



Stability and sensitivity analysis of a nonlinear compartmental model for banking risk contagion

S. Ibnoukady*, , A. Meskaf and K. Allali 

Abstract

In this work, we study a mathematical model that describes the dynamics of a nonlinear financial system and explains how systemic risk propagates within a banking system. The proposed model is formulated as a system of differential equations structured into five compartments: Unexposed, exposed, distressed, recovered, and liquidated. Two mechanisms are incorporated: A direct early-intervention pathway that allows distressed banks to return directly to the unexposed state and a reintegration channel from the liquidated state to the unexposed state. Using the next-generation matrix approach, we derive the risk reproduction number and establish the system's stability properties. We analyze the equilibrium points and prove the global asymptotic stability of the risk-free equilibrium via a Lyapunov function and the local and global asymptotic stability of the persistent-risk equilibrium via the Routh–Hurwitz criterion and a Lyapunov function. Finally, the numerical simulations and the LHS-PRCC sensitivity

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analysis illustrate the effects of contagion, intervention, and reintegration on the dynamics of the system.

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1 Introduction

The financial system is highly interconnected and remains fragile. It is therefore important to study how systemic risk spreads in order to preserve overall stability [29, 30]. Past crises, such as the collapse of Lehman Brothers in 2008, have shown that a local problem can trigger an international chain reaction, disrupt credit markets, and lead to a sharp economic slowdown [4, 25]. The failure of Silicon Valley Bank in 2023 also showed that this problem still exists despite the reforms introduced after previous crises [18]. These events show the vulnerability of banking systems, where financial difficulties can spread through the links and contracts between banks [1]. For this reason, we need reliable tools to describe the evolution of financial contagion [10].

In recent years, mathematical models have helped us understand how risk spreads between banks [30, 2, 17, 26]. These studies show that because bank balance sheets are linked, the failure of one institution can weaken the whole financial system. However, some questions remain about the effects of intervention measures and the ability of banks to recover after a shock [19, 5]. It is therefore useful to develop models that describe these mechanisms more clearly [24, 8].

Systemic risk has received growing attention in recent financial studies. Financial network models explain how financial shocks spread through interbank links and central clearing [10, 29]. Banks are also linked through their balance sheets. Losses can spread from one bank to another through direct exposures, common asset holdings, and fire sales [15]. Macroprudential policies can help reduce systemic risk. However, their effects depend on the measure used and the characteristics of banks [28]. In this work, the proposed compartmental model describes how banks move from one financial state to another. It also studies the effects of early intervention and reintegration.

In this work, we introduce a nonlinear compartmental model for the spread of systemic risk in a banking system. Banks are divided into five compartments: Unexposed (U), Exposed (E), Distressed (D), Recovered (R), and Liquidated (L). The models of Irakoze et al. [16] and Ait Tamerz, Allali, and Meskaf [2] are also based on these five UEDRL compartments. The first

study uses an integer-order system, while the second uses a fractional-order system. Table 1 compares the main contributions of these models and the present study.

Table 1: Comparison of the UEDRL models.

Study	Model	Mathematical contribution	Financial contribution
Irakoze et al. (2023) [16]	Integer-order UEDRL model.	Analysis of the equilibria, their stability, and numerical simulations.	Describes the spread and persistence of systemic risk in a banking system.
Ait Tamerz, Allali, and Meskaf (2024) [2]	Fractional-order UEDRL model.	Extends the UEDRL model using the Caputo derivative and studies its stability.	Takes into account the effect of past financial conditions on systemic risk.
Present study	Integer-order UEDRL model with the transitions $D \rightarrow U$ and $L \rightarrow U$.	Studies the effects of early intervention and reintegration on the risk reproduction number, the equilibria, and system stability.	Represents early intervention and the reintegration of viable banking activities.

The mathematical tools used in this study are standard, but their application depends on the specific structure of our model. We first prove the existence, positivity, and boundedness of the solutions and calculate the risk reproduction number R_0 using the next-generation matrix method [35]. Since the total number of banks is conserved, the Jacobian matrix of the full system has a zero eigenvalue. We therefore remove one variable and study stability on a four-dimensional invariant set. At the persistent-risk equilibrium, the reduced system gives a fourth-degree characteristic polynomial, and the Routh–Hurwitz conditions are verified using the equilibrium relations. We also study the equilibrium points using Lyapunov functions and LaSalle’s invariance principle, where the Lyapunov functions are chosen according to the transfers between the compartments and the return paths to U [22, 31, 34]. Finally, numerical simulations are used to study the effects of early intervention and reintegration on the dynamics of the banking system.

The paper is organized as follows: Section 2 provides the formulation of the model and parameters. Section 3 describes the basic properties, including positivity and boundedness of solutions. Section 4 defines the risk reproduction number R_0 in order to determine whether the crisis disappears or persists. Section 5 focuses on the stability analysis of the financial equilibria. Numerical simulations are presented in Section 6. Section 7 provides the sensitivity analysis. Finally, Section 8 concludes the paper.

2 Mathematical model formulation

This section introduces the mathematical framework used to represent the propagation of financial distress within an interconnected banking system. Institutions are divided into compartments according to their financial condition, which makes it possible to follow their transition from one state to another over time.

According to these assumptions, the evolution of the banking system is described by the following nonlinear system:

$$\begin{cases} \frac{dU}{dt} = -\beta UD + \mu E + \theta R + \delta_3 D + \eta L, \\ \frac{dE}{dt} = \beta UD - (\mu + \gamma)E, \\ \frac{dD}{dt} = \gamma E - (\delta_1 + \delta_2 + \delta_3)D, \\ \frac{dR}{dt} = \delta_1 D - \theta R, \\ \frac{dL}{dt} = \delta_2 D - \eta L. \end{cases} \quad (1)$$

In system (1), exposed banks return to the unexposed class at rate μ or become distressed at rate γ . Distressed banks move to the recovered class at rate δ_1 or to the liquidated class at rate δ_2 . Recovered banks return to the unexposed class at rate θ . The parameters δ_3 and η represent the direct intervention and reintegration mechanisms, respectively. The term βUD represents a mass-action contagion mechanism. From an economic perspective, distressed banks can transmit losses to initially unexposed banks through interbank exposures, common asset holdings, and fire sales [8, 1, 15]. The product UD approximates the number of potential interactions between these two groups, while β represents their average intensity and the probability that such an interaction produces a new exposure. This formulation is therefore suitable as a first approximation for the closed and homogeneous banking system considered here. The compartments U , E , D , R , and L represent unexposed, exposed, distressed, recovered, and liquidated banks, respectively. Liquidated banks may return after acquisition or reorganization [16].

The interactions between these compartments are illustrated in Figure 1. The diagram highlights the main distress and recovery pathways, including the direct intervention mechanism δ_3 and the reintegration mechanism η .

All parameters are assumed to be strictly positive. Table 2 summarises their economic and mathematical roles [13, 30].

