



# An approach to find the optimal design of a 3D damping set for a damped wave equation in spherical coordinates

H. Alimorad, A. Fakharzadeh Jahromi\* and M. Goodarzi

## Abstract

Damped waves play an important role in various aspects of modern life, such as analyzing ocean waves, storage tanks, automotive suspension systems, designing earthquake damping systems, and MRI machines in medicine. Furthermore, analyzing some of these systems is more naturally and efficiently performed in spherical coordinates. Considering these two important points, in this paper, we model the problem of finding the optimal shape of the damping set to minimize the energy of damped waves within a given end time defined in spherical coordinates. The basic idea is to describe the unknown 3D shape using two positive measures in spherical coordinates. This helps us to change the original problem into a linear optimization task that can be solved easily and globally. After explaining and modeling the problem and determining the weak

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solution in spherical coordinates, the measure-theoretical embedding process is also extended for the problem in spherical coordinates. First, for a fixed region, the problem is formulated as an optimal control in variational form and then converted into an infinite-dimensional linear optimization problem. To determine the nearly optimal shape of the damping set, we use an appropriate finite linear programming problem through an iterative method based on the particle swarm optimization algorithm. The advantages of this method over other methods are that it provides a linear treatment for our nonlinear optimization problem, has a global minimizer, and is independent of the initial shape and mesh design. To demonstrate the effectiveness of this method, some illustrative numerical examples are also provided.

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## 1 Introduction and background

In engineering, determining the best position and shape of sensors and actuators that minimize the energy of the system over a given time is a crucial concept [17]. Damped waves exhibit unique characteristics in three-dimensional (3D) space, including simultaneous propagation in all spatial directions, more complex wave interference in different directions, and more sophisticated calculations compared to two-dimensional (2D) cases, which lead to broader applications in realistic simulations. However, there are only a few classical methods for solving optimal shape design (OSD) problems in three dimensions, and they are generally less effective since, in most cases, they are prone to convergence issues. In this regard, the shape of the damping set has the greatest effect on the optimal rate of wave damping.

There are many studies on this optimization problem in the 2D domain. For instance, Münch, Pedregal, and Periago [25] determined the optimal damping set whose damping coefficient was fixed at a given value by applying the first-order gradient descent method. Then, the optimal damping set was determined in the case where the damping coefficient is also unknown by using the level set method [24]. Fakharzadeh, Alimorad, and Beiranvand found the optimal shape of a damping set and an unknown damping parameter simultaneously by using a measure-theoretical method [10].

It is worth noting that alternative methodologies for shape optimization of damped systems have also been explored. For instance, Zuazua [36] investigated geometric control conditions for wave energy decay, while Allaire [4] developed variational approaches for PDE-constrained shape optimization, and Henrot and Pierre [15] developed topological approaches (e.g., homogenization

and Hadamard derivatives) for PDE-constrained shape optimization. These methods, although powerful, often rely on smooth deformations or initial guesses, and may struggle with nonconvex or discontinuous design variables such as characteristic functions of damping sets.

To overcome the limitations of the classical methods mentioned above, measure-theoretical approaches have been developed. In 1999, an embedding method based on Radon measures for solving OSD problems in a 2D domain was established [13]. Other researchers have extended this method successfully for solving OSD problems in two dimensions [10, 14]; they called this method the shape measure. In this context, Mehne and Farahi [22] applied this measure-theoretic framework to transfer the variable domain of an OSD problem into a fixed reference domain, embedding the resulting problem into an infinite-dimensional linear programming (LP) model that was subsequently approximated by a finite-dimensional LP. It is worth noting that a complete overview of studies on optimal shape design using measures is listed in [8]. According to the above-mentioned studies, measure-theoretical methods are flexible, do not require any initial guess, and are not restricted by differentiability, convexity, or linearity of the cost or constraint functions.

Extending such approaches to three dimensions remains challenging. The first attempt to address a 3D symmetric optimal shape in cylindrical coordinates was made by Alimorad and Fakharzadeh [2]. Later, the measure-theoretical method was successfully extended to 3D cylindrical coordinates to solve obstacle problems [1], and to design optimal rotated shapes [7]. Although these studies demonstrated that the measure approach works well in 3D cylindrical domains, many real-world physical problems, such as damped waves in spherical tanks or acoustic fields, are naturally described in spherical coordinates. Solving these problems in cylindrical coordinates requires more complex mathematical formulations and introduces artificial boundaries. Therefore, using the extended measure-theoretical framework in spherical coordinates is necessary for solving such 3D physical problems accurately and easily. In this context, Fakharzadeh and Goodarzi [9] also adapted the shape-measure framework to spherical coordinates for designing nanoparticles, highlighting the growing need for spherical models.

While their approach successfully addressed this specific application, the present work introduces distinct and significant innovations compared to [9]. First, the physical and mathematical nature of the problems differs fundamentally. We recall that nanoparticle design typically involves elliptic or steady-state equations, whereas the damped wave equation is hyperbolic, characterized by finite propagation speeds, wave reflections, and complex interference patterns. Consequently, the energy functional involves both kinetic and potential energies, requiring weak solutions in different Sobolev spaces ( $H^1 \times L^2$ ). Second, the objective of minimizing the total mechanical energy at a strict terminal time for a wave system introduces unique challenges in exact controllability and observability that are absent in steady-state designs. Therefore, the primary

motivation of this work is to bridge the gap between measure-theoretical OSD and complex 3D hyperbolic dynamics in spherical geometries.

Recently, several advanced methods have been developed for 3D shape optimization, which can be grouped into a few main categories. For isogeometric boundary element methods (IGBEM), researchers have combined IGBEM with adjoint variables to optimize 3D acoustic shapes [5, 30]. This method directly connects computer-aided design and numerical analysis, but still relies on gradient-based optimization.

A parametric level set approach was employed to design 3D thermoelastic metamaterials [33]. Furthermore, the level set method was improved by adding local volume constraints to control the shape during optimization [18]. However, these methods require a given initial shape and often fall into local minima.

Adjoint-based methods have been widely utilized to optimize 3D fins in heat exchangers and solar receivers [34, 35], as well as underwater body shapes [21]. Furthermore, this approach has been combined with volume penalization for compressible flows [20]. While highly effective, such techniques require smooth and differentiable objective functions, limiting their application to problems with discontinuous design variables.

Although these methods are powerful, they share four common limitations:

- They rely on gradients and can fall into local minima,
- They require a given initial shape to start,
- They require mesh updates or re-initialization when the shape changes,
- They cannot easily handle discontinuous design variables, such as the characteristic function of a damping set.

Recently, heuristic, meta-heuristic, and intelligent methods such as genetic algorithms (GA), particle swarm optimization (PSO), and artificial neural networks (ANNs) have been widely used to solve complex OSD and optimal control problems [11]. These methods are powerful tools for searching in large solution spaces. However, applying them directly to PDE-constrained shape optimization requires high computational costs and complex mesh updates. In this paper, we combine two methods. First, the measure-theoretical method transforms the complex, nonlinear shape optimization problem into a simple linear programming problem (LPP). Then, the PSO algorithm is used to find the 2D projection of the unknown 3D surface. This combination is very efficient and saves a lot of computing time compared to using only intelligent methods.

To overcome the mentioned challenges, the present work uses a measure-theoretical embedding scheme. This approach offers several distinct advantages:

- No initial shape required: By using positive Radon measures, the method finds the optimal shape without needing an initial geometric guess.
- Avoiding local minima traps: By transferring into a linear programming problem, the method avoids the local minima traps of gradient-based methods.
- Direct 3D formulation: The method works directly in full 3D spherical coordinates without needing artificial symmetry assumptions.
- Mesh independence: Unlike finite element or finite volume methods, this approach does not require mesh deformation and naturally manages discontinuous control variables.

From an optimal control perspective, although measure theory is a well-known mathematical tool, its application to this specific problem introduces a new and simpler computational strategy. In standard shape optimization for wave equations, the control variable (which defines the damping set) is discontinuous, which makes the optimization problem highly nonlinear and very hard to solve. The innovation of our measure-based method has two main parts:

- It handles the discontinuous damping set naturally. We do not need to artificially make the shape smooth or use very fine meshes near the boundaries.
- By using two positive measures, the method turns the complex, nonlinear shape optimization problem into a simple linear one that can be solved globally. This is a major step forward for solving 3D shape design problems for wave equations.

Despite these recent advances in 3D shape optimization, the problem of designing an optimal damping set for damped wave equations in spherical coordinates has not yet been studied. Most existing works focus on continuous boundaries or use coordinate systems that are not suitable for spherical shapes. This important gap motivates us to apply the measure-theoretical method specifically to this problem.

Based on these advantages, this paper focuses on finding the optimal shape of this damping set for the linear damped wave equation. The goal is to minimize the total mechanical energy at a given terminal time. This optimization problem is highly nonconvex and includes a discontinuous control (the characteristic function of the damping region). By using the flexibility of this method, we avoid mesh dependency, require no initial shape, and utilize powerful linearity properties.

To solve this problem, we use Green's formula to derive the weak solution in Sobolev spaces. This variational form naturally incorporates the boundary and initial conditions.

To determine the desired 3D optimal shape of the damping set, the organization of the paper is as follows: Section 2 presents the OSD problem, Section 3 presents its variational formulation, and Section 4 reformulates the problem in spherical coordinates. Section 5 transforms the problem into a classical optimal control problem, while Section 6 presents its measure-space formulation. Section 7 discusses the approximation schemes and methods used to determine the optimal shape, and section 8 presents the numerical examples.

## 2 Classical optimal shape design problem

Let  $T > 0$  and let  $\Sigma \subset \mathbb{R}^3$  be an open and bounded domain of class  $C^2$  with a smooth boundary  $\partial\Sigma$ . Additionally, let  $\sigma$  be a measurable subset of  $\Sigma$ , which is independent of time  $t \in (0, T)$ . We consider the following linear damped wave equation in  $(0, T) \times \Sigma$  with Dirichlet boundary conditions and initial data  $(u_0, u_1)$  [31]:

$$\begin{aligned} \ddot{u} - \Delta u + a(x)\dot{u} &= 0, & (t, x) &\in (0, T) \times \Sigma, \\ u &= 0, & (t, x) &\in (0, T) \times \partial\Sigma, \\ u(0, x) &= u_0(x), \quad \dot{u}(0, x) = u_1(x), & x &\in \Sigma, \end{aligned} \tag{1}$$

where  $\dot{u} = \frac{\partial u}{\partial t}$ ,  $\ddot{u} = \frac{\partial^2 u}{\partial t^2}$  and  $\Delta$  is the Laplacian with respect to the variable  $x$ . We assume that the damping potential  $a(x) \in L^\infty(\Sigma, \mathbb{R}^+)$  has the following form [24]:

$$a(x) = a\chi_\sigma(x), \quad \text{for all } x \in \Sigma, \tag{2}$$

where  $a \in \mathbb{R}^+$  and  $\chi_\sigma$  is the characteristic function of  $\sigma$ . Indeed,  $a(x)$  represents the damping potential in  $\Sigma$  and relation (2) ensures that the damping term  $a(x)\dot{u}$  is active on  $\sigma$ . Moreover, the initial data  $(u_0, u_1)$  denote the initial position and velocity of the membrane whose displacement is governed by system (1) and are independent of both  $\sigma$  and  $a$  [25]. Moreover,  $u(t, x)$  represents the transverse displacement of the membrane at point  $x$  at time  $t$ . For  $(u_0, u_1) \in H_0^1(\Sigma) \times L^2(\Sigma)$ , system (1) is well-posed [31]. In fact, there exists a unique weak solution of (1) in the class  $C((0, T); H_0^1(\Sigma)) \cap C^1((0, T); L^2(\Sigma))$ . According to [29],  $H^1(\Sigma)$  denotes the usual Sobolev space, while  $H_0^1(\Sigma)$  denotes the Sobolev space of order one on  $\Sigma$  whose elements vanish on the boundary of  $\Sigma$ .

**Remark 1.** A Sobolev space is a vector space of functions equipped with a norm that is a combination of  $L^p$ -norms of the function together with its derivatives up to a given order.

The energy at time  $t$  associated with system (1) has the following form [31]:

$$E(\sigma, a, t) = \frac{1}{2} \int_{\Sigma} \left( |\dot{u}|^2 + |\nabla u|^2 \right) dx. \quad (3)$$

Multiplying the first equation in (1) by  $\dot{u}$ , integrating over  $\Sigma$  and using integration by parts, we have from [26]

$$\dot{E}(\sigma, a, t) = - \int_{\Sigma} a(x) |\dot{u}|^2 dx, \quad \text{for all } t > 0.$$

Based on (2) and considering that  $a \in \mathbb{R}^+$ ,  $a(x) \geq 0$  and since  $|\dot{u}|^2 \geq 0$ ,  $\dot{E}(\sigma, a, t) \leq 0$  and hence the energy is a nonincreasing function of time.

