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Research Article

Evaluation of iterative methods for solving nonlinear scalar equations

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Abstract

This study is aimed at performing a comprehensive numerical evaluation of the iterative solution techniques without memory for solving nonlinear scalar equations with simple real roots, in order to specify the most efficient and applicable methods for practical purposes. In this regard, the capabilities of the methods for applicable purposes are be evaluated, in which the ability of the methods to solve different types of nonlinear equations is be studied. First, 26 different iterative methods with the best performance are reviewed. These methods are selected based on performing more than 46000 analyses on 166 different available nonlinear solvers. For the easier application of the techniques, consistent mathematical notation is employed to present reviewed approaches. After presenting the diverse methodologies suggested for solving nonlinear equations, the performances of the reviewed methods are evaluated by solving 28 different nonlinear equations. The utilized test functions, which are selected from the reviewed research works, are solved by all schemes and by assuming different initial guesses. To select the initial guesses, endpoints of five neighboring intervals with different sizes around the root of test functions are used. Therefore, each problem is solved by ten different starting points. In order to calculate novel computational efficiency indices and rank them accurately, the results of the obtained solutions are used. These data include

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the number of iterations, number of function evaluations, and convergence times. In addition, the successful runs for each process are used to rank the evaluated schemes. Although, in general, the choice of the method depends on the problem in practice, but in practical applications, especially in engineering, changing the solution method for different problems is not feasible all the time, and accordingly, the findings of the present study can be used as a guide to specify the fastest and most appropriate solution technique for solving nonlinear problems.

AMS subject classifications (2020): Primary 45D05; Secondary 42C10, 65G99.

Keywords: Nonlinear scalar equations; Iterative method; Efficiency index; Order of convergence; Initial guess; Function evaluation.

1 Introduction

Most of the practical problems in engineering and other fields of science can be modeled by mathematical functions, which are mostly nonlinear. For instance, in engineering applications, nonlinear structural analysis, or computation of three-dimensional stresses require to solve nonlinear equations. Another example of the case in civil engineering practice that requires solving a nonlinear scalar equation is the computation of the torsional-flexural buckling load of steel columns. Similarly, the final step in the mathematical modeling and formulation of many other fields of science is to solve a nonlinear equation. Therefore, a reliable and applicable method for solving nonlinear equations is a necessary tool for scientific research. This device is utilized in performing different science-based activities, such as analysis and design. This need was felt many years ago and consequently, various solution methods are proposed for solving nonlinear equations. From the early works in this field until now, many different schemes are proposed. Some of these techniques are analytical approaches that are limited to special cases of nonlinear equations, but most of them are numerical iterative schemes. An iterative solution technique, as its name indicates, computes the root of a nonlinear function through several iteration cycles by an initial guess. Most of these methods are modifications of the basic earlier techniques, like the Newton method.

The different iterative approach has different convergence order. Order of convergence is an important mathematical quantity that indicates the efficiency of the solver. However, despite this mathematical standpoint, from the practical view, a method with a higher order of convergence is not necessarily the best choice, and the performance of a solver depends on many different factors. On the other hand, the large number of existing iterative methods makes it more difficult to choose a suitable technique for a special applicable problem. Therefore, the main motivation of this study is to provide a clear understanding of the performance of many of these iterative schemes. For this purpose, it is attempted to review many of the basic and well-known as well as newly proposed solution approaches in the first part of this study. Some big questions arise when facing this large number of iterative nonlinear solvers: Which method should be used for solving a given problem? Which approach is the fastest? Which one requires the least computational effort? Do they necessarily converge to the desired response?

The answer to these questions is not simple and certain and depends on many factors, including the problem at hand, the utilized initial guess, number of floating-point arithmetic, and the termination criterion. However, it is assured that no method can solve all the possible problems. The investigators who proposed the iterative solution schemes performed convergence analysis to demonstrate the ability of their methods to find the root of nonlinear functions. Therefore, they proposed a convergence order that, from the mathematical point of view, is an indication of the solution speed. In general, a solution method with a higher convergence order should converge faster to the response. However, in practice, the situation is not as easy as it seems. There is no guarantee that a certain solver can find the roots of a given nonlinear problem. Moreover, it is widely known that higher-order solvers converge faster when the initial guess is close enough to the root. In other words, increasing the order of convergence results in a smaller region of attraction for a certain number of iterations. Therefore, in practical problems where the initial guess may fall in a wide range around the response, a higher-order method is not necessarily superior. In cases, when the selected starting point is far from the root of the function, the higher order of convergence may lead to the inability of the method to find the response within a permissible number of iterations. Even in some cases, the method may diverge.

To the authors' best knowledge, despite a large number of available iterative methods for solving nonlinear scalar equations, there are very limited reviews about these techniques. One of the limited reviews in this field is performed by Babajee and Dauhoo [1]. They investigated the performance of the variants of the Newton method with cubic convergence. They also extended some of these methods to multivariate cases. In another similar study, Varona [27] performed a numerical and graphical comparison between some of the well-known solution methods. However, Varona utilized many different criteria for the evaluation of the solution methods, but his research work is mostly limited to traditional and well-known techniques, and their performance is evaluated by extensive applications during the past decades. Two more recent valuable review studies have also been performed by Cătinaş [2, 3]. Occasionally various researchers propose the same methods independently. This is due to the fact that there are so many iterative techniques available, and this quantity increases very fast every year. Therefore, there is a great need for studies like the present paper to provide useful information for the researchers in this regard to prevent the proposition of the same formulations by different investigators. Another merit of the present research work is to study the effect of the initial guess on the performance of the methods and also evaluate the practical efficiency of different approaches.

