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Stability Analysis of Conformable Fractional Systems

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Abstract

In this paper, we investigate stability analysis of fractional differential systems equipped with the conformable fractional derivatives. Some stability conditions of fractional differential systems are proposed by applying the fractional exponential function and the fractional Laplace transform. Moreover, we check the stability of conformable fractional Lotka-Volterra system with the multi-step homotopy perturbation method to demonstrate the efficiency and effectiveness of the proposed procedure.

Keywords: Stability analysis; Asymptotical stability; Conformable fractional derivative; Lotka-Volterra system.

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order that have recently proved to be valuable tools for the modelling of many physical phenomena, and have been the focus of many studies due to their frequent appearances in various applications, such as physics, biology, finance and fractional dynamics, engineering, signal processing and control theory [8, 12]. Finding solutions to fractional differential

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systems is rather complicated, consequently, the stability results of the fractional differential systems have been the main goal of the previous studies. For example, Matignon considers the stability of fractional differential systems in control processing [11], Deng has studied the stability of fractional differential system with multiple time delays [6] and we have investigated the stability of fractional differential systems with Hilfer derivatives [15]. More novelty, authors in [3, 4, 14] studied stability analysis of distributed order fractional differential equations with respect to the nonnegative density function. There are many definitions of fractional derivatives, such as Riemann-Liouville, Grunwald-Letnikov and Caputo's fractional derivatives [12], which these fractional derivatives do not satisfy the product rule, the quotient rule and the chain rule. In 2014, to overcome these and other difficulties, Khalil et al. [7] introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative. This fractional derivative is theoretically very easier to handle and also obeys some conventional properties that cannot be satisfied by the existing fractional derivatives, for instance, the chain rule [1]. However this fractional derivative has a weakness, which is the fractional derivative of any differentiable function at point zero.

In this paper, we shall describe stability conditions for conformable fractional differential systems. In particular our analysis covers the linear conformable fractional differential systems with commensurate order and in-commensurate order and the nonlinear conformable fractional differential system. At first, stability conditions will be established using fractional exponential function for linear conformable fractional differential systems, corresponding to this result we will derive asymptotic stability for the nonlinear conformable fractional system. Then, we produce sufficient conditions for asymptotical stability of in-commensurate linear conformable fractional differential system by of the fractional Laplace transform and the fractional final value theorem. Finally, we present the multi-step homotopy perturbation method for obtain approximate analytical solution and stability of the conformable fractional Lotka-Volterra system to illustrate the validity of the results.

2 Conformable fractional derivative

Here, some basic definitions and properties of the conformable fractional calculus theory which can be found in [1, 2, 7] are presented.

Definition 1. Let $f:(0,\infty)\to\mathbb{R}$, then, the conformable fractional derivative of f of order α is defined as [7]

$$ {}_tT_\alpha\left(f\right)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1} $$

for all $t > 0$, $\alpha \in (0, 1)$.
If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} {}_tT_\alpha\left(f\right)(t)$ exists, then by definition

$$ {}_tT_\alpha\left(f\right)(0) = \lim_{t\to 0^+} {}_tT_\alpha\left(f\right)(t). \tag{2} $$

The new definition satisfies the properties which are given in the following theorem.

Theorem 1. *Let $\alpha \in (0, 1]$, and f, g be α-differentiable at point t, then [7]*

(i) ${}_tT_\alpha(af + bg) = a\ {}_tT_\alpha(f) + b\ {}_tT_\alpha(g)$, *for all $a, b \in \mathbb{R}$.*

(ii) ${}_tT_\alpha(t^\mu) = \mu\ t^{\mu-\alpha}$, *for all $\mu \in \mathbb{R}$.*

(iii) ${}_tT_\alpha(fg) = f\ {}_tT_\alpha(g) + g\ {}_tT_\alpha(f)$.

(iv) ${}_tT_\alpha(\frac{f}{g}) = \frac{g\ {}_tT_\alpha(f) - f\ {}_tT_\alpha(g)}{g^2}$. *In addition, if f is differentiable, then* ${}_tT_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

In [1] T. Abdeljawad established the chain rule for conformable fractional derivatives as following theorem.

Theorem 2. Let $f : (0, \infty) \to \mathbb{R}$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable; then, one has the following rule

$$ {}_tT_\alpha(fog)(t) = \left({}_tT_\alpha f\right)(g(t))\left({}_tT_\alpha g\right)(t)\ g(t)^{\alpha-1}. \tag{3} $$

If $t = 0$, then

$$ {}_tT_\alpha(fog)(0) = \lim_{t\to 0^+}\left({}_tT_\alpha f\right)(g(t))\left({}_tT_\alpha g\right)(t)\ g(t)^{\alpha-1}. $$

The fractional exponential function plays a very important role in the conformable fractional differential equations. The fractional exponential function $e^{\frac{1}{\alpha}t^\alpha}$, where $0 < \alpha \leq 1$, is defined by the following series representation,

$$ e^{\frac{1}{\alpha}t^\alpha} = \sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\alpha^k k!}. $$

Now, we list here the fractional derivatives of certain functions [7]

(i) ${}_tT_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$,

(ii) ${}_tT_\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos\frac{1}{\alpha}t^\alpha$,

(iii) ${}_tT_\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin\frac{1}{\alpha}t^\alpha$,

(iv) ${}_tT_\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

Definition 2. *(Fractional Integral) [7]* Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in (0, 1]$. Then, the α-fractional integral of f is defined by,

$$ {}_tI_a^\alpha f(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx, \tag{4}$$

if the Riemann improper integral exists.