The parameters δ_3 and η represent two mechanisms used in banking crisis management. The parameter δ_3 represents early intervention, such as recapitalization, debt restructuring, or

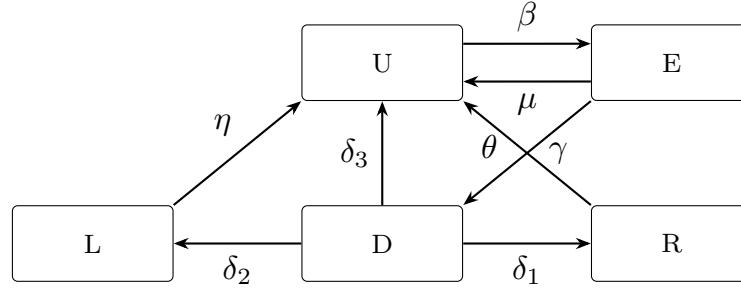


Figure 1: Compartmental structure of the UEDRL banking contagion model.

Table 2: Interpretation of the model parameters.

Parameter	Interpretation
β	Contagion intensity between unexposed and distressed banks, associated with the nonlinear term βUD .
μ	Recovery rate of exposed banks returning directly to the unexposed class.
γ	Deterioration rate of exposed banks moving to the distressed class.
δ_1	Restructuring or resolution rate of distressed banks moving to the recovered class.
δ_2	Liquidation rate of distressed banks temporarily leaving the active system.
δ_3	Direct intervention rate allowing distressed banks to return immediately to the unexposed class.
θ	Reintegration rate of recovered banks into the unexposed class.
η	Reintegration rate of viable banking activities after acquisition or reorganization

a change in management, which may help a distressed bank return from D to U [3]. The parameter η represents the transfer of viable activities from a failed bank to another bank or a bridge bank [32]. Thus, the transition from L to U represents the reintegration of these activities, not the return of the same liquidated bank.

The two new parameters do not have the same mathematical effect. The parameter δ_3 appears in the risk reproduction number because it increases the exit rate from the distressed compartment. The parameter η does not appear in this number, but it makes the liquidated compartment nonterminal and changes the persistent-risk equilibrium.

We assume a closed banking system where the total number of institutions remains constant:

$$N(t) = U(t) + E(t) + D(t) + R(t) + L(t) = N_0. \quad (2)$$

Indeed, by summing the five equations of system (1), we obtain

$$\frac{dN}{dt} = \frac{dU}{dt} + \frac{dE}{dt} + \frac{dD}{dt} + \frac{dR}{dt} + \frac{dL}{dt} = 0. \quad (3)$$

Thus, $N(t) = N(0) = N_0$, which confirms that the total number of banks remains constant.

Finally, defining the state vector

$$X(t) = (U(t), E(t), D(t), R(t), L(t))^{\top},$$

system (1) can be written compactly as

$$\frac{dX}{dt} = F(X),$$

where F is a nonlinear vector field. Since the right-hand side of the system is polynomial in the state variables, it is locally Lipschitz. This property ensures the local existence and uniqueness of solutions and provides the basis for the qualitative analysis developed in the following section [35].

The deterministic model describes the average movement of banks and is used to calculate the risk reproduction number. It assumes that banks are similar and interact in the same way. The term βUD represents a continuous propagation of risk. However, it does not represent sudden changes caused by large losses, fire sales, or the failure of a highly connected bank. Differences in bank size, network structure, and delays in risk transmission are also not included. These aspects may be considered in future studies by using other incidence functions, banking networks, or sudden changes in the model parameters.

3 Basic properties of the model

In this section, we establish the main qualitative properties of system (1). These properties are important because all state variables represent numbers of banks and must therefore remain nonnegative and bounded for all time [33, 34].

Let

$$X(t) = (U(t), E(t), D(t), R(t), L(t))^{\top} \in \mathbb{R}_+^5$$

be the solution of system (1). All parameters are assumed to be strictly positive.

3.1 Local existence and uniqueness

Let $F : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ be the vector field defined by the right-hand side of system (1). Since each component of F is a polynomial function of (U, E, D, R, L) , we have

$$F \in C^\infty(\mathbb{R}^5).$$

Hence, F is locally Lipschitz on $\mathbb{R}^5$. By the classical Cauchy–Lipschitz theorem [7, 21], for any nonnegative initial condition

$$X(0) = (U(0), E(0), D(0), R(0), L(0))^\top \in \mathbb{R}_+^5,$$

there exists a time $T > 0$ and a unique local solution of system (1) on $[0, T)$.

3.2 Positivity of solutions

Since the variables describe banking institutions, it is necessary to show that the nonnegative orthant is positively invariant.

Proposition 1. If $X(0) \in \mathbb{R}_+^5$, then the solution of system (1) satisfies

$$X(t) \in \mathbb{R}_+^5, \quad \text{for all } t \geq 0.$$

Proof. We verify the sign of the vector field on the boundary of the nonnegative orthant. On each boundary face, we obtain

$$\left\{ \begin{array}{l} \frac{dU}{dt} \Big|_{U=0} = \mu E + \theta R + \delta_3 D + \eta L \geq 0, \\ \frac{dE}{dt} \Big|_{E=0} = \beta U D \geq 0, \\ \frac{dD}{dt} \Big|_{D=0} = \gamma E \geq 0, \\ \frac{dR}{dt} \Big|_{R=0} = \delta_1 D \geq 0, \\ \frac{dL}{dt} \Big|_{L=0} = \delta_2 D \geq 0. \end{array} \right.$$

Thus, the vector field points inward or is tangent to the boundary of $\mathbb{R}_+^5$. Therefore, no solution starting from a nonnegative initial condition can leave the nonnegative orthant. Hence, $\mathbb{R}_+^5$ is positively invariant [27, 34]. $\square$

3.3 Conservation and boundedness

From the conservation relation established above, we have

$$N(t) = N(0) = N_0, \quad \text{for all } t \geq 0. \quad (4)$$

This conservation property is standard in closed compartmental systems and is used to define a bounded feasible region for the model [34, 21, 14]. Since all components are nonnegative, it follows that

$$0 \leq U(t), E(t), D(t), R(t), L(t) \leq N_0, \quad \text{for all } t \geq 0. \quad (5)$$

Thus, all solutions remain bounded for all time.

3.4 Invariant feasible region

From the positivity and conservation properties, the natural feasible region of the model is

$$\Omega = \{(U, E, D, R, L) \in \mathbb{R}_+^5 : U + E + D + R + L = N_0\}. \quad (6)$$

If $X(0) \in \Omega$, then positivity gives $X(t) \in \mathbb{R}_+^5$, and the conservation law (4) gives

$$U(t) + E(t) + D(t) + R(t) + L(t) = N_0.$$

Hence,

$$X(t) \in \Omega, \quad \text{for all } t \geq 0.$$

Therefore, Ω is positively invariant for system (1). This follows from the standard invariance arguments used for positive dynamical systems on closed and bounded feasible regions [33, 34].

3.5 Global existence

Since Ω is closed and bounded in $\mathbb{R}^5$, it is compact. Moreover, the vector field F is locally Lipschitz and all solutions starting in Ω remain bounded in Ω . Hence, no finite-time blow-up can occur, and the local solution can be extended to all $t \geq 0$ [7, 34].

Proposition 2. For every initial condition $X(0) \in \Omega$, system (1) admits a unique global solution satisfying

$$X(t) \in \Omega, \quad \text{for all } t \geq 0.$$

4 Equilibria and risk reproduction number

In this section, we determine the equilibrium points of system (1) and derive the risk reproduction number R_0 . This quantity plays an important role in the analysis of financial contagion, since it allows us to distinguish between the disappearance and the persistence of systemic distress.

