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Research Article

Modified hat functions: Application in space-time-fractional differential equations with Caputo derivative

E. Sedighi, O. Baghani*, and H. Azin

Abstract

The present article introduces an operational approach based on modified hat functions to solve the space-time-fractional differential equations in the Caputo sense. In this method, the derivative of the unknown function is considered as a linear combination of modified hat functions. We use the operational matrix of the Riemann–Liouville fractional integral of modified

*Corresponding author

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Elahe Sedighi
Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Iran. e-mail: sedigi913@gmail.com

Omid Baghani
Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Iran. e-mail: omid.baghani@gmail.com, o.baghani@hsu.ac.ir

Hadis Azin
Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Iran. e-mail: h.azin1370@gmail.com

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hat functions to approximate the Caputo fractional derivative in order to reduce the problem to a system of Sylvester equations. The error of the mentioned method is of the order $\mathcal{O}(h^3)$. In addition, we examine several numerical examples to confirm the ability of the proposed approach.

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Keywords: Modified hat functions; Space-time-fractional differential equations; Operational matrix; Caputo fractional derivative.

1 Introduction

Fractional calculus including integral and derivative of arbitrary orders has effectively played an important role in engineering, physics, and various problems modeling during the past decades [26, 6]. The non-locality feature can be the main reason for using fractional calculus in applied problem modeling [21]. This characteristic refers to the property of a dynamical system where the future state of the system is influenced by all of its previous states. The growing usage of fractional differential equations in various applications has sparked significant interest in the development of numerical methods for their solutions. Several approaches have been devised to tackle these problems, as follows:

- Adomian decomposition method: This method involves decomposing fractional differential equations into a series of simpler equations and solving them iteratively. It has been applied in various fields such as physics, engineering, and biology [7, 11].

- Generalized differential transform method: This technique applies the differential transform method to fractional differential equations, enabling the solution through a series expansion. It has been used in solving a wide range of fractional problems [16, 20, 9].

- Variational iteration method: The variational iteration method constructs an iterative sequence of corrections to approximate the solution of a fractional differential equation. It has been successfully employed in various scientific and engineering problems [19].

- Finite difference method: The finite difference method, which is widely used in numerical analysis, discretizes the fractional differential equation on a grid and approximates the derivatives using finite difference formulas. It is a common and versatile approach for solving fractional differential equations [25, 15].

- Homotopy analysis method: The homotopy analysis method constructs a series solution by introducing an auxiliary parameter. By controlling this parameter, the solution can be refined iteratively, leading to accurate approximations for fractional differential equations [10].

- Wavelet method: The wavelet method applies wavelet theory to approximate the solution of fractional differential equations. It provides a robust framework for analyzing and solving fractional problems, particularly those with irregular or discontinuous solutions [4, 13, 5].

These methods offer various approaches to tackle the challenges posed by fractional differential equations, and their suitability depends on the specific characteristics of the problem. The cited references provide more detailed information on each method's theoretical foundations and practical applications. In recent decades, a limited number of algorithms have been proposed to solve numerical space-time-fractional differential problems [24, 12, 27].

In this work, we focus on a type of linear space-time-fractional model as follows:

$$D_x^{\alpha}\mathscr{U}(x,z) + D_z^{\beta}\mathscr{U}(x,z) = F(x,z), \qquad (x,z) \in [0,1]\times[0,1], \tag{1}$$

with the following initial and boundary conditions

$$\begin{aligned} \mathscr{U}(0,z) &= \theta_1(z), \qquad \mathscr{U}(x,0) = \theta_2(x), \\ D_z^1\mathscr{U}(0,z) &= \varphi_1(z), \qquad D_x^1\mathscr{U}(x,0) = \varphi_2(x), \end{aligned} \tag{2}$$

where $D_x^{\alpha}\mathscr{U}(x,z)$ and $D_z^{\beta}\mathscr{U}(x,z)$ are the Caputo fractional derivative with respect to x and z of order α and β with $0 < \alpha, \beta \leq 1$, respectively. Also,

$\mathscr{U}(x,z)$ is the unknown function and $F(x,z), \varphi_1(z), \varphi_2(x), \theta_1(z)$ and $\theta_2(x)$ are known continuous functions.

In the suggested approach of this investigation, the function $\mathscr{U}(x,z)$, which is not known, is expanded using the introduced modified hat functions. By employing integration and the fractional integral operational matrix, the approximations for $\mathscr{U}(x,z), D_x^\alpha \mathscr{U}(x,z)$ and $D_z^\beta \mathscr{U}(x,z)$ will be derived. This described method transforms the primary problem into an algebraic equation, enabling the computation of the unknown coefficients in the expansion.