Theorem 3. *(Integration by parts) [1], Let $f, g : [0, b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_0^b f(t)\ {}_tT_\alpha\,(g)\,(t)\ d\alpha(t) = fg\big|_0^b - \int_0^b g(t)\ {}_tT_\alpha\,(f)\,(t)\ d\alpha(t), \tag{5}$$

where $d\alpha(t) = t^{\alpha-1}dt$.

It is interesting to observe that the α-fractional derivative and the α-fractional integral are inverse of each other as given in [7].

Theorem 4. *(Inverse property). Let $\alpha \in (0, 1]$ and f be a continuous function such that ${}_tI_0^\alpha f$ exists. Then*

$$ {}_tT_\alpha({}_tI_0^\alpha f)(t) = f(t), \qquad for \;\; t \geq 0.$$

Definition 3. *(fractional Laplace transform) [1]* Let $0 < \alpha \leq 1$ and $f : [0, \infty) \to \mathbb{R}$ be real valued function. Then the fractional Laplace transform of order α starting from a of f is defined by,

$$L_\alpha\{f(t)\} = F_\alpha(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\ d\alpha(t), \tag{6}$$

where $d\alpha(t) = t^{\alpha-1}dt$.

Fractional Laplace transform for certain functions are presented as follows [1]

- ◇ $L_\alpha\{1\} = \frac{1}{s}, \quad s > 0.$
- ◇ $L_\alpha\{t^p\} = \frac{\alpha^{p/\alpha}}{s^{1+p/\alpha}}\Gamma(1+\frac{1}{\alpha}), \quad s > 0.$
- ◇ $L_\alpha\{e^{\frac{t^\alpha}{\alpha}}\} = \frac{1}{s-1}, \quad s > 1.$
- ◇ $L_\alpha\{sin\frac{1}{\alpha}t^\alpha\} = \frac{1}{s^2+1}, \quad s > 1.$
- ◇ $L_\alpha\{cos\frac{1}{\alpha}t^\alpha\} = \frac{s}{s^2+1}, \quad s > 1.$

Furthermore, using the properties of the fractional exponential function and integration by parts, we have

$$L_\alpha\{{}_tT_\alpha\left(f\right)\left(t\right)\} = s\ F_\alpha(s) - f(0). \tag{7}$$

Next we prove a fractional version of final value theorem which will be useful is studying stability of conformable fractional systems.

Theorem 5. *(final value theorem) Let $F_\alpha(s)$ be the fractional Laplace transform of the function $f(t)$. If all poles of $sF_\alpha(s)$ are in the open left-half plane, then,*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF_\alpha(s). \tag{8}$$

Proof. By using formula (7) we arrive at the following relation

$$\lim_{s\to 0} L_\alpha\{{}_tT_\alpha\left(f\right)\left(t\right)\} = \lim_{s\to 0}\left(s\ F_\alpha(s) - f(0)\right) = \lim_{s\to 0}\left(s\ F_\alpha(s)\right) - f(0). \tag{9}$$

Consider now the first term only. Since s is independent of t, the order of integrating and taking the limit can be interchanged

$$\begin{aligned} \lim\nolimits_{s\to 0} L_\alpha\{{}_tT_\alpha\left(f\right)\left(t\right)\} &= \lim\nolimits_{s\to 0}\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\ {}_tT_\alpha\left(f\right)\left(t\right)\,d\alpha(t) \\ &= \int_0^\infty \left[\lim\nolimits_{s\to 0} e^{-s\frac{t^\alpha}{\alpha}}\right]\ {}_tT_\alpha\left(f\right)\left(t\right)\,d\alpha(t) \qquad (10) \\ &= \int_0^\infty\ {}_tT_\alpha\left(f\right)\left(t\right)\,d\alpha(t). \end{aligned}$$

Taking the integration in the last term of (10), implies that

$$\int_0^\infty {}_tT_\alpha\left(f\right)\left(t\right)\,d\alpha(t) = \lim_{t\to\infty}\ \left(f(t) - f(0)\right). \tag{11}$$

Finally, using (9) and (11), we have

$$\lim_{s\to 0}\left(s\ F_\alpha(s)\right) - f(0) = \lim_{t\to\infty}\ \left(f(t) - f(0)\right).$$

Neither term on the right-hand side depends on s, so we can remove the limit and simplify, resulting in the final value theorem

$$\lim_{s\to 0} s\, F_\alpha(s) = \lim_{t\to\infty} f(t).$$

□

3 Stability analysis of linear conformable fractional differential system

In this section, we consider the stability of the following linear conformable fractional differential system

$$ {}_tT_\alpha\, x(t) = Ax(t), \quad t > 0, \quad x(0) = x_0, \tag{12}$$

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$, and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ such that $0 < \alpha_i \le 1$, for $i = 1, 2, \ldots, n$.

Remark 1. If $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n$, then system (12) is called a commensurate order system, otherwise system (12) indicates an in-commensurate order system.

Definition 4. The zero solution of linear conformable fractional differential system (12) is said to be stable if, for any initial value x_0, there exists an $\varepsilon > 0$ such that $\|x(t)\| \le \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to \infty$.

Now we state stability theorems from commensurate order system, next produce conditions for asymptotical stability of the in-commensurate order system.

