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Research Article

Mathematical modeling, global stability, sensitivity analysis, and optimal control of merchant adoption dynamics in e-commerce

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Abstract

In this paper, we introduce a novel mathematical model to analyze the dynamics of e-commerce adoption among merchants. The model captures the key factors influencing adoption behaviors and examines both the spread of e-commerce and resistance to its uptake.

Using the next-generation matrix method, we establish the model's well-posedness and derive the basic reproduction number R_0. Two equilibrium points are identified: The rejection

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equilibrium E_0, representing a state of nonadoption, and the widespread adoption equilibrium E_*, representing a state of large-scale e-commerce adoption. A global stability analysis, based on LaSalle's invariance principle and Lyapunov functions, demonstrates that E_0 is globally asymptotically stable if $R_0 \leq 1$, while E_* is globally asymptotically stable when $R_0 > 1$.

Furthermore, the model explores the drivers of e-commerce diffusion and factors contributing to resistance. We formulate the model as an optimal control problem to address these dynamics, introducing control variables aimed at minimizing resistance and promoting widespread adoption. Cost-effectiveness analysis is also performed to evaluate the efficiency of each intervention strategy.

Finally, numerical simulations provide insights into the effectiveness of various policy interventions, illustrating the strategies required to achieve optimal levels of e-commerce adoption.

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Keywords: Mathematical modeling; Stability analysis; E-commerce; Pontryagin maximum principle; Optimal control; Sensitivity analysis.

1 Introduction

E-commerce is widely recognized as a pivotal driver of economic growth at both national and international levels. It provides a dynamic platform for businesses to showcase and sell their products to both domestic and global markets, while offering consumers the convenience of purchasing goods and services from virtually any location at any time. As a result, the growth of e-commerce is expected to significantly reduce the time, energy, and effort required to complete transactions.

Beyond economic benefits, the expansion of e-commerce also yields broader societal impacts. For instance, the shift to digital transactions contributes to lower pollution levels by reducing the need for physical commuting, helps limit the spread of infectious diseases by minimizing in-person interactions, and decreases the frequency of traffic-related accidents by promoting remote shopping alternatives.

E-commerce refers to the buying, selling, and provision of goods and services over the Internet. This process is supported by a diverse range of digital technologies, including computers, smartphones, tablets, and, increasingly, wearable devices. These technologies collectively form a robust infrastructure that enables the seamless execution of commercial activities through online platforms, ensuring real-time communication, secure transactions, and rapid delivery services.

In recent years, the global proliferation of e-commerce has been nothing short of remarkable. However, effective participation in this digital economy requires both reliable communication infrastructure and a high level of proficiency in managing commercial websites. Depending on the nature of interaction, various e-commerce models have emerged, such as business-to-consumer (B2C), business-to-business (B2B), consumer-to-consumer (C2C), and others [6]. Each of these models addresses different segments of the market and plays a complementary role in the digital ecosystem.

The historical development of e-commerce has been shaped by several key milestones. One of the earliest milestones was the introduction of electronic data interchange in the 1960s, which allowed organizations to exchange business documents electronically. In 1969, CompuServe was founded, offering innovative services such as email and online chat that laid the groundwork for future online communication.

A major breakthrough came in 1979, when Michael Aldrich pioneered the concept of online purchasing using videotex technology, marking one of the first true implementations of electronic commerce. The 1990s then witnessed an exponential acceleration in the field: The launch of the Netscape Navigator browser in 1994 significantly enhanced web accessibility for the general public, while the founding of Amazon.com and eBay in 1995 revolutionized the way goods were bought and sold online.

Further important developments include Dell's implementation of direct online sales in 1996 and Alibaba's entry into the market in 1999, which positioned China as a major player in the global e-commerce landscape. More recently, in 2011, Facebook introduced targeted advertising based on user behavior, effectively integrating social media and e-commerce and giving rise to personalized digital marketing strategies [24].

These technological advancements, combined with changing consumer behavior and increased digital literacy, continue to shape the future trajectory of e-commerce as a dominant mode of global trade and interaction.

Related work

The modeling of infectious diseases provides a comprehensive and rigorous framework for analyzing pathogen transmission mechanisms, forecasting the future course of an epidemic, and evaluating strategies for epidemic control. Interestingly, e-commerce models share parallels with biological epidemiological models, as they elucidate transmission processes such as word-of-mouth

or advertising campaigns and the temporal evolution of states within a consumer population, including awareness, interest, and adoption.

Applying these principles to e-commerce allows for greater precision in simulating product diffusion, innovation adoption, and marketing campaign effects. For instance, modeling the spread of a new technological product can help predict its market penetration over time. This analytical approach not only enhances our understanding of the fundamental mechanisms driving market volatility but also enables more accurate prediction of consumer behavior in response to various stimuli.

Integrating epidemiological models into the mathematical and economic modeling of e-commerce presents an opportunity to optimize commercial strategies and anticipate trends in the rapidly evolving digital marketplace.

In [25], the authors modeled e-commerce systems using queuing theory to evaluate the efficiency of a company's transition from a traditional business model to a digital framework. They developed a system of differential equations to analyze key variables, including order flow, processing times, and operational costs. This methodological approach facilitates a precise assessment of the transition's profitability and cost efficiency by accounting for the complexities inherent in transactional dynamics. The study's findings provide valuable insights into the economic implications of digital transformation.

In [15], a compartmental epidemic model was proposed to analyze the propagation of rumours in e-commerce, examining the stability conditions of the equilibrium points. The epidemic threshold R_0 is critical in determining whether rumours become endemic or die out, thereby influencing consumer trust in online transactions.

Li [19] presented a dynamical clustering method for electronic commerce systems, leveraging optimization and game theory to identify and form clusters efficiently. A two-stage game-theoretic framework is used, where the equilibrium contributions of leaders are computed in a sequential-move setting, leading to an efficient optimization of cluster configurations. The proposed dynamical system guarantees rapid convergence with computational complexity nearing linear time for sparse networks, making it suitable for large-scale applications.

In [1], the authors examined how e-commerce policy can enhance inclusive development in the global economy. They highlighted the importance of improving formal institutions, fostering SME participation, and supporting less-developed countries in e-commerce.

Lv and Chen [21] established a time-delayed panic-spreading model with stability analysis and optimal control. They studied the local and global stability of equilibria and used optimal control to minimize the spread of panic through numerical simulations.

Those interested in exploring additional references in this field can refer to [11, 9, 10].

Problem statement

E-commerce has fundamentally transformed global trade; however, the dynamic behavior of merchants toward its adoption remains insufficiently explored in the existing literature. In particular, previous studies have not adequately addressed how merchants who refuse to use e-commerce may influence those who choose to adopt it. To address this gap, we introduce a mathematical model to analyze the dynamic behavior of merchants in relation to e-commerce adoption and rejection. The proposed model considers a compartmental framework comprising five categories: Merchants who are not currently engaged in e-commerce but may potentially adopt it; merchants who actively refuse to adopt e-commerce platforms; merchants who currently use e-commerce in their business activities; merchants who have temporarily discontinued their use of e-commerce; and merchants who have permanently ceased using e-commerce.

The model describes the transitions and interactions among these compartments, thereby providing insights into the factors influencing e-commerce adoption and rejection among merchants. We establish the well-posedness of the model and derive the basic reproduction number R_0 using the next-generation matrix method. Two equilibrium points are identified: The rejection-free equilibrium E_0, representing a state in which e-commerce rejection is absent, and the positive equilibrium E_*, representing the long-term coexistence of the different merchant behavioral states, including e-commerce adoption and rejection.

A global stability analysis is then conducted using Lyapunov functions and LaSalle's invariance principle. We establish that E_0 is globally asymptotically stable when $R_0 \leq 1$, whereas E_* is globally asymptotically stable when $R_0 > 1$. To promote e-commerce adoption and reduce merchant rejection, the model is subsequently formulated as an optimal control problem. Three time-dependent control strategies are introduced to reduce the number of merchants refusing e-commerce and to promote its adoption. The Pontryagin maximum principle is employed to characterize the corresponding optimal control strategies.

Numerical simulations are performed to evaluate the effectiveness of the proposed interventions and to illustrate the optimal strategies for enhancing e-commerce adoption while reducing merchant rejection.

Furthermore, Hassouni, El Bhih, and Balatif [13] presented an application of mathematical modeling and optimal control strategies to mitigate the spread of coronavirus, illustrating the versatility and effectiveness of control theory in epidemiological applications. Additional significant contributions to the field of control theory can be found in [7, 17, 2, 4, 16, 8].

The remainder of this paper is organized as follows. Section 2 introduces the mathematical model developed to analyze merchant behavior in the context of e-commerce. Section 3 presents the stability analysis of the equilibrium points, including both local and global stability. A sensi-

tivity analysis of the parameters appearing in the basic reproduction number is also conducted to identify the most influential factors affecting merchant behavior and to provide insights for the subsequent optimal control analysis. In Section 4, the optimal control problem is formulated, and the existence and characterization of the proposed controls are established. Section 5 presents numerical simulations of the model and its different compartments to evaluate the effectiveness of the proposed intervention strategies. Finally, concluding remarks and possible perspectives are presented in Section 6.

2 Formulation of the mathematical model

This study introduces a novel nonlinear compartmental model, referred to as the $SAWPD$ model, designed to capture the dynamics of e-commerce adoption and disengagement among merchants. The model categorizes the merchant population into five distinct compartments based on their behavioral state with respect to e-commerce. These compartments help to systematically analyze the transition of merchants from traditional commerce to e-commerce and, conversely, the potential regression or withdrawal due to various challenges.

2.1 Mathematical framework of the $SAWPD$ model

The $SAWPD$ model segments the total population of merchants into the following five compartments:

- **Compartment** S: This group includes traditional merchants who have not yet adopted e-commerce. Their reluctance typically stems from a lack of information, insufficient digital literacy, or unfamiliarity with online business models. These merchants represent the entry point into the system.
- **Compartment** A: This compartment consists of merchants who explicitly refuse to adopt e-commerce. Their refusal is often based on negative perceptions, fear of change, or skepticism about the benefits and reliability of digital platforms.
- **Compartment** W: Merchants in this group are active e-commerce users. They have successfully integrated digital tools into their business processes and use online platforms to sell their goods and services. This is the desired state that policy interventions aim to maximize.

