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Research Article

Stability analysis and optimal strategies for controlling a boycotting behavior of a commercial product

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Abstract

In this work, we propose a mathematical model that describes citizens' behavior toward a product, where individuals are generally divided into three main categories: potential consumers, boycotters who abstain from it for

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various reasons, and actual consumers. Therefore, our work contributes to understanding product boycott behavior and the factors influencing this phenomenon. Additionally, it proposes optimal strategies to control boycott behavior and limit its spread, thus protecting product marketing and encouraging consumer reuse.

We use mathematical theoretical analysis to study the local and global stability, as well as sensitivity analysis to identify parameters with a high impact on the reproduction number R_0. Subsequently, we formulate an optimal control problem aimed at minimizing the number of boycotters and maximizing consumer participation. Pontryagin's maximum principle is employed to characterize the optimal controls. Finally, numerical simulations conducted using MATLAB confirm our theoretical results, with a specific application to the case of the boycott of Centrale Danone by several Moroccan citizens in April 2018.

AMS subject classifications (2020): Primary 03C45; Secondary 90C31, 35F21.

Keywords: Modeling a boycott behavior; Local and global stability; Sensitivity analysis; Optimal control problem.

1 Introduction

Boycotting a product is a conscious decision to refrain from buying or using a particular product as a way to express disapproval or disagreement with the company responsible for producing or selling it. This is due to various reasons that may be related to the company's practices, policies, or ethical standards. Boycott behavior may include actively encouraging others to join in abstaining from the product.

Ireland is where the word "boycott" first appeared in the late 1800s. It comes from the name of Captain Charles Boycott, an English land agent who worked in Ireland and rose to prominence as a representative of harsh landlordism. Charles Boycott was singled out by the Irish Land League in 1880 during the Irish Land War for his inequitable treatment of tenants. Boycott was effectively isolated and found it difficult to manage his land as a result of the League's encouragement of other farmers and laborers to refuse to work for or conduct commerce with him. Over time, the act of isolating and refusing to comply has been referred to as a “boycott” and has been employed globally as a means of political protest against unfair behaviors or individuals [15].

Similar to what has been witnessed in many countries of the world, especially North African countries; Morocco has also witnessed the emergence of protest movements and boycotts of many goods and products due to high prices and the decline in purchasing power of citizens. On April 20, 2018, this boycott first appeared on social media. The campaign targeted three main suppliers to Morocco-Centrale Danone (dairy products), Sidi Ali Water brand (bottled water), and African gas stations owned by the Aqua Group (gasoline). As for other Arab countries, such as Egypt, Tunisia, and Jordan, they chose to organize general strikes to demand improved living standards, lower prices, and an end to austerity measures. Morocco has taken a different tack by using the boycott to quietly express its anger at high costs and destitution [7]. Moroccans gathered through this boycott to express their dissatisfaction with high prices and the social and economic conditions in which they live.

This and other similar topics have been the subject of many studies and research projects in the social, economic, and political sciences [7, 11, 19, 9, 4, 1, 21]. However, there are still few mathematical studies and research available on this topic [23, 24, 5, 13, 25].

In this paper, we adopt a compartment modeling approach commonly used in epidemiology to model the spread of product boycott behavior in a population. The compartment model is a widely used approach for explaining the transmission of infectious diseases. In epidemiological models, populations are divided into several categories based on their disease status (i.e., “susceptible,” “infected,” or “removed”), and the process of infection depends on interaction with infected individuals. Likewise, we consider citizens toward a product to be either potential adopters or boycotters and consumers of the product. It closely resembles the phenomenon of contagious, since boycotters have an important impact on prospective consumers not using the product. It is, therefore, reasonable to model product boycotts using the epidemiological approach. Hence, our work contributes to understanding the behavior of product boycotting and the factors influencing this phenomenon. Additionally, in this work, we propose optimal strategies for controlling boycotting behavior and limiting its spread, thereby safeguarding product marketing and encouraging consumer reuse.

We propose a mathematical model that describes citizens' behavior towards consuming a specific product, where individuals are divided into three basic categories: Potential consumers of the product, boycotters who abstain from purchasing, using, and consuming the product for various reasons, and attempt to influence other individuals to adopt boycott behavior for the product, and the class of consumers already using the product. By using Routh–Hurwitz criteria and constructing Lyapunov functions, we study the local and global stability of the equilibriums. We examine the sensitivity analysis of the model parameters in order to determine which parameters significantly affect the reproduction number R_0. Using the theoretical results of optimal control theory, we also propose optimal strategies to encourage potential customers to purchase and use a company's product and to persuade and satisfy boycotters of the product.

The structure of this article is as follows. Section 2 is split into two parts: the first part contains the proposed mathematical model, while the second part contains some of the model's fundamental characteristics. Section 3 is also divided into parts. We start by analyzing the local and global stability after some numerical simulations, and finally, we discuss the problem of the parameter's sensitivity. The optimal control problem for the suggested model is presented in Section 4, where we also provide some results regarding the existence of the optimal controls and use Pontryagin's maximum principle to characterize them. Numerical simulations are also provided in this section. In Section 5, the paper is brought to its conclusion.

2 Mathematical model and fundamental characteristics

2.1 Mathematical model

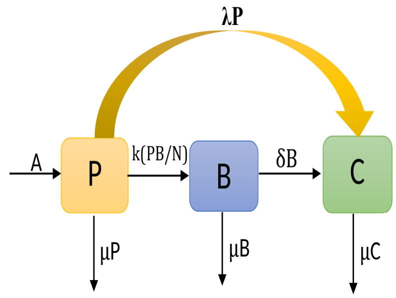


Figure 1: Description of the model.

We consider a mathematical model PBC that captures the behavior of citizens who might use a product, those that boycott the product, and those who consume it. The graphical representation of the proposed model is shown in Figure 1. The total population represented by N is split up into three compartments:

The potential consumers (P) are a category of people that could consume the product. The compartment P is increasing at the rate of A and represents the number of people who can access the product and purchase or consume it. It is decreased when potential consumers become actual consumers at rate λ. It is presumed that potential consumers can also become boycotters of the product at rate k through meaningful interactions with existing boycotters. Finally, the number of potential consumers due to natural death decreases at a rate of μ.