Theorem 6. *The solution to the linear commensurate order system* (12) *is given by*

$$x(t) = x_0\, e^{\frac{1}{\alpha}At^\alpha}, \tag{13}$$

where the solution is assumed to be differentiable on $(0, \infty)$.

Proof. Taking the fractional Laplace transform of (12), by formula (7), we have that

$$sX_\alpha(s) - sx_0 = AX_\alpha(s),$$

so that

$$X_\alpha(s) = x_0(sI - A)^{-1}. \tag{14}$$

Now, by applying the fractional Laplace transform of the fractional exponential function, we obtain the claimed result. □

Definition 5. If all the eigenvalues of A satisfy

$$|\arg(\lambda(A)| > \frac{\pi}{2}, \tag{15}$$

then A is said to be Hurwitz stable. If all the eigenvalues of A satisfy

$$|\arg(\lambda(A)| \geq \pi/2, \tag{16}$$

and if $|\arg(\lambda_{j0}(A)| = \pi/2$, λ_{j0} only correspond to simple elementary divisor of A, then A is said to be quasi-stable.

Theorem 7. *The zero solution of the conformable fractional differential system* (12) *is asymptotically stable if and only if A is a Hurwitz stable, and the zero solution of system* (12) *is stable but not asymptotically stable if and only if A is quasi-stable.*

Proof. According to the relation (13), we have

$$x(t) = x_0 \, e^{\frac{1}{\alpha}At^\alpha}.$$

Let $A = SJS^{-1}$, J is a Jordan canonical form, where J_i, $1 \leq i \leq r$ has the following form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}_{n_i \times n_i}, \qquad \lambda_i \in \mathbb{C},$$

and $\sum_{i=1}^{r} n_i = n$. Then,

$$e^{\frac{1}{\alpha}At^\alpha} = e^{\frac{1}{\alpha}SJS^{-1}t^\alpha} = S\, e^{\frac{1}{\alpha}Jt^\alpha}\, S^{-1},$$

$$e^{\frac{1}{\alpha}Jt^\alpha} = diag\left(e^{\frac{1}{\alpha}J_1t^\alpha}, e^{\frac{1}{\alpha}J_2t^\alpha} \ldots, e^{\frac{1}{\alpha}J_rt^\alpha}\right),$$

$$e^{\frac{1}{\alpha}J_jt^\alpha} = \begin{pmatrix} 1 & \frac{1}{\alpha}t^\alpha & \frac{1}{\alpha^2}\frac{t^{2\alpha}}{2!} & \cdots & \frac{1}{\alpha^{n_j-1}}\frac{t^{(n_j-1)\alpha}}{(n_j-1)!} \\ 0 & 1 & \frac{1}{\alpha}t^\alpha & \cdots & \frac{1}{\alpha^{n_j-2}}\frac{t^{(n_j-2)\alpha}}{(n_j-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1 & \frac{1}{\alpha}t^\alpha \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n_j \times n_j} e^{\frac{1}{\alpha}\lambda_jt^\alpha}.$$

One can easily show that the stability of zero solution of system (12) is determined by the boundedness of $e^{\frac{1}{\alpha}At^\alpha}$, or the boundedness of $e^{\frac{1}{\alpha}Jt^\alpha}$ or the boundedness of all $e^{\frac{1}{\alpha}J_j t^\alpha}$ $j = 1, 2, \ldots, r$ i.e., $|\arg(\lambda_j(A)| \geq \pi/2$, and when $|\arg(\lambda_j(A)| = \pi/2$, $n_j = 1$, i.e., A is quasi-stable. Asymptotical stability of the zero solution of system (12) is given by

$$\lim_{t\to+\infty} e^{\frac{1}{\alpha}At^\alpha} = 0,$$

or

$$\lim_{t\to+\infty} e^{\frac{1}{\alpha}Jt^\alpha} = 0,$$

which is equivalent to

$$\lim_{t\to+\infty} e^{\frac{1}{\alpha}J_j t^\alpha} = 0, \qquad (j = 1, 2, \ldots, r).$$

That is,

$$\lim_{t\to+\infty} \begin{pmatrix} 1 & \frac{1}{\alpha}t^\alpha & \frac{1}{\alpha^2}\frac{t^{2\alpha}}{2!} & \cdots & \frac{1}{\alpha^{n_j-1}}\frac{t^{(n_j-1)\alpha}}{(n_j-1)!} \\ 0 & 1 & \frac{1}{\alpha}t^\alpha & \cdots & \frac{1}{\alpha^{n_j-2}}\frac{t^{(n_j-2)\alpha}}{(n_j-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1 & \frac{1}{\alpha}t^\alpha \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n_j\times n_j} e^{\frac{1}{\alpha}\lambda_j t^\alpha} = 0, \ \forall j = 1, 2, \ldots, r,$$

and thus $|\arg(\lambda_j(A)| > \pi/2$ $(j = 1, 2, \ldots, n)$ implying that A is a Hurwitz matrix. The proof is complete. □

Remark 2. If there is a λ_0 such that $|\arg(\lambda_0)| < \pi/2$, then zero solution of system (12) is unstable.

Remark 3. If A has zero eigenvalue, then zero solution of system (12) is unstable.