The question of which solver is better remained unanswered. Due to the variable nature of different nonlinear problems in the various fields of science and application, giving a definite answer to this question is impossible. This study, it is comprehensively tried to provide a clearer image of the performance of a large number of iterative solution methods without memory. For this purpose, 26 different solution techniques (selected as the best-performing methods among 166 reviewed solvers which are not reported in this manuscript) are used to solve 28 different nonlinear functions by using ten diverse initial guesses for each function. Different initial guesses are used, to investigate their effect of them on the performance of the solution methods. It is worth mentioning that this important effect has been neglected in much of the previous research in this field. To compare the abilities of discussed approaches, a new computational efficiency index is proposed and utilized against the others which were used previously. All solvers are ranked based on the results of presented extensive numerical evaluations. The suggested index has a qualitative-quantitative base and can successfully rank the solution schemes. To indicate the most applicable solver, the results of the new index are compared with those attained by the traditional and well-known efficiency indices. Findings show that the suggested way can better distinguish the performance and efficiency of the nonlinear solvers for practical applications. Finally, according to the obtained results, the reviewed methods are ranked to specify the ones which are more efficient and applicable to be utilized, especially, in engineering practice.

2 Review of the available iterative solvers

In this section, various nonlinear solvers are reviewed briefly and presented in historical order. These methods fall in the category of iterative methods without memory; that is, only the results of the current iteration would be used to determine the next estimation. The iterative formula of each technique is provided for the $n+1$th estimation of the root, assuming that the nth evaluation is available. The process commences by using an initial guess, x_0. Here, $f(x)$ indicate the nonlinear function needs to be solved. The reviewed methods are presented in Table 1, using uniform mathematical notations.

Table 1: Iterative solution methods

No	Method	Iterative Formula
1	Newton	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
2	Ostrowski [22]	$x_{n+1} = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}$
3	Traub–Ostrowski [26]	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)$
4	Jarrat relation [12]	$x_{n+1} = x_n - \frac{1}{2} \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)}{f'(x_n)-3f'(\widetilde{y_n})}$
5	4th order Newton [20]	$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$
6	Three step Newton [4]	$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_n)}$ $x_n^* = y_n - \frac{f(y_n)}{f'(x_n)}$
7	Hansen and Patrick [11]	$x_{n+1} = x_n - \frac{mf(x_n)}{\left(\frac{m+1}{2m}\right)f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}$
8	King method [13]	$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)}{f(x_n)-2f(y_n)}$
9	Kung–Traub [15]	$x_{n+1} = y_n - \frac{f(x_n)f(y_n)}{[f(x_n)-f(y_n)]^2} \frac{f(x_n)}{f'(x_n)}$
10	Potra and Ptak [23]	$x_{n+1} = x_n - \frac{f(x_n)+f(y_n)}{f'(x_n)}$
11	Halley [8]	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{1}{2}\frac{f''(x_n)f(x_n)}{f'(x_n)}}$
12	Dong method [6]	$x_{n+1} = x_n^* - \frac{\frac{m}{m+1}f(x_n)}{\left(1+\frac{1}{m}\right)^m f'(x_n^*) - f'(x_n)}$ $x_n^* = x_n - \frac{m}{m+1}\frac{f(x_n)}{f'(x_n)}$
13	Osada [21]	$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)}$ $+\frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}$
14	Grau and Barrero method [9]	$x_{n+1} = x_n^* - \frac{x_n - y_n}{f(x_n)-2f(y_n)} f(x_n^*)$ $x_n^* = y_n - \frac{x_n - y_n}{f(x_n)-2f(y_n)} f(y_n)$
15	Noor, 1st method [19]	$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_n^*)}$ $x_n^* = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) - 4f^3(x_n)}}$
16	Noor, 2nd method [16]	$x_{n+1} = x_n + 4(x_n^* - x_n)\frac{f(x_n)}{3f(x_n)-2f(x_n^*)}$ $x_n^* = x_n - \frac{1}{2}\frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) + 4f^2(x_n)}}$
17	Nedzhibov method [16]	$x_{n+1} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left(\frac{3f'(\overline{y_n})+f'(x_n)}{3f'(\overline{y_n})-f'(x_n)}\right)$
18	Kou et al. method [14]	$x_{n+1} = x_n - \Big[1 - \frac{3}{4}$ $\frac{\left(f'\widetilde{y_n})-f'(x_n)\right)\left(7f'(\widetilde{y_n})+f'(x_n)\right)}{(3f'(\widetilde{y_n})+5f'(x_n))(2f'(\widetilde{y_n})-f'(x_n))}\Big]\frac{f(x_n)}{f'(x_n)}$
19	Sharma and Guha [25]	$x_{n+1} = x_n^* - \frac{f(x_n)+f(y_n)}{f(x_n)-f(y_n)}\frac{f(x_n^*)}{f'(x_n)}$ $x_n^* = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}$
20	Yun [28]	$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} - \frac{f(x_n^*)}{f'(x_n)}$ $x_n^* = y_n - \frac{f(y_n)}{f'(x_n)}$
21	Fernandez and Aquino method [7]	$x_{n+1} = x_n + \frac{f^2(x_n)}{(f(y_n)-f(x_n))f'(x_n)}$ $-\frac{f^2(y_n)f(x_n)(f(y_n)-3f(x_n))}{(f(y_n)-f(x_n))^2(f(y_n)-2f(x_n))f'(x_n)}$
22	Noor, 3rd method [18]	$x_{n+1} = y_n - \frac{f(y_n)}{f'\left(\frac{x_n+y_n}{2}\right)}$
23	Noor, 4th method [18]	$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'\left(\frac{x_n+x_n^*}{2}\right)}$ $x_n^* = y_n - \frac{f(y_n)}{f'\left(\frac{x_n+y_n}{2}\right)}$
24	Noor, 5th method [18]	$x_{n+1} = x_n^* - \frac{4f(x_n^*)}{f'(x_n)+3f'\left(\frac{x_n+2x_n^*}{3}\right)}$ $x_n^* = y_n - \frac{4f(y_n)}{f'(x_n)+3f'\left(\frac{x_n+2y_n}{3}\right)}$
25	Shah and Noor 1th method [24]	$x_{n+1} = x_n^* - \frac{2f(x_n^*)f'(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}$ $x_n^* = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}$
26	Shah and Noor, 2nd method [24]	$x_{n+1} = x_n^* - \frac{2f(x_n^*)f'(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}$ $x_n^* = x_n^{**} - \frac{2f(x_n^{**})f'(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}$ $x_n^{**} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}$

