



Algebraic properties of the new polyconvolution operator related to the Laplace, Fourier cosine integral transforms and applications

M.K. Nguyen*, L.T.P. Bui and V.T. Tran

Abstract

This work is focused on investigating a new polyconvolution operator with the trigonometric weight function $\beta(y_1) = \cos y_1$ associated with the Laplace and Fourier cosine integral transforms. We give an explicit form of this operator, prove the corresponding factorization identity, and establish some important algebraic properties of it, together with a Titchmarsh-type theorem. Within this analytic setting, we obtain fundamental properties including L_1 -boundedness, Young-type inequalities on mixed L_p spaces, and a norm inequality of Saitoh's type in the $L_p(\mathbb{R}_+, q_i)$ weight spaces. Applications demonstrate the operator's effectiveness in

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solving some new classes of integral equations of Toeplitz plus Hankel-type and of polyconvolution type, a system of equations of polyconvolution type, and an integro-differential equation of Barbashin-type, yielding explicit closed-form solutions.

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1 Introduction

Research on convolutions and generalized convolutions for integral transforms has long played a central role in both theoretical and applied mathematics and in technical fields [1, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 15, 23, 25, 28, 32, 30, 31, 36, 37, 27]. In 1997, with the idea of enlarging the convolution operator, Kakichev [14] advanced a generalized definition related to the polyconvolution operator for $n + 1$ arbitrary integral transforms $G, G_1, G_2, \dots, G_n$ with the weight function $\beta(y_1)$ of functions $g_1, g_2, \dots, g_n$ for which the factorization identity holds:

$$G\left[\overset{\beta}{*}(g_1, g_2, \dots, g_n)\right](y_1) = \beta(y_1)(G_1 g_1)(y_1) \cdot (G_2 g_2)(y_1) \cdots (G_n g_n)(y_1).$$

According to this idea, the polyconvolution operators related to the Fourier, Fourier cosine, Fourier sine, Hartley, and other types of Fourier integral transforms have been presented [16, 18, 19, 20, 21, 22, 29]; see also the polyconvolution operator for the Fourier cosine and Laplace integral transforms [35]. The generalized convolutions generated by the Fourier and Laplace transforms have been studied in [30, 31]. Following these authors, we built and studied a new polyconvolution operator with the weight function $\beta(y_1) = e^{-y_1}$ for a triple of integral transforms, the Laplace and Fourier cosine integral transforms [17]. In this direction, we build and study a new polyconvolution operator with the trigonometric weight function related to the Fourier cosine and Laplace integral transforms. Taking into account the above factorization equality, the general definition related to the polyconvolution operator has the following form:

$$\overset{\beta}{*}(g_1, g_2, \dots, g_n)(x_1) = G^{-1}\left[\beta(\cdot)(G_1 g_1)(\cdot)(G_2 g_2)(\cdot) \cdots (G_n g_n)\right](x_1),$$

with G^{-1} being the inverse operator of the operator G . It is difficult to get an explicit form of a polyconvolution operator when applied to concrete integral transforms. The essence of this work is to introduce several interesting algebraic properties, to prove a Young-type theorem, some

norm inequalities in $L_p(\mathbb{R}_+)$ and a norm inequality of Saitoh's type in the $L_p(\mathbb{R}_+, q_i)$ weight spaces related to this polyconvolution operator, and apply this operator to several nonstandard integral equations and systems of equations of polyconvolution type, obtaining solutions in closed form. For such integral equations, a representation of the solution in closed form is an interesting open problem [3, 23]. Finally, with the help of this operator in subsection 5.4, we study a class of integro-differential equations of Barbashin's type [2, 3]; this is a concrete model arising in mathematical physics, radiation propagation and mathematical biology.

2 Preliminaries

In this section, we recall several known convolution and generalized convolution operators. The convolution operator of functions f_1, h_1 in the space $L_1(\mathbb{R}_+)$ associated with the Laplace operator is introduced in [25]:

$$(f_1 *_L h_1)(x_1) = \frac{1}{\sqrt{2\pi}} \int_0^{x_1} f_1(x_1 - y_1) h_1(y_1) dy_1, \quad x_1 > 0, \quad (1)$$

for which the following identity is satisfied:

$$L(f_1 *_L h_1)(y_1) = (Lf_1)(y_1)(Lh_1)(y_1), \quad y_1 > 0. \quad (2)$$

The symbol L denotes the Laplace integral operator [25]

$$(Lf_1)(y_1) = \int_0^\infty f_1(x_1) e^{-y_1 x_1} dx_1, \quad y_1 > 0.$$

The convolution operator of functions f_1 and h_1 in the space $L_1(\mathbb{R}_+)$ related to the Fourier cosine operator is introduced in [25, 33]:

$$(f_1 *_L h_1)(x_1) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f_1(y_1) [h_1(x_1 + y_1) + h_1(|x_1 - y_1|)] dy_1, \quad x_1 > 0, \quad (3)$$

with the identity

$$F_c(f_1 *_L h_1)(y_1) = (F_c f_1)(y_1)(F_c h_1)(y_1), \quad y_1 > 0, \quad (4)$$

where the Fourier cosine operator is defined by [25, 33]

$$(F_c f_1)(y_1) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_1(x_1) \cos(x_1 y_1) dx_1, \quad y_1 > 0.$$

3 The polyconvolution operator with the trigonometric weight function related to the Fourier cosine and Laplace operators

In this section, we build the new polyconvolution operator with the trigonometric weight function $\beta(y_1) = \cos y_1$ related to the Fourier cosine and Laplace operators. We also study some operational properties of this operator.