Proof. If $\lambda = 0$, then from the proof of Theorem 7, we have

$$\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1} e^{\frac{1}{\alpha}\lambda t^\alpha}\right\}\Bigg|_{\lambda=0} = \frac{t^{(j-1)\alpha}}{(j-1)!\alpha^{(j-1)}}, \ j = 1, 2, \ldots, n_i, \ 1 \leq i \leq r.$$

It is obvious that $\lim_{t\to\infty} \frac{t^{(j-1)\alpha}}{(j-1)!\alpha^{(j-1)}} = \infty$ for $j \geq 1$. Thus, $\lim_{t\to\infty} \|x(t)\| = \infty$. □

Now, we consider an in-commensurate linear conformable fractional differential system

$$\begin{cases} {}_tT_{\alpha_1}\, x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t), \\ {}_tT_{\alpha_2}\, x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t), \\ \quad\vdots \qquad\qquad\qquad \vdots \\ {}_tT_{\alpha_n}\, x_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t), \end{cases} \tag{17}$$

where $x_i(0) = x_{i0}$ and $0 < \alpha_i \leq 1$ for $i = 1, 2, \dots, n$.

We study the stability of system (17) by applying the fractional Laplace transforms on both sides of this system, we have

$$sX_{\alpha_i}(s) - x_{i0} = \sum_{j=1}^{n} a_{ij} X_{\alpha_i}(s), \tag{18}$$

for $i = 1, \dots, n$, where $X_{\alpha_i}(s)$ is the fractional Laplace transform of $x_i(t)$. We can rewrite (18) as follows

$$\Delta(s). \begin{pmatrix} X_{\alpha_1}(s) \\ X_{\alpha_2}(s) \\ \vdots \\ X_{\alpha_n}(s) \end{pmatrix} = x_0. \tag{19}$$

in which

$$\Delta(s) = \begin{pmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \dots & \Delta_{1n}(s) \\ \Delta_{21}(s) & \Delta_{22}(s) & \dots & \Delta_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1}(s) & \Delta_{n2}(s) & \cdots & \Delta_{nn}(s) \end{pmatrix},$$

where

$$\Delta_{ij}(s) = \begin{cases} s - a_{ii} & \text{if } i = j, \\ -a_{ij} & \textit{otherwise}. \end{cases}$$

and $x_0 = (x_{10}, x_{20}, \dots, x_{n0})^T$. For simplicity, we call $\Delta(s)$ a characteristic matrix of (17), moreover $\det(\Delta(s)) = 0$ is the characteristic equation of system (17). Now, we express the main theorem for checking the stability of system (17).

Theorem 8. *If all roots of* $\det(\Delta(s)) = 0$ *have negative real parts, then zero solution system of* (17) *is asymptotically stable.*

Proof. Multiplying s on both sides of (19) gives, we have

$$\Delta(s). \begin{pmatrix} sX_{\alpha_1}(s) \\ sX_{\alpha_2}(s) \\ \vdots \\ sX_{\alpha_n}(s) \end{pmatrix} = sx_0. \tag{20}$$

if all roots of the $\det(\Delta(s)) = 0$ lie in open left half complex plane (i.e. $\Re(s) < 0$), then, we consider (20) in $\Re(s) \geq 0$. In this restricted area, the relation (20) has a unique solution $sX(s) = (sX_{\alpha_1}(s), sX_{\alpha_2}(s), \dots, sX_{\alpha_n}(s))$. Since $\lim_{s\to 0} s = 0$, so we have

$$\lim_{s\to 0, \Re(s)\geq 0} sX_{\alpha_i}(s) = 0, \qquad i = 1, 2, \ldots, n$$

which from the Theorem 5, we get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} (x_1(t), x_2(t), \ldots, x_n(t)) = \lim_{s\to 0} (sX_{\alpha_1}(s), sX_{\alpha_2}(s), \ldots, sX_{\alpha_n}(s)) = 0.$$

The above result shows that system (17) is asymptotically stable. □

The inertia of a matrix is the triplet of the numbers of eigenvalues of A with positive, negative, and zero real parts. Now, we generalize the inertia concept for analyzing the stability of fractional linear system.

Definition 6. The inertia of the system (17) is the triple

$$I_{n(\alpha)}(A) = (\pi_{n(\alpha)}(A), \nu_{n(\alpha)}(A), \delta_{n(\alpha)}(A)),$$

where $\pi_{n(\alpha)}(A), \nu_{n(\alpha)}(A)$ and $\delta_{n(\alpha)}(A)$ are, respectively, the number of roots of $\det(\Delta(s)) = 0$ with positive, negative, and zero real parts.

Theorem 9. *The linear conformable fractional differential system* (17) *is asymptotically stable if any of the following equivalent conditions holds.*
(i) The matrix A is a Hurwitz matrix.
(ii) $\pi_{n(\alpha)}(A) = \delta_{n(\alpha)}(A) = 0$.
(iii) All roots of the characteristic equation of system (17) *satisfy* $|\arg(s)| > \pi/2$.

Proof. According to Theorem 8 and Definition 5, proof can be easily obtained. □

4 Stability of non-linear conformable fractional differential systems

In this section, we will mainly discuss the stability of a nonlinear conformable fractional differential system, which can be described by

$$_tT_\alpha\, X(t) = F(X(t)), \qquad t > 0, \;\; 0 < \alpha \leq 1, \tag{21}$$

with the initial value $X(0) = X_0$, where

$$F(X(t)) = \begin{pmatrix} f_1(x_1(t), x_2(t), \ldots, x_n(t)) \\ f_2(x_1(t), x_2(t), \ldots, x_n(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \ldots, x_n(t)) \end{pmatrix},$$

and $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$.