It is important to note that the strengths and weaknesses of the operational matrix method depend on the specific problem and the implementation details. Consideration should be given to the specific requirements and characteristics of the problem at hand when choosing an appropriate numerical method. The operational matrix method provides efficient solutions for systems of ordinary and partial differential equations. It allows for the conversion of differential equations into algebraic equations, which can be solved using standard matrix techniques. By using higher-order approximation functions or increasing the number of terms in the expansion, the accuracy can be improved. The operational matrix methods are usually straightforward and well-suited for numerical computation. This makes the method relatively easy to implement with computationally efficient. Also, the convergence of the operational matrix method can be affected by the choice of approximation functions or the number of terms used in the expansion. In some cases, convergence may be slow, requiring additional computational resources or modifications to improve accuracy. On the other hand, incorporating boundary conditions into the operational matrix method can be challenging, especially for problems with complex or non-standard boundary conditions. Special techniques or modifications may be required to handle such cases effectively. This methodology has been widely utilized in numerous scholarly papers, showcasing its remarkable effectiveness and accuracy [3, 14, 1, 2, 22].

The structure of this paper is summarized as follows:

- Section 2 begins by presenting essential definitions and mathematical foundations for fractional calculus.

- In section 3, we conduct a thorough examination of modified hat functions.
- Section 4 is dedicated to the development of a numerical technique for addressing space-time-fractional partial differential equations.
- In section 5, an error estimate for the proposed method is derived.
- Moving on to section 6, we apply the devised method to solve a variety of fractional partial differential equations with numerical examples.
- Ultimately, the article concludes in section 7 with a summary of our findings and insights.

2 Fractional calculus

Herein, we express the information required for research work.

Definition 1 (see [23]). Let $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ be real values and let $u : [0,1]^2 \to \mathbb{R}$ be a continuous function. The Caputo fractional derivatives of u concerning x and z are defined as

$$D_x^{\alpha} u(x,z) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-r)^{-\alpha} \frac{\partial u(r,z)}{\partial r} dr, & \alpha \in (0,1), \\ \dfrac{\partial u(x,z)}{\partial x}, & \alpha = 1, \end{cases} \tag{3}$$

and

$$D_z^{\beta} u(x,z) = \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^z (z-s)^{-\beta} \frac{\partial u(x,s)}{\partial s} ds, & \beta \in (0,1), \\ \dfrac{\partial u(x,z)}{\partial z}, & \beta = 1. \end{cases} \tag{4}$$

Definition 2 (see [21]). For $0 < \alpha \leq 1$ and given function h, the Riemann–Liouville fractional integral with order of α is defined by

$$I_z^{\alpha}\ h(z) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^z (z-s)^{\alpha-1} h(s) ds, & \alpha \in (0,1), \\ h(z), & \alpha = 0. \end{cases} \tag{5}$$

Based on the next lemma, Caputo fractional derivatives and Riemann–Liouville fractional integrals can be considered as reciprocal operations.

Lemma 1 (see [18]). If $0 < z$ and $0 < \beta < \alpha \leq m \in \mathbb{N}$, then

$$I_z^{\alpha-\beta}\ D_z^{\alpha} f(z) = D_z^{\beta}\ f(z) - \sum_{l=\lceil\beta\rceil}^{m-1} f^{(l)}(0) \frac{z^{l-\beta}}{\Gamma(l-\beta+1)}, \tag{6}$$

where $\lceil\cdot\rceil$ is the ceiling function.

3 Modified hat functions

We now focus on modified hat functions and some of their properties. In this research study, the modified hat functions are used for approximating space and time variables.

Definition 3 (see [17, 18]). Let the interval $[0, t_f]$ be divided into n subintervals $[jh, (j+1)h]$ for $j = 0, 1, 2, \ldots, n-1$, where $2 \leq n$ and n is an even integer with $h = \frac{t_f}{n}$. The modified hat functions $\{\mathcal{H}_k(z)\}_{k=0}^{n}$ on $[0, t_f]$ are defined as follows:

$$\mathcal{H}_0(z) = \begin{cases} \dfrac{(z-h)(z-2h)}{2h^2}, & 0 \leq z \leq 2h, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

for odd index k with $1 \leq k \leq n-1$,

$$\mathcal{H}_k(t) = \begin{cases} \dfrac{-(z-(k-1)h)(z-(k+1)h)}{h^2}, & (k-1)h \leq z \leq (k+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

for even index k with $2 \leq k \leq n-2$,

$$\mathcal{H}_k(z) = \begin{cases} \dfrac{(z-(k-1)h)(z-(k-2)h)}{2h^2}, & (k-2)h \leq z \leq kh, \\ \dfrac{(z-(k+1)h)(z-(k+2)h)}{2h^2}, & kh \leq z \leq (k+2)h, \\ 0, & \text{otherwise}\ , \end{cases} \tag{9}$$

and

$$\mathcal{H}_n(z) = \begin{cases} \dfrac{(z-(t_f-h))(z-(t_f-2h))}{2h^2}, & t_f - 2h \leq z \leq t_f, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

The modified hat functions exhibit noteworthy characteristics that render them highly advantageous for various applications. Several of these notable properties are as follows:

- These functions satisfy in the Kronecker delta condition as
$$\mathcal{H}_k(jh) = \delta_{kj}.$$
- The sum of the modified hat functions $\{\mathcal{H}_k(z)\}_{k=0}^n$ is one, that is,
$$\sum_{k=0}^{n} \mathcal{H}_k(z) = 1.$$
- The explicit formula for integral of these functions is obtained as follows:

$$\int_0^{t_f} \mathcal{H}_k(z)dz = \begin{cases} \frac{h}{3}, & k = 0, n, \\ \frac{4h}{3}, & k \text{ is odd and } 1 \leq k \leq n-1, \\ \frac{2h}{3}, & k \text{ is even and } \ 2 \leq k \leq n-2. \end{cases} \tag{11}$$

Any function $f(z) \in L^2[0, t_f]$ can be represented via the modified hat functions as

$$f(z) \simeq f_n(z) = \sum_{k=0}^{n} f_k \mathcal{H}_k(z) = F^T \mathscr{H}(z), \tag{12}$$

in which $f_i = f(ih)$ and

$$\mathscr{H}(z) = [\mathcal{H}_0(z),\ \mathcal{H}_1(z),\ \ldots,\ \mathcal{H}_n(z)]^T, \tag{13}$$

$$F = [f_0,\ f_1,\ \ldots,\ f_n]^T. \tag{14}$$

Moreover, every function $y(x, z)$ in $L^2([0, 1]^2)$ can be extended by the modified hat function as

$$y(x, z) \simeq y_n(x, z) = \sum_{k=0}^{n} \sum_{j=0}^{n} y_{kj} \mathcal{H}_i(x) \mathcal{H}_j(z) = \mathscr{H}^T(x) Y \mathscr{H}(z), \tag{15}$$

with $y_{kj} = y(x_k, z_j) = y(kh, jh)$.

Theorem 1 (see [17, 18])**.** For $\gamma > 0$, assume that $\mathscr{H}(z)$ is the vector introduced in the relation (13). Then, we get

$$I_z^{\gamma}\mathscr{H}(z) \simeq \mathcal{R}^{(\gamma)}\mathscr{H}(z), \tag{16}$$

in which $\mathcal{R}^{(\gamma)}$ is an $(n+1)\times(n+1)$ square matrix as

$$\mathcal{R}^{(\gamma)} = \frac{h^{\gamma}}{2\Gamma(\gamma+3)}\begin{pmatrix} 0 & \mathbf{p}_1^{\gamma} & \mathbf{p}_2^{\gamma} & \mathbf{p}_3^{\gamma} & \mathbf{p}_4^{\gamma} & \dots & \mathbf{p}_{n-1}^{\gamma} & \mathbf{p}_n^{\gamma} \\ 0 & \mathbf{q}_0^{\gamma} & \mathbf{q}_1^{\gamma} & \mathbf{q}_2^{\gamma} & \mathbf{q}_3^{\gamma} & \dots & \mathbf{q}_{n-2}^{\gamma} & \mathbf{q}_{n-1}^{\gamma} \\ 0 & \mathbf{w}_{-1}^{\gamma} & \mathbf{w}_0^{\gamma} & \mathbf{w}_1 & \mathbf{w}_2^{\gamma} & \dots & \mathbf{w}_{n-3}^{\gamma} & \mathbf{w}_{n-2}^{\gamma} \\ 0 & 0 & 0 & \mathbf{q}_0^{\gamma} & \mathbf{q}_1^{\gamma} & \dots & \mathbf{q}_{n-4}^{\gamma} & \mathbf{q}_{n-3}^{\gamma} \\ 0 & 0 & 0 & \mathbf{w}_{-1}^{\gamma} & \mathbf{w}_0^{\gamma} & \dots & \mathbf{w}_{n-5}^{\gamma} & \mathbf{w}_{n-4}^{\gamma} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathbf{q}_0^{\gamma} & \mathbf{q}_1^{\gamma} \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathbf{w}_{-1}^{\gamma} & \mathbf{w}_0^{\gamma} \end{pmatrix}, \tag{17}$$

where

$$\mathbf{p}_l^{\gamma} = \begin{cases} \gamma(3+2\gamma), & l=1, \\ l^{\gamma+1}(2l-6-3\gamma) + 2l^{\gamma}(1+\gamma)(2+\gamma) & \\ -(l-2)^{\gamma+1}(2l-2+\gamma), & 2 \le l \le n, \end{cases} \tag{18}$$

$$\mathbf{q}_l^{\gamma} = \begin{cases} 4(1+\gamma), & l=0, \\ 4\big[(l-1)^{\gamma+1}(l+1+\gamma) & \\ -(l+1)^{\gamma+1}(l-1-\gamma)\big], & 1 \le l \le n-1, \end{cases} \tag{19}$$

and

$$\mathbf{w}_l^{\gamma} = \begin{cases} -\gamma, & l=-1, \\ 2^{\gamma+1}(2-\gamma), & l=0, \\ 3^{\gamma+1}(4-\gamma) - 6(2+\gamma), & l=1, \\ (l+2)^{\gamma+1}(2l+2-\gamma) - 6l^{\gamma+1}(2+\gamma) & \\ -(l-2)^{\gamma+1}(2l-2+\gamma), & 2 \le l \le n-2. \end{cases} \tag{20}$$