Definition 1. The polyconvolution operator of functions f_1, g_1, h_1 with the weight function $\beta(y_1) = \cos y_1$ associated with the Fourier cosine and Laplace operators is denoted by $_{F_c, L}^{\beta} \ast (f_1, g_1, h_1)$ and defined by

$$\begin{aligned} &_{F_c, L}^{\beta} \ast (f_1, g_1, h_1)(x_1) \\ &= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} \right. \\ &\quad + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} \\ &\quad \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] f_1(u_1) g_1(v_1) h_1(w_1) du_1 dv_1 dw_1, \quad x_1 > 0. \end{aligned} \quad (5)$$

Theorem 1. Assume that f_1, g_1, h_1 are arbitrary functions in the space $L_1(\mathbb{R}_+)$. Then the operator (5) belongs to the space $L_1(\mathbb{R}_+)$ and satisfies the following property:

$$F_c \left[_{F_c, L}^{\beta} \ast (f_1, g_1, h_1) \right](y_1) = \cos y_1 (F_c f_1)(y_1) (Lg_1)(y_1) (Lh_1)(y_1), \quad (6)$$

for all $y_1 > 0$.

Furthermore, $F_c \left[_{F_c, L}^{\beta} \ast (f_1, g_1, h_1) \right] \in C_0(\mathbb{R}_+)$; here $C_0(\mathbb{R}_+)$ is the space of bounded continuous functions vanishing at infinity. On the other hand, the L_1 -norm inequality of this polyconvolution operator is as follows:

$$\left\| _{F_c, L}^{\beta} \ast (f_1, g_1, h_1) \right\|_{L_1(\mathbb{R}_+)} \leq \|f_1\|_{L_1(\mathbb{R}_+)} \cdot \|g_1\|_{L_1(\mathbb{R}_+)} \cdot \|h_1\|_{L_1(\mathbb{R}_+)}. \quad (7)$$

Finally, when $f_1, g_1, h_1 \in L_2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, the L_2 -inversion formula holds, the equality being understood in the $L_2(\mathbb{R}_+)$ -sense:

$$_{F_c, L}^{\beta} \ast (f_1, g_1, h_1)(x_1) \quad (8)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos y_1 (F_c f_1)(y_1) (Lg_1)(y_1) (Lh_1)(y_1) \cos(x_1 y_1) dy_1.$$

Proof. Using the formula (5), the Tonelli theorem, and the Fubini theorem, we have

$$\begin{aligned} & \left\| \underset{F_c, L}{*}^\beta (f_1, g_1, h_1) \right\|_{L_1(\mathbb{R}_+)} \\ &= \int_0^\infty \left| \underset{F_c, L}{*}^\beta (f_1, g_1, h_1)(x_1) \right| dx_1 \\ &\leq \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} \right. \\ &\quad + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} \\ &\quad \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right| \\ &\quad \times |f_1(u_1)| \cdot |g_1(v_1)| \cdot |h_1(w_1)| du_1 dv_1 dw_1 dx_1 \\ &= \frac{1}{2\pi} \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \\ &\quad + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \left. \right| dx_1 \\ &\quad \cdot \int_0^\infty |f_1(u_1)| du_1 \cdot \int_0^\infty |g_1(v_1)| dv_1 \cdot \int_0^\infty |h_1(w_1)| dw_1. \end{aligned}$$

We consider

$$\begin{aligned} I_1 = \int_0^\infty & \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \\ & \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] dx_1. \end{aligned}$$

Then I_1 can be obtained by pairing the two translated Poisson-type kernels. Let $a = v_1 + w_1 > 0$.

Then

$$\int_0^\infty \left[\frac{a}{a^2 + (u_1 + x_1 + 1)^2} + \frac{a}{a^2 + (u_1 - x_1 + 1)^2} \right] dx_1 = \pi,$$

and the analogous pair involving $u_1 \pm x_1 - 1$ also has integral π . Hence the integral I_1 of the full nonnegative kernel with respect to x_1 equals 2π .

So,

$$\left\| \underset{F_c, L}{*}^\beta (f_1, g_1, h_1) \right\|_{L_1(\mathbb{R}_+)} \leq \|f_1\|_{L_1(\mathbb{R}_+)} \cdot \|g_1\|_{L_1(\mathbb{R}_+)} \cdot \|h_1\|_{L_1(\mathbb{R}_+)} < +\infty.$$

Therefore, we have (7) and $\underset{F_c, L}{*}^\beta (f_1, g_1, h_1) \in L_1(\mathbb{R}_+)$.