- **Compartment** P: This group consists of merchants who have temporarily suspended their e-commerce activities. The suspension may be due to technical issues such as internet outages, platform maintenance, or unresolved user experience problems. These merchants are potentially recoverable with adequate support.

- **Compartment** D: This final compartment represents merchants who have permanently withdrawn from using e-commerce. This decision is typically the result of exposure to cybercrime, data breaches, or persistent technical and operational difficulties. These cases represent a permanent loss to the digital economy.

The transitions between these compartments are governed by a system of nonlinear differential equations, which incorporate key behavioral and technical factors influencing adoption and withdrawal. This model provides a robust analytical framework to study the effectiveness of awareness campaigns, cybersecurity initiatives, and technical support programs in fostering digital transformation among merchants.

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta S(t)A(t) - \alpha S(t) - \mu S(t), \\ \frac{dA(t)}{dt} = \beta S(t)A(t) - \gamma A(t) - \mu A(t), \\ \frac{dW(t)}{dt} = \alpha S(t) + \gamma A(t) + \theta P(t) - \eta W(t) - \delta W(t) - \mu W(t), \\ \frac{dP(t)}{dt} = \eta W(t) - \theta P(t) - \mu P(t), \\ \frac{dD(t)}{dt} = \delta W(t) - \mu D(t), \end{cases} \tag{1}$$

with the initial conditions

$$S(0) \geq 0, \quad A(0) \geq 0, \quad W(0) \geq 0, \quad P(0) \geq 0, \quad D(0) \geq 0,$$

and where $N(t)$ denotes the total number of merchants at time t, given by

$$N(t) = S(t) + A(t) + W(t) + P(t) + D(t).$$

Table 1 summarizes the key parameters used in the mathematical model for e-commerce adoption. Each parameter represents a specific behavioral or structural component influencing merchants' decisions regarding e-commerce uptake or refusal.

The structure and transitions of the proposed e-commerce adoption model are illustrated in Figure 1. The diagram shows the main compartments and the possible transitions between them, which are governed by the corresponding rates of adoption, refusal, recovery, reintegration, and exit.

Table 1: Model parameters for e-commerce adoption among merchants

Parameter	Description
Λ	The number of new merchants entering the market.
α	The proportion of merchants willing to adopt e-commerce.
γ	The proportion of merchants who initially rejected e-commerce but later transitioned to adopting it.
μ	The proportion of merchants who have permanently ceased trading activities.
θ	The proportion of merchants who resumed using e-commerce after overcoming previous challenges.
β	The proportion of merchants who transitioned to rejecting e-commerce due to negative perceptions influenced by external factors.
δ	The proportion of merchants who decided to discontinue using e-commerce as a result of exposure to cybercrime.
η	The proportion of merchants who temporarily halted their use of e-commerce due to encountering specific problems.

2.2 Model's fundamental characteristics

Theorem 1. If the initial conditions satisfy

$$S(0) \geq 0, \quad A(0) \geq 0, \quad W(0) \geq 0, \quad P(0) \geq 0, \quad D(0) \geq 0,$$

then the solutions $S(t)$, $A(t)$, $W(t)$, $P(t)$, and $D(t)$ of system (1) remain nonnegative for all $t \geq 0$.

Proof. From the first equation in system (1), we have

$$\frac{dS(t)}{dt} + \big(\beta A(t) + \alpha + \mu\big)S(t) \geq \Lambda.$$

Since $\Lambda \geq 0$, it follows that

$$\frac{dS(t)}{dt} + \big(\beta A(t) + \alpha + \mu\big)S(t) \geq 0.$$

Multiplying both sides by the integrating factor

$$\exp\left(\int_0^t (\beta A(h) + \alpha + \mu)dh\right),$$

we get

$$\frac{d}{dt}\left(S(t)\exp\left(\int_0^t (\beta A(h) + \alpha + \mu)dh\right)\right) \geq 0, \quad \text{for all } t \geq 0.$$

Integrating yields

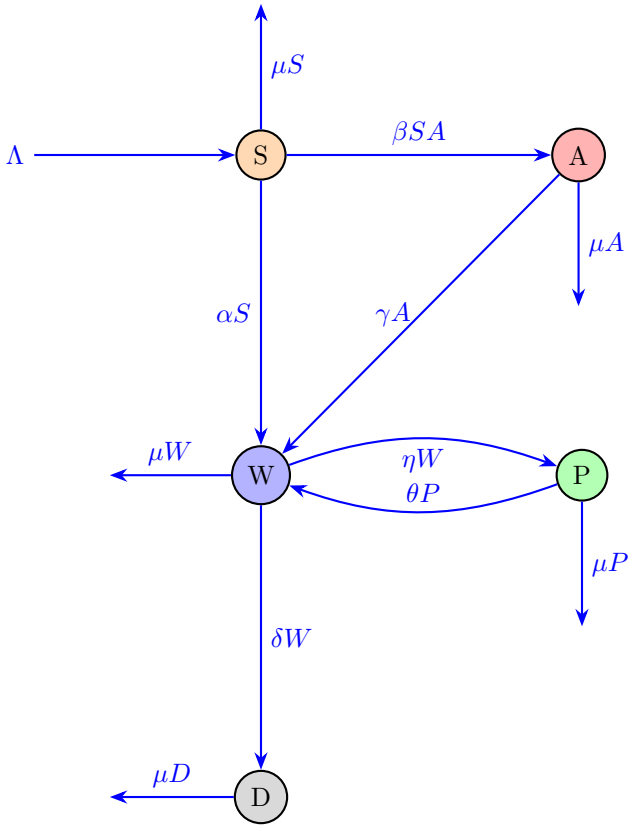


Figure 1: E-commerce merchant flow model

$$S(t) \geq S(0) \exp\left(-\int_0^t (\beta A(h) + \alpha + \mu) dh\right) \geq 0.$$

By applying similar arguments to the other equations, we conclude

$$A(t) \geq 0, \quad W(t) \geq 0, \quad P(t) \geq 0, \quad D(t) \geq 0, \quad \text{for all } t \geq 0.$$

□

2.2.1 Invariant region

In this section, we establish that all solutions of system (1) are bounded within a positively invariant region.

Theorem 2. Define the region

$$\Omega = \Big\{ (S(t), A(t), W(t), P(t), D(t)) \in \mathbb{R}^5_+ \ \Big| \ 0 \leq S(t) + A(t) + W(t) + P(t) + D(t) \leq \frac{\Lambda}{\mu} \Big\}.$$

If the initial conditions satisfy

$$S(0), A(0), W(0), P(0), D(0) \geq 0,$$

then the region Ω is positively invariant under the flow of system (1).

Proof. Let

$$N(t) = S(t) + A(t) + W(t) + P(t) + D(t).$$

Summing the equations of system (1) gives

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t).$$

This implies the differential inequality

$$\frac{dN(t)}{dt} \leq \Lambda - \mu N(t).$$

By standard comparison theorems, we have

$$\limsup_{t \to +\infty} N(t) \leq \frac{\Lambda}{\mu}.$$

Thus,

$$N(t) \leq \frac{\Lambda}{\mu} \quad \text{for all } t \geq 0.$$

Therefore, trajectories starting in Ω remain in Ω for all time, proving that Ω is positively invariant for system (1) [14]. □

2.2.2 Existence of solutions

This subsection addresses the fundamental question of the existence and uniqueness of solutions for the compartmental model proposed in this study. To this end, we state and prove the following theorem.

Theorem 3. System (1), with initial conditions

$$S(0) \geq 0, \quad A(0) \geq 0, \quad W(0) \geq 0, \quad P(0) \geq 0, \quad D(0) \geq 0,$$

admits a unique solution.

Proof. Define

$$X(t) = \begin{pmatrix} S(t) \\ A(t) \\ W(t) \\ P(t) \\ D(t) \end{pmatrix}, \quad \Upsilon(X) = KX + B(X),$$

where

$$K = \begin{bmatrix} -(\alpha+\mu) & 0 & 0 & 0 & 0 \\ 0 & -(\gamma+\mu) & 0 & 0 & 0 \\ \alpha & \gamma & -(\eta+\delta+\mu) & \theta & 0 \\ 0 & 0 & \eta & -(\theta+\mu) & 0 \\ 0 & 0 & \delta & 0 & -\mu \end{bmatrix},$$

$$B(X) = \begin{pmatrix} \Lambda - \beta SA \\ \beta SA \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

To establish uniqueness, we show that Υ is Lipschitz continuous. For any $Z, Y \in \mathbb{R}^5$, we have

$$\|B(Z) - B(Y)\| \leq 2|\beta S_Z A_Z - \beta S_Y A_Y|.$$

By applying the triangle inequality, we have

$$\begin{aligned} |\beta S_Z A_Z - \beta S_Y A_Y| &\leq \beta |S_Z(A_Z - A_Y)| + \beta |A_Y(S_Z - S_Y)| \\ &\leq \beta \|S_Z\| \|A_Z - A_Y\| + \beta \|A_Y\| \|S_Z - S_Y\|. \end{aligned}$$

Since all solutions are bounded in the positively invariant region Ω, where

$$S(t), A(t) \leq \frac{\Lambda}{\mu},$$

we conclude that

$$\|B(Z) - B(Y)\| \leq 2\beta \frac{\Lambda}{\mu} \|Z - Y\| = C\|Z - Y\|, \quad \text{with } C = 2\beta \frac{\Lambda}{\mu}.$$

Moreover, since K is a constant matrix, it is also Lipschitz continuous with Lipschitz constant $\|K\|$. Hence,

$$\|\Upsilon(Z) - \Upsilon(Y)\| \leq G\|Z - Y\|, \quad \text{where } G = \max(C, \|K\|) < \infty.$$

Thus, by the Picard–Lindelöf theorem, system (1) admits a unique solution for the given initial data [22]. □

3 Stability and sensitivity analysis of the model

To gain deeper insights into the dynamics of the proposed compartmental model, we perform a stability analysis and a sensitivity analysis of the model parameters. These analyses are useful for understanding the system dynamics and identifying the parameters that have the greatest influence on e-commerce refusal.