4 Numerical method

This section is devoted to the implementation of a numerical approach for the problem (1)–(2) by the modified hat functions. For this purpose, we have

$$D_x^1(D_z^1\ \mathscr{U}(x,z)) \simeq \mathscr{H}^T(x)A\mathscr{H}(z), \tag{21}$$

where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}. \tag{22}$$

From the relations (2), (16), and (21), we obtain

$$\begin{aligned} D_x^1\mathscr{U}(x,z) &= \int_0^z \Big(D_x^1(D_z^1\ \mathscr{U}(x,z)) \Big) dz + D_x^1 u(x,0) \\ &\simeq \int_0^z \mathscr{H}^T(x)A\mathscr{H}(z)dz + \varphi_2(x) \\ &= \mathscr{H}^T(x)A\mathcal{R}^{(1)}\mathscr{H}(z) + \varphi_2(x). \end{aligned} \tag{23}$$

Similarly, it can be concluded

$$\begin{aligned} D_z^1\mathscr{U}(x,z) &= \int_0^x \Big(D_x^1(D_z^1\ \mathscr{U}(x,z)) \Big) dx + D_z^1 \mathscr{U}(0,z) \\ &\simeq \int_0^z \mathscr{H}^T(x)A\mathscr{H}(z)dx + \varphi_1(z) \\ &= \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathscr{H}(z) + \varphi_1(z). \end{aligned} \tag{24}$$

So, we take the integral from $D_x^1\mathscr{U}(x,z)$ with respect to x of order 1 to make an approximation of $\mathscr{U}(x,z)$ as follows:

$$\begin{aligned} \mathscr{U}(x,z) &\simeq \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(1)}\mathscr{H}(z) + \hat{u}(x,0) + u(0,z) \\ &= \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(1)}\mathscr{H}(z) + \hat{\theta}_2(x) + \theta_1(z), \end{aligned} \tag{25}$$

in which $\hat{\theta}_2(x)$ is $\theta_2(x)$ without its scalar. Replacing the relation (23) in (6), we have

$$\begin{aligned} D_x^\alpha\mathscr{U}(x,z) &= I_x^{1-\alpha}(D_x^1\mathscr{U}(x,z)) - \vartheta_1 \\ &\simeq I_x^{1-\alpha}\Big(\mathscr{H}^T(x)A\mathcal{R}^{(1)}\mathscr{H}(z) + \varphi_2(x) \Big) - \vartheta_1 \\ &= \mathscr{H}^T(x)\big(\mathcal{R}^{(1-\alpha)}\big)^T A\mathcal{R}^{(1)}\mathscr{H}(z) + I_x^{1-\alpha}(\varphi_2(x)) - \vartheta_1, \end{aligned} \tag{26}$$

and

$$\begin{aligned} D_z^{\beta}\mathscr{U}(x,z) &= I_z^{1-\beta}(D_z^1\mathscr{U}(x,z)) - \vartheta_2 \\ &\simeq I_z^{1-\beta}\bigg(\mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathscr{H}(z) + \varphi_1(z)\bigg) - \vartheta_2 \\ &= \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(1-\beta)}\mathscr{H}(z) + I_z^{1-\beta}(\varphi_1(z)) - \vartheta_2, \end{aligned} \tag{27}$$

where $\vartheta_1 = \mathscr{U}_x^{(1)}(0,0)$ and $\vartheta_2 = \mathscr{U}_z^{(1)}(0,0)$ are the known scalars. Therefore, by replacing the relation (26) and (27) in (1), we have

$$\begin{aligned} &(\mathscr{H}^T(x)\big(\mathcal{R}^{(1-\alpha)}\big)^T A\mathcal{R}^{(1)}\mathscr{H}(z) + I_x^{1-\alpha}(\varphi_2(x)) - \vartheta_1) \\ &+ (\mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(1-\beta)}\mathscr{H}(z) + I_z^{1-\beta}(\varphi_1(t)) - \vartheta_2) = F(x,z). \end{aligned} \tag{28}$$

Consider the function $g(x,z)$ as follows:

$$g(x,z) = F(x,z) - I_x^{1-\alpha}\big(\varphi_2(x)\big) - I_z^{1-\beta}\big(\varphi_1(z)\big) + \vartheta_1 + \vartheta_2. \tag{29}$$

Now, we approximate the $g(x,z)$ via the modified hat functions as

$$g(x,z) \simeq \mathscr{H}^T(x)G\mathscr{H}(z), \tag{30}$$

in which G is a square matrix of rank $(n+1)$ with $g_{ij} = g(x_i, z_j)$. So, the relation (28) reduces to (31) as

$$\begin{aligned} \mathscr{H}^T(x)\big(\mathcal{R}^{(1-\alpha)}\big)^T A\mathcal{R}^{(1)}\mathscr{H}(z) + \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(1-\beta)}\mathscr{H}(z) \\ = \mathscr{H}^T(x)G\mathscr{H}(z). \end{aligned} \tag{31}$$

By removing the vectors $\mathscr{H}^T(x)$ and $\mathscr{H}(z)$ from both sides of the equation (31), we will have the following Sylvester-type equation [8]:

$$\big(\mathcal{R}^{(1-\alpha)}\big)^T A\mathcal{R}^{(1)} + \big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(1-\beta)} = G. \tag{32}$$

After solving (32) and replacing the values of a_{ij} in the relation (25), the approximate solution of $\mathscr{U}(x,z)$ is obtained.