Theorem 10. *Let $\hat{X}(t) = (\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t))^T$ is the equilibrium of system (21), i.e. ${}_tT_\alpha \hat{X} = F(\hat{X}) = 0$ and $J = \left(\frac{\partial F}{\partial X}\right)\big|_{X=\hat{X}}$ is the Jacobian matrix at the point $\hat{X}$, then the point $\hat{X}$ is asymptotically stable if and only if J is a Hurwitz matrix.*

Proof. Let $\zeta(t) = X(t) - \hat{X}$, where $\zeta(t) = (\zeta_1(t), \zeta_2(t), \ldots, \zeta_n(t))$ is a small disturbance from a fixed point. Therefore

$$ {}_tT_\alpha\, \zeta(t) = {}_tT_\alpha \left(X(t) - \hat{X} \right), \tag{22}$$

since ${}_tT_\alpha \left(X(t) - \hat{X} \right) = {}_tT_\alpha X(t) - {}_tT_\alpha \hat{X}$, and ${}_tT_\alpha \hat{X} = 0$, thus, we have

$$\begin{aligned} {}_tT_\alpha \zeta(t) = {}_tT_\alpha X(t) &= F(X(t)) = F(\zeta(t) + \hat{X}) \\ &= F(\hat{X}) + J\zeta(t) + \textit{higher order terms} \\ &\approx J\zeta(t). \end{aligned}$$

System (22) can be written as

$$ {}_tT_\alpha \zeta(t) \approx J\zeta(t), \tag{23}$$

with the initial value $\zeta(0) = X_0 - \hat{X}$. The analytical procedure of linearization is based on the fact that if the matrix J has no purely imaginary eigenvalues, then the trajectories of the nonlinear system in the neighborhood of the equilibrium point have the same form as the trajectories of the linear system [17]. Hence, by applying Theorem 7 the linear system (23) is asymptotically stable if and only if all roots of the characteristic function of J satisfy $|\arg(\lambda(J)| > \pi/2$, which implies that the equilibrium $\hat{X}$ of the nonlinear conformable fractional system (21) is as asymptotically stable. □

Remark 4. The nonlinear conformable fractional system (22) in the point $\hat{X}$ is asymptotically stable if and only if $\pi_{n(\alpha)}(J) = \delta_{n(\alpha)}(J) = 0$.

5 The multi-step homotopy perturbation method

The homotopy perturbation method (HPM) was proposed by He [10] in 1999. This method has been used by many mathematicians and engineers to solve various functional equations. Although the HPM yields a solution series which converges very rapidly in most linear and nonlinear equations, in the case of a large time interval t it may produce a large error. To overcome this shortcoming, the multi-step homotopy perturbation method (MHPM) was presented in [5], to solve nonlinear ordinary differential equations. For the convenience of the reader, we will first present a brief account of HPM. Let us consider the following differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{24}$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{25}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.
The operator A can be generally divided into two parts L and N, where L is linear, while N is nonlinear. Therefore Equation (24) can be written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{26}$$

By using homotopy technique, one can construct a homotopy $y(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies

$$H(y,p) = (1-p)[L(y)+L(u_0)]+p[A(y)-f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{27}$$

which is equivalent to

$$H(y,p) = L(y) - L(u_0) + pL(u_0) + p[N(y) - f(r)] = 0, \tag{28}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial guess approximation of Equation (24) which satisfies the boundary conditions. Clearly, we have

$$H(y,0) = L(y) - L(u_0) = 0, \tag{29}$$

$$H(y,1) = A(y) - f(r)] = 0. \tag{30}$$

Thus, the changing process of p from 0 to 1 is just that of $y(r,p)$ from $u_0(r)$ to $y(r)$. In topology this is called deformation and $L(y) - L(u_0)$ and $A(y) - f(r)$ are called homotopic. If, the embedding parameter p, $(0 \leq p \leq 1)$ is considered as a small parameter, applying the classical perturbation technique, we can naturally assume that the solution of Equations (27) and (28) can be given as a power series in p, i.e.,

$$y = y_0 + py_1 + p^2 y_2 + \cdots . \tag{31}$$

According to HPM, the approximation solution of Equation (24) can be expressed as a series of the power of p, i.e.

$$u = \lim_{p \to 1} y = y_0 + y_1 + y_2 + \cdots . \tag{32}$$

Now, consider a general systems of conformable fractional ordinary differential equations

$$\begin{aligned}
{}_tT_{\alpha_1}u_1 + g_1(t,u_1,u_2,\ldots,u_m) &= f_1(t),\\
{}_tT_{\alpha_2}u_2 + g_2(t,u_1,u_2,\ldots,u_m) &= f_2(t),\\
&\vdots\\
{}_tT_{\alpha_m}u_m + g_m(t,u_1,u_2,\ldots,u_m) &= f_m(t),
\end{aligned} \tag{33}$$

subject to the initial conditions

$$u_1(t_0) = c_1, \quad u_2(t_0) = c_2, \; \ldots \; u_m(t_0) = c_m. \tag{34}$$

First, we write the system (33) in the operator form

$$\begin{aligned}
&L(u_1) + N_1(t,u_1,u_2,\ldots,u_m) - f_1(t) = 0,\\
&L(u_2) + N_2(t,u_1,u_2,\ldots,u_m) - f_2(t) = 0,\\
&\quad\vdots\\
&L(u_m) + N_m(t,u_1,u_2,\ldots,u_m) - f_m(t) = 0,
\end{aligned} \tag{35}$$