4.1 Risk-free equilibrium

The risk-free equilibrium corresponds to the situation where no bank is exposed or distressed. In this case,

$$E = 0, \quad D = 0.$$

At equilibrium, from the equations of R and L , we obtain

$$-\theta R = 0, \quad -\eta L = 0.$$

Since $\theta > 0$ and $\eta > 0$, it follows that

$$R = 0, \quad L = 0.$$

Using the conservation law

$$U + E + D + R + L = N_0,$$

we get $U = N_0$. Therefore, the system admits the following risk-free equilibrium:

$$X_0 = (N_0, 0, 0, 0, 0). \tag{7}$$

4.2 Persistent-risk equilibrium

A persistent-risk equilibrium is an equilibrium point at which financial distress remains present in the system. We denote it by

$$X^* = (U^*, E^*, D^*, R^*, L^*),$$

with

$$E^* > 0, \quad D^* > 0.$$

At equilibrium, system (1) gives

$$0 = -\beta U^* D^* + \mu E^* + \theta R^* + \delta_3 D^* + \eta L^*, \quad (8)$$

$$0 = \beta U^* D^* - (\mu + \gamma) E^*, \quad (9)$$

$$0 = \gamma E^* - (\delta_1 + \delta_2 + \delta_3) D^*, \quad (10)$$

$$0 = \delta_1 D^* - \theta R^*, \quad (11)$$

$$0 = \delta_2 D^* - \eta L^*. \quad (12)$$

From (10), (11), and (12), we obtain

$$E^* = \frac{\delta_1 + \delta_2 + \delta_3}{\gamma} D^*, \quad R^* = \frac{\delta_1}{\theta} D^*, \quad L^* = \frac{\delta_2}{\eta} D^*. \quad (13)$$

Moreover, from (9), since $D^* > 0$, we get

$$U^* = \frac{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}{\beta\gamma}. \quad (14)$$

Using the conservation law, we have

$$U^* + E^* + D^* + R^* + L^* = N_0.$$

Substituting (13) gives

$$D^* \left(1 + \frac{\delta_1 + \delta_2 + \delta_3}{\gamma} + \frac{\delta_1}{\theta} + \frac{\delta_2}{\eta} \right) = N_0 - U^*. \quad (15)$$

Hence,

$$D^* > 0 \iff N_0 > U^*.$$

Using (14), this condition becomes

$$\frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)} > 1. \quad (16)$$

This motivates the definition of the risk reproduction number R_0 [9, 35].

4.3 Derivation of the risk reproduction number

To derive R_0 , we use the next-generation matrix approach applied to the exposed and distressed compartments [9, 35]. Around the risk-free equilibrium $X_0 = (N_0, 0, 0, 0, 0)$, the subsystem associated with E and D can be written as

$$\frac{d}{dt} \begin{pmatrix} E \\ D \end{pmatrix} = \mathcal{F}(E, D) - \mathcal{V}(E, D),$$

where

$$\mathcal{F}(E, D) = \begin{pmatrix} \beta N_0 D \\ 0 \end{pmatrix}, \quad \mathcal{V}(E, D) = \begin{pmatrix} (\mu + \gamma)E \\ -\gamma E + (\delta_1 + \delta_2 + \delta_3)D \end{pmatrix}. \quad (17)$$

The corresponding Jacobian matrices at X_0 are

$$F = \begin{pmatrix} 0 & \beta N_0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu + \gamma & 0 \\ -\gamma & \delta_1 + \delta_2 + \delta_3 \end{pmatrix}. \quad (18)$$

Since

$$\det(V) = (\mu + \gamma)(\delta_1 + \delta_2 + \delta_3) > 0,$$

the matrix V is invertible and

$$V^{-1} = \frac{1}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)} \begin{pmatrix} \delta_1 + \delta_2 + \delta_3 & 0 \\ \gamma & \mu + \gamma \end{pmatrix}. \quad (19)$$

Thus, the next-generation matrix is

$$K = FV^{-1} = \frac{\beta N_0}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)} \begin{pmatrix} \gamma & \mu + \gamma \\ 0 & 0 \end{pmatrix}. \quad (20)$$

The risk reproduction number is defined as the spectral radius of K :

$$R_0 = \rho(K). \quad (21)$$

The eigenvalues of K are

$$\lambda_1 = \frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}, \quad \lambda_2 = 0.$$

Therefore,

$$R_0 = \frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}. \quad (22)$$

4.4 Existence of the persistent-risk equilibrium

From (16) and (22), the persistent-risk equilibrium exists if and only if

$$R_0 > 1.$$

More precisely, when $R_0 > 1$, the equilibrium is given by

$$X^* = (U^*, E^*, D^*, R^*, L^*), \quad (23)$$

where

$$U^* = \frac{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}{\beta \gamma}, \quad (24)$$

$$D^* = \frac{N_0 - U^*}{1 + \frac{\delta_1 + \delta_2 + \delta_3}{\gamma} + \frac{\delta_1}{\theta} + \frac{\delta_2}{\eta}}, \quad (25)$$

and

$$E^* = \frac{\delta_1 + \delta_2 + \delta_3}{\gamma} D^*, \quad R^* = \frac{\delta_1}{\theta} D^*, \quad L^* = \frac{\delta_2}{\eta} D^*. \quad (26)$$

Thus, the model has a unique risk-free equilibrium for all parameter values, and a unique persistent-risk equilibrium whenever $R_0 > 1$.

Economically, the condition $R_0 < 1$ means that the propagation of financial distress is not strong enough to persist in the network. In contrast, when $R_0 > 1$, distress can remain in the system and the banking network converges toward a state in which some institutions remain exposed or distressed.

5 Stability analysis

In this section, we study the stability properties of the equilibria obtained in Section 4 [20, 22, 31, 34]. We first analyse the local stability of the risk-free equilibrium. Then, we examine the persistent-risk equilibrium when $R_0 > 1$.

5.1 Local stability of the risk-free equilibrium

The risk-free equilibrium is given by

$$X_0 = (N_0, 0, 0, 0, 0).$$

The Jacobian matrix associated with system (1) is

$$J(X) = \begin{pmatrix} -\beta D & \mu & -\beta U + \delta_3 & \theta & \eta \\ \beta D & -(\mu + \gamma) & \beta U & 0 & 0 \\ 0 & \gamma & -(\delta_1 + \delta_2 + \delta_3) & 0 & 0 \\ 0 & 0 & \delta_1 & -\theta & 0 \\ 0 & 0 & \delta_2 & 0 & -\eta \end{pmatrix}.$$

Evaluating this matrix at X_0 , we obtain

$$J(X_0) = \begin{pmatrix} 0 & \mu & -\beta N_0 + \delta_3 & \theta & \eta \\ 0 & -(\mu + \gamma) & \beta N_0 & 0 & 0 \\ 0 & \gamma & -(\delta_1 + \delta_2 + \delta_3) & 0 & 0 \\ 0 & 0 & \delta_1 & -\theta & 0 \\ 0 & 0 & \delta_2 & 0 & -\eta \end{pmatrix}. \quad (27)$$

Since the total population is conserved, the stability on the feasible region Ω is governed by the exposed-distressed subsystem. The corresponding reduced matrix is

$$A = \begin{pmatrix} -(\mu + \gamma) & \beta N_0 \\ \gamma & -(\delta_1 + \delta_2 + \delta_3) \end{pmatrix}. \quad (28)$$

Its characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_0, \quad (29)$$

where

$$a_1 = (\mu + \gamma) + (\delta_1 + \delta_2 + \delta_3), \quad (30)$$

and

$$a_0 = (\mu + \gamma)(\delta_1 + \delta_2 + \delta_3) - \beta N_0 \gamma. \quad (31)$$

Since all parameters are strictly positive, we have $a_1 > 0$. Moreover,

$$a_0 > 0 \iff \frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)} < 1.$$

Using the expression of the risk reproduction number R_0 , this condition becomes

$$a_0 > 0 \iff R_0 < 1.$$

By the Routh–Hurwitz criterion for second-order polynomials [20], all roots of $P(\lambda)$ have negative real parts if and only if

$$a_1 > 0 \quad \text{and} \quad a_0 > 0.$$

Hence, the risk-free equilibrium is locally asymptotically stable when $R_0 < 1$.