We consider the OSD problem which consists of determining the best position and shape of  $\sigma$  in  $\Sigma$  that minimizes the energy of system (1) at a given end time  $T$ . Without restrictions on  $\sigma$ , the trivial solution is  $\sigma = \Sigma$ . To restrict the volume of  $\sigma$ , we introduce the following constraint [24]:

$$\Sigma_L = \{ \sigma \subset \Sigma, \text{ volume}(\sigma) = L \times \text{volume}(\Sigma) \}, \quad 0 < L < 1. \quad (4)$$

Therefore, the mentioned OSD problem has the following form:

$$\min_{\sigma \in \Sigma_L} E(\sigma, a, T) = \frac{1}{2} \int_{\Sigma} (|\dot{u}|^2 + |\nabla u|^2) dx, \quad (5)$$

where  $u$  is the solution of system (1). The damping potential  $a$  may either be treated as a fixed parameter or considered an unknown variable, while optimizing  $\sigma$  and the aim here is to solve problem (5) using a two-phase optimization approach based on measure theory. For this purpose, first, the problem is transformed into a classical optimal control one in variational form. Next, it is presented in measure space as an infinite-dimensional optimization problem by defining two unknown positive Radon measures.

Although a similar problem was solved using cylindrical coordinates in [2], we prefer to seek the solution in spherical coordinates due to their specific advantages: Natural alignment with smooth boundaries, efficient representation of bounded domains, simple expression of Dirichlet boundary conditions, and convenient formulation of initial value problems—features that are particularly well suited to engineering and physical applications.

At the end of this section, we present three important and fundamental theorems that play a vital role in understanding the concepts introduced in subsequent sections.

**Theorem 1** (Divergence theorem). Suppose that  $\Omega$  is a subset of  $\mathbb{R}^n$  that is compact and has a piecewise smooth boundary  $\Gamma$  (indicated by  $\partial\Omega = \Gamma$ ). If  $F$  is a continuously differentiable vector field defined on a neighborhood of  $\Omega$ , then

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) \, d\Omega = \oint_{\Gamma} (\mathbf{F} \cdot \hat{\mathbf{n}}) \, d\mathbf{l},$$

where the left side is a volume integral over  $\Omega$ , and the right side is the surface integral over the boundary of  $\Omega$  [32].

**Theorem 2** (Stokes' theorem). Let  $S$  be a piecewise smooth, orientable surface bounded by a piecewise smooth closed curve  $C$ , and let  $\mathbf{n}$  denote the unit normal vector to  $S$ . The orientations of  $S$  and  $C$  are assumed to be consistent according to the right-hand rule. In particular, when the fingers of the right hand follow the direction of traversal of  $C$ , the thumb points in the direction of  $\mathbf{n}$ . For a surface forming part of the boundary of a solid region,  $\mathbf{n}$  points outward, and  $C$  is traversed in the positive (counterclockwise) direction when viewed from the side of the normal. The vector differential element of arc length along  $C$  is denoted by  $d\mathbf{l}$ .

Let  $\mathbf{F}$  be a continuously differentiable vector field defined on an open region containing  $S$ . Then

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{l}.$$

The oriented boundary of  $S$  is sometimes denoted by  $\partial S$ . Green's theorem can be viewed as a special case of Stokes' theorem applied to a planar surface [32].

**Theorem 3** (Green's theorem in the plane). Let  $D$  be a region in the  $xy$ -plane bounded by a piecewise smooth closed curve  $C$ . If  $f(x, y)$  and  $g(x, y)$  have continuous first partial derivatives on an open region containing  $D$ , then

$$\int_D \left( \frac{\partial g}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right) dA = \oint_C f(x, y) dx + g(x, y) dy,$$

where  $C$  is oriented counterclockwise.

For a vector field  $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ , when applying Stokes' theorem to a flat surface oriented upward in the  $xy$ -plane, we can identify  $F_1(x, y, 0)$  with  $f(x, y)$  and  $F_2(x, y, 0)$  with  $g(x, y)$ . Since the surface lies in the  $xy$ -plane,  $F_3$  vanishes identically. Furthermore,  $d\mathbf{l}$  is perpendicular to  $\mathbf{k}$  (the unit normal vector pointing in the positive  $z$ -direction), which explains why the third component does not contribute to the integral [32].

### 3 Weak solution

Establishing the existence of a classical solution  $u$  for problem (5) subject to system (1) is difficult. Therefore, weak (generalized) solutions are usually considered [23, 27]. In general, the main steps

presented in this paper could be introduced in this way. First, we state the classical problem as integral equations. Then, by introducing positive Radon measures, we change the problem into a measure space. In the third step, to overcome difficulties such as the existence of solutions, we extend the underlying space of the problem. In this step, we have an infinite linear programming problem in measure spaces. Finally, after solving the related LPP, the path functions and optimal control are calculated. This method has several advantages, such as the automatic existence of a solution, dealing with an alternative linear problem despite the nonlinearity of the original one, and the ability to find a global optimal solution for the problem. In this paper, this method has been extended for finding the optimal shape of damping set in 3D spherical coordinates to obtain the desired 3D unknown surface directly.

To solve (5) subject to system (1), first, the problem must be transformed into its variational form. For this purpose, we consider the following kind of Green's formula [32]:

$$\int_{\Sigma} (u\Delta V - V\Delta u)dx = \int_{\partial\Sigma} (u\frac{\partial V}{\partial n} - V\frac{\partial u}{\partial n})ds,$$

where  $u$  and  $V$  are both twice continuously differentiable on  $\Sigma$  and  $n$  is the outward-pointing unit normal vector of the surface. Let  $V \in H_0^1(\Sigma \times (0, T))$ ; then, it is zero on the boundary  $\partial\Sigma$  and  $\int_{\partial\Sigma} (V\frac{\partial u}{\partial n})ds = 0$ . Additionally, from equation system (1), we have  $u = 0$  on the boundary  $\partial\Sigma$  and  $\int_{\partial\Sigma} (u\frac{\partial V}{\partial n})ds = 0$ . As a result  $\int_{\Sigma} (u\Delta V - V\Delta u)dx = 0$ , and we obtain

$$\int_{\Sigma} V\Delta u \, dx = \int_{\Sigma} u\Delta V \, dx. \quad (6)$$

Similarly, using the following consequence of Green's formula:

$$\int_{\Sigma} (u\Delta V + \nabla u \cdot \nabla V)dx = \int_{\partial\Sigma} u \nabla V \cdot n \, ds,$$

Since  $V \in H_0^1(\Sigma \times (0, T))$  is zero on the boundary  $\partial\Sigma$  and by system (1),  $u = 0$  on the boundary  $\partial\Sigma$ , then,  $\int_{\partial\Sigma} u \nabla V \cdot n \, ds = 0$  and hence we have

$$\int_{\Sigma} u\Delta V \, dx = - \int_{\Sigma} \nabla u \nabla V \, dx. \quad (7)$$

By (6) and (7), the following relation is obtained:

$$\int_{\Sigma} V\Delta u \, dx = - \int_{\Sigma} \nabla u \nabla V \, dx. \quad (8)$$

By multiplying the first equation in (1) with a function  $V \in H_0^1(\Sigma \times (0, T))$  and applying (8), we have

$$\int_{\Sigma} \ddot{u}V \, dx + \int_{\Sigma} \nabla u \cdot \nabla V \, dx + \int_{\Sigma} a(x)\dot{u}V \, dx = 0. \quad (9)$$

Integrating both sides of (9) with respect to  $t$  over  $[0, T]$  gives

$$\int_0^T \int_{\Sigma} \ddot{u}V \, dxdt + \int_0^T \int_{\Sigma} \nabla u \cdot \nabla V \, dxdt + \int_0^T \int_{\Sigma} a(x)\dot{u}V \, dxdt = 0. \quad (10)$$

Double integration by parts with respect to  $t$  from the first term on the left-hand side of (10) shows that

$$\begin{aligned} \int_0^T \int_{\Sigma} \ddot{u}V \, dxdt &= \int_{\Sigma} (\dot{u}(T)V(T) - \dot{u}(0)V(0) - u(T)\dot{V}(T) \\ &\quad + u(0)\dot{V}(0))dx + \int_0^T \int_{\Sigma} u\ddot{V} \, dxdt. \end{aligned} \quad (11)$$

Similarly, integrating the third expression on the left-hand side of (10) gives

$$\begin{aligned} \int_0^T \int_{\Sigma} a(x)\dot{u}V \, dxdt &= \int_{\Sigma} a(x)(u(T)V(T) \\ &\quad - u(0)V(0)) \, dx - \int_0^T \int_{\Sigma} a(x)u\dot{V} \, dxdt. \end{aligned} \quad (12)$$

Now, the initial conditions of system (1) can be substituted into (11) and (12), respectively, to achieve

$$\begin{aligned} \int_0^T \int_{\Sigma} \ddot{u}V \, dxdt &= \int_{\Sigma} (\dot{u}(T)V(T) - u_1(x)V(0) \\ &\quad - u(T)\dot{V}(T) + u_0(x)\dot{V}(0))dx + \int_0^T \int_{\Sigma} u\ddot{V} \, dxdt, \end{aligned} \quad (13)$$

$$\begin{aligned} \int_0^T \int_{\Sigma} a(x)\dot{u}V \, dxdt &= \int_{\Sigma} a(x)(u(T)V(T) \\ &\quad - u_0(x)V(0)) \, dx - \int_0^T \int_{\Sigma} a(x)u\dot{V} \, dxdt. \end{aligned} \quad (14)$$

By applying (13) and (14), equation (10) is transformed as follows:

$$\begin{aligned}
& \int_{\Sigma} \dot{u}(T)V(T) dx - \int_{\Sigma} u(T)\dot{V}(T) dx + \int_0^T \int_{\Sigma} u\ddot{V} dxdt \\
& + \int_0^T \int_{\Sigma} \nabla u \cdot \nabla V dxdt + \int_{\Sigma} a(x)u(T)V(T) dx \\
& - \int_{\Sigma} a(x)u_0(x)V(0) dx - \int_0^T \int_{\Sigma} a(x)u\dot{V} dxdt \\
& = \int_{\Sigma} (u_1(x)V(0) - u_0(x)\dot{V}(0)) dx.
\end{aligned} \tag{15}$$

The function  $u$  which satisfies (15) for each  $V \in H_0^1(\Sigma \times (0, T))$  is called a weak solution of (1). Indeed, Green's formulas allow us to replace (1) with (15), which is simpler for the shape-measure method. Therefore, we change the problem and consider a new one with a different formulation. Additionally, constraint (4) on the volume of  $\sigma$ , can be shown by the following relation:

$$\int_{\sigma} dx = L \int_{\Sigma} dx, \quad 0 < L < 1. \tag{16}$$

Based on Green's theorem, the variational formulation of problem (5) subject to system (1) is revised and can be written in the following form:

$$\begin{aligned}
\min E(\sigma, a, t) &= \frac{1}{2} \int_{\Sigma} (|\dot{u}|^2 + |\nabla u|^2) dx \\
\text{subject to : } & \int_{\Sigma} \dot{u}(T)V(T) dx - \int_{\Sigma} u(T)\dot{V}(T) dx + \int_0^T \int_{\Sigma} u\ddot{V} dxdt \\
& + \int_0^T \int_{\Sigma} \nabla u \cdot \nabla V dxdt + \int_{\Sigma} a(x)u(T)V(T) dx \\
& - \int_{\Sigma} a(x)u_0(x)V(0) dx - \int_0^T \int_{\Sigma} a(x)u\dot{V} dxdt \\
& = \int_{\Sigma} (u_1(x)V(0) - u_0(x)\dot{V}(0)) dx, \\
& \int_{\sigma} dx = L \int_{\Sigma} dx, \quad 0 < L < 1.
\end{aligned} \tag{17}$$

## 4 Expressing the problem in a spherical coordinate system

Based on the abilities of spherical coordinates for simplifying the formulation of some special shapes, we prefer to express and solve problem (5) in spherical coordinates. In this regard, for each  $u(t, \rho, \varphi, \theta) \in C^2(\mathbb{R}^4)$ , we have from [6]

$$\nabla u|_T = \left( \frac{\partial u}{\partial \rho}, \frac{1}{\rho} \frac{\partial u}{\partial \varphi}, \frac{1}{\rho \sin \varphi} \frac{\partial u}{\partial \theta} \right).$$

Therefore, (3) can be expressed as follows:

$$\begin{aligned} E(\sigma, a, T) = & \frac{1}{2} \int_{\Sigma} (|\dot{u}(\rho, \varphi, \theta, T)|^2 + \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \left(\frac{\partial u}{\partial \varphi}\right)^2 \\ & + \frac{1}{\rho^2 \sin^2 \varphi} \left(\frac{\partial u}{\partial \theta}\right)^2) \rho^2 \sin \varphi \, d\rho d\varphi d\theta. \end{aligned}$$