3.1 Computation of the basic reproduction number

The basic reproduction number R_0 represents the average number of new merchants who become refusers of e-commerce due to one refuser introduced into an otherwise refusal-free population. To compute R_0, we use the next-generation matrix approach introduced by Van den Driessche and Watmough [26].

We reformulate system (1) in the form

$$\frac{d\mathbf{X}}{dt} = \mathcal{F}(\mathbf{X}) - \mathcal{V}(\mathbf{X}),$$

where

$$\mathbf{X} = \begin{pmatrix} A \\ W \\ S \end{pmatrix},$$

and

$$\mathcal{F}(\mathbf{X}) = \begin{pmatrix} \beta SA \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{V}(\mathbf{X}) = \begin{pmatrix} (\gamma + \mu)A \\ -\alpha S - \gamma A - \theta P + (\eta + \delta + \mu)W \\ -\Lambda + \beta SA + \alpha S + \mu S \end{pmatrix}.$$

At the **refusal-free equilibrium**

$$E_0 = (A_0, W_0, S_0),$$

where

$$A_0 = 0, \qquad W_0 = 0, \qquad S_0 = \frac{\Lambda}{\alpha + \mu},$$

the Jacobian matrices are given by

$$J_{\mathcal{F}}(E_0) = \begin{pmatrix} \beta S_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\beta\Lambda}{\alpha+\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the next-generation matrix associated with the three state variables (A, W, S) is

$$F = \begin{pmatrix} \frac{\beta\Lambda}{\alpha+\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Jacobian of $\mathcal{V}(\mathbf{X})$ evaluated at E_0 is

$$V = J_{\mathcal{V}}(E_0) = \begin{pmatrix} \gamma + \mu & 0 & 0 \\ -\gamma & \eta + \delta + \mu & -\alpha \\ \frac{\beta\Lambda}{\alpha+\mu} & 0 & \alpha + \mu \end{pmatrix}.$$

Its inverse is

$$V^{-1} = \begin{pmatrix} \frac{1}{\gamma+\mu} & 0 & 0 \\ \frac{\gamma}{(\gamma+\mu)(\eta+\delta+\mu)} & \frac{1}{\eta+\delta+\mu} & \frac{\alpha}{(\alpha+\mu)(\eta+\delta+\mu)} \\ -\frac{\beta\Lambda}{(\alpha+\mu)^2(\gamma+\mu)} & 0 & \frac{1}{\alpha+\mu} \end{pmatrix}.$$

Therefore,

$$FV^{-1} = \begin{pmatrix} \frac{\beta\Lambda}{(\alpha+\mu)(\gamma+\mu)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of FV^{-1} are

$$\lambda_1 = \frac{\beta\Lambda}{(\alpha+\mu)(\gamma+\mu)}, \qquad \lambda_2 = 0, \qquad \lambda_3 = 0.$$

Consequently, the basic reproduction number, which is defined as the spectral radius of FV^{-1}, is

$$R_0 = \rho(FV^{-1}) = \frac{\beta\Lambda}{(\alpha+\mu)(\gamma+\mu)}.$$

3.2 Equilibrium points analysis

The model admits two equilibrium points. The first, E_0, corresponds to the absence of refusal to use e-commerce, while the second, E_*, corresponds to the presence of refusal to use e-commerce among merchants.

3.3 Refusal-free equilibrium point E_0

The equilibrium E_0 corresponds to a situation in which there is no refusal to use e-commerce among merchants. It is given by

$$E_0 = (S_0, A_0, W_0, P_0, D_0),$$

where

$$\begin{aligned}
S_0 &= \frac{\Lambda}{\alpha+\mu},\\
A_0 &= 0,\\
W_0 &= \frac{\alpha\Lambda(\theta+\mu)}{(\alpha+\mu)(\mu^2+\theta\mu+\theta\delta+\mu\delta+\mu\eta)},\\
P_0 &= \frac{\alpha\Lambda\eta}{(\alpha+\mu)(\mu^2+\theta\mu+\theta\delta+\mu\delta+\mu\eta)},\\
D_0 &= \frac{\alpha\Lambda\delta(\theta+\mu)}{\mu(\alpha+\mu)(\mu^2+\theta\mu+\theta\delta+\mu\delta+\mu\eta)}.
\end{aligned}$$

Since $A_0 = 0$, this equilibrium represents the absence of refusal to use e-commerce among merchants.

3.4 Equilibrium point with refusal of e-commerce E_*

The equilibrium E_* corresponds to a situation in which refusal to use e-commerce is present among merchants. It is given by

$$E_* = (S_*, A_*, W_*, P_*, D_*)\,,$$

with

$$\begin{aligned}
S_* &= \frac{\gamma+\mu}{\beta},\\
A_* &= \frac{(\alpha+\mu)(R_0-1)}{\beta},\\
W_* &= \frac{\theta+\mu}{\beta}\cdot\frac{\alpha\mu+\alpha\gamma R_0+\gamma\mu(R_0-1)}{\mu^2+\theta\mu+\theta\delta+\mu\delta+\mu\eta},\\
P_* &= \frac{\dfrac{\alpha(\gamma+\mu)}{\beta}+\dfrac{\gamma(\alpha+\mu)(R_0-1)}{\beta}}{\mu+\dfrac{\mu(\theta+\mu)}{\eta}+\dfrac{\delta(\theta+\mu)}{\eta}},\\
D_* &= \frac{\delta(\theta+\mu)}{\beta\mu}\cdot\frac{\alpha\mu+\gamma\mu(R_0-1)+\alpha\gamma R_0}{\mu^2+\theta\mu+\theta\delta+\mu\delta+\mu\eta}.
\end{aligned}$$

Since

$$A_* = \frac{(\alpha+\mu)(R_0-1)}{\beta},$$

we have

$$A_* > 0 \qquad \text{if and only if} \qquad R_0 > 1.$$

Thus, when $R_0 < 1$, the refusal of e-commerce cannot persist and the equilibrium E_0 describes the absence of refusal. When $R_0 > 1$, refusal to use e-commerce is present and persists at the equilibrium E_*.

3.5 Local stability analysis

In this subsection, we conduct a local stability analysis of the two equilibrium points. We first consider the refusal-free equilibrium E_0.

Theorem 4. The refusal-free equilibrium point E_0 is locally asymptotically stable if $R_0 < 1$.

Proof. We begin by considering the Jacobian matrix J_0, evaluated at the equilibrium point E_0,

$$J_0(E_0) = \begin{bmatrix} -\alpha-\mu & -\Lambda\frac{\beta}{\alpha+\mu} & 0 & 0 & 0 \\ 0 & (R_0-1)(\gamma+\mu) & 0 & 0 & 0 \\ \alpha & \gamma & -\mu-\delta-\eta & \theta & 0 \\ 0 & 0 & \eta & -\theta-\mu & 0 \\ 0 & 0 & \delta & 0 & -\mu \end{bmatrix}.$$

The eigenvalues of this matrix are given by

$$\begin{aligned} v_1 &= -\frac{1}{2}\theta - \mu - \frac{1}{2}\delta - \frac{1}{2}\eta - \frac{1}{2}\sqrt{(\theta-\delta)^2 + 2\theta\eta + 2\delta\eta + \eta^2} < 0, \\ v_2 &= -\alpha - \mu < 0, \\ v_3 &= -\mu < 0, \\ v_4 &= (R_0-1)(\gamma+\mu) < 0 \quad (\text{since } R_0 < 1), \\ v_5 &= \frac{1}{2}\sqrt{(\theta-\delta)^2 + 2\theta\eta + 2\delta\eta + \eta^2} - \left(\mu + \frac{1}{2}\delta + \frac{1}{2}\eta + \frac{1}{2}\theta\right) < 0. \end{aligned}$$

To demonstrate that $v_5 < 0$, we compute

$$\begin{aligned} &\left(\frac{1}{2}\sqrt{(\theta-\delta)^2 + 2\theta\eta + 2\delta\eta + \eta^2}\right)^2 - \left(\mu + \frac{1}{2}\delta + \frac{1}{2}\eta + \frac{1}{2}\theta\right)^2 \\ &\quad = -\mu^2 - \theta\mu - \theta\delta - \mu\delta - \mu\eta < 0. \end{aligned}$$

Since all eigenvalues are negative, the refusal-free equilibrium point E_0 is locally asymptotically stable if $R_0 < 1$. □

We now consider the equilibrium point E_*, which corresponds to the presence of e-commerce refusal among merchants.

Theorem 5. The equilibrium point E_* with e-commerce refusal is locally asymptotically stable if $R_0 > 1$.