If $R_0 > 1$, then $a_0 < 0$, and the matrix A has one positive eigenvalue. Therefore, the risk-free equilibrium becomes unstable.

Theorem 1. The risk-free equilibrium $X_0 = (N_0, 0, 0, 0, 0)$ of system (1) is locally asymptotically stable in Ω if $R_0 < 1$, and unstable if $R_0 > 1$.

Proof. The local dynamics near X_0 is determined by the linearized exposed-distressed subsystem (28). The characteristic polynomial of this subsystem is given by (29). Since $a_1 > 0$ and $a_0 > 0$ if and only if $R_0 < 1$, the Routh–Hurwitz criterion implies that all eigenvalues of the reduced subsystem have negative real parts. Therefore, the risk-free equilibrium is locally

asymptotically stable when $R_0 < 1$. If $R_0 > 1$, then $a_0 < 0$, which implies that one eigenvalue is positive. Hence, X_0 is unstable. $\square$

The previous result has a direct economic interpretation. When $R_0 < 1$, a small initial financial disturbance cannot spread through the network and the system returns to the risk-free state. When $R_0 > 1$, the initial disturbance is amplified, and the banking system may move toward a persistent-risk regime.

5.2 Local stability of the persistent-risk equilibrium

Assume that $R_0 > 1$. Then the persistent-risk equilibrium

$$X^* = (U^*, E^*, D^*, R^*, L^*)$$

exists in Ω and satisfies $E^* > 0$ and $D^* > 0$. Define

$$\alpha = \mu + \gamma, \quad \Delta = \delta_1 + \delta_2 + \delta_3, \quad q = \beta D^* > 0.$$

At equilibrium, we have

$$\beta \gamma U^* = \alpha \Delta. \quad (32)$$

Since the total number of banks is conserved, we use

$$U = N_0 - E - D - R - L$$

and study the reduced system in

$$Y = (E, D, R, L)^\top.$$

The Jacobian matrix of the reduced system at $Y^* = (E^*, D^*, R^*, L^*)^\top$ is

$$J_4(Y^*) = \begin{pmatrix} -(\alpha + q) & \frac{\alpha \Delta}{\gamma} & -q & -q & -q \\ \gamma & -\Delta & 0 & 0 & 0 \\ 0 & \delta_1 & -\theta & 0 & 0 \\ 0 & \delta_2 & 0 & -\eta & 0 \end{pmatrix}. \quad (33)$$

Its characteristic polynomial is

$$p(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \quad (34)$$

where

$$\begin{aligned} a_1 &= \alpha + q + \Delta + \theta + \eta, \\ a_2 &= \Delta\eta + \Delta q + \Delta\theta + \alpha\eta + \alpha\theta + \eta q + \eta\theta + \gamma q + q\theta, \\ a_3 &= \Delta\eta q + \Delta\eta\theta + \Delta q\theta + \alpha\eta\theta + \delta_1\gamma q + \delta_2\gamma q + \eta\gamma q + \eta q\theta + \gamma q\theta, \\ a_4 &= \Delta\eta q\theta + \delta_1\eta\gamma q + \delta_2\gamma q\theta + \eta\gamma q\theta. \end{aligned}$$

Theorem 2. If $R_0 > 1$, then the persistent-risk equilibrium X^* is locally asymptotically stable in Ω .

Proof. Since all parameters are strictly positive and $q > 0$, we have

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0.$$

For the fourth-order polynomial (34), the remaining Routh–Hurwitz conditions are

$$a_1a_2 - a_3 > 0 \quad (35)$$

and

$$a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0. \quad (36)$$

The complete algebraic verification of (35)–(36) is provided in Appendix A. Hence, all eigenvalues of $J_4(Y^*)$ have negative real parts. By the linearization theorem [20], Y^* is locally asymptotically stable for the reduced system. The conservation relation then implies that X^* is locally asymptotically stable in Ω . $\square$

From an economic point of view, in the persistent-risk regime, small changes around the equilibrium remain limited. Under the Routh–Hurwitz conditions, the system gradually returns to the same long-run risk level.

5.3 Global stability of the risk-free equilibrium

We now prove the global asymptotic stability of the risk-free equilibrium in the feasible region Ω when $R_0 < 1$ [22, 31, 34].

Theorem 3. If $R_0 < 1$, then the risk-free equilibrium

$$X_0 = (N_0, 0, 0, 0, 0)$$

is globally asymptotically stable in Ω .

Proof. Consider the Lyapunov function

$$V(E, D) = E + \frac{\mu + \gamma}{\gamma} D. \quad (37)$$

Clearly, $V(E, D) \geq 0$ for all $(E, D) \in \mathbb{R}_+^2$, and

$$V(E, D) = 0 \iff E = D = 0.$$

Differentiating V along the solutions of system (1), we obtain

$$\dot{V} = \dot{E} + \frac{\mu + \gamma}{\gamma} \dot{D}.$$

Using the equations of E and D , this gives

$$\dot{V} = \beta U D - (\mu + \gamma) E + \frac{\mu + \gamma}{\gamma} (\gamma E - (\delta_1 + \delta_2 + \delta_3) D).$$

Hence,

$$\dot{V} = D \left[\beta U - \frac{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}{\gamma} \right]. \quad (38)$$

Since $X(t) \in \Omega$, we have $U(t) \leq N_0$. Therefore,

$$\dot{V} \leq D \left[\beta N_0 - \frac{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}{\gamma} \right].$$

Using the definition of R_0 , we can write

$$\beta N_0 = R_0 \frac{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}{\gamma}.$$

Thus,

$$\dot{V} \leq \frac{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}{\gamma}(R_0 - 1)D. \quad (39)$$

If $R_0 < 1$, then $\dot{V} \leq 0$ in Ω , and $\dot{V} = 0$ implies $D = 0$. From the equation of D ,

$$\dot{D} = \gamma E - (\delta_1 + \delta_2 + \delta_3)D,$$

the condition $D = 0$ gives $\dot{D} = \gamma E$. For the set $\{D = 0\}$ to be invariant, we must also have $E = 0$.