Thus, expressing problem (17) in spherical coordinates yields the following formulation for every  $V \in H_0^1(\Sigma \times (0, T))$ :

$$\begin{aligned} \min E(\sigma, a, T) = & \frac{1}{2} \int_{\Sigma} (|\dot{u}(\rho, \varphi, \theta, T)|^2 + \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \left(\frac{\partial u}{\partial \varphi}\right)^2 \\ & + \frac{1}{\rho^2 \sin^2 \varphi} \left(\frac{\partial u}{\partial \theta}\right)^2) \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\ \text{subject to : } & \int_{\Sigma} \dot{u}(T) V(T) \rho^2 \sin \varphi \, d\rho d\varphi d\theta - \int_{\Sigma} u(T) \dot{V}(T) \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\ & + \int_0^T \int_{\Sigma} u \ddot{V} \rho^2 \sin \varphi \, d\rho d\varphi d\theta dt + \int_0^T \int_{\Sigma} \nabla u \cdot \nabla V \rho^2 \sin \varphi \, d\rho d\varphi d\theta dt \\ & + \int_{\Sigma} a(\rho, \varphi, \theta) u(T) V(T) \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\ & - \int_{\Sigma} a(\rho, \varphi, \theta) u_0(\rho, \varphi, \theta) V(0) \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\ & - \int_0^T \int_{\Sigma} a(\rho, \varphi, \theta) u \dot{V} \rho^2 \sin \varphi \, d\rho d\varphi d\theta dt = Y, \\ & \int_{\sigma} \rho^2 \sin \varphi \, d\rho d\varphi d\theta = L \int_{\Sigma} \rho^2 \sin \varphi \, d\rho d\varphi d\theta, \quad 0 < L < 1, \end{aligned} \quad (18)$$

where

$$Y = \int_{\Sigma} (u_1(\rho, \varphi, \theta) V(0) - u_0(\rho, \varphi, \theta) \dot{V}(0)) \rho^2 \sin \varphi \, d\rho d\varphi d\theta.$$

We integrate the cost and constraint functions in (18) over  $[0, T]$ . Therefore, this problem can be represented in a new variational form for every  $V \in H_0^1(\Sigma \times (0, T))$  as follows:

$$\begin{aligned} \min E(\sigma, a, T) = & \frac{1}{2T} \int_0^T \int_{\Sigma} (|\dot{u}(\rho, \varphi, \theta, T)|^2 + \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \left(\frac{\partial u}{\partial \varphi}\right)^2 \\ & + \frac{1}{\rho^2 \sin^2 \varphi} \left(\frac{\partial u}{\partial \theta}\right)^2) \rho^2 \sin \varphi \, d\rho d\varphi d\theta dt \\ \text{subject to : } & \frac{1}{T} \left( \int_0^T \int_{\Sigma} \dot{u}(T) V(T) \rho^2 \sin \varphi \, d\rho d\varphi d\theta dt \right. \end{aligned} \quad (19)$$

$$\begin{aligned}
& - \int_0^T \int_{\Sigma} u(T) \dot{V}(T) \rho^2 \sin \varphi \, d\rho d\phi d\theta dt \\
& + \int_0^T \int_{\Sigma} u \ddot{V} \rho^2 \sin \varphi \, d\rho d\phi d\theta dt + \int_0^T \int_{\Sigma} \nabla u \cdot \nabla V \rho^2 \sin \varphi \, d\rho d\phi d\theta dt \\
& + \frac{1}{T} \int_0^T \int_{\Sigma} a(\rho, \varphi, \theta) u(T) V(T) \rho^2 \sin \varphi \, d\rho d\phi d\theta dt \\
& - \frac{1}{T} \int_0^T \int_{\Sigma} a(\rho, \varphi, \theta) u_0(\rho, \varphi, \theta) V(0) \rho^2 \sin \varphi \, d\rho d\phi d\theta dt \\
& - \int_0^T \int_{\Sigma} a(\rho, \varphi, \theta) u \dot{V} \rho^2 \sin \varphi \, d\rho d\phi d\theta dt = Y, \\
& \int_{\sigma} \rho^2 \sin \varphi \, d\rho d\phi d\theta - \frac{L}{T} \int_0^T \int_{\Sigma} \rho^2 \sin \varphi \, d\rho d\phi d\theta dt = 0.
\end{aligned} \tag{20}$$

In fact, regarding the new formulation in (19), a 3D shape  $\sigma$  is designed as the damping set in spherical coordinates which is smooth, bounded with a specified volume and placed over the  $\varphi = \frac{\pi}{2}$  plane. It is also supposed that the boundary of  $\sigma$  consists of two parts. The first one is an unknown smooth surface  $S$ , which is defined by the equation  $\varphi = \zeta(\rho, \theta) \in C^2(\mathbb{R}^2)$  and the second part is an unknown region  $D$ , which is the projection of  $S$  in the  $\varphi = \frac{\pi}{2}$  plane (see Figure 1). In addition, a simple smooth closed curve  $\partial D$ , which is represented by the piecewise smooth function  $r = h(\theta)$ ,  $0 \leq \theta \leq 2\pi$  in polar coordinates and passes through a known point  $B = (h(\theta_0), \frac{\pi}{2}, \theta_0)$  surrounds  $D$ ; see Figure 2.

We recall that there exist the following relations for  $D$  and  $\partial D$  in this problem:

$$D = \left\{ (\rho, \frac{\pi}{2}, \theta) \mid 0 \leq \theta \leq 2\pi, \quad 0 \leq \rho \leq h(\theta) \right\}, \tag{21}$$

$$\partial D = \left\{ (\rho, \frac{\pi}{2}, \theta) \mid 0 \leq \theta \leq 2\pi, \quad \rho = h(\theta) \right\}.$$

Here, an unknown smooth surface  $S$  which is defined by the equation  $\varphi = \zeta(\rho, \theta) \in C^2(\mathbb{R}^2)$  is called admissible and  $S_A$  is considered the set containing all such admissible surfaces. Also, since shape  $\sigma$  is bounded, we conclude that there exist scalars  $\varphi_{\min}$ ,  $\varphi_{\max}$  and  $\rho_{\max}$  such that,  $\varphi_{\min} \leq \varphi \leq \varphi_{\max}$  and  $0 \leq \rho \leq \rho_{\max}$ .

## 5 Transferring the OSD problem to a classical optimal control problem

In solving problem (19) by applying a new version of the shape-measure method, first, it must be transformed into a classical optimal control problem. For this purpose, the functions  $(\varphi, u)$

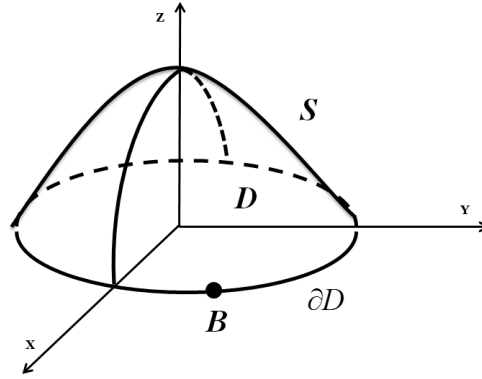


Figure 1: A version of unknown shape.

are considered the trajectory (state) and the vectors  $(\dot{u}, u_\rho, u_\varphi, u_\theta, \zeta_\rho, \zeta_\theta, \zeta_{\rho\theta})$  are considered the control functions (the necessity of defining artificial controls  $\zeta_\rho$ ,  $\zeta_\theta$  and  $\zeta_{\rho\theta}$  will be explained in later sections). The first constraint in (19) specifies the relation between variables  $u$ ,  $\dot{u}$ ,  $u_\rho$ ,  $u_\varphi$  and  $u_\theta$ . In addition, the dependency of the artificial controls  $\zeta_\rho$ ,  $\zeta_\theta$  and  $\zeta_{\rho\theta}$  is demonstrated as follows: Assume that  $h_1(\varphi, \theta)$  and  $h_2(\rho, \varphi)$  are two arbitrary functions of class  $C^1(D \times A)$  (the space of continuously differentiable functions on  $D \times A$ ), by using the chain rule (we remember that  $\varphi = \zeta(\rho, \theta)$ ), we have

$$\frac{\partial h_1}{\partial \rho} = \frac{\partial h_1}{\partial \varphi} \frac{\partial \varphi}{\partial \rho} = \frac{\partial h_1}{\partial \varphi} \zeta_\rho, \quad \frac{\partial h_2}{\partial \theta} = \frac{\partial h_2}{\partial \varphi} \frac{\partial \varphi}{\partial \theta} = \frac{\partial h_2}{\partial \varphi} \zeta_\theta. \quad (22)$$

**Definition 1.** We define the vector  $P = (\varphi, u, \dot{u}, u_\rho, u_\varphi, u_\theta, \zeta_\rho, \zeta_\theta, \zeta_{\rho\theta})$  as admissible if the following conditions hold:

- (i)  $\varphi$  is an absolutely continuous trajectory function that takes its values in a bounded subset of  $\mathbb{R}$ .
- (ii)  $u$  is the weak solution of system (1) that corresponds to (15) and takes its values in a bounded set  $U$ .
- (iii) The control functions  $\dot{u}$ ,  $u_\rho$ ,  $u_\varphi$ ,  $u_\theta$ ,  $\zeta_\rho$ ,  $\zeta_\theta$  and  $\zeta_{\rho\theta}$  are piecewise-continuous and take their values in the bounded subsets  $\dot{U}$ ,  $U_\rho$ ,  $U_\varphi$ ,  $U_\theta$ ,  $F_\rho$ ,  $F_\theta$  and  $F_{\rho\theta}$  in  $\mathbb{R}$ . Moreover, the control functions  $\zeta_\rho$ ,  $\zeta_\theta$  and  $\zeta_{\rho\theta}$  should be Lebesgue measurable.
- (iv) Vector  $P$  satisfies constraints (19) and (22).

By considering the regularity of the initial data  $(u_0, u_1)$ , as explained in Section 2, the existence of a weak solution for (1) is guaranteed. Consequently, the set of all the admissible

vectors is nonempty, which is denoted by  $W$ . By Definition 1, each admissible surface  $S \in S_A$  can be replaced exactly by one admissible vector  $P = (\varphi, u, \dot{u}, u_\rho, u_\varphi, u_\theta, \zeta_\rho, \zeta_\theta, \zeta_{\rho\theta}) \in W$  [14]. In this regard, minimizing the objective function of (19) over  $S_A$  is equivalent to minimizing it over  $W$ . Therefore, we can convert problem (19) to a classical optimal control problem. To embed the solution space, we define regions  $D'$  and  $D''$  as follows:

$$\begin{aligned} D' &= [0, T] \times [0, 2\pi] \times [0, \rho_\Sigma] \times [0, \varphi_\Sigma] \times U \times \dot{U} \times U_\rho \times U_\varphi \times U_\theta, \\ D'' &= D_1 \times F_\rho \times F_\theta \times F_{\rho\theta}, \end{aligned} \quad (23)$$

where  $D_1 = (D \times \{\frac{\pi}{2}\}) \cup ([0, \rho_{\max}] \times [0, 2\pi] \times [0, \frac{\pi}{2}])$  and  $F_\rho$ ,  $F_\theta$  and  $F_{\rho\theta}$  are bounded subsets of  $\mathbb{R}$  which are defined in Definition 1.

## 6 Embedding the solution space

With the help of the shape-measure approach, which is based on an embedding process, first, we reduce the problem to one consisting of minimizing a linear function over a set of positive measures. Afterwards, we approximate the resulting problem using a finite-dimensional linear programming problem for finding the nearly optimal solution. Then, we use the solution of this problem to construct a sufficiently accurate approximate solution for the original problem. As mentioned, the remarkable advantages of this method over other methods include linear treatment for nonlinear shape optimization problems and global optimization independent of the initial shape and mesh design. Some of the most important existing issues regarding this optimal control problem are as follows:

- (i) The possibility of lacking a minimizing admissible vector for  $E$  in  $W$ .
- (ii) Characterizing an admissible minimizer vector for  $E$  in  $W$  could be very difficult, and even it might be more difficult or impossible to estimate numerically.

It appears that these situations may become more favorable if set  $W$  could somehow be made larger. Therefore, we apply the transformation for embedding (19) into another nonclassical problem that has better properties in some respects, allowing us to overcome the mentioned difficulties.