Proof. We have the Jacobian matrix J_* evaluated at E_*:

$$J_*(E_*) = \begin{bmatrix} -R_0(\alpha+\mu) & -(\gamma+\mu) & 0 & 0 & 0 \\ (R_0-1)(\alpha+\mu) & 0 & 0 & 0 & 0 \\ \alpha & \gamma & -(\mu+\delta+\eta) & \theta & 0 \\ 0 & 0 & \eta & -(\theta+\mu) & 0 \\ 0 & 0 & \delta & 0 & -\mu \end{bmatrix}.$$

The characteristic polynomial of $J(E_*)$ is given by

$$P(X) = X^5 + b_1X^4 + b_2X^3 + b_3X^2 + b_4X + b_5,$$

where

$$\begin{aligned}
b_1 =& (\theta + 3\mu + \delta + \eta + R_0(\alpha + \mu)) > 0,\\
b_2 =& 2\mu^2 + 2\theta\mu + \theta\delta + 2\mu\delta + 2\mu\eta + 4\mu^2R_0 + \theta\alpha R_0 + \theta\mu R_0 + 3\alpha\mu R_0\\
&+ \alpha\delta R_0 + \mu\delta R_0 + \alpha\eta R_0 + \mu\eta R_0 + (R_0 - 1)(\alpha\gamma + \alpha\mu + \gamma\mu) > 0,\\
b_3 =& (R_0 - 1)\begin{pmatrix} 2\mu^3 + 3\alpha\mu^2 + 3\gamma\mu^2 + \theta\alpha\gamma + \theta\alpha\mu + \theta\gamma\mu \\ +3\alpha\gamma\mu + \alpha\gamma\delta + \alpha\mu\delta + \gamma\mu\delta + \alpha\gamma\eta + \alpha\mu\eta + \gamma\mu\eta \end{pmatrix}\\
&+ 4\mu^3R_0 + 3\theta\mu^2R_0 + 3\alpha\mu^2R_0 + 3\mu^2\delta R_0 + 3\mu^2\eta R_0\\
&+ \theta\mu\delta + 2\theta\alpha\mu R_0 + \theta\alpha\delta R_0 + \theta\mu\delta R_0 + 2\alpha\mu\delta R_0 + 2\alpha\mu\eta R_0 > 0,\\
b_4 =& -(\alpha + \mu)\left(-(R_0 - 1)\begin{pmatrix} 4\mu^3 + 3\theta\mu^2 + 3\gamma\mu^2 + 3\mu^2\delta \\ +3\mu^2\eta + 2\theta\gamma\mu + \theta\gamma\delta \\ +2\theta\mu\delta + 2\gamma\mu\delta + 2\gamma\mu\eta \end{pmatrix} - \begin{pmatrix} \\ -\mu^3 - \theta\mu^2 - \mu^2\delta - \theta\mu\delta - \mu^2\eta \\ \\ \end{pmatrix}\right) > 0,\\
b_5 =& \mu(R_0 - 1)(\alpha + \mu)(\gamma + \mu)\left(\mu^2 + \theta\mu + \theta\delta + \mu\delta + \mu\eta\right) > 0.
\end{aligned}$$

Using the Routh–Hurwitz criterion, the system is locally asymptotically stable if $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $b_4 > 0$, $b_5 > 0$, $b_1b_2b_3 > b_3^2 + b_1^2b_4$, and

$$(b_1b_4 - b_5)\left(b_1b_2b_3 - b_3^2 + b_1^2b_4\right) > b_5(b_1b_2 - a_3)^2 + b_1b_5^2$$

for $R_0 > 1$.

Finally, if $R_0 > 1$, then the equilibrium point E_* with e-commerce refusal is locally asymptotically stable. □

3.6 Global stability analysis

The objective of this subsection is to examine the global stability of the refusal-free equilibrium E_0 and the equilibrium point E_* associated with e-commerce refusal. The following analysis describes the global stability properties of these two equilibrium points.

Theorem 6. The refusal-free equilibrium E_0 is globally asymptotically stable if $R_0 \leq 1$.

Proof. Let us consider the Lyapunov function $V : \Omega \to \mathbb{R}$, defined as

$$V(S,A) = \frac{1}{2}\left((S - S_0) + A\right)^2 + \frac{\alpha + \gamma + 2\mu}{\beta}A, \tag{2}$$

where S and A represent the state variables, S_0 is a constant, and α, γ, μ, and β are parameters of the system.

The time derivative of $V(S,A)$ is expressed as

$$\dot{V}(S,A) = \left((S - S_0) + A\right)\left(\dot{S} + \dot{A}\right) + \frac{\alpha + \gamma + 2\mu}{\beta}\dot{A}.$$

After performing calculations, we obtain

$$\begin{aligned}\dot{V}(S,A) = &-(\alpha + \mu)\left(S - \frac{\Lambda}{\alpha + \mu}\right)^2 - (\gamma + \mu)A^2 \\ &- (\alpha + \gamma + 2\mu)\frac{(\mu + \gamma)}{\beta}(1 - R_0)A \leq 0.\end{aligned}$$

Thus, $\dot{V}(S,A) = 0 \Longleftrightarrow S = S_0$ and $A = 0$.

Hence, the theorem is proved. Furthermore, by LaSalle's invariance principle [18], the refusal-free equilibrium E_0 is globally asymptotically stable on Ω. □

In the following, we analyze the global stability of the equilibrium point E_* with e-commerce refusal.

Theorem 7. The equilibrium point E_* with e-commerce refusal is globally asymptotically stable if $R_0 > 1$.

Proof. Let us consider the Lyapunov function $V : \Omega \to \mathbb{R}$ defined as

$$V(S,A) = (S - A) - S_* \ln\left(\frac{S}{S_*}\right) + (A - A_*) - A_* \ln\left(\frac{A}{A_*}\right), \tag{3}$$

where S and A are the state variables, and S_* and A_* represent the values of S and A at the equilibrium point E_*.

Taking the time derivative of $V(S,A)$, we obtain

$$\dot{V}(S,A) = \dot{S} + \dot{A} - S_*\frac{\dot{S}}{S} - A_*\frac{\dot{A}}{A}.$$

Further calculations yield

$$\dot{V}(S,A) = -\frac{1}{\beta}\frac{\Lambda}{S(\gamma + \mu)}\left(\gamma + \mu - \beta S\right)^2 \leq 0,$$

which shows that the Lyapunov function is nonincreasing.

Additionally, we find that

$$\dot{V}(S, A) = 0 \Longleftrightarrow S = S_*.$$

Therefore, by LaSalle's invariance principle [18], the equilibrium point E_* with e-commerce refusal is globally asymptotically stable on Ω. □

3.7 Numerical simulation of the model

In this subsection, we present numerical simulations of system (1) to illustrate its dynamical behavior under different parameter settings. The following parameter values are considered:

$$\Lambda = 400{,}000, \quad \mu = 0.39, \quad \alpha = 0.5, \quad \eta = 0.3,$$
$$\gamma = 0.2, \quad \delta = 0.3, \quad \theta = 0.1, \quad \beta = 1 \times 10^{-7}.$$

For these parameter values, the basic reproduction number is

$$R_0 = 0.7618 < 1.$$

Accordingly, the system approaches the refusal-free equilibrium point

$$E_0 = \left(4.4944 \times 10^5,\ 0,\ 2.4195 \times 10^5,\ 1.4813 \times 10^5,\ 1.8612 \times 10^5\right).$$

As shown in Figure 2, the state variables S, A, W, P, and D converge toward the corresponding components of E_0. In particular, the refusal class A decreases toward zero, indicating the disappearance of e-commerce refusal in the long term.

Next, we consider a second scenario in which the transmission rate is increased to

$$\beta = 1.8 \times 10^{-6},$$

while all the other parameter values remain unchanged. In this case, the basic reproduction number satisfies

$$R_0 = 1.33 > 1.$$

The system then approaches the endemic equilibrium point E_* with e-commerce refusal:

$$E_* = \left(3.2778 \times 10^5,\ 1.8352 \times 10^5,\ 2.1598 \times 10^5,\ 1.3223 \times 10^5,\ 1.6614 \times 10^5\right).$$

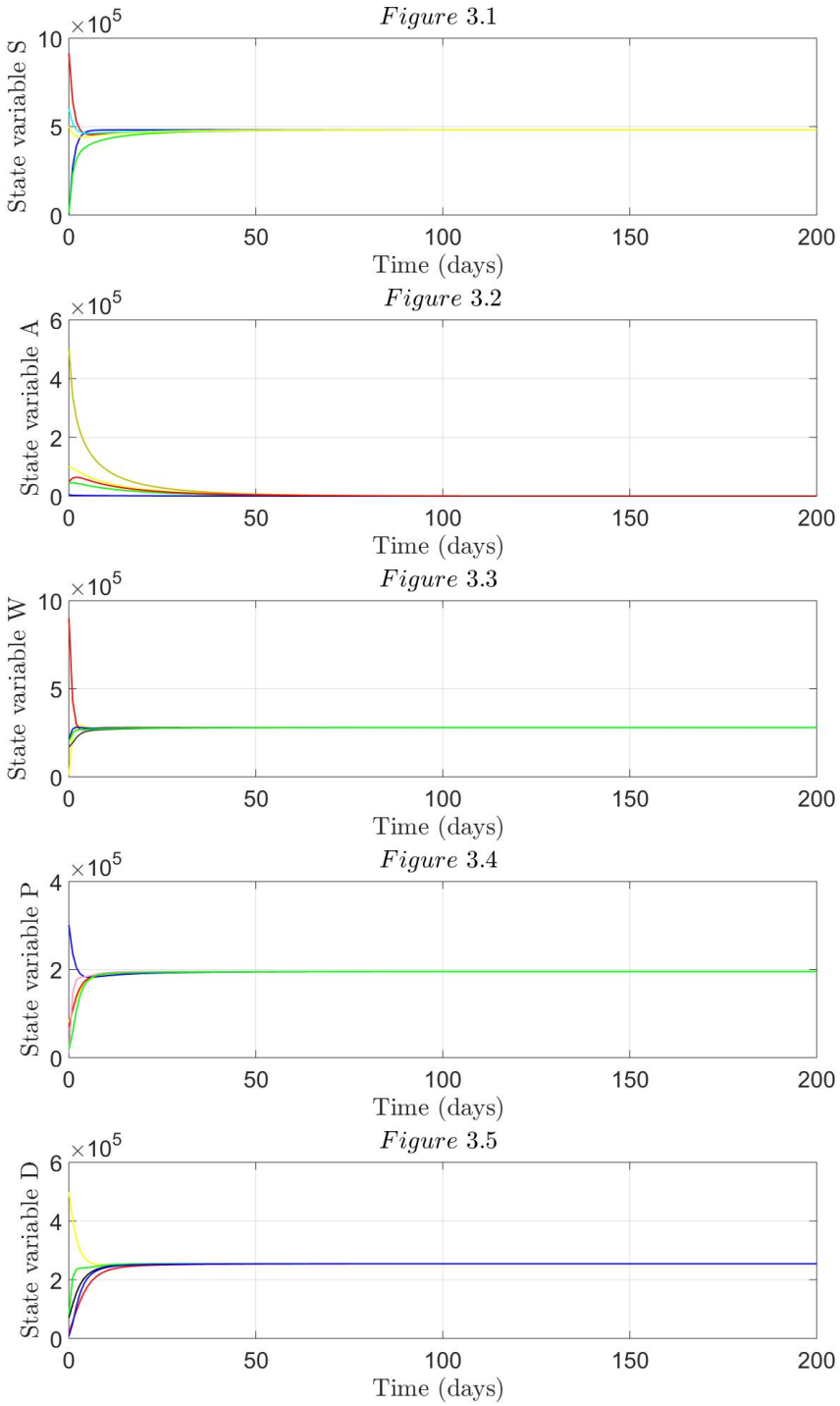


Figure 2: Dynamics converging to the refusal-free equilibrium for e-commerce.