Alternatively, using (5) and the formula $\int_0^\infty e^{-ax_1} \cos y_1 x_1 dx_1 = \frac{a}{a^2 + y_1^2}$, $a > 0$ (see [11, formula 2.13.5]), we get

$$\begin{aligned}
 & \sqrt{\frac{2}{\pi}} \int_0^\infty [\cos y_1 (F_c f_1)(y_1) (Lg_1)(y_1) (Lh_1)(y_1)] \cos y_1 x_1 dy_1 \\
 &= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\cos y_1 \sqrt{\frac{2}{\pi}} \int_0^\infty f_1(u_1) \cos y_1 u_1 du_1 \int_0^\infty g_1(v_1) e^{-y_1 v_1} dv_1 \right. \\
 & \quad \times \left. \int_0^\infty h_1(w_1) e^{-y_1 w_1} dw_1 \right] \cos y_1 x_1 dy_1 \\
 &= \frac{2}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \cos y_1 \cos y_1 u_1 \cos y_1 x_1 \cdot e^{-y_1(v_1+w_1)} \\
 & \quad \times f_1(u_1) g_1(v_1) h_1(w_1) du_1 dv_1 dw_1 dy_1 \\
 &= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty [\cos y_1(u_1 + x_1 + 1) + \cos y_1(u_1 + x_1 - 1) \\
 & \quad + \cos y_1(u_1 - x_1 + 1) + \cos y_1(u_1 - x_1 - 1)] e^{-y_1(v_1+w_1)} \\
 & \quad \times f_1(u_1) \cdot g_1(v_1) \cdot h_1(w_1) du_1 dv_1 dw_1 dy_1 \\
 &= \stackrel{\beta}{*}_{F_c, L} (f_1, g_1, h_1)(x_1).
 \end{aligned}$$

Thus, when $f_1, g_1, h_1 \in L_2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$ the L_2 -inversion formula (8) holds. Combining this with $\stackrel{\beta}{*}_{F_c, L} (f_1, g_1, h_1) \in L_1(\mathbb{R}_+)$, applying the Fourier cosine integral transform to both sides of (8), we obtain the factorization identity (6).

The Riemann–Lebesgue theorem in [26] states that if $f_1 \in L_1(\mathbb{R}^n)$, then $(Ff_1)(y_1) \rightarrow 0$ as $|y_1| \rightarrow +\infty$, and hence $(Ff_1) \in C_0(\mathbb{R}^n)$. This is still true for the Fourier cosine integral transform F_c [33, 11]. So, $F_c[\stackrel{\beta}{*}_{F_c, L} (f_1, g_1, h_1)]$ belongs to $C_0(\mathbb{R}_+)$ by virtue of the Riemann–Lebesgue theorem. $\square$

Proposition 1. Assume that $f_1, g_1, l_1, k_1 \in L_1(\mathbb{R}_+)$. Then, the polyconvolution (5) satisfies the following equalities:

$$a) \quad \stackrel{\beta}{*}_{F_c, L} (f_1, g_1, l_1) = \stackrel{\beta}{*}_{F_c, L} (f_1, l_1, g_1).$$

Furthermore, we denote by D the class of functions on $\mathbb{R}_+$ that are piecewise continuous and satisfy an estimate: $|f_1(t)| \leq Ce^{\alpha t}$ for some $C > 0$, $\alpha \in \mathbb{R}$. Hence, if $f_1, g_1, l_1, k_1 \in L_1(\mathbb{R}_+) \cap D$, we have the following equations:

$$b) \quad \stackrel{\beta}{*}_{F_c, L} [f_1, (g_1 \underset{L}{*} l_1), k_1] = \stackrel{\beta}{*}_{F_c, L} [f_1, g_1, (l_1 \underset{L}{*} k_1)],$$

$$c) \quad \stackrel{\beta}{*}_{F_c, L} [f_1, (g_1 \underset{L}{*} l_1), k_1] = \stackrel{\beta}{*}_{F_c, L} [f_1, k_1, (g_1 \underset{L}{*} l_1)].$$

Proof. First, we prove a). Indeed, from the factorization equality (6), we obtain

$$\begin{aligned} F_c[\overset{\beta}{*}_{F_c,L}(f_1, g_1, l_1)](y_1) &= \cos y_1 (F_c f_1)(y_1)(Lg_1)(y_1)(Ll_1)(y_1) \\ &= \cos y_1 (F_c f_1)(y_1)(Ll_1)(y_1)(Lg_1)(y_1) \\ &= F_c[\overset{\beta}{*}_{F_c,L}(f_1, l_1, g_1)](y_1). \end{aligned}$$

Thus $\overset{\beta}{*}_{F_c,L}(f_1, g_1, l_1) = \overset{\beta}{*}_{F_c,L}(f_1, l_1, g_1)$.

b) Using the factorization properties (2), (6), and $g_1, l_1 \in D \cap L_1(\mathbb{R}_+)$, we can write

$$\begin{aligned} F_c\{\overset{\beta}{*}_{F_c,L}[f_1, (g_1 \underset{L}{*} l_1), k_1]\}(y_1) &= \cos y_1 (F_c f_1)(y_1) L(g_1 \underset{L}{*} l_1)(y_1)(Lk_1)(y_1) \\ &= \cos y_1 (F_c f_1)(y_1)(Lg_1)(y_1)(Ll_1)(y_1)(Lk_1)(y_1) \\ &= \cos y_1 (F_c f_1)(y_1)(Lg_1)(y_1) L(l_1 \underset{L}{*} k_1)(y_1) \\ &= F_c\{\overset{\beta}{*}_{F_c,L}[f_1, g_1, (l_1 \underset{L}{*} k_1)]\}(y_1), \quad y_1 > 0. \end{aligned}$$

Thus, $\overset{\beta}{*}_{F_c,L}[f_1, (g_1 \underset{L}{*} l_1), k_1] = \overset{\beta}{*}_{F_c,L}[f_1, g_1, (l_1 \underset{L}{*} k_1)]$. Similarly, we can prove c). $\square$

Theorem 2 (Titchmarsh-type theorem). Suppose $f_1 \in L_1(\mathbb{R}_+, e^{\alpha x_1})$, $\alpha > 0$ and $g_1, h_1 \in L_1(\mathbb{R}_+)$. If $\overset{\beta}{*}_{F_c,L}(f_1, g_1, h_1)(x_1) = 0$, for all $x_1 > 0$, then either $f_1(x_1) = 0$, for all $x_1 > 0$ or $g_1(x_1) = 0$, for all $x_1 > 0$ or $h_1(x_1) = 0$, for all $x_1 > 0$.