On the largest invariant subset of $\{\dot{V} = 0\}$, we therefore have

$$E = D = 0.$$

Then the equations of R and L reduce to

$$\dot{R} = -\theta R, \quad \dot{L} = -\eta L,$$

which imply

$$R(t) \rightarrow 0, \quad L(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Finally, using the conservation law

$$U + E + D + R + L = N_0,$$

we obtain

$$U(t) \rightarrow N_0.$$

Therefore,

$$X(t) \rightarrow X_0 = (N_0, 0, 0, 0, 0) \quad \text{as } t \rightarrow \infty.$$

By LaSalle's invariance principle [22, 34], the risk-free equilibrium is globally asymptotically stable in Ω . $\square$

Economically, this result means that if $R_0 < 1$, financial distress cannot persist in the banking system. Even if some banks are initially exposed or distressed, the system returns to the risk-free state in the long run.

5.4 Global stability of the persistent-risk equilibrium

Assume that $R_0 > 1$, so that the persistent-risk equilibrium $X^* = (U^*, E^*, D^*, R^*, L^*)$ exists in Ω with $E^* > 0$ and $D^* > 0$. Since $N(t) = N_0$, we substitute $U = N_0 - E - D - R - L$ and work on the reduced system

$$\begin{cases} \dot{E} = \beta(N_0 - E - D - R - L)D - (\mu + \gamma)E, \\ \dot{D} = \gamma E - (\delta_1 + \delta_2 + \delta_3)D, \\ \dot{R} = \delta_1 D - \theta R, \\ \dot{L} = \delta_2 D - \eta L, \end{cases} \quad (40)$$

on $\Omega_4 = \{(E, D, R, L) \in \mathbb{R}_+^4 : E + D + R + L \leq N_0\}$.

Theorem 4. If $R_0 > 1$, then the persistent-risk equilibrium X^* is globally asymptotically stable in Ω .

Proof. Define the Lyapunov function [31]

$$\mathcal{V}(Y) = \left(E - E^* - E^* \ln \frac{E}{E^*}\right) + \frac{\mu + \gamma}{\gamma} \left(D - D^* - D^* \ln \frac{D}{D^*}\right). \quad (41)$$

Since $g(u) = u - 1 - \ln u \geq 0$ for all $u > 0$, we have $\mathcal{V} \geq 0$ with $\mathcal{V}(Y^*) = 0$. Differentiating along (40) and using the equilibrium identities $\beta U^* D^* = (\mu + \gamma)E^*$ and $\gamma E^* = (\delta_1 + \delta_2 + \delta_3)D^*$, we obtain after simplification

$$\dot{\mathcal{V}} = (\mu + \gamma)E^* \left[2 + ab - \frac{ab}{c} - \frac{c}{b} - b\right], \quad (42)$$

where $a = U/U^*$, $b = D/D^*$, $c = E/E^*$. By the AM–GM inequality,

$$\frac{ab}{c} + \frac{c}{b} \geq 2\sqrt{a} \quad \text{and} \quad b + \frac{1}{b} \geq 2,$$

and applying the standard inequality $g(u) \geq 0$ to each term in (42), one obtains [31]

$$2 + ab - \frac{ab}{c} - \frac{c}{b} - b \leq 0 \quad \text{for all } a, b, c > 0, \quad (43)$$

with equality if and only if $a = b = c = 1$, i.e. $U = U^*$, $D = D^*$, $E = E^*$. Hence $\dot{\mathcal{V}} \leq 0$ on Ω_4 .

On the set $\{\dot{\mathcal{V}} = 0\}$, we have $D = D^*$. Substituting into $\dot{R} = \delta_1 D - \theta R$ gives $R(t) \rightarrow R^*$ exponentially, and into $\dot{L} = \delta_2 D - \eta L$ gives $L(t) \rightarrow L^*$ exponentially. By LaSalle's invariance principle [22], every trajectory in Ω converges to X^* . $\square$

When $R_0 > 1$, the system moves toward the persistent-risk equilibrium for any initial condition. Increasing δ_3 above δ_3^* reduces R_0 below one and brings the system back to the risk-free state.

6 Numerical simulations

Numerical simulations are presented to support the analytical results obtained in the previous sections [21, 14, 16]. The objective is to illustrate the role of the risk reproduction number R_0 and to show the effect of the two financial mechanisms introduced in the model, namely the direct intervention parameter δ_3 and the reintegration parameter η .

The simulations are performed for system (1). Unless otherwise stated, the initial condition is

$$(U(0), E(0), D(0), R(0), L(0)) = (90, 5, 3, 1, 1),$$

with $N_0 = 100$. The parameter configurations used in the numerical tests are summarized in Table 3. The first scenario corresponds to $R_0 > 1$, while the second one corresponds to $R_0 < 1$. The last two scenarios are used to examine the effects of δ_3 and η .

Table 3: Numerical scenarios used in the simulations.

Scenario	Main change	Parameter values	R_0	Figure
Persistent-risk regime	High contagion	$\beta = 1.2/N_0$	10	Fig. 2
Risk-free regime	Low contagion	$\beta = 0.01/N_0$	0.083	Fig. 3
Direct intervention	Varying δ_3	$\delta_3 \in \{0.03, \dots, 0.15\}$	decreases with δ_3	Fig. 4
Reintegration	Varying η	$\eta > 0$	unchanged	Fig. 5

For the illustrative simulations, the baseline parameter values are

$$\gamma = 0.5, \quad \mu = 0.1, \quad \delta_1 = 0.05, \quad \delta_2 = 0.02,$$

$$\delta_3 = 0.03, \quad \theta = 0.1, \quad \eta = 0.05.$$

These illustrative parameter values are used unless otherwise specified. They were chosen to show the cases $R_0 > 1$ and $R_0 < 1$ and to study the effects of δ_3 and η . These values do not come from a specific banking data set. In future work, the model could be calibrated using data on bank capital, liquidity, nonperforming loans, and interbank exposures. These indicators

could be used to place each bank in one of the five compartments. The transition parameters could then be estimated from changes in the financial state of banks over time. The parameter β could be estimated by comparing the observed and simulated numbers of exposed and distressed banks. The calibrated model could then be tested over another period [11, 12].

A. Persistent-risk regime

We first consider the case

$$\beta = \frac{1.2}{N_0} = 0.012.$$

Then

$$\mu + \gamma = 0.6, \quad \delta_1 + \delta_2 + \delta_3 = 0.10.$$

Therefore,

$$R_0 = \frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)} = \frac{0.012 \times 100 \times 0.5}{0.6 \times 0.10} = 10.$$

Since $R_0 > 1$, the system is in the persistent-risk regime. The corresponding equilibrium is

$$X^* \approx (9.99, 8.57, 42.89, 21.47, 17.07).$$

Figure 2 shows the time evolution of the five compartments. After a short transition phase, the solution approaches the persistent-risk equilibrium. The distressed class stabilizes near

$$D^* \approx 42.89.$$

This confirms that, when $R_0 > 1$, financial distress remains present in the banking network.

The largest real part of the eigenvalues is approximately

$$\Re(\lambda_{\max}) \approx -0.059.$$

This value is negative, which supports the local stability result obtained from the Routh–Hurwitz criterion. The convergence is slow because this eigenvalue is close to zero.