As illustrated in Figure 3, all state variables converge toward the corresponding components of E_*. In particular, the refusal class A converges to a positive value, indicating that e-commerce refusal persists in the merchant population over time.

These numerical simulations illustrate the influence of the basic reproduction number R_0 on the long-term dynamics of the model. When $R_0 < 1$, e-commerce refusal decreases over time and the system converges to the refusal-free equilibrium E_0. In contrast, when $R_0 > 1$, the refusal class persists and the system approaches the equilibrium E_*, where the number of refusing merchants remains positive. These results emphasize the importance of controlling the mechanisms that contribute to the transmission of e-commerce refusal in order to promote e-commerce adoption among merchants.

3.8 Sensitivity analysis of model parameters

Sensitivity analysis is a key tool used to evaluate how variations in model parameters influence the system's output. In particular, it allows us to identify which parameters have the greatest impact on the basic reproduction number R_0, which governs the spread of e-commerce refusal among merchants [3]. In this subsection, we assess the effect of each parameter in system (1) on R_0.

Definition

The normalized forward sensitivity index of a variable v with respect to a parameter p is defined by

$$\Delta_p^v = \frac{\partial v}{\partial p} \cdot \frac{p}{v}.$$

For our model, the basic reproduction number is given by

$$R_0 = \beta \cdot \frac{\Lambda}{(\alpha+\mu)(\gamma+\mu)}.$$

Computation of sensitivity indices

Applying the definition, the sensitivity indices of R_0 with respect to each parameter are

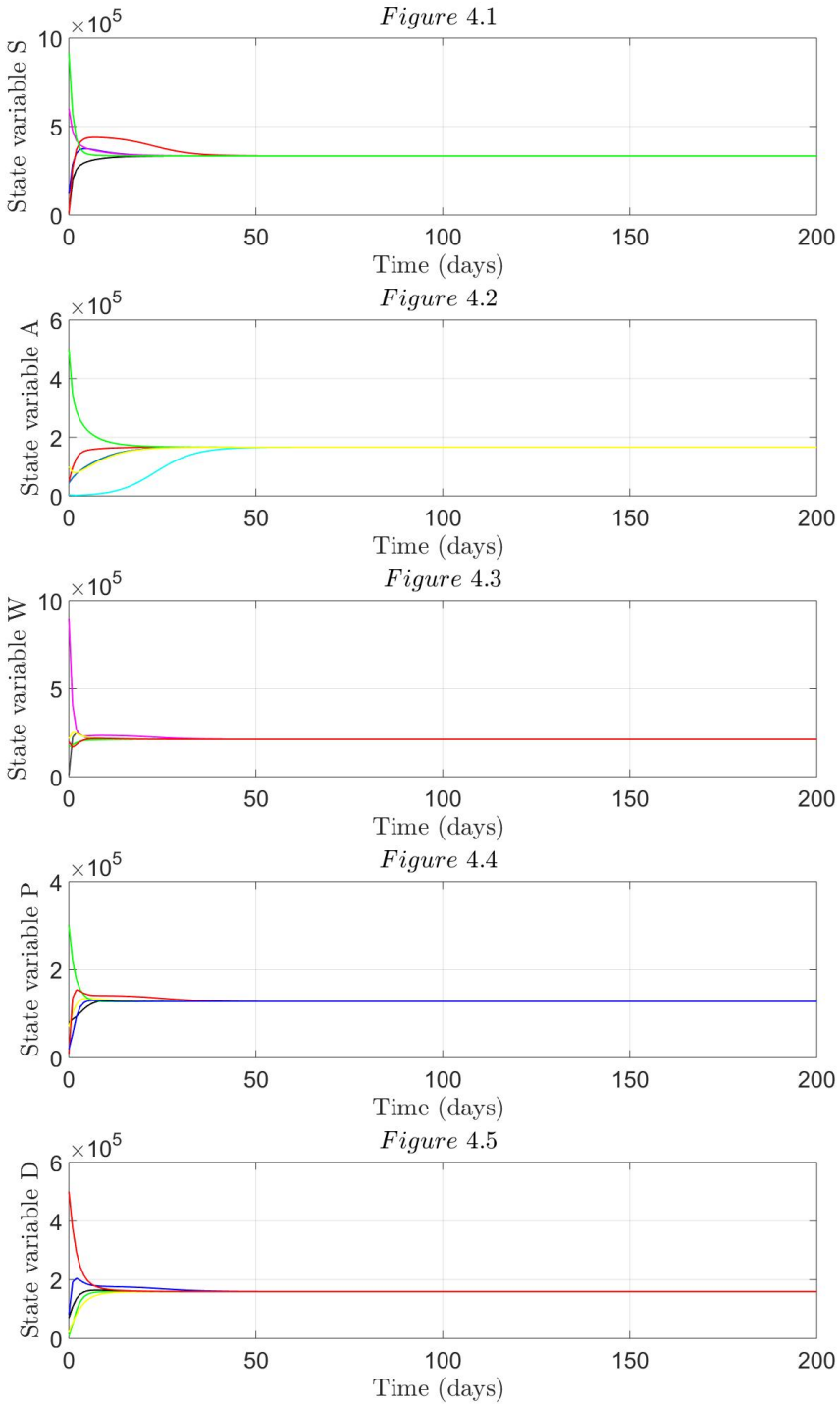


Figure 3: Dynamics converging to the equilibrium with e-commerce refusal.

$$\begin{cases} \Delta_{\Lambda}^{R_0} = 1, \\ \Delta_{\beta}^{R_0} = 1, \\ \Delta_{\alpha}^{R_0} = -\dfrac{\alpha}{\alpha+\mu} < 0, \\ \Delta_{\gamma}^{R_0} = -\dfrac{\gamma}{\gamma+\mu} < 0, \\ \Delta_{\mu}^{R_0} = -\dfrac{\mu(\alpha+\gamma+2\mu)}{(\alpha+\mu)(\gamma+\mu)} < 0. \end{cases} \tag{4}$$

Interpretation

The sign of each sensitivity index indicates the direction of its effect on R_0. A positive sensitivity index, such as $\Delta_{\Lambda}^{R_0} = 1$, means that increasing the corresponding parameter increases R_0. For instance, a 10% increase in β leads to a 10% increase in R_0.

In contrast, a negative sensitivity index, such as $\Delta_{\alpha}^{R_0} < 0$, implies that increasing the corresponding parameter reduces R_0. For example, raising α by 10% reduces R_0 proportionally to

$$\frac{10\% \cdot \alpha}{\alpha+\mu}.$$

The sensitivity indices are illustrated in Figure 4, which provides a graphical representation of the influence of the model parameters on R_0.

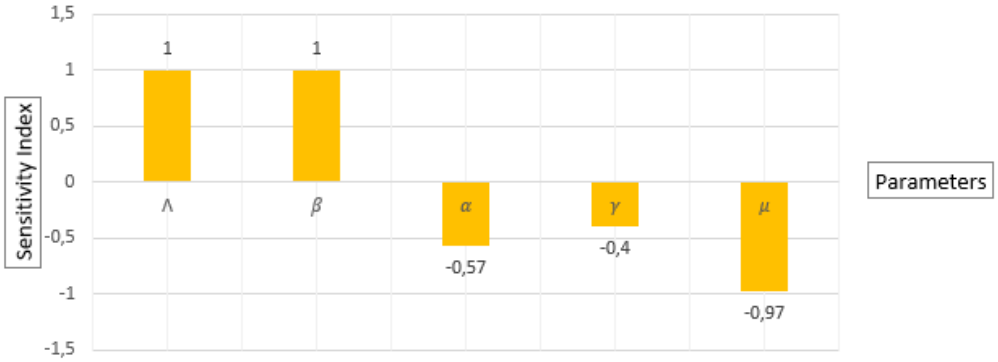


Figure 4: Sensitivity indices of R_0 with respect to model parameters.

Remark 1. The sensitivity analysis shows that the recruitment rate Λ and the rejection influence rate β increase R_0, thereby promoting the persistence of e-commerce refusal. In contrast,

increasing either the willingness rate α or the recovery rate γ decreases R_0, thereby contributing to the reduction of e-commerce refusal. The reintegration rate θ does not directly affect R_0, since it does not appear in its expression.

These findings motivate the introduction of three time-dependent control functions:

- $u_1(t)$: awareness campaigns,
- $u_2(t)$: cybersecurity support,
- $u_3(t)$: technical assistance.

By targeting the main factors and processes influencing merchant behavior, these control strategies provide a practical approach to reducing e-commerce refusal and promoting digital adoption among merchants.

4 Formulation of the optimal control problem

In this section, we formulate an optimal control problem aimed at increasing the number of merchants who utilize e-commerce and reducing those who refrain from its use. The proposed control strategies are incorporated into a mathematical model to achieve this objective.

4.1 Mathematical model with control variables

E-commerce enables online store owners to showcase their products and services to customers at any time of the day or night, throughout the week. Furthermore, they can conduct transactions both locally and internationally. Moreover, it allows them to communicate with a diverse customer base, facilitated by the ongoing global digital revolution, which increases customer reach.

To augment the number of merchants who utilize e-commerce to fulfill their daily requirements and curtail the number of those who eschew its use, we introduce three control variables:

- $u_1(t)$: Advertising campaigns across all visual and audio communication channels, including social media, to encourage stores to adopt e-commerce and motivate merchants to utilize it.