Proof. Indeed, from the hypothesis $\overset{\beta}{*}_{F_c,L}(f_1, g_1, h_1)(x_1) = 0$, for all $x_1 > 0$ and the factorization identity (6), we have

$$\cos y_1 (F_c f_1)(y_1)(Lg_1)(y_1)(Lh_1)(y_1) = 0, \quad \text{for all } y_1 > 0. \quad (9)$$

On the other hand, we have

$$\begin{aligned} \left| \frac{d^n}{dy_1^n} [\cos(x_1 y_1) f_1(x_1)] \right| &= \left| f_1(x_1) \cdot x_1^n \cdot \cos\left(x_1 y_1 + n \frac{\pi}{2}\right) \right| \\ &\leq |e^{-\alpha x_1} \cdot x_1^n| \cdot |e^{\alpha x_1} \cdot f_1(x_1)| \leq \frac{n!}{\alpha^n} \cdot |e^{\alpha x_1} \cdot f_1(x_1)|. \end{aligned}$$

Here, we use the estimation

$$0 \leq e^{-\alpha x_1} \cdot x_1^n = e^{-\alpha x_1} \cdot \frac{(\alpha x_1)^n}{n!} \cdot \frac{n!}{\alpha^n} \leq e^{-\alpha x_1} \cdot e^{\alpha x_1} \cdot \frac{n!}{\alpha^n} = \frac{n!}{\alpha^n}.$$

Since $f_1 \in L_1(\mathbb{R}_+, e^{\alpha x_1}) \subset L_1(\mathbb{R}_+)$, we conclude that $(F_c f_1)$ is analytic in $\mathbb{R}_+$. Moreover, $(Lg_1)(y_1)$, $(Lh_1)(y_1)$ are analytic in $\mathbb{R}_+$. Together with (9), it implies that either $f_1(x_1) = 0$, for all $x_1 > 0$, or $g_1(x_1) = 0$, for all $x_1 > 0$, or $h_1(x_1) = 0$, for all $x_1 > 0$. $\square$

4 Inequalities related to the polyconvolution operator

4.1 Norm inequality in $L_p(\mathbb{R}_+, e^{-\alpha x_1})$

The standard norms are defined as follows:

$$\begin{aligned} \|f_1\|_{L_1(\mathbb{R}_+)} &= \int_0^\infty |f_1(x_1)| dx_1; \\ \|f_1\|_{L_1(\mathbb{R}_+, q)} &= \int_0^\infty |f_1(x_1)| q(x_1) dx_1, \quad q(x_1) > 0, \text{ for all } x_1 > 0; \\ \|f_1\|_{L_p(\mathbb{R}_+)} &= \left[\int_0^\infty |f_1(x_1)|^p dx_1 \right]^{1/p}; \\ \|f_1\|_{L_p(\mathbb{R}_+, q)} &= \left[\int_0^\infty |f_1(x_1)|^p q(x_1) dx_1 \right]^{1/p}, \quad q(x_1) > 0, \text{ for all } x_1 > 0. \end{aligned}$$

In this subsection, we establish the norm inequalities for the polyconvolution (5) in $L_p(\mathbb{R}_+, e^{-\alpha x_1})$ with $1 \leq p \leq \infty$.

Theorem 3. Let $f_1 \in L_p(\mathbb{R}_+)$, $g_1 \in L_q(\mathbb{R}_+)$, $h_1 \in L_r(\mathbb{R}_+)$ be such that $p, q, r > 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Then the polyconvolution (5) is bounded in $L_s(\mathbb{R}_+, e^{-\alpha x_1})$ when $s > 1$, $\alpha > 0$ and the following inequality holds:

$$\left\| \stackrel{\beta}{*}_{F_c, L} (f_1, g_1, h_1) \right\|_{L_s(\mathbb{R}_+, e^{-\alpha x_1})} \leq \frac{1}{\alpha^{1/s}} \|f_1\|_{L_p(\mathbb{R}_+)} \|g_1\|_{L_q(\mathbb{R}_+)} \|h_1\|_{L_r(\mathbb{R}_+)}. \quad (10)$$

Proof. From the proof of Theorem 1, we have the following inequality:

$$\begin{aligned} \left| \stackrel{\beta}{*}_{F_c, L} (f_1, g_1, h_1)(x_1) \right| &\leq \int_0^\infty \int_0^\infty \int_0^\infty |f_1(u_1)| |g_1(v_1)| |h_1(w_1)| du_1 dv_1 dw_1, \\ &x_1 \in \mathbb{R}_+. \end{aligned}$$

Let p_1, q_1, r_1 be the conjugate exponents of p, q, r and let

$$\begin{aligned} M &= |g_1(v_1)|^{\frac{q}{p_1}} \cdot |h_1(w_1)|^{\frac{r}{p_1}}, \\ N &= |h_1(w_1)|^{\frac{r}{q_1}} \cdot |f_1(u_1)|^{\frac{p}{q_1}}, \\ P &= |f_1(u_1)|^{\frac{p}{r_1}} \cdot |g_1(v_1)|^{\frac{q}{r_1}}. \end{aligned}$$

We see that $MNP = |f_1(u_1)| \cdot |g_1(v_1)| \cdot |h_1(w_1)|$.