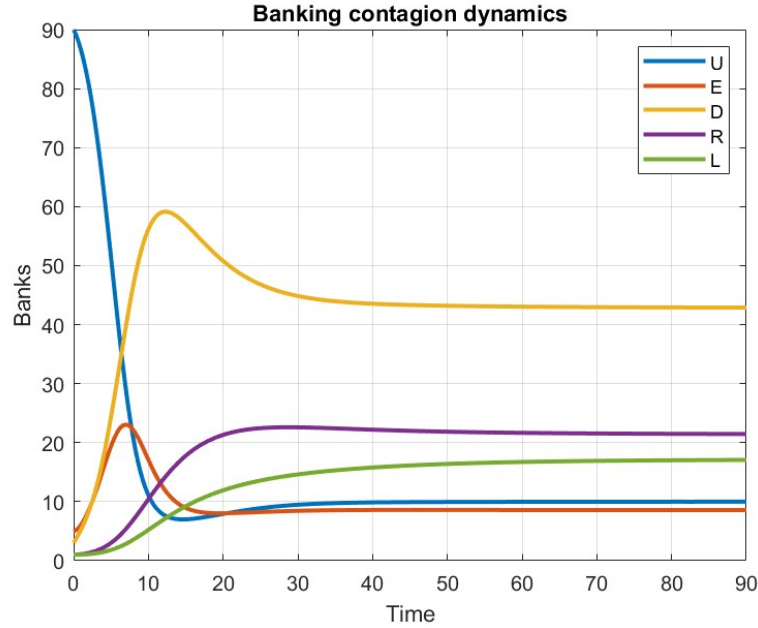


Figure 2: Time evolution of the banking system in the persistent-risk regime ($R_0 > 1$).

B. Risk-free regime

We now reduce the contagion intensity and take

$$\beta = \frac{0.01}{N_0} = 0.0001.$$

Keeping all other parameters unchanged, we obtain

$$R_0 = \frac{0.0001 \times 100 \times 0.5}{0.6 \times 0.10} \approx 0.083 < 1.$$

Thus, the system is in the risk-free regime. According to the global stability result proved in Section 5, the solution converges to the risk-free equilibrium.

Figure 3 confirms this result. The exposed and distressed classes decrease toward zero, while the unexposed class tends to N_0 . Numerically, the solution approaches

$$(99.94, 0, 0.002, 0.007, 0.051),$$

which is close to the risk-free equilibrium.

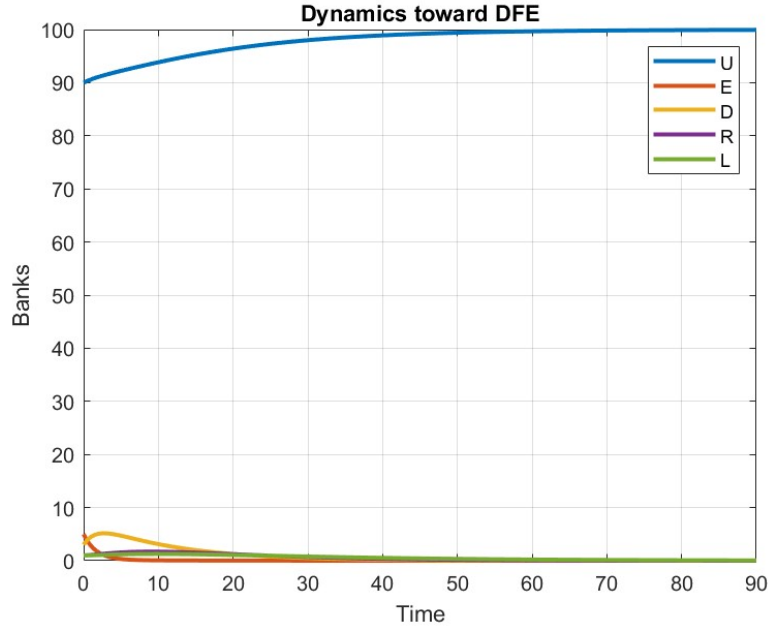


Figure 3: Time evolution of the banking system in the risk-free regime ($R_0 < 1$).

In this regime, the crisis remains limited. The maximum value of the distressed class is approximately

$$D_{\max} \approx 5.16,$$

after which $D(t)$ decreases toward zero. This agrees with the Lyapunov–LaSalle result established for $R_0 < 1$.

C. Effect of direct intervention

The parameter δ_3 represents the direct intervention mechanism from the distressed class to the healthy class. It appears in the denominator of the risk reproduction number:

$$R_0 = \frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}.$$

Hence, increasing δ_3 decreases R_0 and reduces the propagation of financial distress.

The critical value of δ_3 is obtained from $R_0 = 1$ as follows:

$$\delta_3^* = \frac{\beta N_0 \gamma}{\mu + \gamma} - (\delta_1 + \delta_2). \quad (44)$$

For the persistent-risk parameter set, we get

$$\delta_3^* = \frac{0.012 \times 100 \times 0.5}{0.6} - (0.05 + 0.02) = 0.93.$$

Thus, values of δ_3 below 0.93 keep the system in the persistent-risk regime, while values above this level move the system toward the risk-free regime.

Figure 4 shows the effect of δ_3 on the distressed compartment. When δ_3 increases, the level of distress decreases. In particular, the long-run value of $D(t)$ decreases from about 43 for $\delta_3 = 0.03$ to nearly 30 for $\delta_3 = 0.15$. This confirms the stabilizing role of direct intervention.

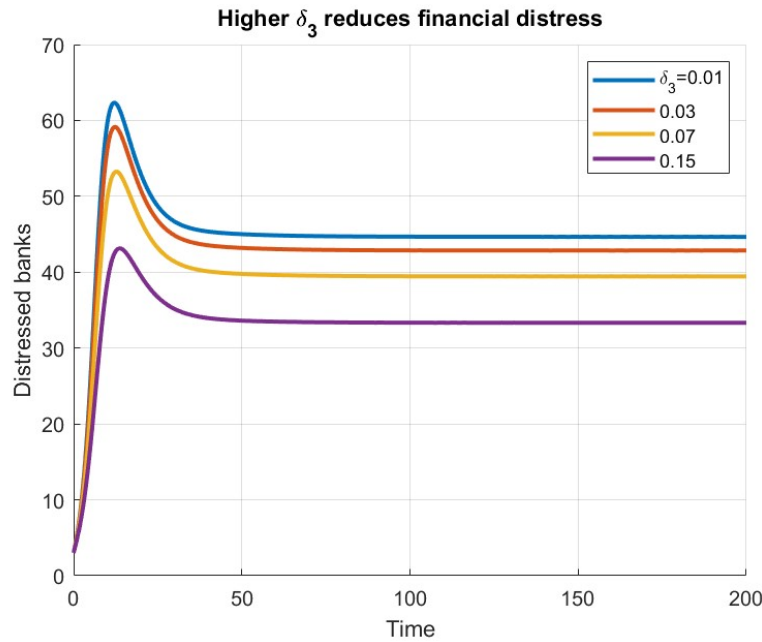


Figure 4: Effect of the direct intervention parameter δ_3 on the distressed compartment.

From a financial point of view, increasing δ_3 means that regulators intervene before the distress spreads further in the network. This mechanism reduces systemic risk directly, because it acts on R_0 .

D. Effect of reintegration from liquidation

The parameter η represents the reintegration of liquidated institutions into the unexposed class. Unlike δ_3 , it does not appear in the expression of R_0 . Therefore, η does not directly determine whether the system is in the risk-free or persistent-risk regime. However, it affects the final distribution of banks.