5 Error estimation

In this section, we derive an error estimate for the proposed method. For this issue, we first recall the following theorem from the reference [17].

Theorem 2. If the function $f(\cdot) \in C^3[0, t_f]$ is approximated by the set of modified hat functions as $f_n(t) = \sum_{i=0}^{n} f(ih)\mathcal{H}_i(t)$, then $|f(t) - f_n(t)| = \mathcal{O}(h^3)$, where $2 \le n$ and n is an even integer with $h = \frac{t_f}{n}$.

Let sufficiently smooth function $u(x, z)$ be the exact solution of the original problem (1)–(2). Assume that $u(x, z)$ can be separated as a product of two functions $F(x)$ and $G(z)$, that is,

$$u(x, z) = F(x)G(z).$$

Now, by approximating $F(x)$ and $G(z)$ using modified hat functions, we get

$$F(x) = F^T \mathscr{H}(x) + \mathcal{O}(h^3), \quad G(z) = G^T \mathscr{H}(z) + \mathcal{O}(h^3),$$

where $F_i := F(x_i)$ and $G_j := G(z_j)$. So $u(x, z) = \mathscr{H}^T(x) F G^T \mathscr{H}^T(z) + \mathcal{O}(h^3)$. This shows that the error of a two-dimensional function estimated by modified hat functions is of the order $\mathcal{O}(h^3)$. With this explanation and by (21), we have thus derived the following relation:

$$|D_x^1(D_z^1 \mathscr{U}(x, z)) - \mathscr{H}^T(x) A \mathscr{H}(z)| = \mathcal{O}(h^3).$$

Now from (25), since the functions $\hat{\theta}_2(x)$ and $\theta_1(z)$ are known, by ignoring the operational matrix error, we have

$$\begin{aligned} &|\mathscr{U}(x, z) - \mathscr{U}_n(x, z)| \\ &= |I_z^1(I_x^1(D_x^1(D_z^1 \mathscr{U}(x, z)))) - \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A \mathcal{R}^{(1)} \mathscr{H}(z)| \\ &= \mathcal{O}(h^3). \end{aligned} \tag{33}$$

This shows that the convergence rate of the suggested method is $\mathcal{O}(h^3)$.

6 Illustrative examples

In this section, we examine the presented numerical method to solve the problem (1)–(2) and compare the results with their exact solutions using Maple2020 software. In each example, the error E_n is defined by

$$E_n(\mathscr{U}) = \frac{1}{n^2}\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\Big(\mathscr{U}(ih, jh) - \mathscr{U}_n(ih, jh)\Big)^2\right)^{\frac{1}{2}}, \tag{34}$$

where $\mathscr{U}$ and $\mathscr{U}_n$ are the exact and numerical solutions, respectively. Also, the error function is plotted for each example by $|\mathscr{U}(x,z) - \mathscr{U}_n(x,z)|$. All examples are numerically approximated with the help of *Maple* 2020 software on a laptop with CPU 3.1 GHz and Core i5.

Example 6.1. Consider the non-homogeneous space-time fractional problem with the following form:

$$D_x^{\frac{1}{4}}\mathscr{U}(x,z) + D_z^{\frac{1}{4}}\mathscr{U}(x,z) = F(x,z), \quad x,z \in [0,1], \tag{35}$$

in which

$$F(x,z) = \frac{4(x^{\frac{3}{4}}z + xz^{\frac{3}{4}})}{3\Gamma(\frac{3}{4})}, \tag{36}$$

with

$$D_x^1\mathscr{U}(0,z) = D_z^1\mathscr{U}(x,0) = \mathscr{U}(0,z) = \mathscr{U}(x,0) = 0. \tag{37}$$

The exact solution is $\mathscr{U}(x,z) = xz$. So, it can be written that

$$I_x^{\frac{3}{4}}(\varphi_2(x)) = I_z^{\frac{1}{4}}(\varphi_1(z)) = \vartheta_1 = \vartheta_2 = 0. \tag{38}$$

According to the relations (17) and (30) with $n = 2$, we get

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{2\ 2^{\frac{1}{4}}}{3\ \Gamma(\frac{3}{4})} & \dfrac{4(\frac{1}{2} + \frac{2^{\frac{1}{4}}}{2})}{3\ \Gamma(\frac{3}{4})} \\ 0 & \dfrac{4(\frac{1}{2} + \frac{2^{\frac{1}{4}}}{2})}{3\ \Gamma(\frac{3}{4})} & \dfrac{8}{3\ \Gamma(\frac{3}{4})} \end{pmatrix},$$
$$\mathcal{R}^{(1)} = \begin{pmatrix} 0 & \frac{5}{24} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{1}{24} & \frac{1}{6} \end{pmatrix}, \quad \mathcal{R}^{(\frac{3}{4})} = \begin{pmatrix} 0 & \dfrac{27\ 2^{\frac{1}{4}}}{32\ \Gamma(\frac{15}{4})} & \dfrac{9}{16\ \Gamma(\frac{15}{4})} \\ 0 & \dfrac{7\ 2^{\frac{1}{4}}}{4\ \Gamma(\frac{15}{4})} & \dfrac{3}{\Gamma(\frac{15}{4})} \\ 0 & -\dfrac{3\ 2^{\frac{1}{4}}}{16\ \Gamma(\frac{15}{4})} & \dfrac{5}{4\ \Gamma(\frac{15}{4})} \end{pmatrix}. \tag{39}$$