In this table, m stands for the multiplicity of the roots. In addition, it should be noted that in the relations including $\pm$ sign in the denominator, the sign should be selected so as to maximize absolute value of the denominator. The utilized intermediate variables used in the above-mentioned relations are defined as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

$$\widetilde{y_n} = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

$$\overline{y_n} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}. \tag{3}$$

3 Efficiency and performance evaluations

The order of convergence is an important mathematical feature of a nonlinear solver, and the higher order of convergence is an indicator of the better performance of the methods from a mathematical standpoint. However, it is well known that increasing the order of convergence reduces the size of attraction intervals of Newton-type solution methods. The attraction interval is an interval around the root of the function that if the initial guess falls in this interval, the required iteration for converging to the root would be less than a specific number. In the higher-order methods, it is necessary to utilize initial guesses that are closer to the root of the function, which is practically difficult, because, in some cases, the range of possible responses is not known beforehand.

The comparison between different methods is not an easy task and depends on many factors, mostly the context in which the method is going to be used. However, the concentration of this study is on scalar nonlinear equations with real simple roots. In this regard, previous investigators proposed some efficiency indices for this purpose. The most well-known efficiency index is the one proposed by Traub [22]. Many of the reviewed researches utilized this index, which is defined by equation (4), to evaluate the performance of the proposed techniques:

$$EI = p^{\frac{1}{q}}. \tag{4}$$

In this relation, p is the convergence order and q is the number of function evaluations per step (NFE). There is also another common index, which is called the informational index, and is defined as follows:

$$EI_I = \frac{p}{q}. \tag{5}$$

In both indices, the higher value of the index is considered a sign of better performance. This is a widely accepted concept by mathematicians. However, from the practical point of view, for instance, for an engineer who aims to solve a nonlinear equation to find the response to a practical design problem, it raises some questions. For example, two different methods with the same order of convergence and the total number of function evaluations have the same values of efficiency indices, but it is obvious that their performances are not necessarily similar. To show this complexity, the efficiency indices for the evaluated methods are calculated by the authors and shown in Table 2. These results present the convergence order and NFE, as well as, the values of the two introduced efficiency indices.

Table 2: Efficiency indices of the solution methods

No	Function Name	Convergence order	NFE	Informational Index($\frac{p}{q}$)	Efficiency Index($P^{\frac{1}{q}}$)
1	Newton	2	2	1.000	1.414
2	Ostrowski [22]	4	3	1.333	1.587
3	Traub-Ostrowski [26]	4	3	1.333	1.587
4	Jarrat relation [12]	4	3	1.333	1.587
5	4th order Newton [20]	4	4	1.000	1.414
6	Three step Newton [4]	4	4	1.000	1.414
7	Hansen and Patrick [11]	3	3	1.333	1.442
8	King method [13]	4	3	1.333	1.587
9	Kung–Traub [15]	8	3	2.667	2.000
10	Potra and Ptak [23]	3	3	1.000	1.442
11	Halley [8]	2	3	0.667	1.260
12	Dong method [6]	2	2	1.000	1.414
13	Osada [21]	3	3	1.000	1.442
14	Grau and Barrero method [9]	6	4	1.500	1.565
15	Noor, 1st method [19]	4	4	1.000	1.414
16	Noor, 2nd method [19]	4	3	1.333	1.587
17	Nedzhibov method [16]	3	3	1.000	1.442
18	Kou et al. method [14]	4	3	1.333	1.587
19	Sharma and Guha [25]	6	4	1.500	1.565
20	Yun [28]	4	3	1.333	1.587
21	Fernandez and Aquino method [7]	4	3	1.333	1.587
22	Noor, 3rd method [18]	3	4	0.750	1.316
23	Noor, 4th method [18]	3	4	0.750	1.316
24	Noor, 5th method [18]	3	4	0.750	1.316
25	Shah and Noor 1th method [24]	4	4	1.000	1.414
26	Shah and Noor, 2nd method [24]	5	5	1.000	1.380