From the equilibrium equations, we have

$$L^* = \frac{\delta_2}{\eta} D^*.$$

This relation shows that increasing η reduces the long-term number of liquidated institutions.

Figure 5 shows the effect of η on the liquidated class. When η increases, the liquidated compartment decreases faster and reaches a lower final level. For example, when $\eta = 0.05$ and $D^* \approx 42.89$, we obtain

$$L^* \approx \frac{0.02}{0.05} \times 42.89 \approx 17.16,$$

which is consistent with the numerical equilibrium.

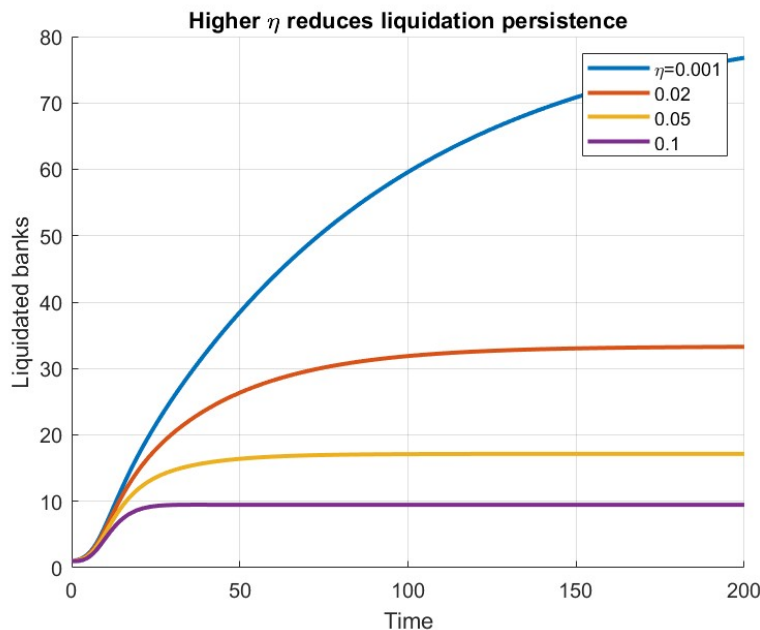


Figure 5: Effect of the reintegration parameter η on the liquidated compartment.

This result shows that η mainly improves the reintegration process after liquidation. It reduces the number of liquidated banks and accelerates reintegration, but it does not reduce R_0 directly.

E. Discussion

The numerical results confirm the analytical role of R_0 . When $R_0 < 1$, the exposed and distressed compartments vanish and the system approaches the risk-free equilibrium. When $R_0 > 1$, the system approaches a persistent-risk state in which distress remains present.

The parameter δ_3 acts directly on R_0 , whereas η improves reintegration without changing R_0 . In practice, a higher value of δ_3 may represent faster and more effective early supervisory actions, such as the activation of recovery plans, recapitalization, restrictions on distributions, reductions in risky exposures, or changes in management [3]. These actions could be triggered by deteriorating capital, liquidity, asset-quality, or interbank-exposure indicators. The parameter η may instead represent the transfer of viable activities to an acquiring institution or a bridge bank [32]. Thus, the model represents the combined speed and effectiveness of regulatory intervention rather than prescribing a single policy instrument. These simulations confirm the analytical results but do not constitute an empirical validation against an observed banking crisis. The model assumes that all banks have the same size and interact in the same way. However, large banks and highly connected banks may have a stronger effect on the spread of financial risk. This is related to the “Too Big to Fail” problem, since the distress of a systemically important bank may affect a large part of the banking system. The present model describes the average behavior of banks and does not distinguish systemically important institutions. Bank size and interbank connections may be included in future work [1].

7 Local and global sensitivity analysis

In this section, we study the sensitivity of the risk reproduction number R_0 with respect to the model parameters. This analysis helps identify the parameters that have the strongest influence on the propagation of systemic risk in the banking network.

Following [6], the normalized sensitivity index of R_0 with respect to a parameter p is defined by

$$S_p = \frac{\partial R_0}{\partial p} \frac{p}{R_0}. \quad (45)$$

Recall that

$$R_0 = \frac{\beta N_0 \gamma}{(\mu + \gamma)(\delta_1 + \delta_2 + \delta_3)}. \quad (46)$$

By direct differentiation, we obtain

$$\begin{aligned} S_\beta &= 1, & S_{N_0} &= 1, \\ S_\gamma &= \frac{\mu}{\mu + \gamma}, & S_\mu &= -\frac{\mu}{\mu + \gamma}, \\ S_{\delta_j} &= -\frac{\delta_j}{\delta_1 + \delta_2 + \delta_3}, & j &= 1, 2, 3. \end{aligned} \quad (47)$$

The local sensitivity analysis considers one parameter at a time. To obtain a more complete comparison, we also performed a global sensitivity analysis using Latin hypercube sampling (LHS) and partial rank correlation coefficients (PRCCs), following Marino et al. [23]. The seven parameters appearing in R_0 were varied simultaneously and independently within $\pm 20\%$ of their reference values. A total of 5000 parameter sets were generated. The results are summarized in Table 4.

Table 4: Local and global sensitivity results for the risk reproduction number R_0 .

Parameter	Reference value	Local index	PRCC	p -value
β	0.012	1.0000	0.9653	< 0.001
N_0	100	1.0000	0.9651	< 0.001
γ	0.5	0.1667	0.5165	< 0.001
μ	0.1	-0.1667	-0.4976	< 0.001
δ_1	0.05	-0.5000	-0.8710	< 0.001
δ_2	0.02	-0.2000	-0.5751	< 0.001
δ_3	0.03	-0.3000	-0.7178	< 0.001

Figure 6 presents the PRCC values obtained from the global sensitivity analysis.

The local and global analyses give the same directions of influence. The parameters β , N_0 , and γ have positive effects on R_0 , whereas μ , δ_1 , δ_2 , and δ_3 have negative effects. All PRCC values are statistically significant at $p < 0.001$.

The parameters β and N_0 have the strongest positive influence, with PRCC values equal to 0.9653 and 0.9651, respectively. This shows that an increase in contagion intensity or in the size

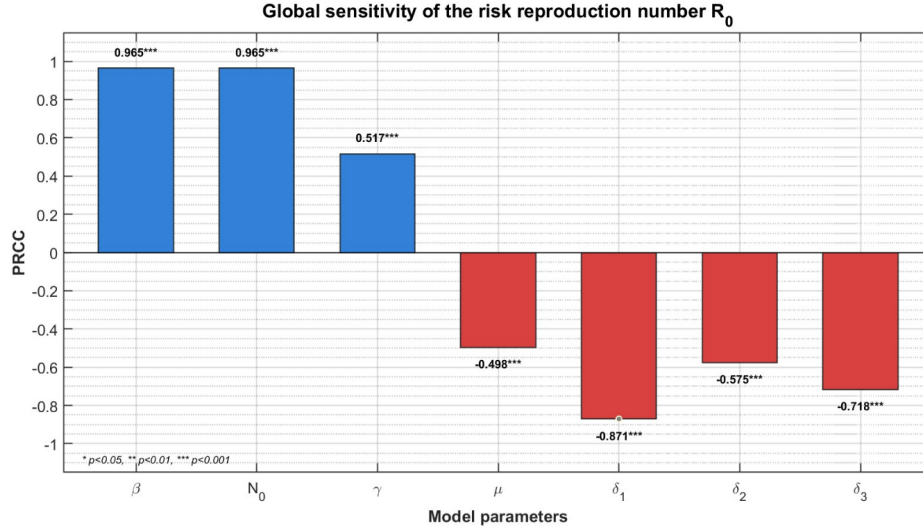


Figure 6: PRCCs of the risk reproduction number R_0 obtained from 5000 Latin hypercube samples. The parameters were varied independently within $\pm 20\%$ of their reference values. All PRCCs are statistically significant ($p < 0.001$).

of the banking system strongly increases the propagation of systemic risk. The parameter γ also increases R_0 , but its influence is smaller.