Then, we can rewritten (31) as

$$\mathscr{H}^T(x)\big(\mathcal{R}^{(\frac{3}{4})}\big)^T A\mathcal{R}^{(1)}\mathscr{H}(z) + \mathscr{H}^T(x)\big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(\frac{3}{4})}\mathscr{H}(z) = \mathscr{H}^T(x)G\mathscr{H}(z)), \tag{40}$$

which reduces to

$$\big(\mathcal{R}^{(\frac{3}{4})}\big)^T A\mathcal{R}^{(1)} + \big(\mathcal{R}^{(1)}\big)^T A\mathcal{R}^{(\frac{3}{4})} = G. \tag{41}$$

We solve (41) using Maple software and obtain the approximation $\mathscr{U}(x,z)$ with the relation (25). The exact and numerical solutions with $n=2$ are shown in Figure 1. The numerical solution with $n=8$ is displayed in Figure 2. Also, error functions with $n=2$ and $n=8$ are shown in Figure 3 in which $E_2(\mathscr{U}) = 0.00086$ and $E_8(\mathscr{U}) = 0$. We list the absolute error for different values of n in Example 6.1 in Table 1.

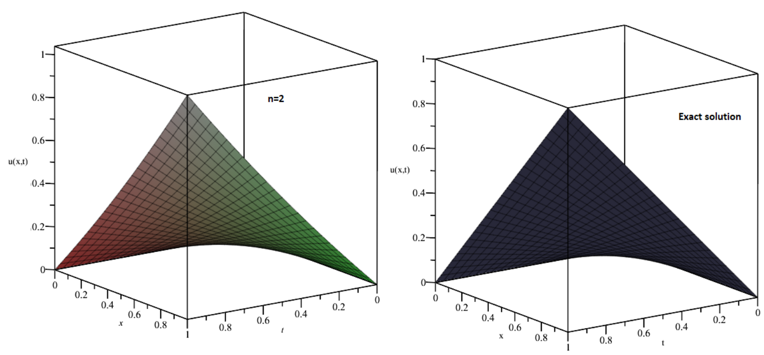

Figure 1: The exact and numerical solutions with $n=2$ for Example 6.1.

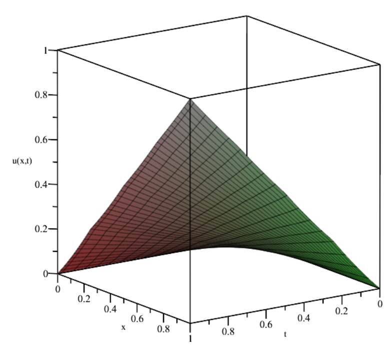

Figure 2: Numerical solution with $n=8$ for Example 6.1.

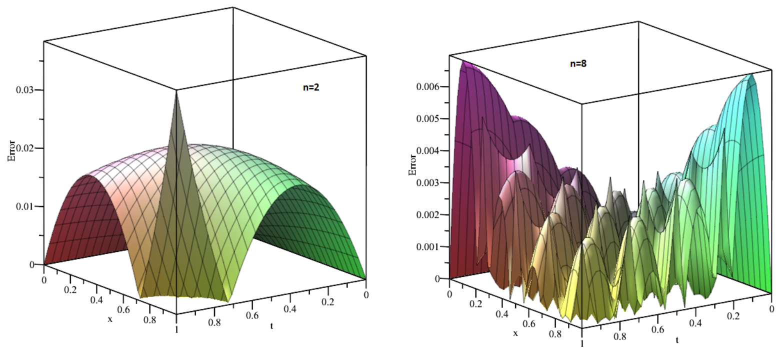

Figure 3: Error function $|\mathscr{U}(x,z) - \mathscr{U}_n(x,z)|$ with $n = 2$ and $n = 8$ for Example 6.1.