The boycotters (B) who abstain from purchasing, using, and consuming the product for various reasons, and attempt to influence other individuals to adopt boycott behavior for the product. This compartment is increased through effective contact with potential consumers who stop using the product as a result, at a rate k. It is lowered, either by natural death at a rate of μ when boycotters change their opinions about the product and become new actual consumers, at a rate δ.

The actual consumers (C) are those who buy and consume the product. When boycotters change their position and start using the product, the consumer's compartment is increased at a rate of δ. Likewise, it increases at a rate λ when potential consumers are convinced to use the product. Natural death reduces it at the rate μ.

The numbers of people in each of the three classes at time t are represented by the variables $P(t)$, $B(t)$, and $C(t)$, respectively. Time can be measured in years, months, days, or other intervals depending on how frequently survey studies are conducted as needed.

The equation $N(t) = P(t) + B(t) + C(t)$ represents the overall population size at time t. We suppose in this work that N is constant. This model's dynamics are controlled by the nonlinear system of differential equations below.

$$\begin{cases} \dot{P} = A - k\dfrac{PB}{N} - (\mu + \lambda)P, \\ \dot{B} = k\dfrac{PB}{N} - (\delta + \mu)B, \\ \dot{C} = \lambda P + \delta B - \mu C, \end{cases} \tag{1}$$

where $P(0) \geq 0$, $B(0) \geq 0$, and $C(0) \geq 0$ are the given initial states.

2.2 Fundamental characteristics

Since system (1) reflects the population of humans, it is necessary to demonstrate that all of the system's solutions with positive initial data are bounded and will remain positive for all times $t > 0$. The following lemma and theorem will establish this.

2.2.1 The positivity of the model's solutions

Theorem 1. If the initial conditions are positive, that is, $P(0) \geq 0$, $B(0) \geq 0$, and $C(0) \geq 0$, then the solutions $P(t)$, $B(t)$, and $C(t)$ of system (1) are positive for all $t \geq 0$.

Proof. The first equation of system (1) indicates that

$$\frac{dP(t)}{dt} + \left(k\frac{B(t)}{N} + (\lambda + \mu)\right) P(t) \geq 0. \tag{2}$$

Multiplying the inequality (2) by

$$\exp\left[\int_0^t \left(k\frac{B(v)}{N} + (\lambda + \mu)\right) dv\right],$$

we have

$$\frac{dP(t)}{dt}\exp\left[\int_0^t (kB(v)/N + (\lambda+\mu))\,dv\right]$$
$$+\,[kB(t)/N + (\lambda+\mu)].P(t)\exp\left[\int_0^t (kB(v)/N + (\lambda+\mu))\,dv\right] \geq 0.$$

Then,

$$\frac{d}{dt}\left[P(t)\exp\left[\int_0^t \left(k\frac{B(v)}{N} + (\lambda+\mu)\right)dv\right]\right] \geq 0. \tag{3}$$

Integrating (3) gives

$$P(t) \geq P(0)\exp\left[\int_0^t \left(-k\frac{B(v)}{N} - (\lambda+\mu)\right)dv\right].$$

So, the solution $P(t)$ is positive.

Likewise, utilizing system (1)'s second and third equations, we have

$$B(t) \geq B(0)\exp\left[-(\delta+\mu)t\right] \geq 0$$

and

$$C(t) \geq C(0)\exp(-\mu t) \geq 0.$$

Thus, we can observe that system (1)'s solutions $P(t)$, $B(t)$, and $C(t)$ are positive for all $t \geq 0$. □

2.2.2 Invariant region

Lemma 1. If the initial conditions are positive, that is, $P(0) \geq 0$, $B(0) \geq 0$, and $C(0) \geq 0$, then the region Ω defined by
$\Omega = \left\{(P(t), B(t), C(t)) \in \mathbb{R}^3_+, P(t) + B(t) + C(t) \leq \dfrac{A}{\mu}\right\}$
is positive invariant for system (1).

Proof. When we sum up the system equations (1), we get

$$\frac{dN}{dt} \leq A - (\lambda+\mu)N.$$

Then,

$$N(t) \leq N(0) + At + \int_0^t -\mu N(v)dv.$$

Using a Gronwall lemma, we get

$$N(t) \leq N(0)\exp(-\mu t) + \frac{A}{\mu}(1 - \exp(-\mu t)),$$

for the population's initial values as a whole. So, $\lim_{t\to\infty} \sup N(t) = \dfrac{A}{\mu}$. For system (1), it suggests that the region Ω is a positively invariant set.

Thus, the dynamics of the system must be considered in the set Ω. □

3 Analysis of stability and model parameter sensitivity

This section investigates the system (1)'s stability behavior at both a boycotted equilibrium point and a boycott-free equilibrium point. System (1) possesses the subsequent two equilibrium points:

(1) boycott-free equilibrium given by $E_0 = \left(\dfrac{A}{\lambda+\mu}, 0, \dfrac{\lambda A}{\mu(\lambda+\mu)}\right)$. The situation in which there are no boycotters in the population.

(2) boycotted equilibrium point, if $R_0 > 1$, given by $E^* = (P^*, B^*, C^*)$, where $P^* = \dfrac{A(\delta+\mu)}{\mu k}$, $B^* = \dfrac{A(\lambda+\mu)(R_0-1)}{\mu k}$, and $C^* = \dfrac{\lambda A(\delta+\mu)^2 + \delta A(\lambda+\mu)(\delta+\mu)(R_0-1)}{\mu^2(\delta+\mu)k}$. This equilibrium reflects the situation in which a product boycott becomes widespread among the populace.