Among the parameters with negative effects, δ_1 has the largest absolute PRCC value, followed by δ_3 and δ_2 . Thus, these intervention mechanisms play an important role in reducing financial contagion. In particular, the negative PRCC of δ_3 confirms that direct early intervention can reduce the propagation of financial distress. The parameter μ also reduces R_0 , but it has the smallest global influence among the parameters considered.

Since the parameter distributions have not yet been estimated from real banking data, the intervals used in this analysis are exploratory. Future work will consider parameter ranges estimated from banking data and the global sensitivity of other model outputs.

8 Conclusion

Several mathematical models of financial contagion are inspired by epidemiological models, with some adjustments to represent the functioning of banking systems. In this work, we followed a different approach. We kept the compartmental structure but modified two points that we consider important. First, a distressed bank does not always follow the complete recovery process. Regulators can intervene early and return it to a sound condition. Second, liquidation is not

necessarily the final step, since viable banking activities may return to the system after acquisition or reorganization. These observations led us to introduce the parameters δ_3 and η into the UEDRL model.

The mathematical analysis allowed us to determine the role of the risk reproduction number in separating the two possible regimes. We proved that the risk-free equilibrium is globally asymptotically stable when $R_0 < 1$. When $R_0 > 1$, a persistent-risk equilibrium exists, and we established its local and global asymptotic stability. The numerical simulations confirm these analytical results. The local and global sensitivity analyses also show that the contagion rate and the size of the banking system have the strongest positive influence on R_0 , while the intervention parameters δ_1 , δ_2 , and δ_3 have negative effects. In particular, δ_3 acts directly on R_0 , whereas η changes the long-run distribution of banks without affecting this number.

From a regulatory perspective, δ_3 may represent the combined speed and effectiveness of early supervisory actions, such as the activation of recovery plans, recapitalization, restrictions on distributions, reductions in risky exposures, or changes in management. However, the model remains a simplified representation of a banking system. It assumes a closed and homogeneous population, mass-action contagion, constant parameters, and illustrative parameter values. It does not explicitly account for differences in bank size, heterogeneous interbank connections, or time-dependent regulatory decisions. Future work will therefore consider calibration with bank-level data, heterogeneous banking networks, time-dependent interventions, and empirical validation over different periods.

A Algebraic verification of the Routh–Hurwitz conditions

We verify here the two Routh–Hurwitz conditions used in Section 5.2. Let

$$s = \alpha + q + \Delta, \quad r = \theta + \eta, \quad z = \theta\eta,$$

and define

$$b = q(\Delta + \gamma), \quad c = \gamma q(\delta_1 + \delta_2), \quad d = \gamma q(\delta_1\eta + \delta_2\theta).$$

The coefficients of the characteristic polynomial can then be written as

$$a_1 = s + r, \quad a_2 = b + sr + z, \quad a_3 = br + sz + c, \quad a_4 = bz + d. \quad (48)$$

Indeed, the characteristic polynomial of the reduced Jacobian admits the following compact form:

$$\begin{aligned}
p(\lambda) &= (\lambda + \theta)(\lambda + \eta) [\lambda^2 + s\lambda + b] \\
&\quad + \gamma q [\delta_1(\lambda + \eta) + \delta_2(\lambda + \theta)].
\end{aligned} \tag{49}$$

Expanding (49) gives precisely the four coefficients in (48).

For the first Routh–Hurwitz determinant, we obtain

$$\begin{aligned}
a_1 a_2 - a_3 &= (s + r)(b + sr + z) - (br + sz + c) \\
&= sb + s^2 r + sr^2 + rz - c.
\end{aligned} \tag{50}$$

Using the definitions of b and c , this becomes

$$\begin{aligned}
a_1 a_2 - a_3 &= s^2 r + sr^2 + rz + sq\Delta \\
&\quad + \gamma q (s - \delta_1 - \delta_2).
\end{aligned} \tag{51}$$

Moreover,

$$s - \delta_1 - \delta_2 = \alpha + q + \delta_3 > 0.$$

Hence every term on the right-hand side of (51) is strictly positive, and therefore

$$a_1 a_2 - a_3 > 0.$$

For the second determinant, let

$$H_1 = a_1 a_2 - a_3.$$

Then

$$a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 = a_3 H_1 - a_1^2 a_4. \tag{52}$$

Substituting (48) and (50) into (52) gives

$$\begin{aligned}
H_2 &:= a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 \\
&= (br + sz + c)(sb + s^2 r + sr^2 + rz - c) \\
&\quad - (s + r)^2 (bz + d).
\end{aligned} \tag{53}$$

To verify the sign of H_2 , we substitute

$$s = \alpha + \Delta + q, \quad \Delta = \delta_1 + \delta_2 + \delta_3,$$

and collect the resulting expression according to the powers of q . This gives the exact decomposition

$$H_2 = P_0 + qP_1 + q^2P_2 + q^3P_3, \quad (54)$$

where

$$P_0 = \theta\eta(\theta + \eta)(\alpha + \Delta)(\alpha + \Delta + \theta)(\alpha + \Delta + \eta) > 0. \quad (55)$$

The coefficients P_1, P_2, P_3 are obtained by exact polynomial collection after using

$$\alpha = \mu + \gamma, \quad \Delta = \delta_1 + \delta_2 + \delta_3.$$

Their signs were checked without assigning numerical values to the parameters. For completeness, the exact symbolic verification is given below in SageMath syntax:

```
S.<q> = PolynomialRing(
    PolynomialRing(QQ,
        names=('mu','gamma','d1','d2','d3','theta','eta'))
)
mu,gamma,d1,d2,d3,theta,eta = S.base_ring().gens()

alpha = mu + gamma
Delta = d1 + d2 + d3
s = alpha + Delta + q
r = theta + eta
z = theta*eta

b = q*(Delta + gamma)
c = gamma*q*(d1 + d2)
d = gamma*q*(d1*eta + d2*theta)

a1 = s + r
a2 = b + s*r + z
a3 = b*r + s*z + c
a4 = b*z + d

H2 = a1*a2*a3 - a3^2 - a1^2*a4

for k in range(4):
    Pk = H2[k]
    assert all(v > 0 for v in Pk.dict().values())
```

After collection, P_1, P_2, P_3 contain, respectively, 169, 132, and 41 distinct monomials, and all their coefficients are strictly positive. Hence,

$$P_i > 0, \quad i = 0, 1, 2, 3.$$

Since $q = \beta D^* > 0$, relation (54) yields

$$a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0.$$

Therefore, the two remaining Routh–Hurwitz conditions are satisfied.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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