subject to the initial conditions (34), where $L\left(u_l\right) ={}_t T_\alpha \left(u_l\right)$ is a linear operator and $N_1, N_2, \ldots, N_m$ are the nonlinear operators. To apply the MHPM, we first construct a homotopy for system (35) as follows

$$\begin{aligned}
&L(u_1) - L(v_1) + pL(v_1) + p[N_1(u_1,u_2,\ldots,u_m) - f_1(t)] = 0,\\
&L(u_2) - L(v_2) + pL(v_2) + p[N_2(u_1,u_2,\ldots,u_m) - f_2(t)] = 0,\\
&\quad\vdots\\
&L(u_m) - L(v_m) + pL(v_m) + p[N_m(u_1,u_2,\ldots,u_m) - f_m(t)] = 0,
\end{aligned} \tag{36}$$

where $v_1, v_2, \ldots, v_m$ are initial approximations which satisfying the given conditions. Let us take the initial approximations as follows

$$\begin{aligned}
&u_{1,0}(t) = v_1(t) = u_1(t_0) = c_1,\\
&u_{2,0}(t) = v_2(t) = u_2(t_0) = c_2,\\
&\quad\vdots\\
&u_{m,0}(t) = v_m(t) = u_m(t_0) = c_m,
\end{aligned} \tag{37}$$

and

$$\begin{aligned} u_1(t) &= u_{1,0}(t) + pu_{1,1}(t) + p^2u_{1,2}(t) + p^3u_{1,3}(t)\ldots, \\ u_1(t) &= u_{2,0}(t) + pu_{2,1}(t) + p^2u_{2,2}(t) + p^3u_{2,3}(t)\ldots, \\ &\vdots \\ u_m(t) &= u_{m,0}(t) + pu_{m,1}(t) + p^2u_{m,2}(t) + p^3u_{m,3}(t)\ldots, \end{aligned} \tag{38}$$

where $u_{i,j}$, $(i = 1, 2, \ldots, m; j = 1, 2, \ldots)$ are functions yet to be determined. Substituting (38) into (36) and arranging the coefficients of the same powers of p, we get

$$\begin{aligned} &L(u_{1,1}) + L(v_1) + N_1(u_{1,0}, u_{2,0}, \ldots, u_{m,0}) - f_1 = 0, \quad u_{1,1}(t_0) = 0, \\ &L(u_{2,1}) + L(v_2) + N_2(u_{1,0}, u_{2,0}, \ldots, u_{m,0}) - f_2 = 0, \quad u_{2,1}(t_0) = 0, \\ &\quad\vdots \\ &L(u_{m,1}) + L(v_m) + N_m(u_{1,0}, u_{2,0}, \ldots, u_{m,0}) - f_m = 0, \quad u_{m,1}(t_0) = 0, \\ &L(u_{1,2}) + N_1(u_{1,1}, u_{2,1}, \ldots, u_{m,1}) = 0, \quad u_{1,2}(t_0) = 0, \\ &L(u_{2,2}) + N_1(u_{1,1}, u_{2,1}, \ldots, u_{m,1}) = 0, \quad u_{2,2}(t_0) = 0, \\ &\quad\vdots \\ &L(u_{m,2}) + N_m(u_{1,1}, u_{2,1}, \ldots, u_{m,1}) = 0, \quad u_{m,2}(t_0) = 0, \end{aligned} \tag{39}$$

etc. We solve the above systems of equations for the unknowns $u_{i,j}$, $(i = 1, 2, \ldots, m; j = 1, 2, \ldots)$ by applying the inverse operator

$$L^{-1}(.) = \int_0^t (.)\ d\alpha(t). \tag{40}$$

Therefore, according to HPM the n-term approximations for the solutions of (33) can be expressed as

$$\begin{aligned} \phi_{1,n}(t) &= u_1(t) = \lim_{p\to 1} u_1(t) = \textstyle\sum_{k=0}^{n-1} u_{1,k}(t), \\ \phi_{2,n}(t) &= u_2(t) = \lim_{p\to 1} u_2(t) = \textstyle\sum_{k=0}^{n-1} u_{2,k}(t), \\ &\vdots \\ \phi_{m,n}(t) &= u_m(t) = \lim_{p\to 1} u_m(t) = \textstyle\sum_{k=0}^{n-1} u_{m,k}(t), \end{aligned} \tag{41}$$

The solution obtained by HPM is not valid for large t. A simple way of ensuring validity of the approximations for large t is to treat the algorithm of HPM in a sequence of intervals choosing the initial approximations as

$$\begin{aligned}
u_{1,0}(t) &= v_1(t) = u_1(t^*) = c_1^*, \\
u_{2,0}(t) &= v_2(t) = u_2(t^*) = c_2^*, \\
&\vdots \\
u_{m,0}(t) &= v_m(t) = u_m(t^*) = c_m^*,
\end{aligned} \tag{42}$$

where t^* is the left-end point of each subinterval. Now we solve (42) for the unknowns $u_{i,j}$, $(i = 1, 2, \dots, m; j = 1, 2, \dots)$ by applying the inverse linear operator

$$L^{-1}(.) = \int_{t^*}^{t} (.)\, d\alpha(t). \tag{43}$$

In order to carry out the iterations in every subinterval of equal length $\triangle t$, $[0, t_1)$, $[t_1, t_2)$,
$[t_2, t_3), \dots,$ $[t_{j-1}, t)$, we would need to know the values of the following

$$u_{1,0}^*(t) = u_1(t^*), \quad u_{2,0}^*(t) = u_2(t^*), \dots, u_{m,0}^*(t) = u_m(t^*). \tag{44}$$