Here, R_0 is the basic reproduction number given by

$$R_0 = \frac{\mu k}{(\lambda+\mu)(\delta+\mu)}.$$

In the field of epidemiology, the basic reproduction number R_0 denotes the mean quantity of secondary infections among a fully susceptible population.

This threshold, as it relates to our work, denotes the mean number of prospective consumers that a boycotter will convince not to use the product during his interaction time.

In fact, if we assume that $x = (B, C, P)$, then the system (1) may be expressed as

$$\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{W}(x),$$

where

$$\mathcal{F}(x)=\begin{pmatrix} k\dfrac{BP}{N} \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathcal{W}(x)=\begin{pmatrix} (\delta+\mu)B \\ -\lambda P-\delta B+\mu C \\ -A+k\dfrac{BP}{N}+(\lambda+\mu)P \end{pmatrix}.$$

At the free equilibrium E_0, the Jacobian matrices of $\mathcal{F}(x)$ and $\mathcal{W}(x)$ are

$$D\mathcal{F}(E_0)=\begin{pmatrix} F_{2\times 2} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$D\mathcal{W}(E_0)=\begin{pmatrix} W_{2\times 2} & & 0 \\ & & -\lambda \\ \dfrac{\mu k}{(\lambda+\mu)} & 0 & 0 \end{pmatrix},$$

where

$$F=\begin{pmatrix} \dfrac{\mu k}{(\lambda+\mu)} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$W=\begin{pmatrix} \delta+\mu & 0 \\ -\delta & \mu \end{pmatrix}.$$

At last, we have

$$R_0=\rho\left(FW^{-1}\right)=\frac{\mu k}{(\lambda+\mu)(\delta+\mu)}.$$

3.1 Analysis of local stability

This section examines the boycotted equilibrium's and the boycott-free equilibrium's local stability.

Theorem 2. If $R_0 < 1$, then the boycott-free equilibrium E_0 is locally asymptotically stable; if $R_0 > 1$, then E_0 is unstable.

Proof. At E_0, the Jacobian matrix is provided by

$$J(E_0) = \begin{pmatrix} -(\lambda+\mu) & -\dfrac{\mu k}{(\lambda+\mu)} & 0 \\ 0 & \dfrac{\mu k}{(\lambda+\mu)} - (\delta+\mu) & 0 \\ \lambda & \delta & -\mu \end{pmatrix}.$$

Consequently, eigenvalues of $J(E_0)$'s characteristic equation are

$$\begin{aligned} \zeta_1 &= -\mu, \\ \zeta_2 &= -(\lambda+\mu), \\ \zeta_3 &= (\delta+\mu)(R_0-1). \end{aligned}$$

Clearly, the first and second eigenvalues ζ_1 and ζ_2, respectively, are negative. The third eigenvalue is also negative supplied that $R_0 < 1$.

We deduce that the boycott-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$, while it is unstable if $R_0 > 1$. □

After that, we assert the following theorem to ascertain the stability of the boycotted equilibrium E^*.

Theorem 3. The boycotted equilibrium E^* is locally asymptotically stable if $R_0 \geq 1$.

Proof. At E^*, the Jacobian matrix is provided by

$$J(E^*) = \begin{pmatrix} -k\dfrac{B^*}{N^*} - (\lambda+\mu) & -k\dfrac{P^*}{N^*} & 0 \\ k\dfrac{B^*}{N^*} & k\dfrac{P^*}{N^*} - (\delta+\mu) & 0 \\ \lambda & \delta & -\mu \end{pmatrix},$$

which is provided by its characteristic equation

$$\zeta^3 + a_1\zeta^2 + a_2\zeta + a_3 = 0,$$

where

$$a_1 = \frac{k\mu}{(\mu+\lambda)} + \mu,$$

$$a_2 = \frac{k\mu^2}{(\mu+\lambda)} + (\mu+\lambda)(\mu+\delta)(R_0-1),$$

$$a_3 = \frac{k\mu^3}{(\mu+\lambda)N} + \mu(\mu+\lambda)(\mu+\delta)(R_0-1).$$

Applying the Routh–Hurwitz criterion [6], if $a_1 > 0$, $a_3 > 0$, and $a_1a_2 > a_3$, then system (1) is locally asymptotically stable.

Therefore, if $R_0 \geq 1$, then E^* is locally asymptotically stable. □

3.2 Analysis of global stability

The boycotted equilibrium E^* and boycott-free equilibrium E_0 of the model (1), respectively, constitute the global asymptotic stability that we are now concerned with.

Theorem 4. The free equilibrium E_0 of system (1) is globally asymptotically stable on Ω if $R_0 \leq 1$.

Proof. Consider the Lyapunov function $V_1 : \Omega \longrightarrow \mathbb{R}$ in the manner mentioned below:

$$V_1(P,B) = \frac{1}{2}\left[(P-P^0)+B\right]^2 + \frac{(\lambda+\delta+2\mu)N}{k}B.$$

Computing the time derivation of V_1, we get

$$\dot{V}_1(P,B) = (P-P^0+B)\left[A-(\lambda+\mu)P-(\delta+\mu)B\right] + \frac{(\lambda+\delta+2\mu)N}{k}\dot{B}. \quad (4)$$

Due to $A = P^0(\lambda+\mu)$, (4) becomes

$$\begin{aligned}\dot{V}_1(P,B) &= (P-P^0+B)[-(\lambda+\mu)(P-P^0)-(\delta+\mu)B] \\ &\quad + \frac{(\lambda+\delta+2\mu)N}{k}\dot{B} \\ &= -(\lambda+\mu)\left(P-P^0\right)^2 - (\delta+\mu)B^2 \\ &\quad -(\lambda+\delta+2\mu).\frac{(\delta+\mu)N}{k}(1-R_0)B.\end{aligned}$$

Consequently, $\dot{V}_1(P,B) \leq 0$ for $R_0 \leq 1$.

Moreover, if $R_0 \leq 1$, then $\dot{V}_1(P,B) = 0$ is equivalent to $B = 0$ and $P = P^0$.