Using Fubini's theorem, we get

$$\begin{aligned} \|M\|_{L_{p_1}(\mathbb{R}_+^3)}^{p_1} &= \int_0^\infty \int_0^\infty \{ |g_1(v_1)|^{q/p_1} |h_1(w_1)|^{r/p_1} \}^{p_1} dv_1 dw_1 \\ &= \int_0^\infty |g_1(v_1)|^q \left(\int_0^\infty |h_1(w_1)|^r dw_1 \right) dv_1 \\ &= \int_0^\infty |g_1(v_1)|^q \|h_1\|_{L_r(\mathbb{R}_+)}^r dv_1 = \|g_1\|_{L_q(\mathbb{R}_+)}^q \|h_1\|_{L_r(\mathbb{R}_+)}^r. \end{aligned}$$

Similarly, we get

$$\|N\|_{L_{q_1}(\mathbb{R}_+^3)}^{q_1} = \|f_1\|_{L_p(\mathbb{R}_+)}^p \|h_1\|_{L_r(\mathbb{R}_+)}^r; \quad \|P\|_{L_{r_1}(\mathbb{R}_+^3)}^{r_1} = \|f_1\|_{L_p(\mathbb{R}_+)}^p \|g_1\|_{L_q(\mathbb{R}_+)}^q. \quad (11)$$

Since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, we have $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} = 1$. Using Hölder's inequality and (11), we obtain

$$\begin{aligned} \left| \underset{F_c, L}{\overset{\beta}{*}}(f_1, g_1, h_1) \right| &\leq \int_0^\infty \int_0^\infty \int_0^\infty MNP du_1 dv_1 dw_1 \\ &\leq \left(\int_0^\infty \int_0^\infty \int_0^\infty M^{p_1} du_1 dv_1 dw_1 \right)^{1/p_1} \\ &\quad \times \left(\int_0^\infty \int_0^\infty \int_0^\infty N^{q_1} du_1 dv_1 dw_1 \right)^{1/q_1} \\ &\quad \times \left(\int_0^\infty \int_0^\infty \int_0^\infty P^{r_1} du_1 dv_1 dw_1 \right)^{1/r_1} \\ &= \|M\|_{L_{p_1}(\mathbb{R}_+^3)} \|N\|_{L_{q_1}(\mathbb{R}_+^3)} \|P\|_{L_{r_1}(\mathbb{R}_+^3)} \\ &= \left(\|g_1\|_{L_q(\mathbb{R}_+)}^{q/p_1} \|h_1\|_{L_r(\mathbb{R}_+)}^{r/p_1} \right) \left(\|f_1\|_{L_p(\mathbb{R}_+)}^{p/q_1} \|h_1\|_{L_r(\mathbb{R}_+)}^{r/q_1} \right) \\ &\quad \times \left(\|f_1\|_{L_p(\mathbb{R}_+)}^{p/r_1} \|g_1\|_{L_q(\mathbb{R}_+)}^{q/r_1} \right) \\ &= \|f_1\|_{L_p(\mathbb{R}_+)} \|g_1\|_{L_q(\mathbb{R}_+)} \|h_1\|_{L_r(\mathbb{R}_+)}. \end{aligned}$$

Since $\int_0^\infty e^{-\alpha x_1} dx_1 = \frac{1}{\alpha}$, $\alpha > 0$, we have

$$\begin{aligned} &\left\| \underset{F_c, L}{\overset{\beta}{*}}(f_1, g_1, h_1) \right\|_{L_s(\mathbb{R}_+, e^{-\alpha x_1})}^s \\ &= \int_0^\infty e^{-\alpha x_1} \left| \underset{F_c, L}{\overset{\beta}{*}}(f_1, g_1, h_1)(x_1) \right|^s dx_1 \end{aligned}$$

$$\begin{aligned} &\leq \int_0^\infty e^{-\alpha x_1} \|f_1\|_{L_p(\mathbb{R}_+)}^s \|g_1\|_{L_q(\mathbb{R}_+)}^s \|h_1\|_{L_r(\mathbb{R}_+)}^s dx_1 \\ &= \frac{1}{\alpha} \|f_1\|_{L_p(\mathbb{R}_+)}^s \|g_1\|_{L_q(\mathbb{R}_+)}^s \|h_1\|_{L_r(\mathbb{R}_+)}^s. \end{aligned}$$

So, we obtain

$$\left\| \left[\begin{smallmatrix} \beta \\ * \\ F_c, L \end{smallmatrix} (f_1, g_1, h_1) \right] \right\|_{L_s(\mathbb{R}_+, e^{-\alpha x_1})} \leq \frac{1}{\alpha^{1/s}} \|f_1\|_{L_p(\mathbb{R}_+)} \|g_1\|_{L_q(\mathbb{R}_+)} \|h_1\|_{L_r(\mathbb{R}_+)}.$$