But, in general, we do not have these information at our clearance except at the initial point $t^* = t_0$. A simple way for obtaining the necessary values could be by means of the previous n-term approximations $\phi_{1,n}$, $\phi_{2,n}, \dots, \phi_{m,n}$ of the preceding subinterval given by (41), i.e.

$$u_{1,0}^* \simeq \phi_{1,n}(t^*), \quad u_{2,0}^* \simeq \phi_{2,n}(t^*), \dots, u_{m,0}^* \simeq \phi_{m,n}(t^*). \tag{45}$$

6 An illustrative example

The following example is presented to illustrate the effectiveness and applicability of the proposed stability criteria.

The Lotka-Volterra equations, also known as the predator-prey (or parasite-host) equations, are a pair of first order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact on each other, one is a predator and the other is its prey. They were proposed independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926 [9, 16]. The conformable fractional Lotka-Volterra system is described by the following nonlinear fractional conformable differential equations [13]

$$\begin{aligned}
{}_tT_{\alpha_1} x_1(t) &= x_1(t) \left(r - a\, x_1(t) - b\, x_2(t)\right), \\
{}_tT_{\alpha_2} x_2(t) &= x_2(t) \left(-d + c\, x_1(t)\right),
\end{aligned} \qquad 0 < \alpha_1, \alpha_2 \le 1, \tag{46}$$

with the initial values $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, where $x_1, x_2 \ge 0$ are prey and predator densities, respectively, and all constants r, a, b, c and d are positive. This system has three equilibrium $E_1 = (0, 0)$, $E_2 = (\frac{r}{a}, 0)$ and

$E_3 = (\frac{d}{c}, \frac{cr-ad}{cb})$. The Jacobian matrix for equilibria $E^* = (x^*, y^*)$ is defined as

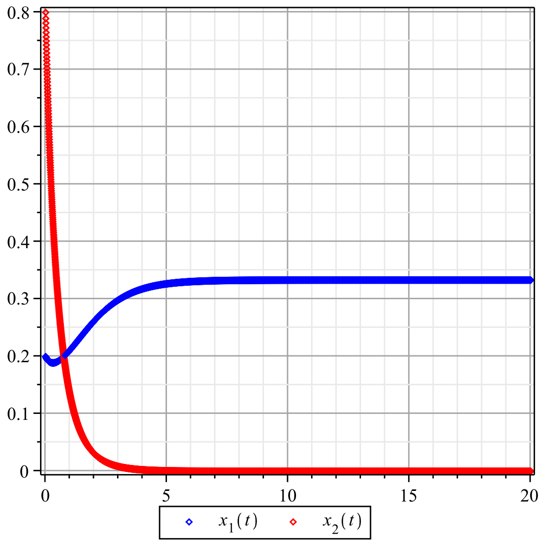

Figure 1: The equilibrium point E_2 of the system (46) with $(\alpha_1, \alpha_2) = (0.95, 0.9)$ and $(r, a, b, c, d) = (1, 3, 1, 2, 2)$ is asymptotically stable

$$J = \begin{bmatrix} r - 2ax^* - by^* & -bx^* \\ cy^* & -d + cx^* \end{bmatrix}. \tag{47}$$

When $(r, a, b, c, d) = (1, 3, 1, 2, 2)$ the system equilibrium points are $E_1 = (0, 0)$, $E_2 = (\frac{1}{3}, 0)$ and $E_3 = (1, -2)$. Corresponding eigenvalues for equilibrium point E_1 are $\lambda_1 = 1$, $\lambda_2 = -2$, since $I_{n(\alpha)}(J) = (1, 1, 0)$, hence the equilibrium point E_1 is unstable. For equilibrium point E_2 the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \frac{-4}{3}$, because $I_{n(\alpha)}(J) = (0, 2, 0)$, thus the equilibrium point is asymptotically stable and the equilibrium point E_3 is unstable, because $I_{n(\alpha)}(J) = (1, 1, 0)$. Let us consider the following parameters of the system (46), $(r, a, b, c, d) = (3, 0.5, 2, 2, 2.5)$, for these parameters the system (46) has three equilibrium points $E_1 = (0, 0)$, $E_2 = (6, 0)$ and $E_3 = (1.25, 1.1875)$ and their corresponding inertias are $I_{n(\alpha)}(J) = (1, 1, 0)$ for E_1, E_2 and $I_{n(\alpha)}(J) = (0, 2, 0)$ for E_3, thus the equilibrium point E_3 is asymptotically stable.

Figure 1 shows that system (46) with parameters $(r, a, b, c, d) = (1, 3, 1, 2, 2)$ and $(\alpha_1, \alpha_2) = (0.95, 0.9)$ is asymptotically stable in the equilibrium point E_2. From Figure 2 we can see that the system (46) is asymptotically stable in the equilibrium point E_2, with parameters $(r, a, b, c, d) = (3, 0.5, 2, 2, 2.5)$ and $(\alpha_1, \alpha_2) = (0.95, 0.85)$.