## 6.1 Characteristics of admissible vectors

To solve problem (19) using the embedding process [28], first, we point out some characteristics of the admissible vector in  $W$ . Assume that  $B$  is an open ball containing  $D \times A$  (region  $D$  is defined in (21)) and that  $C'(B)$  is the space of real-valued continuously differentiable functions such that their first and second derivatives are continuous and bounded on  $B$ , for  $\Psi \in C'(B)$ , we define the function  $\Psi^g \in C(D'')$  as

$$\Psi^g(\rho, \varphi, \theta, \zeta_\rho, \zeta_\theta, \zeta_{\rho\theta}) = (\Psi_{\theta\varphi} - \Psi_\theta \cot \varphi) \frac{\zeta_\rho}{\zeta_\theta} - \Psi_{\rho\varphi} - \frac{1}{\rho} \Psi_\varphi.$$

The function  $\Psi$  has continuous partial derivatives and according to Definition 1, the control functions  $\zeta_\rho$ ,  $\zeta_\theta$  and  $\zeta_{\rho\theta}$  are Lebesgue measurable. As a result,  $\Psi^g$  is integrable on  $D \times A$  [29] and by using Stokes' theorem [32], for  $F = \nabla(\rho \Psi)$ , we have

$$\begin{aligned} \oint_{\partial S} F dR &= \iint_S \nabla \times F \cdot n d\sigma = \iint_D \nabla \times F \cdot \frac{\nabla K}{|\nabla K|} \frac{|\nabla K|}{|\nabla K \cdot p|} dA \\ &= \iint_D ((\Psi_{\theta\varphi} - \Psi_\theta \cot \varphi) \frac{\zeta_\rho}{\zeta_\theta} - \Psi_{\rho\varphi} - \frac{1}{\rho} \Psi_\varphi) \rho d\rho d\theta \equiv \rho_0 \Delta_\Psi, \end{aligned} \quad (24)$$

where  $\rho_0 \equiv h(\theta_0)$  and since  $\partial D$  is a closed curve,  $\Delta_\Psi = \Psi(\rho_0, \frac{\pi}{2}, 2\pi) - \Psi(\rho_0, \frac{\pi}{2}, 0)$ . Additionally,  $\Psi_\theta$ ,  $\Psi_\varphi$  and  $\Psi_{\theta\varphi}$ ,  $\Psi_{\rho\varphi}$  are the first and second order partial derivatives of  $\Psi$ .

The curl of the arbitrary function  $F = F_\rho \hat{\rho} + F_\varphi \hat{\varphi} + F_\theta \hat{\theta}$  in spherical coordinates is defined from [6] as follows:

$$\begin{aligned} \text{curl } F &= \frac{1}{\rho^2 \sin \varphi} \begin{vmatrix} \hat{\rho} & \rho \hat{\varphi} & \rho \sin(\varphi) \hat{\theta} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\varphi & \rho \sin(\varphi) F_\theta \end{vmatrix} \\ &= \frac{1}{\rho \sin \varphi} \left( \frac{\partial}{\partial \varphi} (F_\theta \sin \varphi) - \frac{\partial F_\varphi}{\partial \theta} \right) \hat{\rho} \\ &\quad + \frac{1}{\rho} \left( \frac{1}{\sin \varphi} \frac{\partial F_\rho}{\partial \theta} - \frac{\partial}{\partial \rho} (\rho F_\theta) \right) \hat{\varphi} + \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho F_\varphi) - \frac{\partial F_\rho}{\partial \varphi} \right) \hat{\theta}, \end{aligned}$$

where  $\hat{\rho}$ ,  $\hat{\varphi}$  and  $\hat{\theta}$  are the unit vectors in increasing directions of  $\rho$ ,  $\varphi$  and  $\theta$ , respectively and  $F_\rho$ ,  $F_\varphi$  and  $F_\theta$  are the components of field  $F$ .

Assume that  $\wp(D^\circ)$  is the space of infinitely differentiable real-valued functions with compact support in  $D^\circ$  (the set of interior points of  $D$ ). We consider  $\gamma(\rho, \theta) \in \wp(D^\circ)$  and define  $\Theta \in C(D'')$  as follows:

$$\begin{aligned}
\Theta(\rho, \varphi, \theta, \zeta_\rho, \zeta_\theta, \zeta_{\rho\theta}) = & (2 + \frac{(\varphi - \frac{\pi}{2}) \cos \varphi}{\sin \varphi} - \frac{1}{\sin \varphi} - \frac{1}{\rho \sin \varphi}) \frac{\zeta_\rho \gamma_\theta \sin \varphi}{\zeta_\theta} \\
& + (\frac{-2}{\rho} + \sin \varphi + 1) \gamma_\rho \\
& + \frac{(\frac{-1}{\rho} + \sin \varphi)}{\zeta_\theta} (\zeta_{\rho\theta} \gamma + \gamma_{\rho\theta} (\varphi - \frac{\pi}{2})) \\
& + (\frac{\sin \varphi + 1}{\rho} + \zeta_\rho \cos \varphi) \gamma + \frac{1}{\rho \zeta_\theta} \sin(\varphi) (\varphi - \frac{\pi}{2}) \gamma_\theta.
\end{aligned}$$

Here we have  $\gamma_\theta$  and  $\gamma_\rho$  as the first-order partial derivatives of  $\gamma$  and  $\gamma_{\rho\theta}$  as the second-order partial derivative of  $\gamma$ . Since every infinitely differentiable function is absolutely continuous because it satisfies the Lipschitz condition, by Definition 1, the trajectory function  $\varphi$  is absolutely continuous, and  $(\varphi - \frac{\pi}{2}), \gamma(\rho, \theta)$  is also an absolutely continuous function. Therefore, its partial derivatives are analytically integrable [29].

Now, if we consider  $\Psi_1(\rho, \varphi, \theta) = (\varphi - \frac{\pi}{2})\gamma(\rho, \theta)$  and define

$$F \equiv (\Psi_{1\rho}, \Psi_{1\theta}, \Psi_{1\theta}) = (\zeta_\rho \gamma + \gamma_\rho (\varphi - \frac{\pi}{2}), \gamma, \zeta_\theta \gamma + (\varphi - \frac{\pi}{2}) \gamma_\theta),$$

then by Stokes' theorem we have

$$\begin{aligned}
\oint_{\partial S} F \cdot dR &= \iint_S \nabla \times F \cdot n \, d\sigma = \iint_D \frac{(\nabla \times F) \cdot \nabla K}{|\nabla K \cdot p|} \, dA \\
&= \iint_D \frac{(\nabla \times F) \cdot \nabla K}{\frac{\zeta_\theta}{\rho \sin \varphi}} \, dA = \iint_D \Theta \, \rho \, d\rho \, d\theta = 0.
\end{aligned} \tag{25}$$

The value of  $\gamma$  on  $\partial D$  is zero because function  $\gamma$  has compact support in  $D^\circ$  [29], and regarding the definition of  $F$ , the value of  $F$  on  $\partial D$  is also equal to zero. Additionally, assuming that  $C_1(D'')$  is the space of all functions in  $C(D'')$  depending only on the variables  $\rho$  and  $\theta$ , we have

$$\int_S \frac{\zeta_\theta}{\sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{\zeta_\theta^2}{\rho^2 \sin^2 \varphi}} \rho \sin \varphi} g(\rho, \theta) \, d\sigma = \iint_D g(\rho, \theta) \rho \, d\rho \, d\theta \equiv a_g, \tag{26}$$

for all  $g \in C_1(D'')$ , where  $a_g$  is the Lebesgue integral of  $g(\rho, \theta)$  over  $D$ .

## 6.2 Transformation

Suppose that  $D'$  and  $D''$  are defined as in (23). For any admissible vector  $P$  in  $W$ , we define the linear, positive functionals  $\Gamma_p$  and  $\Upsilon_p$  on  $C(D')$  and  $C(D'')$  of all continuous real-valued

functions as

$$\begin{aligned}\Gamma_P(F) &= \int_0^T \int_{\Sigma} F \rho^2 \sin \varphi d\rho d\phi d\theta dt, \quad \text{for all } F \in C(D'), \\ \Upsilon_P(I) &= \iint_D I \rho d\rho d\theta, \quad \text{for all } I \in C(D'').\end{aligned}$$

To enlarge the set  $W$ , we denote the space of all positive Radon measures on  $D'$  and  $D''$  by  $M^+(D')$  and  $M^+(D'')$ , respectively. Using Riesz's representation theorem [29], there exist unique positive Radon measures  $\mu \in M^+(D')$  and  $\nu \in M^+(D'')$  such that

$$\Gamma(F) = \int_0^T \int_{\Sigma} F d\mu \equiv \mu(F), \quad \text{for all } F \in C(D'), \quad (27)$$

$$\Upsilon(I) = \iint_D I d\lambda \equiv \nu(I), \quad \text{for all } I \in C(D''). \quad (28)$$

According to [28], these transformations are injective. Consequently, any admissible element can be displayed as (27) or (28) by a unique pair of measures, say  $(\mu, \nu)$ , in a subset  $W$  of  $M^+(D') \times M^+(D'')$ . Therefore, one can transform problem (19) into a measure space as follows:

$$(\varphi, u, \dot{u}, u_\rho, u_\varphi, u_\theta, \zeta_\rho, \zeta_\theta, \zeta_{\rho\theta}) \in P \longmapsto (\mu, \nu) \in M^+(D') \times M^+(D'').$$

Given the definitions of the measures in (27) and (28), it is necessary to rewrite constraint (16) with respect to only one of the corresponding integrals,  $\mu$  or  $\nu$ . To this end, and following [6, 32], we can apply the divergence theorem (see Theorem 1) by regarding

$$\begin{aligned}\nabla K &= (\zeta_\rho, \frac{-1}{\rho}, \frac{1}{\rho \sin \varphi} \zeta_\theta), \\ dS &= \frac{|\nabla K|}{|\nabla K \cdot p|} dA = \frac{\sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho^2 \sin^2 \varphi} \zeta_\theta^2}}{\zeta_\theta} \rho \sin(\varphi) \rho d\rho d\theta.\end{aligned}$$

Suppose that the equation of surface  $S$  is given by  $K \equiv \zeta(\rho, \theta) - \varphi = 0$ , where  $dS$  is the differential element of the surface area and  $p = (0, 0, 1)$  is a unit vector normal to  $D$ . It is important to mention that the area element is given by  $dA = \rho \sin(\varphi) d\rho d\theta$ . Thus, we have  $dA = \rho d\rho d\theta$  because in  $D$  we have  $\varphi = \frac{\pi}{2}$ . Hence, the calculation can be simplified by converting the integral over  $\sigma$  in (19) into an integral over  $D$  with  $\partial\sigma = S \cup D$  using the divergence theorem [32] as follows:

$$\int_{\sigma} \operatorname{div} F dV = \oint_{S \cup D} F \cdot n dS$$

$$= \iint_D F \cdot \frac{\nabla K}{|\nabla K|} \frac{\sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho^2 \sin^2 \varphi} \zeta_\theta^2}}{\zeta_\theta} \rho \sin(\varphi) \rho d\rho d\theta,$$

where  $n$  is an outward-pointing unit normal vector of the boundary  $\sigma$ . Since the gradient vector of  $K$  in spherical coordinates is  $\nabla K = (\zeta_\rho, \frac{-1}{\rho}, \frac{1}{\rho \sin \varphi} \zeta_\theta)$ , we have  $|\nabla K| = \sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho^2 \sin^2 \varphi} \zeta_\theta^2}$ . Also,  $\operatorname{div} F$  in spherical coordinates is calculated by the following formula:

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 F_\rho) + \frac{1}{\rho \sin \varphi} \left( \frac{\partial F_\theta}{\partial \theta} \right) + \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} (\sin \varphi F_\varphi).$$

Because  $\int_\sigma dV$  is the volume constraint, we consider the vector function  $F$  so that its divergence is a constant number. If we substitute  $F = \rho \hat{\rho} + \rho \hat{\varphi} - \theta \rho \cos(\varphi) \hat{\theta}$  in the above statement, from the definition of divergence in spherical coordinates [6], we have  $\operatorname{div} F = 3$  (with  $\hat{\rho}$ ,  $\hat{\varphi}$  and  $\hat{\theta}$  as unit vectors in increasing directions of  $\rho$ ,  $\varphi$  and  $\theta$ , respectively). We conclude

$$\begin{aligned} & \iint_D F \cdot \frac{\nabla K}{|\nabla K|} \frac{\sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho^2 \sin^2 \varphi} \zeta_\theta^2}}{\zeta_\theta} \rho \sin(\varphi) \rho d\rho d\theta \\ &= \iint_D (\rho, \rho, -\theta \rho \cos(\varphi)) \cdot \left( \frac{(\zeta_\rho, \frac{-1}{\rho}, \frac{1}{\rho \sin \varphi} \zeta_\theta)}{\sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho^2 \sin^2 \varphi} \zeta_\theta^2}} \right) \\ & \quad \frac{\sqrt{\zeta_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho^2 \sin^2 \varphi} \zeta_\theta^2}}{\zeta_\theta} \rho \sin(\varphi) \rho d\rho d\theta. \end{aligned}$$