Table 1: The absolute error for different values of n in Example 6.1

(x,t)	$n=2$	$n=4$	$n=6$	$n=8$
$(0,0)$	0	0	0	0
$(0.2,0.2)$	0.0058	0.0042	0.0025	0.00060
$(0.4,0.4)$	0.017	0.0024	0.00035	0.0026
$(0.6,0.6)$	0.022	0.00079	0.0019	0.0023
$(0.8,0.8)$	0.0096	0.010	0.0041	0.000048
$(1,1)$	0.038	0.013	0.0059	0.0027

Example 6.2. Consider the following fractional space-time fractional model:

$$D_x^{\frac{1}{3}}\mathscr{U}(x,z) + D_z^{\frac{1}{2}}\mathscr{U}(x,z) = F(x,z), \quad 0 \le x, z \le 1, \tag{42}$$

where

$$F(x,z) = \frac{\Gamma(3)x^{\frac{5}{3}}}{\Gamma(\frac{8}{3})} + \frac{\Gamma(3)z^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, \tag{43}$$

with

$$D_z^1\mathscr{U}(0,z) = 2z,\ D_x^1\mathscr{U}(x,0) = 2x,\ \mathscr{U}(0,z) = z^2,\ \mathscr{U}(x,0) = x^2. \tag{44}$$

The exact solution is $\mathscr{U}(x,z) = x^2 + z^2$. Now, we calculate the following values:

$$\begin{cases} I_x^{\frac{2}{3}}(\varphi_2(x)) = I_x^{\frac{2}{3}}(2x) = \dfrac{2\Gamma(2)}{\Gamma(\frac{8}{3})}x^{\frac{5}{3}}, \\ I_z^{\frac{1}{2}}(\varphi_1(z)) = I_z^{\frac{1}{2}}(2z) = \dfrac{2\Gamma(2)}{\Gamma(\frac{5}{2})}z^{\frac{3}{2}}, \\ \vartheta_1 = \vartheta_2 = 0. \end{cases} \tag{45}$$

Therefore, we get

$$g(x,z) = \left(\frac{\Gamma(3)x^{\frac{5}{3}}}{\Gamma(\dfrac{8}{3})} + \frac{\Gamma(3)z^{\frac{3}{2}}}{\Gamma(\dfrac{5}{2})}\right) - \frac{2\Gamma(2)}{\Gamma(\frac{8}{3})}x^{\frac{5}{3}} - \frac{2\Gamma(2)}{\Gamma(\frac{5}{2})}z^{\frac{3}{2}} = 0.$$

Equation (32) is rewritten as follows:

$$\left(\mathcal{R}^{(\frac{2}{3})}\right)^T A\mathcal{R}^{(1)} + \left(\mathcal{R}^{(1)}\right)^T A\mathcal{R}^{(\frac{1}{2})} = 0. \tag{46}$$

After solving Sylvester equation (46), based on (25), we obtain $A = 0$ and $\mathscr{U}(x,z) = x^2 + z^2$, which is the exact solution. The exact and numerical solutions with $n = 2$ are plotted in Figure 4. Moreover, the error function is shown in Figure 5 in which $E_2(\mathscr{U}) = 0$.

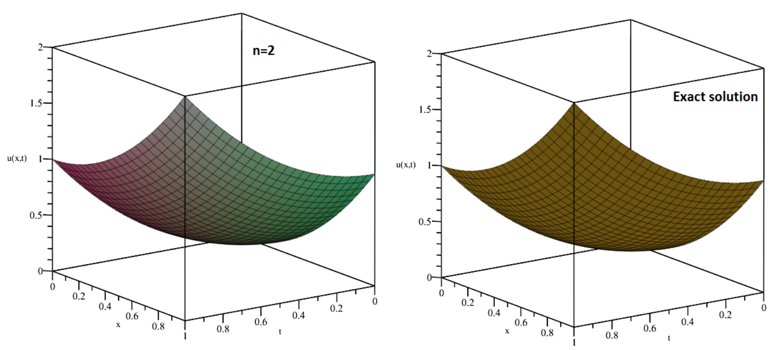

Figure 4: The exact and numerical solutions with $n = 2$ for Example 6.2.

Example 6.3. Consider the following non-homogeneous fractional space-time problem:

$$D_x^{\frac{1}{2}}\mathscr{U}(x,z) + D_z^{\frac{1}{3}}\mathscr{U}(x,z) = F(x,z), \quad (x,z) \in [0,1]^2, \tag{47}$$

with

$$F(x,z) = \frac{(3t^5\sqrt{\Pi}x)}{4} + \frac{729z^{\frac{14}{3}}}{308\Gamma(\frac{2}{3})}, \tag{48}$$

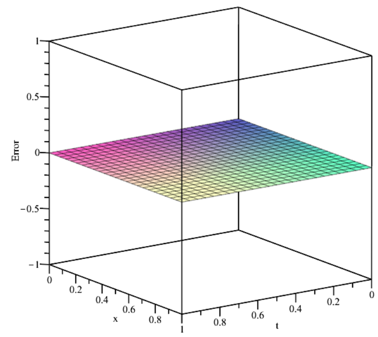

Figure 5: Error function with $n = 2$ for Example 6.2.

in which

$$D_z^1 \mathscr{U}(0,t) = D_x^1 \mathscr{U}(x,0) = \mathscr{U}(0,z) = \mathscr{U}(x,0) = 0. \tag{49}$$

The exact solution is $\mathscr{U}(x,z) = x^{\frac{3}{2}} t^5$. By calculating the operational matrices of the orders $\dfrac{1}{2}$ and $\dfrac{2}{3}$ and then substituting them into (32), it can be obtained a system of linear equations to calculate the unknown coefficients. Figure 6 shows the approximate and exact solutions of the fractional space-time equation for $n = 10$.