Thus, Theorem 4 has been proved, and we can now say that by LaSalle's invariance principle [12], the boycott-free equilibrium E_0 is globally asymptotically stable on Ω. □

The global asymptotic stable theorem for the boycott-free equilibrium E^* is then presented as follows.

Theorem 5. The boycotted equilibrium E^* of system (1) is globally asymptotically stable on Ω if $R_0 > 1$.

Proof. Consider the Lyapunov function $V_2 : \Omega \longrightarrow \mathbb{R}$ in the manner mentioned below:

$$\begin{aligned} V_2(P,B) = Y_1 \left[P - P^* \left(1 + ln \left(\frac{P}{P^*} \right) \right) \right] \\ + Y_2 \left[B - B^* \left(1 + ln \left(\frac{B}{B^*} \right) \right) \right], \end{aligned}$$

where Y_1 and Y_2 are positive constants to be chosen later.

Computing the time derivation of V_2, we get

$$\begin{aligned} \dot{V}_2(P,B) = \frac{k}{N}(Y_1 - Y_2)(B - B^*)(P - P^*) \\ - AY_1 \frac{(P - P^*)^2}{PP^*}. \end{aligned}$$

For $Y_1 = Y_2 = 1$, we get

$$\dot{V}_2(P,B) = -A\frac{(P - P^*)^2}{PP^*} \leq 0,$$

and
$\dot{V}_2(P,B) = 0$ is equivalent to $P = P^*$.
Thus, Theorem 5 has been proved, and we can now say that by LaSalle's invariance principle [12], the boycotted equilibrium E^* is globally asymptotically stable on Ω. □

3.3 Sensitivity analysis of the model's parameters

Sensitivity analysis is widely used to determine which parameters significantly affect the reproduction number R_0 or to evaluate a model's resilience to parameter values. In the context of [2, 14, 18], a sensitivity analysis of the model (1) is conducted.

Definition 1. If ξ is a variable that depends differently on t, then its normalized forward sensitivity index (S.I.) is defined as follows:

$$\Upsilon_{\xi}^{t} = \frac{\partial \xi}{\partial t} . \frac{t}{\xi}.$$

Specifically, the following are the computerized S.I.s of the fundamental reproduction number R_0 concerning the model parameters:

$$\begin{cases} \Upsilon_{\mu}^{R_0} = \dfrac{\partial R_0}{\partial \mu} . \dfrac{\mu}{R_0} = \dfrac{\lambda\delta - \mu^2}{(\lambda+\mu)(\delta+\mu)}, \\ \Upsilon_{k}^{R_0} = \dfrac{\partial R_0}{\partial k} . \dfrac{k}{R_0} = 1, \\ \Upsilon_{\delta}^{R_0} = \dfrac{\partial R_0}{\partial \delta} . \dfrac{\delta}{R_0} = -\dfrac{\delta}{\delta+\mu}, \\ \Upsilon_{\lambda}^{R_0} = \dfrac{\partial R_0}{\partial \lambda} . \dfrac{\lambda}{R_0} = -\dfrac{\lambda}{\lambda+\mu}. \end{cases}$$

A positive value of the S.I., that is $\Upsilon_{k}^{R_0}$ indicates that an increase (decrease) in the value of each parameter in this instance results in a proportional increase (decrease) in the basic reproduction number of the disease. Conversely, the negative sign of S.I. suggests that an increase (decrease) in the value of each of the parameters leads to a corresponding decrease (increase) in the basic reproduction number R_0. As an illustration, $\Upsilon_{k}^{R_0} = 1$ implies that a 15% increase or decrease in the effective contact rate k will result in a 15% increase or reduction in the basic reproduction number R_0. In Table 1, we present the sensitivity indices of all model parameters.

Therefore, sensitivity analysis provides information on the appropriate intervention tactics to stop and manage the emergence of a product boycott across the communities outlined in the model (1).

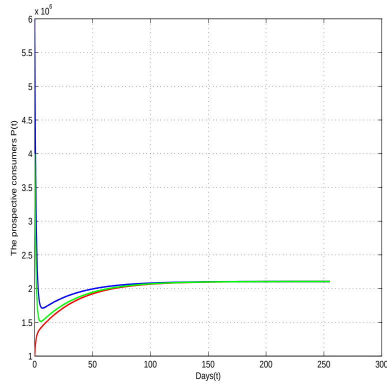
Table 1: Description and S.I. of parameters

Parameter	Description	Value	S.I.
μ	The natural death rate	0.053	+0.077
k	The effective impact rate of boycotters	0.4	+1
δ	The rate at which boycotters convert to actual consumers	0.01	−0.15
λ	The rate of transition of potential consumers to actual consumers	0.6	−0.91

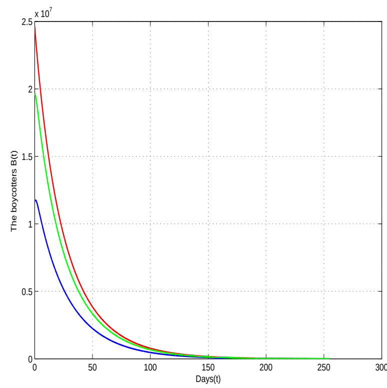
3.4 The numerical simulation

To support our theoretical findings on the stability analysis of the system (1), we provide some numerical simulations in this part. Certain simulation parameters are taken from [16, 17, 10, 20], which discusses the Moroccans' boycott of Centrale Danone [7]. The boycott lasted approximately one year, starting on the 20th of April 2018 [17], so we take $t_f = 255$ days. Each of the two sections of our testing is intended to demonstrate a different feature of the design. First, we aim to test Theorem 4, which states that the boycott free equilibrium E_0 of system (1) is globally stable on Ω. We choose parameters $A = 1375244$, $\lambda = 0.6$, $\delta = 0.01$, $\mu = 0.053$, $k = 0.4$, and $N = 25950000$, and we note that the parameter's model units in this work are in days.