□

4.2 The Young-type theorem related to this operator

Theorem 4. Suppose that $p, q, r, m > 1$ satisfy the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{m} = 3$. If $f_1 \in L_p(\mathbb{R}_+)$, $g_1 \in L_q(\mathbb{R}_+)$, $h_1 \in L_r(\mathbb{R}_+)$, $k_1 \in L_m(\mathbb{R}_+)$, then the following estimate holds:

$$\begin{aligned} &\left| \int_0^\infty \left[\begin{smallmatrix} \beta \\ * \\ F_c, L \end{smallmatrix} (f_1, g_1, h_1)(x_1) \cdot k_1(x_1) \right] dx_1 \right| \\ &\leq \|f_1\|_{L_p(\mathbb{R}_+)} \cdot \|g_1\|_{L_q(\mathbb{R}_+)} \cdot \|h_1\|_{L_r(\mathbb{R}_+)} \cdot \|k_1\|_{L_m(\mathbb{R}_+)}. \end{aligned} \quad (12)$$

Proof. From (5) and the proof of Theorem 1, we have the estimate

$$\begin{aligned} &\left| \int_0^\infty \left[\begin{smallmatrix} \beta \\ * \\ F_c, L \end{smallmatrix} (f_1, g_1, h_1)(x_1) \cdot k_1(x_1) \right] dx_1 \right| \\ &\leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |f_1(u_1)| \cdot |g_1(v_1)| \cdot |h_1(w_1)| \cdot |k_1(x_1)| dw_1 dv_1 du_1 dx_1. \end{aligned} \quad (13)$$

Let p_1, q_1, r_1, m_1 be the conjugate exponents of p, q, r, m , respectively. Then

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{q} + \frac{1}{q_1} = \frac{1}{r} + \frac{1}{r_1} = \frac{1}{m} + \frac{1}{m_1} = 1.$$

From the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{m} = 3$, we obtain that

$$\begin{cases} \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} = 1, \\ p\left(\frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1}\right) = q\left(\frac{1}{p_1} + \frac{1}{r_1} + \frac{1}{m_1}\right) = r\left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{m_1}\right) \\ \quad = m\left(\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1}\right) = 1. \end{cases} \quad (14)$$

Putting

$$\begin{aligned} K_1 &= |g_1(v_1)|^{\frac{q}{p_1}} \cdot |h_1(w_1)|^{\frac{r}{p_1}} \cdot |k_1(x_1)|^{\frac{m}{p_1}}, \\ K_2 &= |f_1(u_1)|^{\frac{p}{q_1}} \cdot |h_1(w_1)|^{\frac{r}{q_1}} \cdot |k_1(x_1)|^{\frac{m}{q_1}}, \\ K_3 &= |k_1(x_1)|^{\frac{m}{r_1}} \cdot |f_1(u_1)|^{\frac{p}{r_1}} \cdot |g_1(v_1)|^{\frac{q}{r_1}}, \\ K_4 &= |f_1(u_1)|^{\frac{p}{m_1}} \cdot |g_1(v_1)|^{\frac{q}{m_1}} \cdot |h_1(w_1)|^{\frac{r}{m_1}}. \end{aligned}$$

From (14), we obtain

$$K_1 \cdot K_2 \cdot K_3 \cdot K_4 = |f_1(u_1)| \cdot |g_1(v_1)| \cdot |h_1(w_1)| \cdot |k_1(x_1)|. \quad (15)$$

From (15), we have

$$\begin{aligned} & \left| \int_0^\infty \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1)(x_1) \cdot k_1(x_1) \right] dx_1 \right| \\ & \leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty K_1 \cdot K_2 \cdot K_3 \cdot K_4 dx_1 du_1 dv_1 dw_1. \end{aligned}$$

Since $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{m_1} = 1$, using Hölder's inequality, we have the following estimate:

$$\begin{aligned} & \left| \int_0^\infty \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1)(x_1) \cdot k_1(x_1) \right] dx_1 \right| \\ & \leq \left(\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |K_1|^{p_1} dx_1 du_1 dv_1 dw_1 \right)^{\frac{1}{p_1}} \\ & \quad \cdot \left(\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |K_2|^{q_1} dx_1 du_1 dv_1 dw_1 \right)^{\frac{1}{q_1}} \\ & \quad \cdot \left(\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |K_3|^{r_1} dx_1 du_1 dv_1 dw_1 \right)^{\frac{1}{r_1}} \\ & \quad \cdot \left(\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |K_4|^{m_1} dx_1 du_1 dv_1 dw_1 \right)^{\frac{1}{m_1}} \\ & = \|K_1\|_{L_{p_1}(\mathbb{R}_+^4)} \cdot \|K_2\|_{L_{q_1}(\mathbb{R}_+^4)} \cdot \|K_3\|_{L_{r_1}(\mathbb{R}_+^4)} \cdot \|K_4\|_{L_{m_1}(\mathbb{R}_+^4)}. \end{aligned} \quad (16)$$

Furthermore, for $g_1 \in L_q(\mathbb{R}_+)$, $h_1 \in L_r(\mathbb{R}_+)$, $k_1 \in L_m(\mathbb{R}_+)$, along with the help of Fubini theorem, we have

$$\begin{aligned}
 \|K_1\|_{L_{p_1}^{p_1}(\mathbb{R}_+^4)} &= \int_0^\infty \int_0^\infty \int_0^\infty |g_1(v_1)|^q \cdot |h_1(w_1)|^r \cdot |k_1(x_1)|^m dv_1 dw_1 dx_1 \\
 &\leq \int_0^\infty |g_1(v_1)|^q dv_1 \cdot \int_0^\infty |h_1(w_1)|^r dw_1 \cdot \int_0^\infty |k_1(x_1)|^m dx_1 \\
 &= \|g_1\|_{L_q(\mathbb{R}_+)}^q \cdot \|h_1\|_{L_r(\mathbb{R}_+)}^r \cdot \|k_1\|_{L_m(\mathbb{R}_+)}^m.
 \end{aligned}$$