- $u_2(t)$: Provision of insurance for all shops against potential cybercrime exposure and creation of safer websites for commercial shops to protect against cybercrime, thereby motivating merchants to use e-commerce.

- $u_3(t)$: Establishment of dedicated teams to train and educate merchants on how to manage e-commerce sites, especially concerning internet and maintenance issues.

By incorporating these control variables into the initial system (1), we obtain the following controlled dynamical system:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta S(t)A(t) - \alpha S(t) - \mu S(t) - u_1(t)S(t), \\ \dfrac{dA(t)}{dt} = \beta S(t)A(t) - \gamma A(t) - \mu A(t) - u_2(t)A(t), \\ \dfrac{dW(t)}{dt} = \alpha S(t) + \gamma A(t) + \theta P(t) - (\eta + \delta + \mu)W(t) \\ \qquad\qquad + u_1(t)S(t) + u_2(t)A(t) + u_3(t)P(t), \\ \dfrac{dP(t)}{dt} = \eta W(t) - \theta P(t) - \mu P(t) - u_3(t)P(t), \\ \dfrac{dD(t)}{dt} = \delta W(t) - \mu D(t). \end{cases} \tag{5}$$

The initial conditions are given by

$$S(0) \geq 0, \quad A(0) \geq 0, \quad W(0) \geq 0, \quad P(0) \geq 0, \quad D(0) \geq 0.$$

4.2 Objective functional

Our aim is to minimize the total number of merchants who refuse to use e-commerce over a fixed time horizon $[t_0, t_f]$. This is achieved by optimizing the control variables $u_1(t)$, $u_2(t)$, and $u_3(t)$ in the model (5). The objective functional to be minimized is defined as

$$\begin{aligned} &J(u_1, u_2, u_3) \\ &\quad = A(t_f) + P(t_f) - W(t_f) \\ &\qquad + \int_{t_0}^{t_f} \left[A(t) + P(t) - W(t) + \frac{C_1}{2}u_1^2(t) + \frac{C_2}{2}u_2^2(t) + \frac{C_3}{2}u_3^2(t) \right] dt, \end{aligned} \tag{6}$$

where $C_1, C_2, C_3 > 0$ are weight parameters representing the cost associated with implementing each control strategy.

4.3 Admissible control set

We seek optimal controls $u_1^*(t)$, $u_2^*(t)$, and $u_3^*(t)$ such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{(u_1,u_2,u_3)\in U} J(u_1, u_2, u_3), \tag{7}$$

where the set of admissible controls U is defined by

$$\begin{aligned} U = \{(u_1(t), u_2(t), u_3(t)) \,|\, & 0 \le u_1(t) \le 1,\ 0 \le u_2(t) \le 1, \\ & 0 \le u_3(t) \le 1,\ \text{for } t \in [t_0, t_f]\}\,. \end{aligned} \tag{8}$$

4.4 Problem statement

The optimal control problem can be formally stated as follows.

Problem

Find control functions $u_1^*(t)$, $u_2^*(t)$, and $u_3^*(t)$ belonging to the admissible control set U that minimize the objective functional $J(u_1, u_2, u_3)$ given by (6), subject to the controlled system dynamics (5) and initial conditions.

The solution to this optimal control problem will provide the optimal implementation of the proposed strategies to maximize the adoption of e-commerce among merchants while minimizing the associated costs.

4.5 Existence of an optimal control solution

This subsection establishes the existence of an optimal control (u_1^*, u_2^*, u_3^*) that minimizes the cost functional J. The proof is formulated to meet the standards expected by specialists in control theory.

Theorem 8. An optimal control $(u_1^*, u_2^*, u_3^*) \in U$ exists such that

$$J(u_1^*, u_2^*, u_3^*) = \inf_{(u_1,u_2,u_3)\in U} J(u_1, u_2, u_3).$$

Proof. To demonstrate the existence of an optimal control, we utilize standard results from the theory of optimal control as outlined in [12]. The following conditions must be satisfied:

(C1) The set of admissible controls U and the associated state trajectories are nonempty.

(C2) The control set U is convex and closed.

(C3) The right-hand side of the state system is continuous and satisfies linear growth with respect to the state and control variables.

(C4) The cost functional $J(u_1, u_2, u_3)$ is convex with respect to the controls and lower semi-continuous.

Verification of Condition C1: The nonemptiness of U and the existence of corresponding state trajectories follow from the well-posedness of the control system (5), as established in [20].

Verification of Condition C2: By definition, the control set U consists of all measurable functions $(u_1(t), u_2(t), u_3(t))$ such that $0 \le u_i(t) \le 1$ for all $t \in [t_0, t_f]$ and $i = 1, 2, 3$. This set is convex and closed in the space of measurable functions.

Verification of Condition C3: The right-hand side of the system (5) is linear in the control variables and satisfies Lipschitz continuity with respect to the state variables. Additionally, the state variables are known to be bounded due to the modeling and practical considerations inherent in the model, as discussed in the analysis of system (1).

Verification of Condition C4: The cost functional comprises a linear combination of the state variables and quadratic terms in the control variables:

$$\begin{aligned} & J(u_1, u_2, u_3) \\ &= \phi\big(S(t_f), A(t_f), W(t_f), P(t_f), D(t_f)\big) \\ &\quad + \int_{t_0}^{t_f} \left[L\big(A(t), W(t), P(t)\big) + \frac{C_1}{2} u_1^2(t) + \frac{C_2}{2} u_2^2(t) + \frac{C_3}{2} u_3^2(t) \right] dt, \end{aligned}$$

where ϕ and L are linear functions of the state variables. The integrand is convex in the controls due to the quadratic terms, and the functional is lower semicontinuous with respect to the weak convergence of controls in $L^2([t_0, t_f])$.

Having verified all necessary conditions, we invoke the standard existence theorem (see [12]) to conclude that an optimal control (u_1^*, u_2^*, u_3^*) exists in U that minimizes the cost functional J. □

4.6 Characterization of the optimal controls

In this subsection, we derive the necessary conditions for optimality using Pontryagin's maximum principle [23]. By applying this principle to the control system (5) and the cost functional (6), we transform the problem into minimizing the Hamiltonian $\mathcal{H}$ with respect to the control variables u_1, u_2, and u_3.

The Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H} = A(t) + P(t) - W(t) + \frac{C_1}{2}u_1^2(t) + \frac{C_2}{2}u_2^2(t) + \frac{C_3}{2}u_3^2(t) + \sum_{i=1}^{5}\lambda_i(t)f_i(t),$$

where $\lambda_i(t)$ are the adjoint variables and $f_i(t)$ are the right-hand sides of the state equations in (5),

$$\begin{cases} f_1(t) = \Lambda - \beta S(t)A(t) - \alpha S(t) - \mu S(t) - u_1(t)S(t), \\ f_2(t) = \beta S(t)A(t) - \gamma A(t) - \mu A(t) - u_2(t)A(t), \\ f_3(t) = \alpha S(t) + \gamma A(t) + \theta P(t) - (\eta + \delta + \mu)W(t) \\ \qquad\quad + u_1(t)S(t) + u_2(t)A(t) + u_3(t)P(t), \\ f_4(t) = \eta W(t) - \theta P(t) - \mu P(t) - u_3(t)P(t), \\ f_5(t) = \delta W(t) - \mu D(t). \end{cases}$$

Theorem 9. Given the optimal controls (u_1^*, u_2^*, u_3^*) and the corresponding state trajectories $S^*(t)$, $A^*(t)$, $W^*(t)$, $P^*(t)$, and $D^*(t)$, there exist absolutely continuous adjoint variables $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, $\lambda_4(t)$, and $\lambda_5(t)$ satisfying the following adjoint equations almost everywhere on $[t_0, t_f]$:

$$\begin{cases} \dfrac{d\lambda_1}{dt} = \lambda_1 \left(\beta A^*(t) + \alpha + \mu + u_1^*(t)\right) - \lambda_2 \beta A^*(t) - \lambda_3 \left(\alpha + u_1^*(t)\right), \\ \dfrac{d\lambda_2}{dt} = 1 + \lambda_1 \beta S^*(t) + \lambda_2 \left(\beta S^*(t) + \gamma + \mu + u_2^*(t)\right) - \lambda_3 \left(\gamma + u_2^*(t)\right), \\ \dfrac{d\lambda_3}{dt} = -1 + \lambda_3 \left(\eta + \delta + \mu\right) - \lambda_4 \eta - \lambda_5 \delta, \\ \dfrac{d\lambda_4}{dt} = 1 + \lambda_3 \left(\theta + u_3^*(t)\right) + \lambda_4 \left(\theta + \mu + u_3^*(t)\right), \\ \dfrac{d\lambda_5}{dt} = \lambda_5 \mu. \end{cases} \tag{9}$$

The transversality conditions at the final time t_f are

$$\begin{cases} \lambda_1(t_f) = 0, \\ \lambda_2(t_f) = 1, \\ \lambda_3(t_f) = -1, \\ \lambda_4(t_f) = 1, \\ \lambda_5(t_f) = 0. \end{cases} \tag{10}$$

Moreover, the optimal controls are given by

$$\begin{cases} u_1^*(t) = \min\left\{1, \max\left\{0, \dfrac{\left(\lambda_1(t) - \lambda_3(t)\right) S^*(t)}{C_1}\right\}\right\}, \\ u_2^*(t) = \min\left\{1, \max\left\{0, \dfrac{\left(\lambda_2(t) - \lambda_3(t)\right) A^*(t)}{C_2}\right\}\right\}, \\ u_3^*(t) = \min\left\{1, \max\left\{0, \dfrac{\left(\lambda_4(t) - \lambda_3(t)\right) P^*(t)}{C_3}\right\}\right\}. \end{cases} \tag{11}$$

Proof. Applying Pontryagin's maximum principle, the Hamiltonian $\mathcal{H}$ is constructed as given above. The adjoint equations are derived by taking the negative partial derivatives of the Hamiltonian with respect to the state variables. Specifically, for $i = 1, \ldots, 5$,

$$\frac{d\lambda_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i},$$

where $x_1 = S$, $x_2 = A$, $x_3 = W$, $x_4 = P$, and $x_5 = D$.