It is concluded from this comprehensive study that many of the available methods have the same value as the efficiency indices, but as will be revealed in the coming sections, the solved problems show very different performances for these techniques. Moreover, the effect of the starting point on the performance of a solution method is not included in these indices. Accordingly, it seems that these indices are not sufficient to judge about the applicability of them in solving different types of applied problems, such as engineering problems.

In most of the reviewed research, after the proposition of the method, mathematical proofs about the order of convergence of the suggested schemes are provided. In some of these studies, numerical evaluations were performed, in which; a limited number of test functions were solved, and the obtained results were compared with some other nonlinear solution techniques. The number of iterations, the total number of function evaluations, solution time, computational order of convergence, residual error of the function value, and difference of the last two estimations were the parameters, which were usually recorded in the numerical evaluations. Some of the available solution methods are able to calculate roots of nonlinear functions with very high accuracy, for example 10^{-100}. However, in practical applications, such high accuracy is not required. Instead, a robust method should be able to compute the response of different types of nonlinear functions within the least possible number of iterations. Moreover, the exact ranges of the responses for some practical purposes, such as nonlinear structural problems, are not known beforehand, or it is difficult to estimate such ranges in highly nonlinear problems in structural engineering. Therefore, a robust method must be able to solve a problem with a random initial guess. Therefore, some investigators attempted to study the effects of different initial guesses on the performance of the suggested methods by using assessing the basin of attractions [27, 5, 17, 10].

For the applicable purposes, a method is considered desirable if it can solve different types of nonlinear equations and by using diverse initial guesses. Moreover, a powerful solution technique requires a fewer number of function evaluations and computational time. Accordingly, to rank the reviewed methods and specify the most efficient techniques for applicable purposes, a thorough numerical evaluation program is necessary. Such a responsibility is defined and undertaken in the following of this study.

4 Test functions

To study the performance of these methods, 28 different scalar nonlinear functions are solved. These test equations, which are selected from the reviewed research works, are listed in Table 3. All of these nonlinear equations are famous benchmark problems. As was mentioned previously, the exact ranges of the responses for some practical purposes, such as nonlinear structural problems, are not known beforehand, or it is difficult to estimate such ranges in highly nonlinear problems in structural engineering. Therefore, a robust process must be able to find the response even in the cases that the initial guess is not close to the root. To investigate accurately the effect of the selected starting point on the performance of the approaches, each test function is solved ten times by using ten different initial guesses. These starting points are selected as the endpoints of a symmetrical interval around the roots of the functions. Five intervals, namely, $[x^* - 0.1, x^* + 0.1], [x^* - 1, x^* + 1], [x^* - 10, x^* + 10], [x^* - 100, x^* + 100]$ and $[x^* - 1000, x^* + 1000]$ are selected. They are named very small, small, medium, large, and very large neighboring intervals. As mentioned in Table 3, x^* represents the root of the function.

Table 3: Test functions

No.	Test Functions	x^*
1	$f(x) = x^3 + 4x^2 - 15$	1.63198
2	$f(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	-1.20764
3	$f(x) = \cos(x) - x$	0.73908
4	$f(x) = x^3 + 1$	-1.00000
5	$f(x) = 2xe^{-5} + 1 - 2e^{-5x}$	0.13826
6	$f(x) = 2xe^{-10} + 1 - 2e^{-10x}$	0.06931
7	$f(x) = \sin^{-1}\left(x^2 - 1\right) - \frac{x}{2} + 1$	0.59481
8	$f(x) = x^5 + 23x - 6$	0.26082
9	$f(x) = x^3 + 4x^2 - 10$	1.36500
10	$f(x) = ln\left(x^2 + x + 2\right) - x + 1$	4.15200
11	$f(x) = e^{(-x^2+x+2)} - 1$	-1.00000
12	$f(x) = x^5 + x^4 + 4x^2 - 15$	1.34700
13	$f(x) = x^5 + x - 10000$	6.30800
14	$f(x) = (x-1)^3 - 1$	2.00000
15	$f(x) = x^3 - 10$	2.15440
16	$f(x) = x^3 - 2x - 5$	2.09450
17	$f(x) = (x-1)^3 - 2$	2.25992
18	$f(x) = e^{(x^2+7x-30)} - 1$	3.00000
19	$f(x) = (x+2)e^x - 1$	-0.44280
20	$x^3 - e^{-x}$	0.77290
21	$f(x) = e^{(-x^2+x+2)} - \cos(x+1) + x^3 + 1$	-1.0000
22	$f(x) = x^3 + 4x^2 - 25$	2.03500
23	$f(x) = \sin^2(x) + x$	0.00000
24	$f(x) = \tan^{-1}(x) - 1$	1.55740
25	$f(x) = x^3 - \cos(x) + 2$	-1.17250
26	$f(x) = x^3 + 4x^2 + 8x + 8$	-2.00000
27	$f(x) = x^2 - (1-x)^5$	0.34595
28	$f(x) = \mathrm{xlog}(\mathrm{x}) - 1.2$	2.74064

