.74
Present method 1		
$M = 5$	0.00527460682730895	0.55
$M = 6$	0.00529464663721832	0.67
$M = 7$	0.00530275422863559	0.71
Present method 2		
$M = 5$	0.00530817821841613	0.06
$M = 6$	0.00530851397972318	0.10
$M = 7$	0.00530847110421464	0.13

**Example 5.** Consider the problem of transferring containers from a ship to a cargo truck [31]. The container crane is driven by a hoist motor and a trolley drive motor. The aim is to minimize the swing during and at the end of the transfer. After appropriate normalization, we summarize the problem as follows:

$$J = 4.5 \int_0^1 (x_3^2(t) + x_6^2(t)) dt$$

subject to



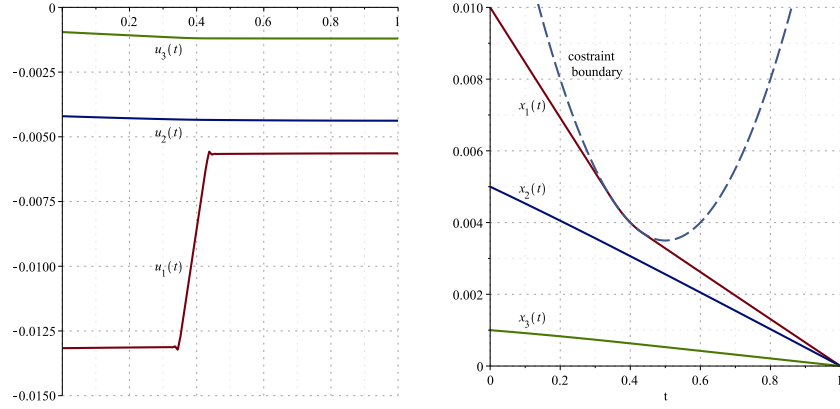


Figure 5: Control and state variables and constraint boundary for Example 4 using method 2 with  $M = 7$

$$\begin{aligned}
 \dot{x}_1(t) &= 9x_4(t), \\
 \dot{x}_2(t) &= 9x_5(t), \\
 \dot{x}_3(t) &= 9x_6(t), \\
 \dot{x}_4(t) &= 9(u_1(t) + 17.2656x_3(t)), \\
 \dot{x}_5(t) &= 9u_2(t), \\
 \dot{x}_6(t) &= \frac{-9(u_1(t) + 27.0756x_3(t) + 2x_5(t)x_6(t))}{x_2(t)},
 \end{aligned}$$

where

$$\begin{aligned}
 x(0) &= [0, 22, 0, 0, -1, 0]^T, \\
 x(1) &= [10, 14, 0, 2.5, 0, 0]^T,
 \end{aligned}$$

and

$$\begin{aligned}
 |u_1(t)| &\leq 2.83374, \quad t \in [0, 1], \\
 -0.80865 &\leq u_2(t) \leq 0.71265, \quad t \in [0, 1],
 \end{aligned}$$

with continuous state inequality constraints,

$$\begin{aligned}
 |x_4(t)| &\leq 2.5, \quad t \in [0, 1], \\
 |x_5(t)| &\leq 1.0, \quad t \in [0, 1].
 \end{aligned}$$

In Table 5, we compare the solution obtained using the proposed two methods with the method of [9] and [28].

Table 5: Results for Example 5

Method	$J$	CPUTime
Method of [9]		
$m = 5$	$0.5366 \times 10^{-2}$	2.589
$m = 7$	$0.53614 \times 10^{-2}$	2.607
$m = 9$	$0.53610895 \times 10^{-2}$	3.002
$m = 11$	$0.5361102700 \times 10^{-2}$	3.021
Hybrid of block-pulse and Bernoulli [28]		
$N = 2, M = 2$	$0.593000 \times 10^{-2}$	1.904
$N = 2, M = 3$	$0.528915 \times 10^{-2}$	2.125
$N = 2, M = 4$	$0.521421 \times 10^{-2}$	2.305
$N = 2, M = 5$	$0.521411 \times 10^{-2}$	2.663
Present method 1		
$M = 5$	$0.498574175174882 \times 10^{-2}$	1.815
$M = 6$	$0.503885802644245 \times 10^{-2}$	1.963
$M = 7$	$0.511514185733751 \times 10^{-2}$	2.025
Present method 2		
$M = 5$	$0.516049181578648 \times 10^{-2}$	1.579
$M = 6$	$0.515021009757565 \times 10^{-2}$	1.708
$M = 7$	$0.515021009428266 \times 10^{-2}$	1.869

## 6 Conclusion

In this paper we presented two numerical methods for solving nonlinear constrained quadratic optimal control problems. Two methods are based upon the linear B-spline functions. Also several test problems were used to see the applicability and efficiency of the method. The obtained results show that when the state variables are unknown at the endpoints, then method 1 is more accurate than method 2 but in all problems method 2 is faster than method 1. In total, our two methods are more accurate than existing methods in the literature.

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## دو روش عددی برای حل مسائل کنترل بهینه درجه دوم با قيود غیرخطی با استفاده از توابع بی-اسپلاین خطی

یوسف ادیسی تبریز<sup>۱</sup>، مهرداد لکستانی<sup>۲</sup> و عقیده حیدری<sup>۱</sup>

<sup>۱</sup> دانشگاه پیام نور تهران، گروه ریاضی

<sup>۲</sup> دانشگاه تبریز، دانشکده علوم ریاضی

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**چکیده :** این مقاله به ارائه ی دو روش عددی برای حل مسائل کنترل بهینه ای می پردازد که دارای تابع هدف درجه دوم می باشند. همچنین ویژگی های توابع بی-اسپلاین بیان می گردد. دو ماتریس عملیاتی انتگرال مرتبط با روش ها معرفی می شوند. سپس از این ماتریسها استفاده می شود تا حل مسئله کنترل بهینه درجه دوم با قيود غیرخطی به حل مسئله برنامه ریزی غیرخطی تبدیل گردد. در انتها چندین مثال کاربردی برای نمایش کارایی و راستی آزمایی روشهای مذکور بیان می شود.

**کلمات کلیدی :** مسائل کنترل بهینه؛ تابع بی-اسپلاین خطی؛ ماتریس عملیاتی انتگرال؛ روش هم مکانی .