As a result,

$$\int_\sigma dV = \frac{1}{3} \iint_D \frac{(\rho \zeta_\rho - 1 - \theta \zeta_\theta \cot \varphi) \rho \sin \varphi}{\zeta_\theta} \rho d\rho d\theta. \quad (29)$$

Therefore, the volume constraint (the last constraint in (19)) would be rewritten in the following way:

$$\frac{1}{3} \iint_D \frac{(\rho \zeta_\rho - 1 - \theta \zeta_\theta \cot \varphi) \rho \sin \varphi}{\zeta_\theta} \rho d\rho d\theta = \frac{L}{T} \int_0^T \int_\Sigma \rho^2 \sin \varphi d\rho d\phi d\theta dt.$$

### 6.3 Converting the problem into a measure space

Regarding relations (22), (24), (25), (26) and (29), the known properties of admissible elements of  $P$  and the definitions of measures  $\mu$  and  $\nu$  in (27) and (28), problem (19) can now be expressed as follows based on unknown measures  $\nu$  and  $\mu$ :

$$\begin{aligned}
 \min E(\mu, \nu) &= \frac{1}{2T} \mu(|\dot{u}(\rho, \varphi, \theta, T)|^2 + (\frac{\partial u}{\partial \rho})^2 \\
 &\quad + \frac{1}{\rho^2} (\frac{\partial u}{\partial \varphi})^2 + \frac{1}{\rho^2 \sin^2 \varphi} (\frac{\partial u}{\partial \theta})^2) \\
 \text{subject to :} \quad &\nu(\Psi^g) = \rho_0 \Delta_\Psi, \quad \text{for all } \Psi \in C'(B), \\
 &\nu(\Theta) = 0, \quad \text{for all } \gamma \in \wp(D^0), \\
 &\nu(g(\rho, \theta)) = a_g, \quad \text{for all } g \in C_1(D''), \\
 &\frac{1}{T} \mu(\dot{u}(T)V(T) - u(T)\dot{V}(T)) + \mu(u\ddot{V} - au\dot{V}) \\
 &\quad - \mu(\nabla u \cdot \nabla V) + \frac{1}{T} \mu(au(T)V(T)) - \frac{1}{T} \mu(au_0 V(0)) \\
 &\quad = Y, \quad \text{for all } V \in H_0^1(\Sigma \times (0, T)), \\
 &\nu(F_0) - \frac{3L}{T} \mu(1) = 0, \\
 &\nu(L_1) = 0, \quad \text{for all } h_1(\varphi, \theta) \in C^1(D \times A), \\
 &\nu(L_2) = 0, \quad \text{for all } h_2(\rho, \varphi) \in C^1(D \times A),
 \end{aligned} \tag{30}$$

where  $L_1 = \frac{\partial h_1}{\partial \rho} - \frac{\partial h_1}{\partial \varphi} \zeta_\rho$ ,  $L_2 = \frac{\partial h_2}{\partial \theta} - \frac{\partial h_2}{\partial \varphi} \zeta_\theta$ ,  $F_0 = \frac{(\rho \zeta_\rho - 1 - \theta \zeta_\theta \cot \varphi) \rho \sin \varphi}{\zeta_\theta}$ , and

$$Y = \int_{\Sigma} (u_1(\rho, \varphi, \theta)V(0) - u_0(\rho, \varphi, \theta)\dot{V}(0)) \rho^2 \sin \varphi d\rho d\varphi d\theta.$$

We recall that the theoretical measure problem (30) is linear with respect to the two measures  $\mu$  and  $\nu$  even though the initial problem is nonlinear.

The space  $M^+(D') \times M^+(D'')$  is a linear space which will become a locally convex topological vector space when the weak\* topology is given [28].

The proof of the following existence theorem can be found in [8, 3, 28].

**Theorem 4.** a)  $\tilde{Q} \subseteq M^+(D') \times M^+(D'')$  is compact under the weak\* topology on  $M^+(D') \times M^+(D'')$

b) The objective function  $E(\mu, \nu)$  of problem (30) is continuous.

c) There exists a pair of measures  $(\mu^*, v^*)$  which is an optimal solution of problem (30) in the set  $\tilde{Q} \subset M^+(D') \times M^+(D'')$ ; that is, for every  $(\mu, v) \in \tilde{Q}$ , we have

$$E(\mu^*, v^*) \leq E(\mu, v).$$

Even though (30) has an optimal solution in  $\tilde{Q}$ , it is still very difficult to obtain the exact solution due to the fact that the underlying spaces are not finite-dimensional because the number of equations is not finite and the unknowns are measures. Therefore, it is completely acceptable to search for a suboptimal solution. Thus, first, by choosing suitable dense subsets in the appropriate spaces and then, by choosing a finite number of them, the problem is approximated by semi-finite linear programming.

## 7 Approximation schemes

Measure-theoretical problem (30) is an infinite-dimensional LPP and hence, it is not easy to solve. In this section, we shall apply approximation schemes to find a nearly optimal solution of (30) by solving a finite-dimensional linear program. In this regard, first the function  $(\mu, v) \rightarrow E(\mu, v)$  should be minimized, but over a subset of  $\tilde{Q}$  that satisfies only a finite number of constraints of (30) called  $\tilde{Q}(M_1, M_2, R_1, R_2, M_3, M_4, M_5)$ . To achieve this goal, we select countable sets of functions whose linear combinations are dense in the appropriate spaces. Therefore, we consider total function sets  $\{\Psi_i, i \in N\}$ ,  $\{\gamma_h, h \in N\}$ ,  $\{g_{sk}, s, k \in N\}$ ,  $\{V_j, j \in N\}$  and  $\{h_{1l_1}, h_{2l_2}, l_1, l_2 \in N\}$  in spaces  $C'(B)$ ,  $\wp(D^\circ)$ ,  $C_1(D'')$ ,  $H_0^1(\Sigma \times (0, T))$  and  $C^1(D \times A)$ , respectively and select a finite number of them to achieve the following approximated problem for (30):

$$\begin{aligned} \min E(\mu, v) &= \frac{1}{2T} \mu(|\dot{u}(\rho, \varphi, \theta, T)|^2) + \left(\frac{\partial u}{\partial \rho}\right)^2 \\ &\quad + \frac{1}{\rho^2} \left(\frac{\partial u}{\partial \varphi}\right)^2 + \frac{1}{\rho^2 \sin^2 \varphi} \left(\frac{\partial u}{\partial \theta}\right)^2 \\ \text{subject to :} \quad &v(\Psi_i^g) = \rho_0 \Delta \Psi_i, \quad i = 1, 2, \dots, M_1, \\ &v(\Theta_h) = 0, \quad h = 1, 2, \dots, M_2, \\ &v(g_{sk}(\rho, \theta)) = a_{g_{sk}}, \quad s = 1, 2, \dots, R_1, \quad k = 1, 2, \dots, R_2, \\ &\frac{1}{T} \mu(\dot{u}(T)V_j(T) - u(T)\dot{V}_j(T)) + \mu(u\ddot{V}_j - au\dot{V}_j) \\ &\quad - \mu(\nabla u \cdot \nabla V_j) + \frac{1}{T} \mu(au(T)V_j(T)) - \frac{1}{T} \mu(au_0 V_j(0)) \end{aligned}$$

$$\begin{aligned}
&= Y, \quad j = 1, 2, \dots, M_3, \\
&v(F_0) - \frac{3L}{T}\mu(1) = 0, \\
&v(L_{1l_1}) = 0, \quad l_1 = 1, 2, \dots, M_4, \\
&v(L_{2l_2}) = 0, \quad l_2 = 1, 2, \dots, M_5.
\end{aligned} \tag{31}$$

According to [28], if  $M_1, M_2, R_1, R_2, M_3, M_4$  and  $M_5$  tend to infinity, the solution of problem (31) is a very good approximation for the main problem (30). Also, from [28], we can characterize the pair of optimal measures of (27)–(28) as

$$\mu^* = \sum_{n=1}^{M_3+1} \hat{\alpha}_n^* \delta(Z_n^*), \quad v^* = \sum_{m=1}^{M_1+M_2+(R_1 \times R_2)+M_4+M_5+1} \hat{\beta}_m^* \hat{\delta}(z_m^*), \tag{32}$$

with  $\hat{\alpha}_n^*, \hat{\beta}_m^* \geq 0$  and  $Z_n^* \in D', z_m^* \in D''$ . Here  $\delta(t)$  and  $\hat{\delta}(t)$  are unitary atomic measure supported by singleton sets  $\{t\}$ . Then, if we substitute (32) in (31), it becomes equivalent to a nonlinear optimization problem whose unknowns are the coefficients  $\hat{\alpha}_n^*, \hat{\beta}_m^*$  and their supports  $\{Z_n^*\}$  and  $\{z_m^*\}$ ,  $n = 1, 2, \dots, M_3 + 1$  and  $m = 1, 2, \dots, M_1 + M_2 + (R_1 \times R_2) + M_4 + M_5 + 1$ .

If we approximate the supports  $\{Z_n^*\}$  and  $\{z_m^*\}$  by introducing the support points in a countable dense subset of  $D'$  and  $D''$ , respectively (referred to as  $W_1$  and  $W_2$ ), we will obtain the corresponding finite LP as desired. Therefore, one can transform (31) into a finite linear programming problem with unknown coefficients  $\hat{\alpha}_n$  and  $\hat{\beta}_m$  as follows:

$$\begin{aligned}
\min E(\hat{\alpha}, \hat{\beta}, D) &= \frac{1}{2T} \sum_{n=1}^{N_1} \hat{\alpha}_n (|\dot{u}(Z_n)|^2 + (\frac{\partial u(Z_n)}{\partial \rho_n})^2 \\
&\quad + \frac{1}{\rho_n^2} (\frac{\partial u(Z_n)}{\partial \varphi_n})^2 + \frac{1}{\rho_n^2 \sin^2 \varphi_n} (\frac{\partial u(Z_n)}{\partial \theta_n})^2) \\
\text{subject to :} \quad &\sum_{m=1}^{N_2} \hat{\beta}_m \Psi_i^g(z_m) = \rho_0 \Delta_{\Psi i}, \quad i = 1, 2, \dots, M_1, \\
&\sum_{m=1}^{N_2} \hat{\beta}_m \Theta_h(z_m) = 0, \quad h = 1, 2, \dots, M_2, \\
&\sum_{m=1}^{N_2} \hat{\beta}_m g_{sk}(z_m) = 0, \quad s = 1, 2, \dots, R_1, \quad k = 1, 2, \dots, R_2, \\
&\frac{1}{T} \sum_{n=1}^{N_1} \hat{\alpha}_n (\dot{u}(Z_n) \dot{V}_j(Z_n) - u(Z_n) \ddot{V}_j(Z_n)) \\
&\quad + \sum_{n=1}^{N_1} \hat{\alpha}_n (u(Z_n) \ddot{V}_j(Z_n) - \dot{u}(Z_n) \dot{V}_j(Z_n))
\end{aligned}$$

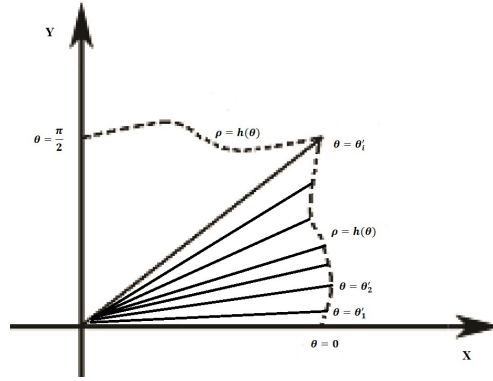
$$\begin{aligned}
& - \sum_{n=1}^{N_1} \hat{\alpha}_n \nabla u(Z_n) \cdot \nabla V_j(Z_n) + \frac{1}{T} \sum_{n=1}^{N_1} \hat{\alpha}_n a u(Z_n) V_j(Z_n) \\
& - \frac{1}{T} \sum_{n=1}^{N_1} \hat{\alpha}_n a u_0(Z_n) V_j(0) = Y, \quad j = 1, 2, \dots, M_3, \\
& \sum_{m=1}^{N_2} \hat{\beta}_m F_0(z_m) - \frac{3L}{T} \sum_{n=1}^{N_1} \hat{\alpha}_n = 0, \\
& \sum_{m=1}^{N_2} \hat{\beta}_m L_{1l_1}(z_m) = 0, \quad l_1 = 1, 2, \dots, M_4, \\
& \sum_{m=1}^{N_2} \hat{\beta}_m L_{2l_2}(z_m) = 0, \quad l_2 = 1, 2, \dots, M_5, \\
& \hat{\alpha}_n \geq 0, \quad n = 1, 2, \dots, M_3 + 1, \\
& \hat{\beta}_m \geq 0, \quad m = 1, 2, \dots, M_1 + M_2 + (R_1 \times R_2) + M_4 + M_5 + 1,
\end{aligned} \tag{33}$$

where  $W_1$  and  $W_2$  are the countable dense subsets of  $D'$  and  $D''$ , respectively, and  $N_1$  and  $N_2$  are sufficiently large numbers. Sets  $\{Z_n, n = 1, 2, \dots, M_3 + 1\} \subset W_1$  and  $\{z_m, m = 1, m = 2, \dots, M_1 + M_2 + (R_1 \times R_2) + M_4 + M_5 + 1\} \subset W_2$  are constructed by dividing appropriate intervals in the definitions of  $D'$  and  $D''$  (see (23)) into a finite number of equal subintervals, respectively.