It must be noted that due to the described approach for the selection of the starting points of the iterative process, the nonlinear functions, with only one root, are selected for this study. The number of iterations, the total number of function evaluations, and convergence time are recorded for each run that reached the root within the admissible number of iterations. In this study, the permissible number of iterations is assumed to be 1000. It is noteworthy that the admissible tolerance for the convergence is selected as equal to 10^{-10}. This value is mostly more than the necessary value for applicable engineering problems.

5 Proposed efficiency index

In addition to the ability to solve different nonlinear problems for dissimilar initial guesses, a powerful solution technique should be able to compute the response by utilizing a lesser number of function evaluations or computation time. Since the performance of a method for different types of nonlinear problems is not uniform, it is difficult to select more efficient methods directly based on the recorded parameters for each test function, including convergence time and the total number of function evaluations. Comprehensive numerical experiences inform the authors that more sophisticated approaches are necessary. In this study, a computational efficiency index in the following form is suggested:

$$EI_c = \alpha + \beta + 70\left(\frac{i_{\max} - i}{i_{\max} - i_{\min}}\right), \tag{6}$$

where EI_c is the computational index that can be computed for any of the recorded values, including convergence time or total number of function evaluations. The parameters α and β are calculated by the following relations:

$$\alpha = \begin{cases} 0 & \text{if the method diverged,} \\ 10 & \text{if the solver converged to the response,} \end{cases} \tag{7}$$

$$\beta = \begin{cases} 0 & \text{if the method didn't converge within the admissible} \\ & \quad \text{number of iterations,} \\ 20 & \text{if the method converged within the admissible} \\ & \quad \text{number of iterations.} \end{cases} \tag{8}$$

In equation (6), i stands for the selected parameter, which can be the number of function evaluations or the solution time. For a given starting point, $i_{\max}$ and $i_{\min}$ are the maximum and the minimum values of the selected parameter for the different solution methods. Because the number of the maximum allowable iterations is equal to 1000, the maximum and minimums are specified for the methods that have converged to the response in the admissible number of iterations. Therefore, the third term in the right-hand side of equation (6) is equal to 0 for the approaches that have diverged or cannot compute the response within the permissible iterations. It must be noted that the mentioned values for different parameters in the suggested efficiency index are found and proposed by the authors. In fact, the computational efficiency index and the related parameters are selected in the process of extensive numerical experiences as a tool for providing meaningful results. These are not based on any special mathematical concept. For various solution techniques, these indices are computed in a test function and for each initial guess. Then, the mean value of each neighboring interval is calculated by averaging the values of the initial guesses corresponding to the start and end point of the intervals. Finally, the averages of the derived values for all the test functions are computed.

It is evident that the suggested index, which varies between 0 and 100, has three different phases. These phases are specified by four boundary values of 0, 10, 30, and 100. The average efficiency index equal to 0 indicates that the solution method is not able to solve the nonlinear problems at all. Obviously, it is an extreme case that will not happen for any of the available solution techniques in the average results, because a solver, no matter how weak, is able to solve some type of nonlinear problems. The value of 10 demonstrates that a solution technique can converge to the response of the nonlinear problem but with iterations more than the admissible number. Therefore, if the average computational efficiency index for a solution method in a given neighboring interval falls between 0 and 10, it is an indication that the technique is probably not able to solve the problem on the condition that the initial guess is close to the endpoints of that interval. This probability is higher if the value is closer to 0. It must be noted that the term “probably” in the previous statement is of extreme importance because a solution method does not demonstrate the same performance for different types of nonlinear functions. On the other hand, the calculated indices in this study are derived based on solving a limited number of test functions. Therefore, it is neither possible nor logical to make a “certain” statement.

The next boundary value is 30, which characterizes the borderline between the probability of convergence to the response with fewer and more iterations than the permissible number of iterations. Therefore, the value of the efficiency index between 10 and 30 is the sign of the ability of the solution technique to converge to the root of the nonlinear function for a given neighboring interval. However, the number of required iterations is expected to be more than the admissible iterations. The values closer to 10 indicate a lower probability of convergence. Hence, the methods which have efficiency indices in the range of $(10, 30)$ are not numerically efficient, but there is an acceptable probability that they are able to solve nonlinear problems.

Finally, the last boundary value is 100, which shows that a solution method is the most efficient one, among the evaluated solution techniques. The values of efficiency indices between 30 and 100 show that there is a high probability of solving diverse nonlinear problems by the corresponding solution technique for the given neighboring. A technique is deemed more efficient if its average efficiency indices are closer to 100. It must be stated that the mentioned values are selected by the authors to provide a clear image of the performance of the methods. Obviously, it is possible to choose different boundaries for the three phases of the proposed index.