So, we have the norm estimate K_1 as follows:

$$\|K_1\|_{L_{p_1}(\mathbb{R}_+^4)} \leq \|g_1\|_{L_q(\mathbb{R}_+)}^{\frac{q}{p_1}} \cdot \|h_1\|_{L_r(\mathbb{R}_+)}^{\frac{r}{p_1}} \cdot \|k_1\|_{L_m(\mathbb{R}_+)}^{\frac{m}{p_1}}. \quad (17)$$

Similarly, we get

$$\|K_2\|_{L_{q_1}(\mathbb{R}_+^4)} \leq \|h_1\|_{L_r(\mathbb{R}_+)}^{\frac{r}{q_1}} \cdot \|k_1\|_{L_m(\mathbb{R}_+)}^{\frac{m}{q_1}} \cdot \|f_1\|_{L_p(\mathbb{R}_+)}^{\frac{p}{q_1}}, \quad (18)$$

$$\|K_3\|_{L_{r_1}(\mathbb{R}_+^4)} \leq \|k_1\|_{L_m(\mathbb{R}_+)}^{\frac{m}{r_1}} \cdot \|f_1\|_{L_p(\mathbb{R}_+)}^{\frac{p}{r_1}} \cdot \|g_1\|_{L_q(\mathbb{R}_+)}^{\frac{q}{r_1}}, \quad (19)$$

$$\|K_4\|_{L_{m_1}(\mathbb{R}_+^4)} \leq \|f_1\|_{L_p(\mathbb{R}_+)}^{\frac{p}{m_1}} \cdot \|g_1\|_{L_q(\mathbb{R}_+)}^{\frac{q}{m_1}} \cdot \|h_1\|_{L_r(\mathbb{R}_+)}^{\frac{r}{m_1}}. \quad (20)$$

From (16)–(20), we obtain

$$\begin{aligned}
 &\left| \int_0^\infty \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1)(x_1) \cdot k_1(x_1) \right] dx_1 \right| \\
 &\leq \|f_1\|_{L_p(\mathbb{R}_+)} \cdot \|g_1\|_{L_q(\mathbb{R}_+)} \cdot \|h_1\|_{L_r(\mathbb{R}_+)} \cdot \|k_1\|_{L_m(\mathbb{R}_+)}.
 \end{aligned}$$

□

4.3 Norm inequality of Saitoh's type in the $L_p(\mathbb{R}_+, q_i)$ weight spaces

Saitoh [24] derived, on the whole line $\mathbb{R}$, a weight $L_p(\mathbb{R}, |q_i|)$ norm inequality related to the Fourier convolution operator, which has the form

$$\left| (P_1 q_1) *_F (P_2 q_2) (q_1 *_F q_2)^{\frac{1}{p}-1} \right| \leq \|P_1\|_{L_p(\mathbb{R}, |q_1|)} \|P_2\|_{L_p(\mathbb{R}, |q_2|)},$$

with $p > 1$, where q_i are nonvanishing functions and $P_i \in L_p(\mathbb{R}, |q_i|)$, for all $i = 1, 2$; see also [7] for new convolutions and the corresponding norm inequalities. Here, the norm of P_i in the weight space $L_p(\mathbb{R}, |q_i|)$ is understood as

$$\|P_i\|_{L_p(\mathbb{R}, |q_i|)} = \left[\int_{-\infty}^{+\infty} |P_i(x_1)|^p |q_i(x_1)| dx_1 \right]^{\frac{1}{p}}, \quad i = 1, 2.$$

In this section, we develop the analogue of this inequality on the half-line $\mathbb{R}_+$, for the polyconvolution operator (5); accordingly, all the spaces below are weight spaces $L_p(\mathbb{R}_+, q_i)$ over $\mathbb{R}_+$.

Theorem 5. Suppose that q_1, q_2, q_3 are nonvanishing positive functions such that the polyconvolution operator $\stackrel{\beta}{*}_{F_c, L}(q_1, q_2, q_3)$ is well-defined. For any $P_i \in L_p(\mathbb{R}_+, q_i)$, with $p > 1$, the following $L_p(\mathbb{R}_+, q_i)$ weight inequality holds:

$$\begin{aligned} & \left\| \stackrel{\beta}{*}_{F_c, L}(P_1 q_1, P_2 q_2, P_3 q_3) \stackrel{\beta}{*}_{F_c, L}(q_1, q_2, q_3)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\ & \leq \|P_1\|_{L_p(\mathbb{R}_+, q_1)}^p \|P_2\|_{L_p(\mathbb{R}_+, q_2)}^p \|P_3\|_{L_p(\mathbb{R}_+, q_3)}^p. \end{aligned} \quad (21)$$