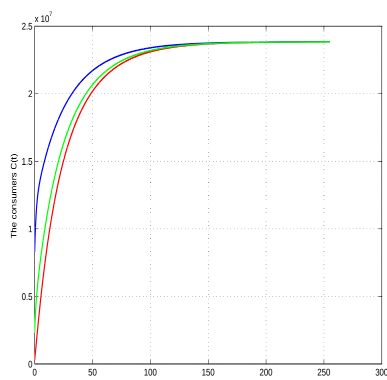
From Figure 2, where $R_0 < 1$, we can easily observe the global stability of the equilibrium $E_0 = (2.106 \times 10^6, 0, 2.384 \times 10^7)$ such that the variables (P), (B), and (C) converge to the equilibrium point E_0.



(a)



(b)



(c)

Figure 2: The convergence of the solutions to the equilibrium point E_0.

The second series of tests (see Figure 3) simulates the spread of boycott behavior of Central Danone's company in the population as a result of high prices and low product quality [7]. By chosen $A = 1375244$, $\lambda = 0.2$, $\delta = 0.01$, $\mu = 0.053$, $k = 0.6$, we have $R_0 > 1$. Then, according to Theorem 5, the equilibrium $E^* = (2.724 \times 10^6, 1.089 \times 10^7, 1.233 \times 10^7)$ is globally stable.

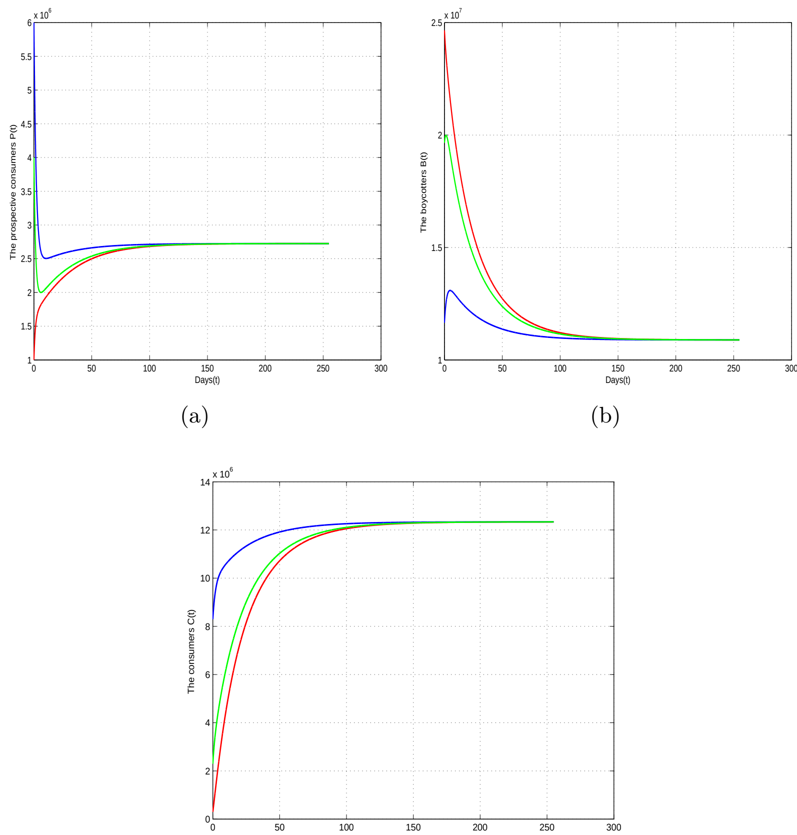


Figure 3: The convergence of the solutions to the equilibrium point E^*.

4 The Problem of optimal control

4.1 Problem synopsis

By targeting its products with boycott campaigns, each producing company aims to protect the level of sales of its product and maintain its loyal customers. To achieve this, appropriate strategies must be adopted to ensure that the number of consumers $C(t)$ is maximized and the number of interrupters $B(t)$ is minimized over the period $[t_0, t_f]$. For that, we propose in this work two controls. The control u_1 represents the effort made to introduce the product and encourage possible consumers to use it by promoting it through social media, including promotions and offers. The control u_2 indicates the efforts to deal with boycotters by understanding the reasons for their position and striving to meet their demands, including developing the product, gradually lowering its price over time, and providing incentives to the product's users. Thus, we consider our controlled mathematical model:

$$\begin{cases} \dot{P}(t) = A - k\dfrac{P(t)B(t)}{N} - (\mu+\lambda)P(t) - u_1(t)P(t), \\ \dot{B}(t) = k\dfrac{P(t)B(t)}{N} - (\delta+\mu)B(t) - u_2(t)B(t), \\ \dot{C}(t) = \lambda P(t) + \delta B(t) - \mu C(t) + u_1(t)P(t) + u_2(t)B(t), \end{cases} \tag{5}$$

with the initial conditions $P_0 \geq 0$, $C_0 \geq 0$, and $B_0 \geq 0$.

The problem is to minimize the objective functional,

$$J(u_1,u_2) = B(t_f) - C(t_f) + \int_{t_0}^{t_f} [B(v) - C(v) + \frac{M_1}{2}u_1^2(v) + \frac{M_2}{2}u_2^2(v)]dv, \tag{6}$$

where t_f is the final time, and the parameters M_1 and M_2 are the strictly positive cost coefficients; they are selected to weigh the relative importance of u_1 and u_2 at time t.

In other words, we look for the optimal controls u_1 and u_2 such that

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U_{ad}^2} J(u_1,u_2),$$

where U_{ad} is the set of admissible controls defined by

$$U_{ad} = \{u_i(t) : 0 \leq u_i \leq 1, \text{ for i} = 1, 2, \text{ and t} \in [t_0, t_f]\}.$$

4.2 Optimal controls' existence

Fleming and Rishel's result (see [8, Corollary 4.1]) can be used to determine whether the optimal controls exist.