6 The obtained results

The test functions presented in Table 2 are solved by the nonlinear solution methods listed in Table 1, and the previously mentioned parameters are recorded for each initial guess. To shorten the paper, the recorded values for each problem are not included in the text. The recorded values are used to calculate the performance criteria which were introduced in the previous section. Table 4 presents the total number of failed runs, as well as, the success ratio for each method.

Table 4: The success ratio of different solution methods

Rank	**Function Name**	**Number of failed runs**	**Success ratio (%)**
1	Halley	32	88.57
2	Traub–Ostrowski	33	88.21
2	Hansen and Patrick	33	88.21
3	Ostrowski	34	87.86
3	King method	34	87.86
3	Shah and Noor, 1st method	36	87.14
4	Shah and Noor, 2nd method	36	87.14
4	Grau and Barrero method	37	86.79
5	Noor, 2nd method	37	86.79
5	4th order Newton	38	86.43
6	Newton	39	86.07
7	Kou et al. method	39	86.07
7	Noor, 4th method	39	86.07
7	Jarrat relation	40	85.71
8	Osada	40	85.71
8	Noor, 5th method	40	85.71
8	Three step Newton method	41	85.36
9	Kung-Traub	41	85.36
9	Dong method	41	85.36
9	Yun	41	85.36
9	Noor, 3rd method	41	85.36
9	Potra and Ptak	42	85.00
10	Noor, 1st method	42	85.00
10	Nedzhibov method	42	85.00
10	Sharma and Guha	42	85.00
10	Fernandez and Aquino method	42	85.00

According to the total number of function evaluations, the average computational efficiency indices are listed in Table 5. These outcomes include separate results for each neighboring interval, and an overall average.

Table 5: The average computational efficiency indices based on the total number of function evaluations for different solution methods

No	Function Name	Neighboring interval					
		$[x^* - 1000, x^* + 1000]$	$[x^* - 100, x^* + 100]$	$[x^* - 10, x^* + 10]$	$[x^* - 1, x^* + 1]$	$[x^* - 0.1, x^* + 0.1]$	**Average**
1	Newton	73.101	76.014	83.273	87.158	96.147	83.139
2	Ostrowski [22]	73.071	77.045	88.475	89.494	96.845	84.986
3	Traub–Ostrowski [26]	71.914	77.424	89.328	89.494	96.845	85.001
4	Jarrat relation [12]	71.999	75.304	83.486	87.640	96.845	83.055
5	4th order Newton [20]	71.777	74.519	81.233	86.306	95.763	81.919
6	Three step Newton [4]	71.226	71.920	81.602	80.263	93.961	79.794
7	Hansen and Patrick [11]	70.477	74.770	91.034	92.024	96.400	84.941
8	King method [13]	73.071	77.045	88.475	89.494	96.845	84.986
9	Kung–Traub [15]	71.784	73.546	83.822	87.438	96.822	82.683
10	Potra and Ptak [23]	71.396	74.216	81.595	82.334	93.190	80.546
11	Halley [8]	70.503	73.860	92.453	92.457	96.845	85.223
12	Dong method [6]	71.204	75.192	84.251	90.540	97.812	83.800
13	Osada [21]	68.974	71.690	80.842	84.379	94.221	80.021
14	Grau and Barrero method [9]	73.621	74.829	84.297	87.235	96.235	83.243
15	Noor, 1st method [19]	66.743	67.428	80.845	88.717	90.966	78.940
16	Noor, 2nd method [19]	28.192	60.246	86.995	86.501	88.122	70.011
17	Nedzhibov method [16]	70.384	73.727	83.486	87.640	96.845	82.417
18	Kou et al. method [14]	72.420	78.026	85.926	86.146	96.863	83.876
19	Sharma and Guha [25]	71.552	75.038	82.283	83.326	96.227	81.685
20	Yun [28]	72.189	73.530	82.751	80.935	95.024	80.886
21	Fernandez and Aquino method [7]	72.002	73.790	81.968	87.158	96.488	82.281
22	Noor, 3rd method [18]	72.489	74.307	82.175	81.684	93.060	80.743
23	Noor, 4th method [18]	73.215	75.330	82.800	84.895	95.768	82.402
24	Noor, 5th method [18]	71.891	75.321	82.977	84.743	95.763	82.139
25	Shah and Noor, 1th method [24]	68.756	73.769	85.590	90.221	95.794	82.826
26	Shah and Noor, 2nd method [24]	66.049	70.799	88.613	89.720	94.820	82.000

Finally, the average computational efficiency indices based on the convergence time are presented in Table 6. The attained results will be discussed in the next section.

7 Discussion about the results

The first interesting finding according to the calculated success ratio is that the Halley approach, which is one of the basic solution techniques, performs better than all the other schemes, including those which are proposed newly and those which have a higher order of convergence. The other remarkable observation is the outstanding performance of the classical Newton method, which is ranked 6, among 26 reviewed procedures. Its success ratio is only about 2 percent less than the Halley technique! Based on the findings, most of the traditional solution techniques are ranked among the top solvers, according to the success ratio, while many of the recently proposed iterative approaches perform poorly.