On the other hand, we know that a curve can usually be approximated with broken lines. So suppose that  $\partial D$  is approximated by a number  $M$  of its points (corners of broken lines belonging to  $\partial D$ ) which will be called the  $M$ -representation of  $D$ . We can assign fixed -components such as  $\theta_i = \theta'_i, i = 1, 2, \dots, M$ , to the points in the  $M$ -representation set to obtain a fixed number  $M$  for shape variables, without loss of generality. Therefore, we can characterize each admissible  $M$ -representation set, denoted as  $D_M$ , by  $\rho_1 \sin \varphi_1, \rho_2 \sin \varphi_2, \dots, \rho_M \sin \varphi_M$ . So, we can determine each region of  $D \in D_M$ , by a set of finite points  $(\theta_m, \rho_m \sin \varphi_m), m = 1, 2, \dots, M$ . Then, we can find the nearly optimal value for  $E(\hat{\alpha}^*, \hat{\beta}^*, D)$  as a function of variables  $\rho_1, \rho_2, \dots, \rho_M$  and  $\varphi_1, \varphi_2, \dots, \varphi_M$ , for a given  $D \in D_M$ , by solving (33). As a result, we can determine the following vector function:

$$J : (\rho_1, \rho_2, \dots, \rho_M, \varphi_1, \varphi_2, \dots, \varphi_M) \in \mathbb{R}^{2M} \mapsto E(\hat{\alpha}^*, \hat{\beta}^*, D) \in \mathbb{R}. \tag{34}$$

The global minimizer of this function can be identified using a suitable optimization algorithm, such as PSO, such that each time the algorithm needs to evaluate  $J$ , a version of problem (33) must be solved. In this regard, the optimal coefficients  $\hat{\alpha}_j^*$  and  $\hat{\beta}_j^*$  are characterized. Whenever the function  $J$  is minimized, the minimizer  $(\rho_1^*, \rho_2^*, \dots, \rho_M^*, \varphi_1^*, \varphi_2^*, \dots, \varphi_M^*)$  and consequently,

Figure 2: An example of unknown domain  $D$ .

the optimal region  $D^*$  are obtained. Then, related to the optimal region  $D^*$ , optimal points  $(\theta_m^*, \rho_m^*, \varphi_m^*)$  corresponding to optimal coefficients  $\hat{\beta}_m^* > 0$  are obtained.

We might find some outliers among these optimal points [16] as a result of using approximation scheme (we recall that an outlier is an observation that deviates significantly from other observations). These outlier points may cause some difficulties in achieving the smoothness of the optimal shape. Therefore, they can be rejected from the set of optimal points if we employ a suitable outlier detection algorithm [1], such as the LoOP algorithm in [19]. Then, by applying a smooth curve fitting, we can achieve a smooth surface with a shape that is nearly optimal for the specified damping set.

## 7.1 Introducing total sets

In this subsection, we will explain how to select total functions for problem (30). Consider  $M_1$  functions  $\Psi_i \in C'(B)$  as monomials of the form  $\rho^{s_1} \varphi^{s_2} \theta^{s_3}$ , where  $s_1, s_2, s_3 \in \{0, 1, 2, \dots\}$ , and  $M_2$  functions  $\gamma_h \in \wp(D^\circ)$  as

$$\begin{aligned} \gamma_{1i} &= (\rho - h(\theta))(\sin(i\pi\theta)), \quad i = 1, 2, 3, \dots, \\ \gamma_{2i} &= (\rho - h(\theta))(\cos(i\pi\theta)), \quad i = 1, 2, 3, \dots, \\ \gamma_{3i} &= (\rho - h(\theta))(\sin(i\pi\theta))(\cos(i\pi\theta)), \quad i = 1, 2, 3, \dots \end{aligned}$$

where the function  $h(\theta)$  represents the boundary of  $D$ . Now, we will introduce functions which depend only on  $\rho$  and  $\theta$ . Without loss of generality, we assume that the origin is an interior point of  $D$  [12]. A finite number of angles  $\theta = \theta_i$ ,  $i = 1, 2, \dots, M$  from the uniform dense subset

of  $[0, 2\pi]$  would be considered such that  $0 = \theta_1 \leq \theta_2 \leq \dots \leq \theta_M = 2\pi$ . Then, region  $D$  can be divided into  $M - 1$  parts by half-lines  $\theta = \theta_i$ . Also, the  $i$ th part of  $D$  ( $i = 1, 2, \dots, M - 1$ ), called  $D_i$ , can be approximated by a sector of the circle with radius  $R_i = \frac{\rho_i \sin \varphi_i + \rho_{i+1} \sin \varphi_{i+1}}{2}$ , when  $\theta_i \leq \theta \leq \theta_{i+1}$ . Then  $(\rho_1^*, \rho_2^*, \dots, \rho_M^*)$  and  $(\varphi_1^*, \varphi_2^*, \dots, \varphi_M^*)$  can be obtained optimally corresponding to  $(\theta_1, \theta_2, \dots, \theta_M)$ . Obviously, if the number of angles is large enough, the union of  $D_i$ 's can effectively approximate  $D$ . Now, we define function  $g_{sk} \in C_1(D'')$  as

$$g_{sk}(\rho, \theta) = \begin{cases} 1, & \theta \in J_{1s} \quad \text{and} \quad \rho \in J_{2k}, \\ 0, & O.W. \end{cases},$$

where  $J_{1s} = \left[ \frac{(s-1)\theta_i}{R_1}, \frac{s\theta_i}{R_1} \right)$ ,  $s = 1, 2, \dots, R_1$  and  $J_{2k} = \left[ \frac{(k-1)R_i}{R_2}, \frac{kR_i}{R_2} \right)$  for  $k = 1, 2, \dots, R_2$ . So, we have

$$\begin{aligned} \iint_D g_{sk}(\rho, \theta) \rho d\rho d\theta &= \int_{\rho_{k-1}}^{\rho_k} \int_{\theta_{s-1}}^{\theta_s} \rho d\theta d\rho \\ &= \frac{1}{2}(\theta_s - \theta_{s-1})(\rho_k^2 - \rho_{k-1}^2) \equiv a_{g_{sk}}. \end{aligned}$$

Moreover,  $M_4$  functions  $h_1(\varphi, \theta) \in C^1(D \times A)$  are taken as the polynomials with respect to  $\varphi$  and  $\theta$ , and  $M_5$  functions  $h_2(\rho, \varphi) \in C^1(D \times A)$  as products of different powers of  $\rho$  and  $\varphi$ . From [28], the linear combinations of these functions are identically dense in appropriate spaces.

In this paper, the PSO algorithm is used in Step 3 of the algorithm (Subsection 7.2) and in Section 8 to minimize the objective function  $J$ . Specifically, PSO searches for the optimal coordinates of  $M$  boundary points in spherical coordinates to draw the 2D region  $D_M$  of the damping set. Compared to other heuristic methods like GA, PSO has several clear advantages. First, it is very simple to implement and needs fewer parameters. Second, it has a faster convergence speed because it updates positions directly without complex operations like crossover and mutation. Third, it requires less computer memory. Finally, PSO works very well for continuous problems, which perfectly fits our task of finding continuous geometric boundaries in a 3D space.

## 7.2 Algorithm of the proposed approach

We present an algorithmic path for the solution method based on the discussions in Sections 6 and 7 to demonstrate how our method could be applied to solve problem (5). In this regard, using the following five steps helps us identify the nearly optimal shape for the damping set.

**Step 1:** Select  $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8$  and  $n_9$  points in the given sets  $[0, T]$ ,  $[0, 2\pi]$ ,  $[0, \rho_\Sigma]$ ,  $[0, \varphi_\Sigma]$ ,  $U$ ,  $\dot{U}$ ,  $U_\rho$ ,  $U_\varphi$  and  $U_\theta$ , respectively. Thus  $N_1 = \prod_{i=1}^9 n_i$  number of 9-dimensional cells is constructed. Then, an arbitrary point  $Z_j$ ,  $j = 1, 2, \dots, N_1$ , is selected in each cell. Similarly, consider  $m_1, m_2, m_3, m_4, m_5$  and  $m_6$  points in the intervals  $[0, 2\pi]$ ,  $[0, \frac{\pi}{2}]$ ,  $[0, \rho_\sigma]$ ,  $F_\rho$ ,  $F_\theta$  and  $F_{\rho\theta}$ , respectively. So  $N_2 = m_1.m_2.m_3.m_4.m_5.m_6$  number of 6-dimensional cells is determined. In each cell, an arbitrary point  $z_j$ ,  $j = 1, 2, \dots, N_2$  is selected (note that sets  $\dot{U}$ ,  $U_\rho$ ,  $U_\varphi$ ,  $U_\theta$ ,  $F_\rho$ ,  $F_\theta$  and  $F_{\rho\theta}$  can be chosen by trial and error).

**Step 2:** We select functions  $\Psi_i(z)$ ,  $\gamma_h(z)$ ,  $g_{sk}(z)$ ,  $V_j(z)$ ,  $h_{1v}(z)$  and  $h_{2l}(z)$  in appropriate spaces as mentioned in Subsection 7.1, for a given initial region  $D$  and for fixed numbers  $M_1, M_2, R_1, R_2, M_3, M_4$  and  $M_5$ . Now, we can set up the finite LPP (33) with  $N_1 + N_2$  variables and  $N' \equiv M_1 + M_2 + (R_1 \times R_2) + M_3 + M_4 + M_5 + 1$  constraints.

**Step 3:** To solve problem (33), we apply a two-phase optimization approach for the given  $M$  and the specified number of iterations. First, we establish function  $J$  defined by (34). Then, we minimize it using a standard optimization method like those discussed in [11], such as a random search method or PSO algorithm. The algorithm produces  $M$  new points in each iteration as  $(\theta_m, \rho_m \sin \varphi_m)$ ,  $m = 1, 2, \dots, M$  which determine an  $M$ -representation region  $D_M$  as explained in Section 7. A value for  $J$  can be obtained for each  $D_M$  by running its related LP in (33). Therefore, we determine the optimal value of the energy for system (1) and the optimal coefficients  $\hat{\alpha}^*$ 's and  $\hat{\beta}^*$ 's simultaneously by minimizing  $J$  based on the optimal values  $(\rho_1^*, \rho_2^*, \dots, \rho_M^*)$  and  $(\varphi_1^*, \varphi_2^*, \dots, \varphi_M^*)$ .

**Step 4:** We characterize the indices  $n$  for the optimal values  $\hat{\beta}_1^*, \hat{\beta}_2^*, \dots, \hat{\beta}_{N_2}^*$  resulting from Step 3 so that the coefficient  $\hat{\beta}_n^*$  (related to the  $n$ -th extreme point) is positive and takes  $\rho = \rho_n^*$ ,  $\theta = \theta_n^*$ ,  $\varphi = \varphi_n^*$ ,  $\zeta_\rho = \zeta_{\rho_n}^*$ ,  $\zeta_\theta = \zeta_{\theta_n}^*$  and  $\zeta_{\rho\theta} = \zeta_{\rho\theta_n}^*$  (in fact,  $\rho_n^*, \theta_n^*, \varphi_n^*, \zeta_{\rho_n}^*, \zeta_{\theta_n}^*$  and  $\zeta_{\rho\theta_n}^*$  are the related components associated with point  $z_n^*$  corresponding to  $\hat{\beta}_n^* > 0$ ). Consequently, we can determine the nearly optimal shape using points  $(\rho_i^*, \varphi_i^*, \theta_i^*)$ .

**Step 5:** We apply the outlier detection algorithm to remove outliers from the set of points obtained in Step 4. Subsequently, to achieve a smooth surface resembling the nearly optimal damping set, we perform a smooth curve fitting using the MATLAB software toolbox on the obtained nearly optimal  $N'$  points.