Theorem 6. Take into consideration the system (5) with control problem. An optimal control $(u_1^*, u_2^*) \in U_{ad}^2$ exists such that

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U_{ad}^2} J(u_1, u_2)$$

if all of the following conditions hold:

1. The set of corresponding state variables and the controls is nonempty.
2. The U_{ad} control set is closed and convex.
3. A linear function in the state and control variables bounds the state system's right side.
4. The integrand $L(P, B, C, u_1, u_2)$ of the objective functional is convex on U_{ad} and there exist constants c_1, $c_2 > 0$, and $\epsilon > 1$ such that:

$$L(P, B, C, u_1, u_2) \geq -c_1 + c_2 \left(|u_1|^2 + |u_2|^2\right)^{\epsilon/2}.$$

Proof. **Condition 1.** To prove that the set of corresponding state variables and the controls is nonempty, a simplified version of an existing result (see [3, Theorem 7.1.1]) is used.

Let $\dot{P} = O_P(t; P, B, C)$, $\dot{B} = O_B(t; P, B, C)$, and $\dot{C} = O_C(t; P, B, C)$, where O_P, O_B, and O_C from the equations system (5)' right-hand side.

Let $u_i(t) = c_i$ for $i = 1, 2$ for some constants, and because all parameters are constants, and P, B, and C are continuous, then O_P, O_B, and O_C are also continuous.

Moreover, the partial derivatives $\frac{\partial O_P}{\partial P}$, $\frac{\partial O_P}{\partial B}$, $\frac{\partial O_P}{\partial C}$, $\frac{\partial O_B}{\partial P}$, $\frac{\partial O_B}{\partial B}$, $\frac{\partial O_B}{\partial C}$, and $\frac{\partial O_C}{\partial P}$, $\frac{\partial O_C}{\partial B}$, $\frac{\partial O_C}{\partial C}$ are all continuous. Consequently, there exists a unique solution (P, B, C) that fulfills the initial conditions.

Therefore, the set of corresponding state variables and the controls is nonempty, and Condition 1 is satisfied.

Condition 2. By definition, U_{ad} is closed. Take any controls $v_1, v_2 \in U_{ad}$ and $\varepsilon \in [0,1]$, then $0 \leq \varepsilon v_1 + (1-\varepsilon)v_2$.

Moreover, we note that $\varepsilon v_1 \leq \varepsilon$ and $(1-\varepsilon)v_2 \leq (1-\varepsilon)$. Then $\varepsilon v_1 + (1-\varepsilon)v_2 \leq \varepsilon + (1-\varepsilon) = 1$.

Therefore, $0 \leq \varepsilon v_1 + (1-\varepsilon)v_2 \leq 1$, for all $v_1, v_2 \in U_{ad}$ and $\varepsilon \in [0,1]$. Hence, U_{ad} is convex and Condition 2 is fulfilled.

Condition 3. Using the differential equations system (5), we get

$$\frac{dN}{dt} \leq A - \mu N.$$

So,

$$\lim_{t\to\infty} \sup N(t) \leq \frac{A}{\mu}.$$

As a result, every solution for model (5) is bounded.

Thus, there exist positive constants R_1, R_2, and R_3 such that for all $t \in [t_0, t_f]$,

$$P(t) \leq R_1,$$
$$B(t) \leq R_2,$$
$$C(t) \leq R_3.$$

We take into consideration

$$\begin{cases} O_P = \dot{P}(t) \leq A, \\ O_B = \dot{B}(t) \leq kP(t) - u_2(t)B(t), \\ O_C = \dot{C}(t) \leq \lambda P(t) + \delta B(t) + u_1(t)R_1 + u_2(t)R_2 \end{cases}$$

Then, system (5) can be rewritten in a matrix from as

$$O(t; P, B, C) \leq \bar{A} + BX(t) - RU(t),$$

where $O(t; P, B, C) = \begin{bmatrix} O_P \ O_B \ O_C \end{bmatrix}^T$, $\bar{A} = \begin{bmatrix} A \ 0 \ 0 \end{bmatrix}^T$, $X(t) = \begin{bmatrix} P \ B \ C \end{bmatrix}^T$, $U(t) = \begin{bmatrix} u_1 \ u_2 \end{bmatrix}^T$, and

$$B = \begin{bmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ \lambda & \delta & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & B \\ -P & -B \end{bmatrix}.$$

The control vector and state variable vector are given by a linear function. Consequently, we are able to write

$$\begin{aligned} \|O(t; P, B, C)\| &\leq \|\bar{A}\| + \|B\|\|X(t)\| + \|R\|\|U(t)\| \\ &\leq \psi + \Phi \left(\|X(t)\| + \|U(t)\|\right), \end{aligned}$$

where $\psi = \|\bar{A}\|$ and $\Phi = \max\left(\|B\|, \|R\|\right)$.

As a result, we can observe that the sum of the state and control vectors bounds the right side. Consequently, condition 3 is met.

Condition 4. The integrand in the objective functional (6) is convex on U_{ad}. The goal is to demonstrate that there exist constants $c_1, c_2 > 0$ and $\epsilon > 1$ such that the integrand $L(P, B, C, u_1, u_2)$ of the objective functional satisfies

$$\begin{aligned} L(P, B, C, u_1, u_2) &= B(t) - C(t) + \frac{M_1}{2} u_1^2 + \frac{M_2}{2} u_2^2 \\ &\geq -c_1 + c_2 \left(|u_1|^2 + |u_2|^2\right)^{\epsilon/2}. \end{aligned}$$

Since the state variables are bounded, let $\epsilon = 2$, $c_1 = 2 \sup_{t \in [t_0, t_f]} (B, C)$, and $c_2 = \inf(\frac{M_1}{2}, \frac{M_2}{2})$. Subsequently, it implies that

$$L(P, B, C, u_1, u_2) \geq -c_1 + c_2 \left(|u_1|^2 + |u_2|^2\right)^{\epsilon/2}.$$

□

4.3 The optimal controls' characterization

In this part, we make use of Pontryagin's maximal principle [22]. The key idea is to use the adjoint function to produce the Hamiltonian function by connecting the differential equations system to the objective function. This idea transfers the problem of finding the control to optimize the objective functional subject to the state of differential equations with initial condition and then finds the control to optimize the Hamiltonian pointwise (concerning the control).