The derived results for efficiency indices in Tables 5 and 6 provide the opportunity to study the effect of the starting point on the performance of reviewed methods. Before presenting a further discussion about the attained results, it seems necessary to rank the reviewed method based on the obtained outcomes. For the criteria of the success ratio, the presented results in Table 4 are ranked in descending order. As was expected, most of the reviewed techniques have computational efficiency indices higher than 90 for a very small neighboring interval. This is a sign of fast convergence because the starting point of the iterative process is very close to the response. The general trend of efficiency index variation shows that the efficiency of the methods reduces by increasing the distance of the initial guess from the root of the test functions. It is interesting to note that the traditional solution techniques, such as Newton and Traub–Ostrowski perform as well as the newly proposed higher-order schemes and are even better than many of them. As it was expected, the efficiency indices of convergence time and the total number of function evaluations are compatible with each other.

According to the computed indices, the Halley technique is one of the best approaches. An ironic and astonishing finding is that according to the three considered criteria, many of the basic and traditional nonlinear solution techniques, such as, Halley, Hansen and Patrick, Ostrowski, Newton, Traub–Ostrowski, and King are among the best solvers. Obviously more detailed discussion is possible about the performance of the reviewed methods according to the results of the comprehensive numerical evaluation undertaken in this study. For example, it is possible to study the effect of the type of nonlinear functions on the performance of the methods. It is noteworthy that this study, and the presented results can be useful means for the future investigator in the field of nonlinear solution techniques, as well as, the scientists and engineers who seek to select a powerful method for solving practical

Table 6: The average computational efficiency indices based on convergence time for different solution methods

No	Function Name	Neighboring interval					
		$[x^* - 1000, x^* + 1000]$	$x^* - 100, x^* + 100]$	$[x^* - 10, x^* + 10]$	$[x^* - 1, x^* + 1]$	$[x^* - 0.1, x^* + 0.1]$	Average
1	Newton	74.451	78.192	84.341	89.797	98.603	85.077
2	Ostrowski [22]	73.344	77.217	88.019	90.528	97.874	85.396
3	Traub-Ostrowski [26]	72.058	76.861	87.860	89.844	97.132	84.751
4	Jarrat relation [12]	72.668	76.066	83.829	88.956	97.912	83.886
5	4th order Newton [20]	73.845	77.997	83.910	90.135	99.157	85.009
6	Three step Newton [4]	73.439	75.250	83.178	84.405	96.630	82.580
7	Hansen and Patrick [11]	71.501	76.920	92.294	94.485	98.558	86.751
8	King method [13]	73.139	77.217	87.849	90.193	97.622	85.204
9	Kung–Traub [15]	72.074	72.994	82.945	87.763	96.997	82.555
10	Potra and Ptak [23]	73.066	75.573	82.470	84.502	94.799	82.082
11	Halley [8]	71.503	76.377	93.845	94.867	98.964	87.111
12	Dong method [6]	71.666	75.657	84.771	91.766	98.810	84.534
13	Osada [21]	70.783	75.918	83.053	88.715	97.950	83.284
14	Grau and Barrero method [9]	74.486	76.892	85.450	89.366	98.458	84.930
15	Noor, 1st method [19]	68.174	68.515	80.645	90.791	92.626	80.150
16	Noor, 2nd method [19]	28.416	53.152	83.386	84.333	86.913	67.240
17	Nedzhibov method [16]	71.299	75.571	84.629	89.877	98.974	84.070
18	Kou et al. method [14]	73.062	79.502	86.988	88.036	98.742	85.266
19	Sharma and Guha [25]	71.808	74.764	81.532	83.318	96.303	81.545
20	Yun [28]	73.311	74.854	83.280	83.158	96.527	82.226
21	Fernandez and Aquino method [7]	71.627	71.338	79.678	84.718	94.360	80.344
22	Noor, 3rd method [18]	74.152	77.893	84.388	85.932	97.004	83.874
23	Noor, 4th method [18]	74.284	77.414	83.957	87.609	98.373	84.327
24	Noor, 5th method [18]	72.752	77.210	84.164	87.422	98.264	83.963
25	Shah and Noor, 1th method [24]	69.782	75.328	86.200	91.905	97.637	84.170
26	Shah and Noor, 2nd method [24]	66.809	71.563	88.311	90.772	96.203	82.732

nonlinear equations. In order to evaluate the performance of future solution techniques in a more realistic and applicable manner, the proposed efficiency index can be used along with the previous indices.

8 Conclusion

In this study, 26 different iterative nonlinear solution techniques for solving nonlinear scalar equations were reviewed and evaluated numerically. For this purpose, 28 different nonlinear problems, which were selected from a review of the previous studies in this field, were solved by the discussed techniques. To study the effects of the starting point on the performance of solvers, each problem was solved by assuming different initial guesses. The selected starting points for the iterative process were the endpoints of the symmetrical neighboring interval around the root of the solved functions. In each run of a solver, the recorded parameters include the total number of function evaluations and convergence time.