### 7.3 Theoretical convergence and parameter sensitivity

To address the sensitivity of the proposed method to parameters such as mesh size and the number of basis functions (which determine the number of constraints and variables in the linear

programming formulation), we rely on the theoretical foundations established in the measure-theoretic framework. According to the fundamental work [10] on the linear programming approach to 2D shape design, as both the number of constraints and the number of variables in the embedded linear program increase (corresponding respectively to a smaller mesh size and using more basis functions), the solution of the finite-dimensional LP converges globally to the weak solution of the original infinite-dimensional problem. Based on this concept, in our 3D spherical formulation, this theoretical guarantee fact ensures that reducing the mesh size or increasing the number of functions, strictly improves the convergence to the optimal shape of the damping set. However, increasing the number of constraints and variables makes the linear programming problem much harder to solve. Therefore, in our numerical examples, we chose a good balance between accuracy and computation time. The selected mesh size and number of functions provide a good approximation of the optimal shape. At the same time, they ensure a reasonable computation time and global convergence.

## 8 Numerical examples

To illustrate the application of the algorithm described in the previous section, this section presents two numerical simulations for problem (1). It is worth noting that the examples presented here are extracted from the 2D example provided in [24].

**Example 1.** Consider the following 3D damped wave equation, which is defined on a spherical domain  $\Sigma$  centered at the origin with a radius of 2 in the time interval  $[0, 1]$ . For all  $(t, \rho, \varphi, \theta) \in (0, T) \times \Sigma$ , we have

$$\begin{aligned} \ddot{u} - \left( \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\cot \varphi}{\rho^2} \frac{\partial u}{\partial \varphi} + \frac{1}{\rho^2 \sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2} \right) + a(\rho, \varphi, \theta) \dot{u} &= 0, \\ u &= 0, \quad (t, \rho, \varphi, \theta) \in (0, T) \times \partial \Sigma, \\ u_0(\rho, \varphi, \theta) &= \sin(\pi \rho \sin \varphi \cos \theta) \sin(\pi \rho \sin \varphi \sin \theta) \sin(\pi \rho \cos \varphi), \\ u_1(\rho, \varphi, \theta) &= 0, \end{aligned}$$

Additionally, we assume that  $a(\rho, \varphi, \theta) = 10$  and  $L = \frac{1}{10}$ . The aim is to obtain the optimal region  $\sigma \subset \Sigma$  such that the energy of the system in (1) is minimized at the given end time  $T = 1$ . We also impose the following additional conditions required by the proposed method:

$$u(T = 1) = \dot{u}(T = 1) = 1, \quad (1.6, \frac{\pi}{2}, 0) \in \partial D, \quad F_\rho = F_\theta = F_{\rho\theta} = [-5, 5], \quad U = \dot{U} = [0, 1], \\ U_\rho = [0, \frac{3}{2}] \text{ and } U_\varphi = U_\theta = [0, 2].$$

For discretization, we consider  $N_1 = 3^7 \times 6 \times 5$  number of points in  $D'$  and  $N_2 = 6 \times 17 \times 8^4$  number of points in  $D''$ . To discretize  $D' = [0, T] \times [0, 2\pi] \times [0, \rho_\Sigma] \times [0, \varphi_\Sigma] \times U \times \dot{U} \times U_\rho \times U_\varphi \times U_\theta$ , we choose  $N_1$  points in this set by selecting

1. 3 points in  $[0, T]$  for  $t$ :  $0, \frac{1}{2}, 1$ ;
2. 3 points in  $[0, 2\pi]$  for  $\theta$ :  $0, \pi, 2\pi$ ;
3. 6 points in  $[0, 2]$  for  $\rho$ :  $0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, 2$ ;
4. 5 points in  $[0, \varphi_\Sigma]$  for  $\varphi$ :  $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ ;
5. 3 points in  $U$  for  $u$ :  $0, \frac{1}{2}, 1$ ;
6. 3 points in  $\dot{U}$  for  $\dot{u}$ :  $0, \frac{1}{2}, 1$ ;
7. 3 points in  $U_\rho$  for  $u_\rho$ :  $0, \frac{3}{4}, \frac{3}{2}$ ;
8. 3 points in  $U_\varphi$  for  $u_\varphi$ :  $0, 1, 2$ ;
9. 3 points in  $U_\theta$  for  $u_\theta$ :  $0, 1, 2$ ;

Also, to discretize  $D'' = D_1 \times F_\rho \times F_\theta \times F_{\rho\theta}$ , we choose  $N_2$  points in these sets by selecting

1. 6 points in  $[0, \rho_{max}]$  for  $\rho$ :  $0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, 2$ ;
2. 17 points in  $[0, 2\pi]$  for  $\theta$ :  $0, \frac{2\pi}{16}, \frac{4\pi}{16}, \frac{6\pi}{16}, \frac{8\pi}{16}, \frac{10\pi}{16}, \frac{12\pi}{16}, \frac{14\pi}{16}, \pi, \frac{18\pi}{16}, \frac{20\pi}{16}, \frac{22\pi}{16}, \frac{24\pi}{16}, \frac{26\pi}{16}, \frac{28\pi}{16}, \frac{30\pi}{16}, 2\pi$ ;
3. 8 values in  $F_\rho$  for  $f_\rho$ :  $-5, \frac{-25}{7}, \frac{-15}{7}, \frac{-5}{7}, \frac{5}{7}, \frac{15}{7}, \frac{25}{7}, 5$ ;
4. 8 values in  $F_\theta$  for  $f_\theta$ :  $-5, \frac{-25}{7}, \frac{-15}{7}, \frac{-5}{7}, \frac{5}{7}, \frac{15}{7}, \frac{25}{7}, 5$ ;
5. 8 values in  $F_{\rho\theta}$  for  $f_{\rho\theta}$ :  $-5, \frac{-25}{7}, \frac{-15}{7}, \frac{-5}{7}, \frac{5}{7}, \frac{15}{7}, \frac{25}{7}, 5$ .

We then set up the corresponding LP (33) by considering the total functions in the same way as expressed in Section 7 for  $M_1 = 3$ ,  $M_2 = 6$ ,  $R_1 = 3$ ,  $R_2 = 6$ ,  $M_3 = 2$ , and  $M_4 = M_5 = 3$ . Therefore, the resulting LPP has  $((3^7 \times 6 \times 5) + (6 \times 17 \times 8^4))$  variables and 36 constraints.

We use the PSO algorithm to determine the optimal values

$$(\rho_1^*, \dots, \rho_M^*, \varphi_1^*, \dots, \varphi_M^*)$$

by implementing the modified simplex method in MATLAB 7.11. After 30 iterations, the optimal points  $(\rho_m^*, \varphi_m^*, \theta_m^*)$  of the optimal shape and the optimal value  $E^*(\hat{\alpha}, \hat{\beta}, D) = 5.941 \times 10^{-18}$  are obtained.

The obtained solution exhibits a very small objective value of  $5.941 \times 10^{-18}$ . Moreover, the objective value shows no significant improvement after 30 iterations, indicating that the PSO search has stabilized. To further examine the sensitivity of the result to the random initialization, we performed multiple independent runs, all of which converged to the same objective value. These observations provide numerical evidence for the stability and near-optimality of the obtained solution.

The convergence performance of the PSO algorithm is illustrated in Figure 3. As can be seen, the algorithm demonstrates excellent convergence characteristics, achieving a final objective function value of  $5.941 \times 10^{-18}$  after 30 iterations.

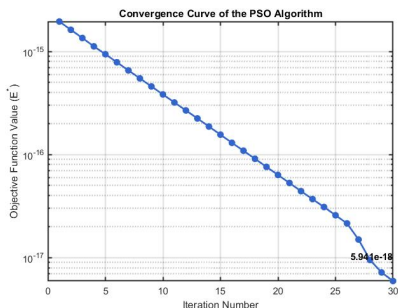
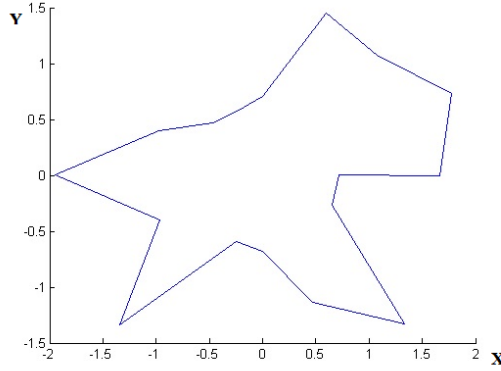
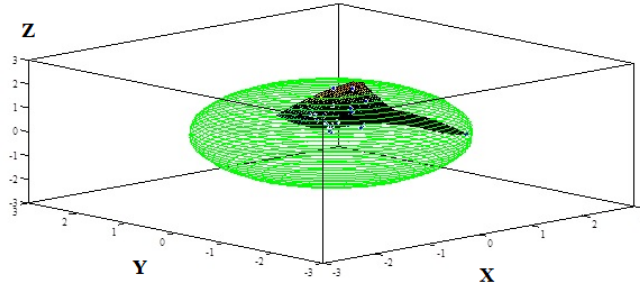


Figure 3: Convergence curve of the PSO algorithm in Example 1.

From Step 3 of the algorithm, the 2D shadow of the unknown surface  $\sigma$  is obtained, as shown by the broken lines in Figure 4. The unknown surface  $\sigma$  is displayed in Figure 5, obtained without imposing the symmetry property. Moreover, the complete 3D damping region is constructed by considering the symmetry property and plotting contour curves of spheres through the optimal points, as illustrated in Figure 6.

A detailed description of the figures for Example 1 is provided in Table 1.

Figure 4: The nearly optimal region  $D$  in Example 1.Figure 5: The nearly optimal surface of damping set in the given domain  $\Sigma$  in Example 1.

**Example 2.** In this example, all conditions and functions are the same as in the previous example, with the difference that the damping coefficient  $a$  is also unknown here.

To solve (33), we use an iterative method that consists of two stages. In the first stage, by considering a constant value for the variable  $a$ , function  $J_1 : [0, 1] \rightarrow \mathbb{R}$  is defined as  $J_1(D_M) = E^*(\alpha, \beta, D_M)$ . Then, in the second stage, this function is optimized using one of the standard optimization algorithms (like linear search). By doing so, for each value of  $D_M$ , a value for the initial objective function is obtained. The set of values obtained for the initial objective function constructs the new objective function. We recall that every time this function is calculated, a linear programming problem similar to (33) with the  $D_M$  value identified by this function must be solved.

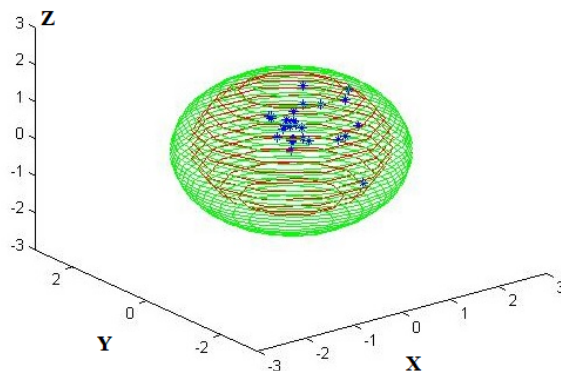


Figure 6: The nearly optimal points that determine the unknown damping set within the domain  $\Sigma$  in Example 1.

Table 1: Detailed description of the figures for Example 1.

| Figure   | Description                                                                                                                                                                                                                                                                     |
|----------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Figure 4 | The 2D shadow (projection) of the unknown surface that defines the damping region $\sigma$ , obtained by the shape optimization method. The exact shape of this shadow is not known a priori and is determined by minimizing the energy functional of the 3D wave equation.     |
| Figure 5 | The unknown damping region $\sigma$ inside the spherical region $\Sigma$ with radius $R = 2$ . This figure shows the entire damping region and how it is positioned within the computational domain.                                                                            |
| Figure 6 | Optimal points in 3D space obtained by a linear programming problem scheme. These points determine the unknown surface $\sigma$ defining the wave damping region. The surface is visualized using contour lines inside the large sphere $\Sigma$ to make it clearly observable. |

If we denote the optimal value of  $J_1$  as  $D_M^*$ , then  $J_1^*(D_M) \equiv E^*(\alpha, \beta, a, D_M^*)$ . Now, the function  $J_2 : [0, 1] \rightarrow \mathbb{R}$  is calculated as  $J_2(a) = J_1^*(D_M^*)$  in the second stage. Therefore, by optimizing the functions  $J_1$  and  $J_2$  and solving the linear programming problem (33), the unknowns  $a^*$ ,  $D_M^*$ ,  $\hat{\alpha}_1^*$ ,  $\hat{\alpha}_2^*$ ,  $\dots$ ,  $\hat{\alpha}_N^*$  and  $\hat{\beta}_1^*$ ,  $\hat{\beta}_2^*$ ,  $\dots$ ,  $\hat{\beta}_n^*$  are determined simultaneously.