The Hamiltonian H in time t is defined as

$$H(t) = B(t) - C(t) + \frac{M_1}{2}u_1^2(t) + \frac{M_2}{2}u_2^2(t) + \sum_{i=1}^{3}\zeta_i f_i,$$

where f_i represents the right side of the ith state variable's differential equations system (5).

Theorem 7. Given an optimal control $u^* = (u_1^*, u_2^*) \in U_{ad}^2$ and corresponding solutions P^*, B^*, and C^* of corresponding state system (5), there exist adjoint functions ζ_1, ζ_2, and ζ_3 fulfilling

$$\begin{cases} \dot{\zeta}_1 = \zeta_1\{k\frac{B(t)}{N} + \lambda + \mu + u_1(t)\} - \zeta_2 k\frac{B(t)}{N} - \zeta_3\{\lambda + u_1(t)\}, \\ \dot{\zeta}_2 = -1 + \zeta_1 k\frac{P(t)}{N} - \zeta_2\{k\frac{P(t)}{N} - \delta - \mu - u_2(t)\} - \zeta_3\{\delta + u_2(t)\}, \\ \dot{\zeta}_3 = 1 + \zeta_3\mu. \end{cases} \tag{7}$$

At the time t_f, given the transversality conditions, we have

$$\zeta_1(t_f) = 0,$$
$$\zeta_2(t_f) = 1,$$
$$\zeta_3(t_f) = -1.$$

Moreover, the optimal controls $u_1^*(t)$ and $u_2^*(t)$ for $t \in [t_0, t_f]$ are provided by

$$u_1^*(t) = \min\left\{1, \max\left\{0, \frac{1}{M_1}P(t)(\zeta_1(t) - \zeta_3(t))\right\}\right\}, \tag{8}$$

$$u_2^*(t) = \min\left\{1, \max\left\{0, \frac{1}{M_2}B(t)(\zeta_3(t) - \zeta_2(t))\right\}\right\}. \tag{9}$$

Proof. The Hamiltonian H in time t, is defined by

$$\begin{aligned} H(t) = {} & B(t) - C(t) + \frac{M_1}{2}u_1^2(t) + \frac{M_2}{2}u_2^2(t) \\ & + \zeta_1\{A - k\frac{B(t)P(t)}{N} - (\lambda + \mu)P(t) - u_1(t)P(t)\} \\ & + \zeta_2\{k\frac{B(t)P(t)}{N} - (\delta + \mu)B(t) - u_2(t)B(t)\} \end{aligned}$$

$$+\zeta_3\{\lambda P(t) + \delta B(t) - \mu C(t) + u_1(t)P(t) + u_2(t)B(t)\}.$$

Using Pontryagin maximum principle, one may obtain the transversality conditions and adjoint equations, for $t \in [t_0, t_f]$, such that

$$\begin{aligned}
\zeta_1(t_f) &= 0, & \dot{\zeta}_1(t) &= -\frac{\partial H}{\partial P},\\
\zeta_2(t_f) &= 1, & \dot{\zeta}_2(t) &= -\frac{\partial H}{\partial B},\\
\zeta_3(t_f) &= -1, & \dot{\zeta}_3(t) &= -\frac{\partial H}{\partial C}.
\end{aligned}$$

The optimality condition can be used to solve the optimal controls $u_1^*(t)$ and $u_2^*(t)$ for $t \in [t_0, t_f]$. We have

$$\begin{aligned}
\frac{\partial H}{\partial u_1} &= M_1 u_1(t) + \zeta_1(t)\{-P(t)\} + \zeta_3(t)\{P(t)\} = 0,\\
\frac{\partial H}{\partial u_2} &= M_2 u_2(t) + \zeta_2(t)\{-B(t)\} + \zeta_3(t)\{B(t)\} = 0.
\end{aligned}$$

That is,

$$\begin{aligned}
u_1(t) &= \frac{1}{M_1} P(t)(\zeta_1(t) - \zeta_3(t)),\\
u_2(t) &= \frac{1}{M_2} B(t)(\zeta_3(t) - \zeta_2(t)).
\end{aligned}$$

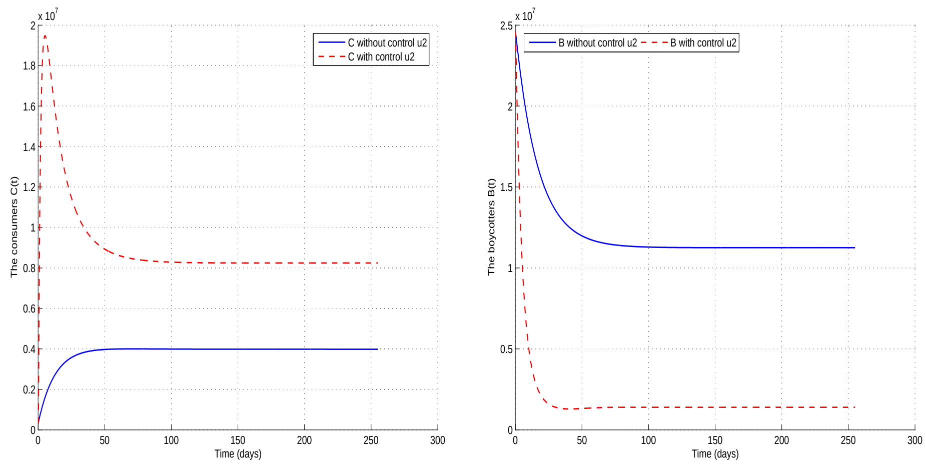
It is easy to obtain $u_1^*(t)$ and $u_2^*(t)$ in the form of (8) and (9) by the bounds in U_{ad} of the controls. □

4.4 The numerical simulation

We have starting conditions for the state variables and terminal conditions for the adjoints in our control problem. The optimality system is a two-point boundary value problem with discrete boundary conditions at periods step $i = t_0$ and $i = t_f$. We solve the optimality system iteratively by solving the adjoint system backward after solving the state system forward. We first estimate the controls in the first iteration. Next, we adjust the controls before the subsequent iteration based on the characterization. We continue until the next iteration converges. The following data are used to create and compile a code in MATLAB: $A = 900000$, $\lambda = 0.05$, $\delta = 0.011$, $\mu = 0.053$, $k = 0.95$, and $N = 25950000$ (the parameter's model units are in days).