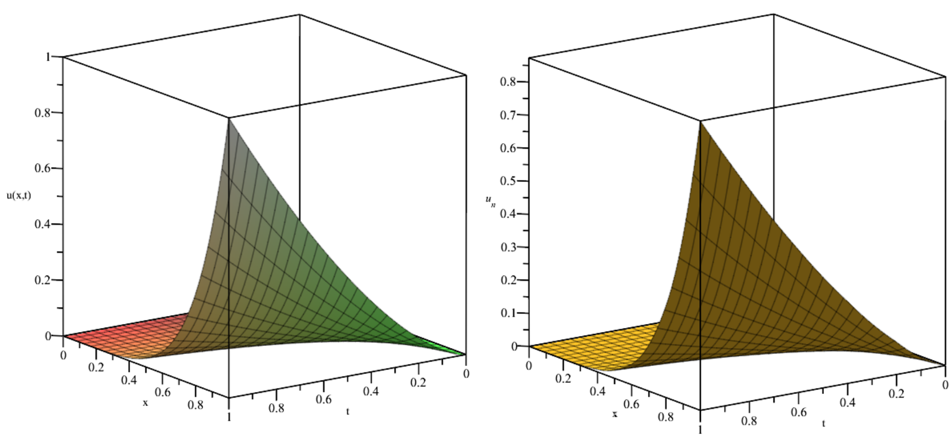

Figure 6: The exact and numerical solutions with n=10 for Example 6.3.

Example 6.4. At the end, consider the space-time-fractional differential equation as

$$D_x^{\alpha}\mathscr{U}(x,z)+D_z^{\beta}\mathscr{U}(x,z)=F(x,z),\quad (x,z)\in[0,1]^2, \tag{50}$$

with the term source

$$F(x,z)=\frac{\Gamma(3)x^{2-\alpha}(z^2+1)}{\Gamma(3-\alpha)}+\frac{\Gamma(3)z^{2-\beta}(x^2+1)}{\Gamma(3-\beta)}, \tag{51}$$

in which

$$D_z^1\mathscr{U}(0,t)=2z,\ D_x^1\mathscr{U}(x,0)=2x,\ \mathscr{U}(0,z)=z^2+1,\ \mathscr{U}(x,0)=x^2+1. \tag{52}$$

The analytical solution is $\mathscr{U}(x,z)=(x^2+1)(z^2+1)$ with $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$. So, we obtain

$$g(x,z)=\frac{\Gamma(3)x^{2-\alpha}(z^2)}{\Gamma(3-\alpha)}+\frac{\Gamma(3)z^{2-\beta}(x^2)}{\Gamma(3-\beta)}.$$

Now, for $n=2$, we get

$$G=\begin{pmatrix}0 & 0 & 0\\ 0 & 0.2648 & 0.9036\\ 0 & 0.8642 & 2.8337\end{pmatrix}. \tag{53}$$

By replacing (53) in (32), this equation can be solved. The analytical and numerical solutions with $n=2$ are shown in Figure 7 in which $E_2(\mathscr{U})=0.0159$.

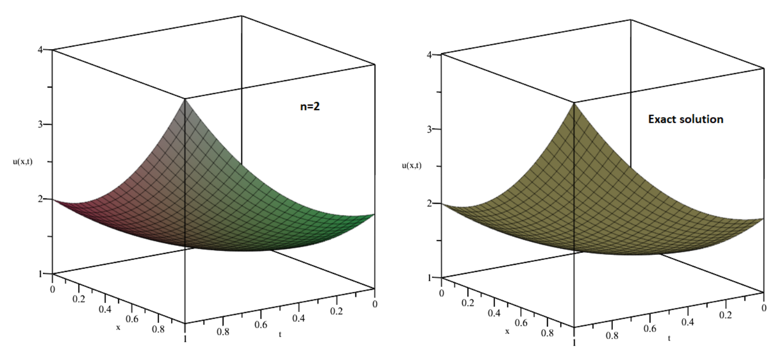

Figure 7: The exact and numerical solutions with $n=2$ for Example 6.4.

7 Conclusion

In the present article, a numerical algorithm for solving space-time-fractional problems was given. Using the derivative in the Caputo sense and the Riemann–Liouville integral, we approximated the existing functions by modified hat functions. By replacing approximation with the main problem and simplifying it, the fractional equation was reduced to a system of algebraic equations. In the presented method, the linear space-time-fractional differential equation becomes a system of Sylvester equations, which is very easy to solve. The coefficients of the basic functions can be easily calculated, but it takes more time to find solutions closer to the real value. The solutions obtained from solving the mentioned system in (25) are an approximate solution of the main problem (1)–(2). The proposed approach has been used for several examples. In each example, the approximation function $\mathscr{U}_n(x,z)$ and the error from its exact solution were calculated. The results obtained demonstrated that the proposed technique has the capability to solve numerically the space-time-fractional partial differential problem.

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