Calculating these derivatives yields the adjoint equations (9). The transversality conditions (10) are obtained from the terminal conditions of the cost functional.

The optimality conditions require that the Hamiltonian be minimized with respect to the control variables almost everywhere on $[t_0, t_f]$. Taking the partial derivatives of $\mathcal{H}$ with respect to u_1, u_2, and u_3 and setting them to zero gives

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial u_1} = C_1 u_1^*(t) - \lambda_1(t) S^*(t) + \lambda_3(t) S^*(t) = 0, \\ \dfrac{\partial \mathcal{H}}{\partial u_2} = C_2 u_2^*(t) - \lambda_2(t) A^*(t) + \lambda_3(t) A^*(t) = 0, \\ \dfrac{\partial \mathcal{H}}{\partial u_3} = C_3 u_3^*(t) - \lambda_3(t) P^*(t) + \lambda_4(t) P^*(t) = 0. \end{cases}$$

Solving for the controls, we obtain expressions for $u_1^*(t)$, $u_2^*(t)$, and $u_3^*(t)$ as given in (11). These expressions are then projected onto the admissible control set U by applying the bounds $0 \leq u_i(t) \leq 1$.

□

5 Numerical simulations and discussion

In this section, we present numerical simulations of the optimal control strategies implemented in the proposed model using MATLAB. The simulations aim to illustrate the dynamic effects of the control interventions on the spread and mitigation of e-commerce rejection among merchants.

Initial conditions:

$$S(0) = 120{,}000, \quad A(0) = 100{,}000, \quad W(0) = 200{,}000,$$
$$P(0) = 80{,}000, \quad D(0) = 500{,}000.$$

Parameter values used:

$$\Lambda = 400{,}000, \quad \mu = 0.39, \quad \alpha = 0.5, \quad \eta = 0.3,$$

$$\gamma = 0.2, \quad \delta = 0.3, \quad \theta = 0.1, \quad \beta = 1.8 \times 10^{-7}.$$

These parameter settings ensure that the basic reproduction number R_0 exceeds one, indicating the persistence of e-commerce rejection in the absence of control. The upcoming simulation results analyze how appropriately designed control policies can reduce resistance and promote digital adoption across the merchant population.

5.1 Strategy u_1: Promotion through advertising

Figure 5 illustrates the effect of implementing the control strategy $u_1(t)$, which focuses on promoting the importance of e-commerce through advertising. The figure compares the dynamics of the active e-commerce merchant group $W(t)$ with and without the control u_1. It is evident that the application of u_1 leads to a significant increase in the size of the group $W(t)$ over time. This demonstrates the positive impact of advertising on encouraging merchants to adopt e-commerce.

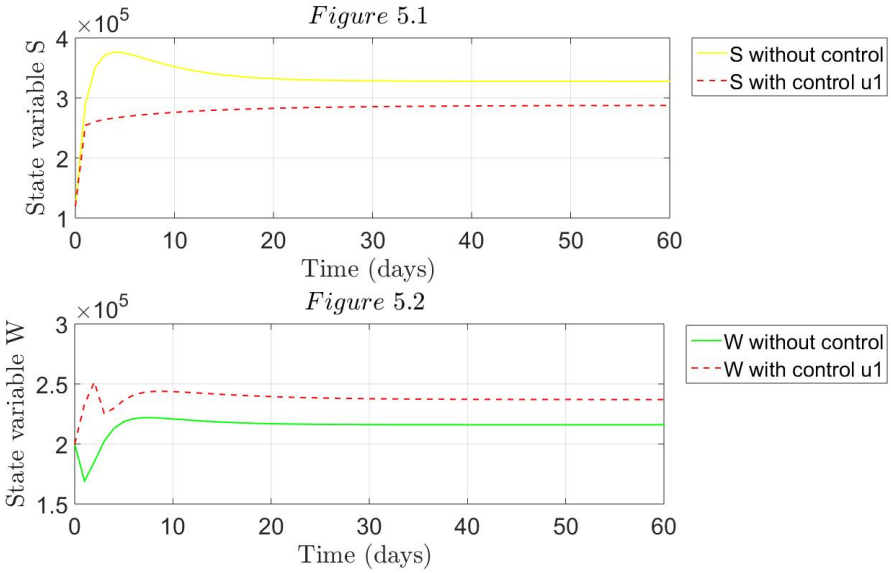


Figure 5: Dynamics of $W(t)$ with and without control u_1.

5.2 Strategy u_2: Cybercrime insurance provision

In Figure 6, we examine the effects of the control strategy $u_2(t)$, which involves providing cybercrime insurance to merchants. The figure shows the changes in the size of the willing merchant group $W(t)$ compared to the scenario without control u_2. The results indicate that implementing u_2 increases the size of $W(t)$ over time, while the size of the e-commerce-refusal merchant group $A(t)$ decreases. This suggests that offering cybercrime insurance effectively encourages merchants to transition from being merely aware to actively willing to use e-commerce.

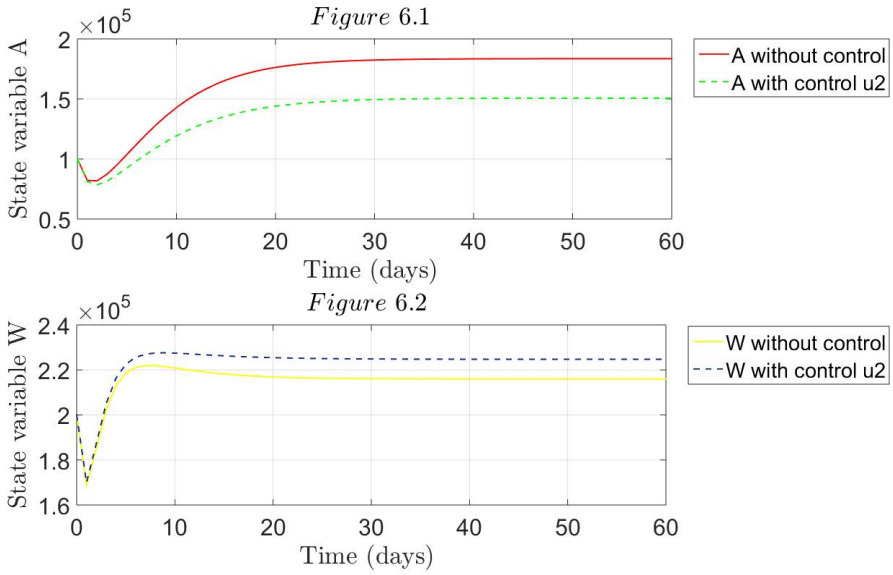


Figure 6: Dynamics of $W(t)$ and $A(t)$ with and without control u_2.

5.3 Strategy u_3: Training and support services

Figure 7 depicts the impact of the control strategy $u_3(t)$, which focuses on providing training and support services to merchants. The figure compares the dynamics of the active e-commerce merchant group $W(t)$ with and without the control u_3. The application of u_3 results in an increase in $W(t)$ over time, while the size of the temporarily inactive merchants group $P(t)$ decreases. This indicates that offering training and support accelerates the transition of merchants from being in the process of adopting e-commerce to becoming willing users.

5.4 Combined Strategies u_1, u_2, and u_3

Finally, Figure 8 shows the effects of implementing all three control strategies simultaneously. The figure compares the dynamics of the willing merchant group $W(t)$ with and without the combined controls. The results demonstrate a significant increase in $W(t)$ when all controls are applied. Concurrently, the sizes of the susceptible group $S(t)$, the e-commerce-refusal group $A(t)$, and the group of merchants in process $P(t)$ decrease over time. This suggests that the combined application of advertising, cybercrime insurance, and training services has a synergistic effect, greatly enhancing the adoption of e-commerce among merchants.

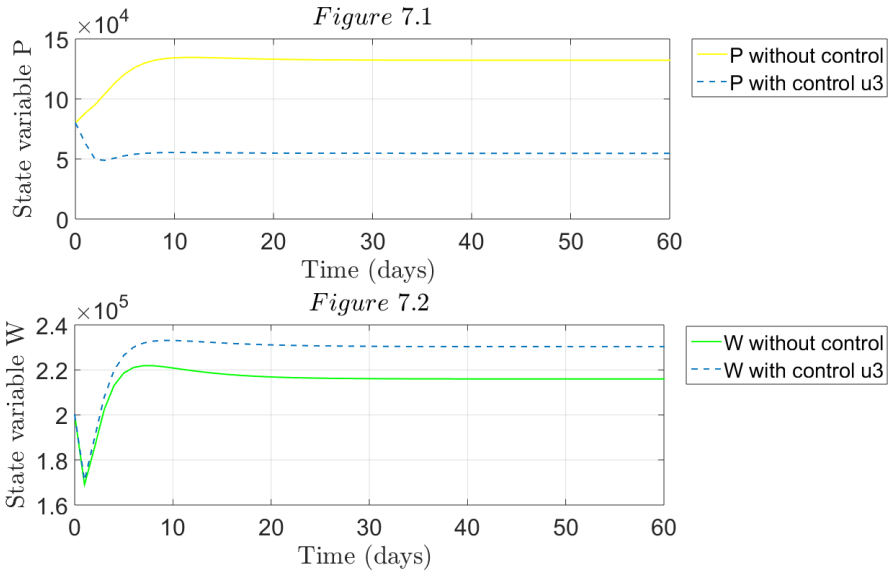


Figure 7: Dynamics of $W(t)$ and $P(t)$ with and without control u_3.

5.5 Cost-effectiveness analysis

To evaluate the efficiency of various intervention strategies aimed at increasing e-commerce adoption among merchants, we employ the average cost-effectiveness ratio (ACER) as a comparative metric [5]. The ACER quantifies the average cost required to prevent a single instance of e-commerce refusal; that is, to stop a merchant from entering or remaining in the refusal compartment A.