To compare the mentioned methods and rank them according to their performances, three different criteria, namely, success ratio, computational efficiency index of the total number of function evaluations, and also the solution time's efficiency index were defined. The first criterion was a simple ratio of the successful runs to the total number of runs for each scheme. The other two criteria were compared based on a new computational efficiency index, which was suggested by the authors in order to provide a clear picture of the procedure performances. The reason for proposing this new index was the inability of the classical efficiency indices, such as the well-known Traub index, in distinguishing between the performance of different solution methods that have the same order of convergence and the number of function evaluations per step. The solver is considered more efficient if its index is closer to 100. The comprehensive obtained results showed that the higher order of convergence is not necessarily a sign of better performance. Moreover, it is observed that many of the most powerful solution techniques are among the old and traditional approaches, such as Halley, Hansen and Patrick, Ostrowski, and even Newton. Astonishingly, it is found that some of the newly presented solvers are not as operational as the old ones. According to the performances of 26 different iterative techniques, the first four effective procedures for solving nonlinear equations can be ranked as follows: 1. Halley, 2. Traub–Ostrowski, 3. Ostrowski 4. Hansen and Patrick.

References

[1] Babajee, D. and Dauhoo, M. *An analysis of the properties of the variants of Newton's method with third order convergence*, Appl. Math. Comput. 183(1) (2006), 659–684.

[2] Cătinaş, E. *How many steps still left to x^*?*, SIAM Rev. 63(3) (2021), 585–624.

[3] Cătinaş, E. *A survey on the high convergence orders and computational convergence orders of sequences*, Appl. Math. Comput., 343 (2019), 1–20.

[4] Cheney, E.W. and Kincaid, D.R. *Numerical analysis: mathematics of scientific computing*, 1996: Brooks/Cole Publ.

[5] Chun, C. and Neta, B. *Comparison of several families of optimal eighth order methods*, Appl. Math. Comput., 274 (2016), 762–773.

[6] Dong, C. *A family of multiopoint iterative functions for finding multiple roots of equations*, Int. J. Comput. Math. 21(3-4) (1987), 363–367.

[7] Ferngndez-Torres, G. and Vgsquez-Aquino, J. *Three new optimal fourth-order iterative methods to solve nonlinear equations*, Adv. Numer. Anal. 2013, Art. ID 957496, 8 pp.

[8] Gander, W. *On Halley's iteration method*, Amer. Math. Monthly, 92(2) (1985), 131–134.

[9] Grau, M. and Daaz-Barrero, J.L. *An improvement to Ostrowski root-finding method*, Appl. Math. Comput., 173(1) (2006), 450–456.

[10] Gutiérrez, J.M., Magreg M.A. and Varona, J.L. *The "Gauss-Seidelization" of iterative methods for solving nonlinear equations in the complex plane*, Appl. Math. Comput., 218(6) (2011), 2467–2479.

[11] Hansen, E. and Patrick, M. *A family of root finding methods*, Numer. Math. 27(3) (1976/77), 257–269.

[12] Jarratt, P. *Some fourth order multipoint iterative methods for solving equations*, Math. Comput., 20(95) (1966), 434–437.

[13] King, R.F. *A family of fourth order methods for nonlinear equations*, SIAM J. Numer. Anal. 10(5) (1973), 876–879.

[14] Kou, J., Li, Y. and Wang, X. *A composite fourth-order iterative method for solving non-linear equations*, Appl. Math. Comput., 184(2) (2007), 471–475.

[15] Kung, H. and Traub, J.F.*Optimal order of one-point and multipoint iteration*, J. Assoc. Comput. Mach. 21(4) (1974), 643–651.

[16] Nedzhibov, G.H. and Petkov, M.G. *On a family of iterative methods for simultaneous extraction of all roots of algebraic polynomial*, Appl. Math. Comput., 162(1) (2005), 427–433.

[17] Neta, B., Chun, C. and Scott, M.*Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations*, Appl. Math. Comput., 227 (2014), 567–592.

[18] Noor, M.A., Waseem, M., Noor, K.I., Ali, M.A. *New iterative technique for solving nonlinear equations*, Appl. Math. Comput., 256 (2015), 1115–1125.

[19] Noor, M.A., Ahmad, F. and Javeed, S. *Two-step iterative methods for nonlinear equations*, Appl. Math. Comput., 181(2) (2006), 1068–1075.

[20] Ortega, J.M. and Rheinboldt, W.C. *Iterative solution of nonlinear equations in several variables*, Reprint of the 1970 original. Classics in Applied Mathematics, 30. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

[21] Osada, N. *An optimal multiple root-finding method of order three*, J. Comput. Appl. Math. 51(1) (1994), 131–133.

[22] Ostrowski, A.M. *Solution of Equations and Systems of Equations*, Pure and Applied Mathematics, Vol. IX. Academic Press, New York-London, 1960 ix+202 pp.

[23] Potra, F. and Ptak, V. *Nondiscrete induction and iterative processes*, Research Notes in Mathematics, 103. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[24] Shah, F.A. and Noor, M.A. *Some numerical methods for solving nonlinear equations by using decomposition technique*, Appl. Math. Comput., 251 (2015), 378–386.

[25] Sharma, J.R. and Guha, R.K. *A family of modified Ostrowski methods with accelerated sixth order convergence*, Appl. Math. Comput., 190(1) (2007), 111–115.

[26] Traub, J. *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[27] Varona, J.L. *Graphic and numerical comparison between iterative methods*, Math. Intell. 24(1) (2002), 37–47.

[28] Yun, J.H. *A note on three-step iterative method for nonlinear equations*, Appl. Math. Comput., 202(1) (2008), 401–405.

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