In this step, we have used the random search algorithm to minimize the function  $J_1$  and the PSO algorithm to optimize the function  $J_2$ . The optimal value of  $a^* = 24.3624$  and the optimal value of the energy function is 10.323. The optimal region for this example is shown in Figure 7. In [24], for the 2D case and  $T = 1$ , the optimal value for the variable  $a = 29.09$  was found, and

with 2000 iterations using the level set method, the optimal energy value was determined to be 12.56.

As mentioned in this reference, the level set method is entirely dependent on its initial shape, and changing the initial shape will alter the optimal value of the objective function. However, the shape-measure method presented in this paper is independent of the initial shape or value, which is one of the advantages of this method for solving such problems. It is worth noting that in random search and PSO algorithms, the upper and lower bounds for the parameters  $a$  and  $D_M$  need to be determined. Therefore, when using these algorithms, we considered the unknown  $a$  within the range  $[1, 50]$  and the unknown  $D_M$  based on the shape of the area  $\Sigma$  within the range  $[0, 2]$ .

The convergence behavior of the proposed algorithm is illustrated in Figure 8. The objective function ( $E^*$ ) represents the total energy functional that we aim to minimize through optimal design of the damping coefficient distribution.

Figure 7 illustrates the optimal surfaces obtained in the 3D domain. Unlike Figure 6, where only the shape of the damping region was unknown, in this example, the damping coefficient is also an unknown parameter to be identified. The figure displays the optimal surfaces that most efficiently reduce the wave energy to zero. The contour lines represent the level sets of these surfaces within the spherical domain  $\Sigma$ .

The large region  $\Sigma$  is a regular sphere with radius  $R = 2$  centered at the origin. The damping region  $\sigma \subset \Sigma$  is an unknown subdomain obtained from the optimization process. The goal is to find the optimal shape and coefficient of the damping region such that the total energy of the 3D wave equation decays exponentially to zero.

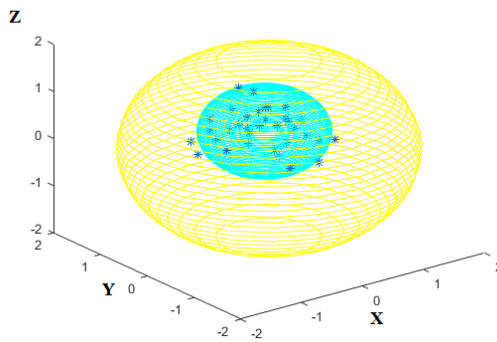


Figure 7: The nearly optimal shape of damping set with an unknown damping coefficient in Example 2.

To study the damping behavior, we draw the wave function  $u(t, \rho, \phi, \theta)$  with the optimal damping  $a^* = 24.3624$ . Figure 9 shows the wave surface at  $z = 0.5$ , which is equivalent to

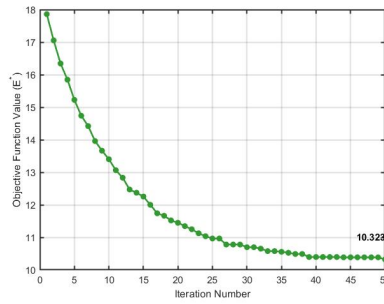
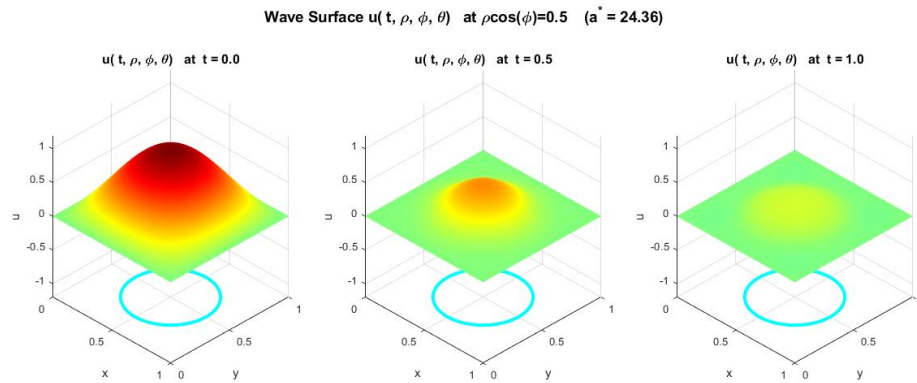


Figure 8: Convergence curve of the PSO algorithm in Example 2.

$\rho \cos \phi = 0.5$ , at three times:  $t = 0$ ,  $t = 0.5$ , and  $t = 1.0$ . The cyan circle on the base shows the damping region. At  $t = 0$ , the wave has a large peak at the center. At  $t = 0.5$ , the wave becomes smaller. At  $t = 1.0$ , the wave is very small; this means that the optimal damping reduces the wave energy well.

Figure 9: Wave surface  $u(t, \rho, \phi, \theta)$  at  $\rho \cos \phi = 0.5$  for three times: (a)  $t = 0$ , (b)  $t = 0.5$ , and (c)  $t = 1.0$  in Example 2.

## 9 Conclusion

We present and apply an extended embedding method to design the shape of a 3D damping set in order to minimize the energy associated with the system of the linear damped wave system. The problem was solved using a two-phase optimization approach based on measure theory. The new method has transformed the classical OSD problem into a nonclassical measure-theoretical problem, which can be formulated as a linear programming problem in measure

space. Besides simplifying the original problem, this method offers several advantages over others: Linear treatment for nonlinear problems, global optimization, time efficiency, and independence from the initial shape and mesh design or additional assumptions on the functions involved in the classical main problem. Therefore, the proposed method can be considered highly suitable for designing the optimal shape in 3D.

As a proposal for future work, the presented method can be extended to solve optimal shape design problems for nonlinear wave equations. Another interesting direction is to apply this method to design optimal damping sets in domains where the wave speed or material properties vary spatially. Furthermore, combining this measure-theoretical approach with intelligent methods, such as neural networks, could help reduce the computational time required to solve complex 3D shape optimization problems.

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## Conflicts of Interest

The authors declare no conflicts of interest.

## References

- [1] Alimorad, H. and Fakharzadeh, A. J. *A new approach for solving the obstacle problems in three-dimensions*, Optik, 276 (2023) 170646.
- [2] Alimorad, H. and Fakharzadeh, A. J. *Finding the optimal place of sensors for a 3D damped wave equation by using measure approach*, Control Optim. Appl. Math. 4(1) (2019) 15–36.
- [3] Alimorad, H. and Fakharzadeh, A. J. *A theoretical measure technique for determining 3D symmetric nearly optimal shapes with a given center of mass*, Comput. Math. & Math. Phys. 57(7) (2017) 1225–1240.
- [4] Allaire, G. *Shape Optimization by the Homogenization Method*, Springer, 2007.

- [5] Chen, L. L., Lian, H., Liu, Z., Chen, H. B., Atroshchenko, E. and Bordas, S. P. A. *Structural shape optimization of three dimensional acoustic problems with isogeometric boundary element methods*, Comput. Methods Appl. Mech. Eng. 355 (2019) 926–951.
- [6] Cheng, D. K. *Field and Wave Electromagnetics*, 2nd ed., Addison-Wesley, 1989.
- [7] Fakharzadeh J, A., Alimorad, H. and Tuobaei, S. *A linear technique for designing optimal rotated shapes*, Comput. Appl. Math. 41 (2022) 421.
- [8] Fakharzadeh J, A. and Alimorad, H. *A review of theoretical measure approaches in optimal shape problems*, Int. J. Numer. Anal. Model. 16(4) (2019) 543–574.
- [9] Fakharzadeh J, A. and Goodarzi, M. *General optimal shape of plasmonic nano particles based absorption coefficient using a linearization approach*, Optik 140 (2017) 1069–1080.
- [10] Fakharzadeh J, A., Alimorad, H. and Beiranvand, A. *A linearization technique for optimal design of the damping set with internal dissipation*, Iran. J. Numer. Anal. Optim. 6(1) (2016) 1–29.
- [11] Fakharzadeh J, A., Goodarzi, M. and Karimyar, M. J. *On efficiency of some meta-heuristic optimization algorithm in shape-measure method*, The first national conference on meta-heuristic algorithms and their applications in science and engineering, Fereydunkenar, 2014.
- [12] Fakharzadeh J, A. and Jafarpour, F. N. *A new discretization method for solving an optimal vibrating controlled system on an arbitrary domain*, J. Vib. Control 19(14) (2012) 2092–2099.
- [13] Fakharzadeh J, A. and Rubio, J. E. *Global solution of optimal shapes design problems*, J. Anal. Appl. 18(1) (1999) 143–155.
- [14] Farahi, M. H., Borzabadi, A. H., Mehneh, H. H. and Kamyad, A. V. *Measure theoretical approach for optimal shape design of a nozzle*, J. Appl. Math. Comput. 17(1-2) (2005) 315–328.
- [15] Henrot, A. and Pierre, M. *Shape Variation and Optimization: A Geometrical Analysis*, EMS Publishing House, 2018.
- [16] Hawkins, D. *Identification of Outliers*, Chapman and Hall, London, 1980.
- [17] Hébrard, P. and Henrot, A. *Optimal shape and position of the actuators for the stabilization of a string*, Syst. Control Lett. 48(3-4) (2003) 199–209.

- [18] Hesse, S. H., Leidinger, L. F., Kremheller, J., Lukaszewicz, D. H. J. A. and Duddeck, F. *Shape optimization with the level-set-method using local volume constraints*, Struct. Multi-discip. Optim. 57(1) (2018) 115–130.
- [19] Kriegel, H.-P., Kroger, P., Schubert, E. and Zimek, A. *LoOP: Local outlier probabilities*, In Proc ACM Conference on Information and Knowledge Management (CIKM), Hong Kong, China, 2009, 2–6.
- [20] Liu, M. and Hasegawa, Y. *Adjoint-based shape optimization for compressible flow based on volume penalization method*, Eng. Comput. 41 (2025) 989–1010.
- [21] Lu, C., Liu, J. and Guo, A. *Three-dimensional shape optimization of a submerged body under wave diffraction*, Phys. Fluids 35(10) (2023) 107116.
- [22] Mehne, H. H. and Farahi, M. H. *Transformation to a fixed domain in LP modelling for a class of optimal shape design problems*, Iran. J. Numer. Anal. Optim. 9(1) (2019) 1–16.
- [23] Mikhailov, V. P. *Partial Differential Equations*, MIR Publisher, Moscow, 1978.
- [24] Münch, A. *Optimal internal dissipation of a damped wave equation using a topological approach*, Int. J. Appl. Math. Comput. Sci. 19(1) (2009) 15–37.
- [25] Münch, A., Pedregal, P. and Periago, F. *Optimal design of the damping set for the stabilization of the wave equation*, J. Differ. Equ. 231(1) (2006) 331–358.
- [26] Münch, A., Pedregal, P. and Periago, F. *Optimal internal stabilization of the linear system of elasticity*, Arch. Ration. Mech. Anal. 193(1) (2009) 171–193.
- [27] Pironneau, O. *Optimal Shape Design for Elliptic Systems*, Springer, New York, 1983.
- [28] Rubio, J. E. *Control and Optimization: The Linear Treatment of Nonlinear Problems*, Manchester University Press, Manchester, U.K., 1986.
- [29] Rudin, W. *Real and Complex Analysis*, 2nd edn., Tata McGraw-Hill Publishing Co Ltd., New Delhi, 1983.
- [30] Takahashi, T., Sato, D., Isakari, H. and Matsumoto, T. *A shape optimisation with the isogeometric boundary element method and adjoint variable method for the three-dimensional Helmholtz equation*, Comput.-Aided Des. 143 (2022) 103126.

- [31] Tcheugoué Tébou, L. R. *Stabilization of the wave equation with localized nonlinear damping*, J. Differ. Equ. 145 (1998) 502–524.
- [32] Thomas George, B. and Finney Ross, L. *Calculus and Analytic Geometry*, 9th edn., Addison-Wesley, United States of America, 1998.
- [33] Vineyard, E. and Gao, X. L. *Topology and shape optimization of 2D and 3D micro-architected thermoelastic metamaterials using a parametric level set method*, CMES-Comput. Model. Eng. Sci. 127(3) (2021) 819–854.
- [34] Wang, K., Zhang, X. Y., Zhang, Z. D. and Min, C. H. *Three-dimensional shape optimization of fins in a printed circuit recuperator using S-CO<sub>2</sub> as the heat-transfer fluid*, Int. J. Heat Mass Transf. 192 (2022) 122910.
- [35] Wang, K., Liu, Y. J., Zhang, Z. D., Zhang, X., Fan, Y. H. and Min, C. H. *Three-dimensional shape optimization of fins for application in compact supercritical CO<sub>2</sub> solar receivers*, Int. J. Heat Mass Transf. 221 (2024) 125013.
- [36] Zuazua, E. *Propagation, observation, and control of waves approximated by finite difference methods*, SIAM Rev. 47(2) (2005) 197–243.