Mathematically, the ACER is defined by

$$\text{ACER} = \frac{T_c}{T_a}, \tag{12}$$

where

- T_c denotes the total cost of implementing the intervention strategy over the time horizon $[0, T]$,
- T_a represents the total number of avoided refusals, calculated as the difference in the cumulative number of merchants in compartment A between the uncontrolled and controlled scenarios.

The number of refusals avoided, T_a, is given by

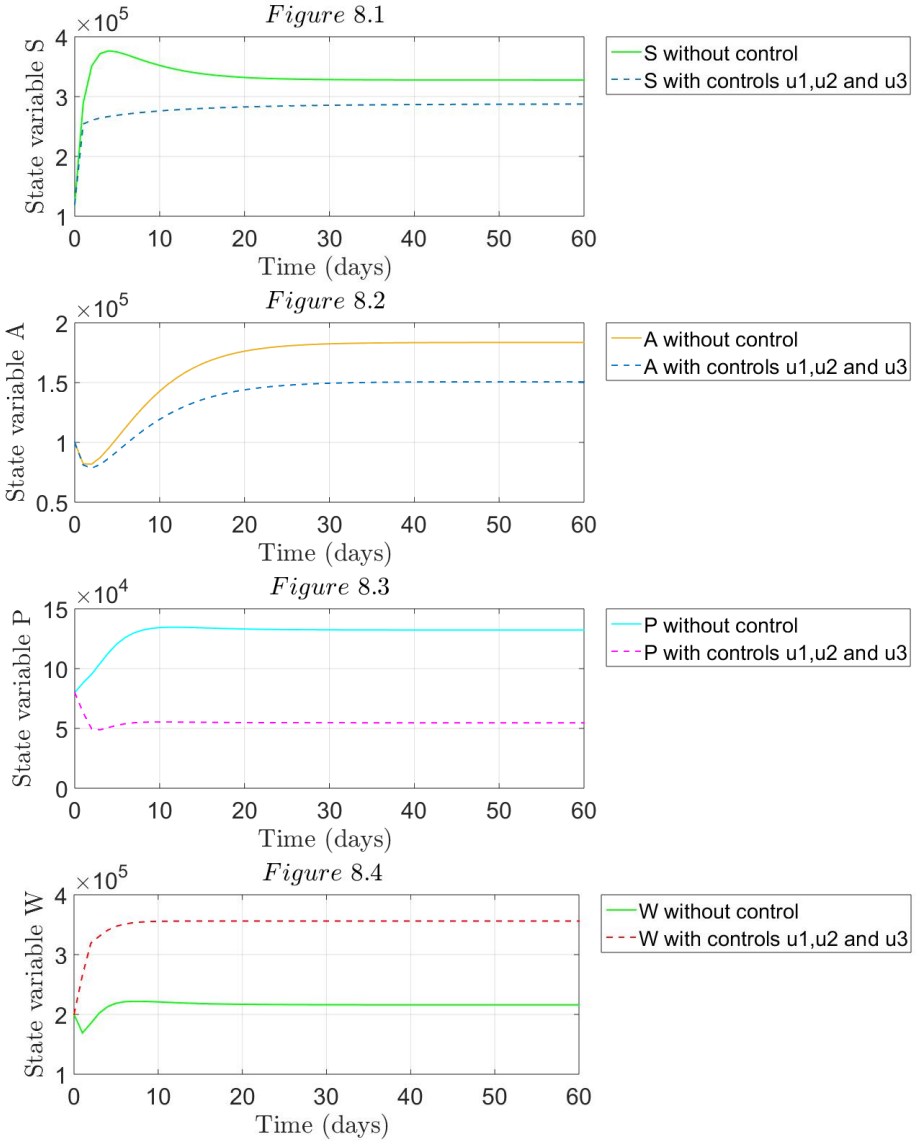


Figure 8: Dynamics of $W(t)$, $S(t)$, $A(t)$, and $P(t)$ with and without controls u_1, u_2, and u_3.

$$T_a = \int_0^T A_{\text{without}}(t)\,dt - \int_0^T A_{\text{with}}(t)\,dt,$$

where $A(t)$ denotes the number of merchants in the refusal compartment at time t, with and without intervention, respectively.

The total intervention cost T_c, assuming constant control efforts, is expressed as

$$T_c = \int_0^T \left(C_1 u_1^2(t) + C_2 u_2^2(t) + C_3 u_3^2(t)\right) dt,$$

where $u_1(t)$, $u_2(t)$, and $u_3(t)$ represent awareness campaigns, cybersecurity reinforcement, and technical training efforts, respectively. The unit costs of these interventions are given by $C_1 = 6$, $C_2 = 8$, and $C_3 = 9$.

Numerical results and interpretation

Table 2 presents the cost-effectiveness outcomes for four different intervention scenarios, each employing constant control values throughout the simulation period. The table reports the total number of avoided refusals T_a, the total cost T_c, and the corresponding ACER for each strategy.

Table 2: Cost-effectiveness analysis results for different scenarios

Scenario	T_a **(avoided refusals)**	T_c **(cost \$)**
Scenario 1	1.7004×10^5	28.80
Scenario 2	3.6109×10^5	86.40
Scenario 4	3.6617×10^5	321.00
Scenario 3	2.5455×10^2	205.80

The following observations can be made:

- **Scenario 1** is the most cost-effective, with an ACER of

$$\text{ACER}_{\text{S1}} = \frac{28.80}{1.7004 \times 10^5} \approx 0.000169,$$

  indicating that a relatively small investment can yield a significant reduction in refusals.

- **Scenario 2** achieves the highest reduction in refusals while maintaining a low ACER,

$$\text{ACER}_{\text{S2}} = \frac{86.40}{3.6109 \times 10^5} \approx 0.000239,$$

  which demonstrates a strong trade-off between cost and impact.

- **Scenario 4** is effective in reducing refusals but incurs a significantly higher cost,

$$\text{ACER}_{\text{S4}} = \frac{321.00}{3.6617 \times 10^5} \approx 0.000877,$$

making it less cost-efficient compared to Scenarios 1 and 2.

- **Scenario 3** is the least cost-effective, with a high ACER,

$$\text{ACER}_{\text{S3}} = \frac{205.80}{2.5455 \times 10^2} \approx 0.8085,$$

indicating substantial cost for very limited impact.

Remark 2. Among the evaluated strategies, **Scenario 1** proves to be the most cost-efficient, making it especially valuable in resource-constrained environments. Conversely, for objectives centered on maximizing the total number of refusals avoided, **Scenario 2** delivers the most substantial outcome, while still preserving a favorable cost-effectiveness ratio. These findings offer crucial guidance for decision-makers in optimizing intervention design and resource allocation.

5.6 Discussion

The numerical simulations demonstrate the effectiveness of the proposed control strategies in fostering e-commerce adoption among merchants. Each intervention individually contributes positively to increasing the number of merchants willing to adopt e-commerce.

- **Advertising** (u_1) plays a crucial role in raising awareness and motivating merchants to consider e-commerce.
- **Cybercrime insurance** (u_2) addresses security-related concerns, encouraging merchants to progress from awareness to a genuine willingness to adopt.
- **Training and support** (u_3) helps overcome technical and operational barriers, facilitating the transition toward active adoption.

When combined, these control measures yield a synergistic effect, significantly enhancing adoption rates compared to individual strategies. The observed reduction in the sizes of the susceptible, aware, and in-process merchant compartments indicates that more merchants are advancing swiftly toward active e-commerce participation.

These results highlight the importance of implementing comprehensive and integrated policies that simultaneously raise awareness, mitigate risks, and provide technical support. Such a multifaceted approach is essential for addressing the diverse concerns and obstacles faced by merchants, thereby substantially increasing e-commerce adoption and generating considerable economic benefits.

6 Conclusion

In this paper, we developed a mathematical model to analyze the dynamics of e-commerce adoption among merchants, accounting for both adoption and resistance behaviors. The model considers different merchant states, including traditional merchants, merchants refusing e-commerce, active e-commerce merchants, temporarily inactive merchants, and merchants leaving the market. Using the next-generation matrix method, we derived the basic reproduction number R_0 and identified two equilibrium points: The rejection-free equilibrium E_0 and the persistent-refusal equilibrium E_*. The global stability analysis showed that E_0 is globally asymptotically stable when $R_0 \leq 1$, whereas E_* is globally asymptotically stable when $R_0 > 1$, under the corresponding feasibility conditions. Thus, $R_0 = 1$ represents the threshold between the disappearance and persistence of refusal behavior. From a behavioral perspective, $R_0 < 1$ indicates that refusal tends to disappear over time, whereas $R_0 > 1$ allows refusal to persist within the merchant population.

We further formulated the model as an optimal control problem to reduce merchant resistance and promote e-commerce adoption. Using Pontryagin's maximum principle, we derived the necessary optimality conditions and characterized the optimal control strategies.

Numerical simulations confirmed the theoretical results and demonstrated the effectiveness of the proposed interventions. In particular, the combined application of advertising, cybercrime insurance, and training support was the most effective strategy among those considered, increasing e-commerce adoption while reducing merchant resistance. The cost-effectiveness analysis also provided a comparison of the intervention strategies and identified efficient approaches for promoting adoption.

The sensitivity analysis further identified the parameters with the greatest influence on merchant resistance and e-commerce adoption, highlighting the importance of awareness, training, and cybersecurity measures. These findings provide useful guidance for policymakers and stakeholders in designing targeted interventions.

Overall, the proposed framework integrates equilibrium and global stability analysis, sensitivity analysis, optimal control, and cost-effectiveness analysis to provide a quantitative approach for understanding merchant behavior and evaluating e-commerce interventions.

Nevertheless, the model relies on simplifying assumptions, such as population homogeneity and deterministic dynamics. Future work could incorporate merchant heterogeneity, network effects, stochastic dynamics, time-dependent parameters, and empirical data for calibration and validation, thereby improving the practical applicability of the model.

Author Contributions

All authors have made equal contributions to the writing of this article. The final manuscript was read and approved by all authors.

Availability of Data

Data are available upon request.

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Conflicts of Interest

The authors declare no conflicts of interest.

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