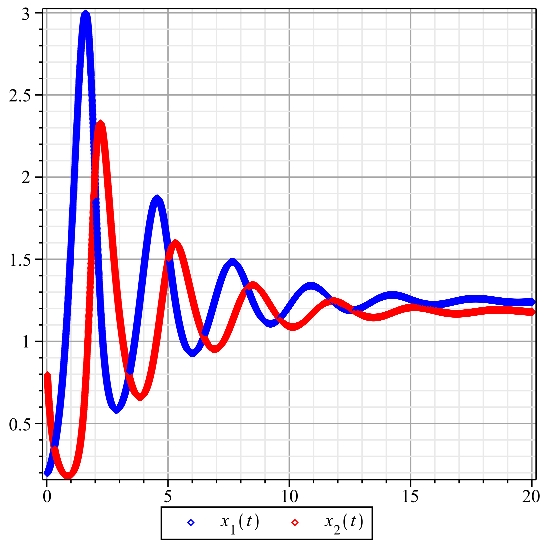

Figure 2: The equilibrium point E_3 of the system (46) with $(\alpha_1, \alpha_2) = (0.95, 0.85)$ and $(r, a, b, c, d) = (3, 0.5, 2, 2, 2.5)$ is asymptotically stable

In system (46) when we take $a = 0$, we obtain modified the fractional conformable Lotka-Volterra system

$$\begin{aligned} {}_tT_{\alpha_1} x_1(t) &= r x_1(t) - b\, x_1(t) x_2(t), \\ {}_tT_{\alpha_2} x_2(t) &= x_2(t)\left(-d + c\, x_1(t)\right). \end{aligned} \tag{48}$$

System (48) has two equilibrium points: $E_1 = (0, 0)$, $E_2 = (\frac{d}{c}, \frac{r}{b})$.

The Jacobian matrix of the system (48), evaluated at the equilibrium $E^* = (x^*, y^*)$, is given by

$$J = \begin{bmatrix} r - by^* & -bx^* \\ cy^* & -d + cx^* \end{bmatrix}. \tag{49}$$

The eigenvalues of the Jacobian matrix (48) evaluated at all equilibrium points show that all equilibria are unstable. Since, for the equilibrium point E_1 we obtain $\lambda_1 = r$ and $\lambda_2 = -d$, for the equilibrium point E_2 we get $\lambda_{1,2} = \pm\, i\sqrt{rd}$. All these eigenvalues satisfy the condition for the system to be unstable $\big(I_{n(\alpha)}(J) = (1, 1, 0)\big)$. Figure 3 shows that the system (48) with parameters $(r, b, c, d) = (1, 1, 4, 2)$ and $(\alpha_1, \alpha_2) = (0.97, 0.97)$ is unstable.

All the results are calculated by using the computer algebra package Maple. The term-number of MHPM series solutions is fixed $N = 4$ and the time step size $h = 0.1$, with the initial conditions $(x_1(0), x_2(0)) = (0.2, 0.8)$.

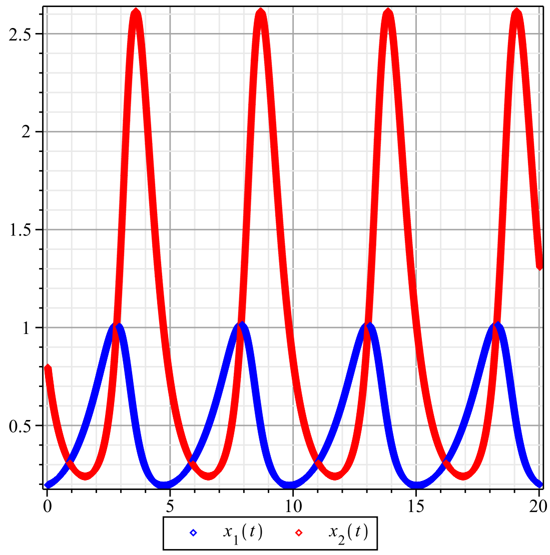

Figure 3: System (48) with $(\alpha_1, \alpha_2) = (0.97, 0.97)$ and $(r, b, c, d) = (1, 1, 4, 2)$ is unstable

7 Conclusion

In this paper, we have studied the stability analysis of fractional differential systems. Fractional derivatives are described by conformable fractional derivatives. At first, the stability conditions established by fractional exponential function for commensurate linear fractional differential systems. Then, we proved a fractional version of final value theorem and by using this theorem we proposed sufficient conditions on the asymptotical stability for in-commensurate linear fractional differential system. The numerical simulations of conformable fractional Lotka-Volterra system are used to illustrate our main result. Although this paper just focuses on the linear systems and non-linear systems but, the problems remain open for the multi-order fractional differential systems, the time-delayed fractional differential systems and distributed order fractional systems. This will be the investigation goal of future works.

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تحليل پايداري سيستم هاي كسري منطبق شدني

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چكيده : در اين مقاله، به بررسي تجزيه و تحليل پايداري سيستم هاي ديفرانسيل كسري شامل مشتق كسري منطبق شدني مي پردازيم. برخي از شرايط پايداري سيستم هاي ديفرانسيل كسري با استفاده از تابع نمايي كسري و تبديل لاپلاس كسري ارائه شده است. به علاوه براي نشان دادن اثربخشي و كارايي فرآيند ذكرشده، پايداري دستگاه كسري منطبق شدني لوتكا-ولترا را با روش چندگامي آشفتگي هوموتوپي بررسي مي كنيم.

كلمات كليدی : تحليل پايداری؛ پايداری مجانبی؛ مشتق كسری منطبق شدنی؛ دستگاه لوتكا-ولترا .