Strategy 1: Correcting fallacies and restoring confidence in the product.

In this strategy, we concentrate the efforts, using optimal control u_2, to deal with boycotters by understanding the reasons for their position and striving to meet their demands, including developing the product, addressing negative rumors, restoring their confidence, and gradually lowering its price.



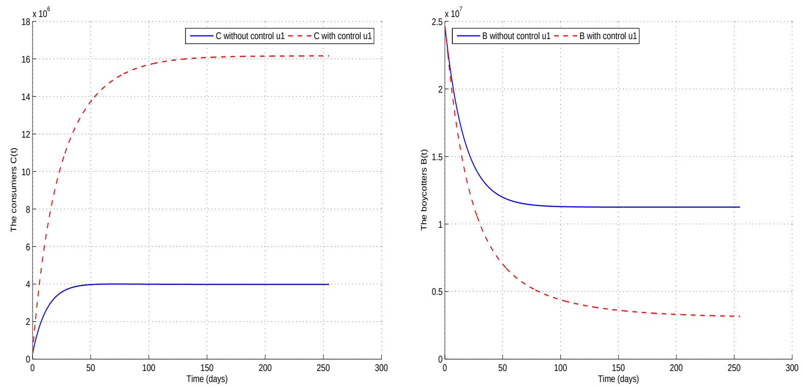
(a) (b)

Figure 4: Optimal consumers and boycotters with and without control u_2^*.

From Figures 4a and 4b, we can see that the number of consumers of the product has grown from 4×10^6 to 8.243×10^6. Also, the number of boycotters has decreased from 1.125×10^7 to 1.392×10^6.

Strategy 2: Publicity and Marketing.

Using the optimal control u_1, this strategy focuses on stimulating and compelling advertising to encourage and motivate potential consumers to use the product and protect them from being affected by boycott campaigns.



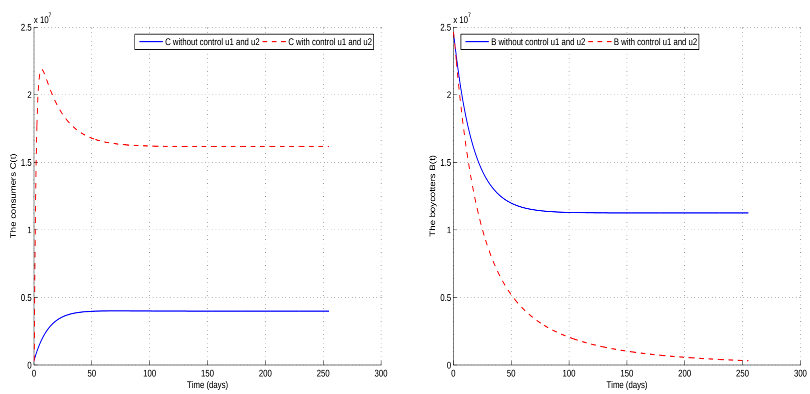
(a) (b)

Figure 5: Optimal consumers and boycotters with and without control u_1^*.

From Figures 5a and 5b, we observe that the number of consumers increased from 4×10^6 to 1.616×10^7. Also, the number of boycotters decreased from 1.125×10^7 to 3.174×10^6.

Strategy 3: Encouraging and motivating potential consumers and targeting boycotters.

This strategy aims to improve the numerical outcomes of cases 1 and 2 by activating simultaneously the two optimal controls u_1 and u_2.



(a) (b)

Figure 6: Optimal consumers and boycotters with and without controls u_1^* and u_2^*.

From Figure 6a, we show clearly that the number of consumers grows significantly from 4×10^6 to 1.617×10^7. Also, Figure 6b shows that the number of boycotters dropped from 1.125×10^7 to 3.255×10^5, which means the proposed strategy is more effective when we combine two optimal controls u_1^* and u_2^*.

Ultimately, we deduce that the suggested approach becomes more successful when we combine the two optimal controls, u_1^* and u_2^*.

Therefore, we observe that as the number of boycotters decreases, their influence on potential consumers diminishes as well, and thus trust is renewed between the product and the consumer. Consequently, a large number of consumers tend to give the product another try, often in a shorter time. This is particularly evident with the introduction of marketing campaigns that address misconceptions, highlight new features of the product, and interact positively with citizens' demands to respect reasonable prices.

5 Conclusion

In this research, we proposed a mathematical model that describes the boycott behavior of citizens regarding a product. We studied the stability analysis of the equilibriums of the proposed model, as well as the sensitivity analysis, in order to know more about the parameters that have a high impact on the reproduction number R_0. Using the results of optimal control theory, we have presented optimal strategies to persuade boycotters of a product to retract their position and thus reduce their influence on potential consumers. We ultimately concluded that both the number of boycotters and their influence on potential consumers decreased, leading to a renewal of trust between the product and the consumer. Consequently, a large number of consumers tend to give the product another try, often in a shorter period, especially with the launch of marketing campaigns that rectify misconceptions, highlight new features of the product, and interact positively with citizens' demands to respect reasonable prices. This study could have other interesting extensions, such as studying stochastic stability and optimal control in a stochastic version of our model through stochastic outcomes. This approach provides an additional degree of realism compared to its deterministic counterpart via a stochastic differential equation and includes the effect of a fluctuating environment.

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