Proof. From Definition 1, we have

$$\begin{aligned} & \left\| \stackrel{\beta}{*}_{F_c, L}(P_1 q_1, P_2 q_2, P_3 q_3) \stackrel{\beta}{*}_{F_c, L}(q_1, q_2, q_3)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\ & = \int_0^\infty \left| \stackrel{\beta}{*}_{F_c, L}(P_1 q_1, P_2 q_2, P_3 q_3)(x_1) \right|^p \left| \stackrel{\beta}{*}_{F_c, L}(q_1, q_2, q_3)(x_1) \right|^{1-p} dx_1 \\ & = \frac{1}{2\pi} \int_0^\infty \left\{ \left| \int_0^\infty \int_0^\infty \int_0^\infty \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \right. \\ & \quad \left. \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] \right. \\ & \quad \left. \times (P_1 q_1)(u_1) (P_2 q_2)(v_1) \times (P_3 q_3)(w_1) du_1 dv_1 dw_1 \right|^p \\ & \quad \times \left| \int_0^\infty \int_0^\infty \int_0^\infty \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \\ & \quad \left. \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] \right. \\ & \quad \left. \times q_1(u_1) q_2(v_1) q_3(w_1) du_1 dv_1 dw_1 \right|^{1-p} \Big\} dx_1. \end{aligned} \quad (22)$$

Alternatively, using Hölder's inequality with q the conjugate exponent of p , we get

$$\left| \int_0^\infty \int_0^\infty \int_0^\infty \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right.$$

$$\begin{aligned}
 & + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \Big] \\
 & \times (P_1 q_1)(u_1)(P_2 q_2)(v_1)(P_3 q_3)(w_1) du_1 dv_1 dw_1 \Big| \\
 \leq & \left\{ \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \\
 & + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \Big| \\
 & \times |P_1(u_1)|^p q_1(u_1) \\
 & \times |P_2(v_1)|^p q_2(v_1) |P_3(w_1)|^p q_3(w_1) du_1 dv_1 dw_1 \Big\}^{\frac{1}{p}} \\
 & \times \left\{ \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \\
 & + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \Big| \\
 & \times q_1(u_1) q_2(v_1) q_3(w_1) du_1 dv_1 dw_1 \Big\}^{\frac{1}{q}}. \tag{23}
 \end{aligned}$$

From (22) and (23), we obtain

$$\begin{aligned}
 & \left\| \left(\beta_{F_c, L}^* (P_1 q_1, P_2 q_2, P_3 q_3) \right) \beta_{F_c, L}^* (q_1, q_2, q_3)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\
 & \leq \frac{1}{2\pi} \int_0^\infty \left\{ \left[\int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \right. \\
 & + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \Big| \\
 & \times |P_1(u_1)|^p q_1(u_1) \\
 & \times |P_2(v_1)|^p q_2(v_1) |P_3(w_1)|^p q_3(w_1) du_1 dv_1 dw_1 \Big] \\
 & \times \left[\int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \\
 & + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \Big|
 \end{aligned}$$

$$\begin{aligned}
& \times q_1(u_1)q_2(v_1)q_3(w_1) du_1 dv_1 dw_1 \Big]^{\frac{p}{q}} \Big\} \\
& \times \left[\int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \\
& \quad \left. \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right| \right. \\
& \quad \left. \times q_1(u_1)q_2(v_1)q_3(w_1) du_1 dv_1 dw_1 \right]^{1-p} dx_1.
\end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $\frac{p}{q} + 1 - p = 0$. Therefore, we have

$$\begin{aligned}
& \left\| \beta_{F_c, L}^* (P_1 q_1, P_2 q_2, P_3 q_3) \beta_{F_c, L}^* (q_1, q_2, q_3)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\
& \leq \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \\
& \quad \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right| \\
& \quad \times |P_1(u_1)|^p |q_1(u_1)| |P_2(v_1)|^p |q_2(v_1)| |P_3(w_1)|^p |q_3(w_1)| du_1 dv_1 dw_1 dx_1.
\end{aligned}$$

From the supposition $P_i \in L_p(\mathbb{R}_+, q_i)$, $i = 1, 2, 3$, using Fubini's theorem, we get

$$\begin{aligned}
& \left\| \beta_{F_c, L}^* (P_1 q_1, P_2 q_2, P_3 q_3) \beta_{F_c, L}^* (q_1, q_2, q_3)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}^p \\
& \leq \frac{1}{2\pi} \left[\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \right. \\
& \quad \left. \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right| dx_1 \right] \\
& \quad \times \left[\int_0^\infty |P_1(u_1)|^p |q_1(u_1)| du_1 \right] \\
& \quad \times \left[\int_0^\infty |P_2(v_1)|^p |q_2(v_1)| dv_1 \right] \left[\int_0^\infty |P_3(w_1)|^p |q_3(w_1)| dw_1 \right] \\
& = \|P_1\|_{L_p(\mathbb{R}_+, q_1)}^p \|P_2\|_{L_p(\mathbb{R}_+, q_2)}^p \|P_3\|_{L_p(\mathbb{R}_+, q_3)}^p.
\end{aligned}$$

□

5 Applications to solving integral equations, systems of integral equations and an integro-differential equation of Barbashin-type

In this section, we recall the Wiener–Lévy theorem: If f is the Fourier integral transform of a function in $L(\mathbb{R})$, $\Phi(u)$ is an analytic function on the value region of f and $\Phi(0) = 0$. Then $\Phi[f(x)]$ is the Fourier integral transform of a function in $L(\mathbb{R})$.

Remark 1. The Wiener–Lévy theorem also holds when f is the Fourier cosine integral transform of a function in $L(\mathbb{R}_+)$.

We unitize the Banach algebra in which expressions of the form

$$\frac{F_c g}{1 + F_c g}$$

are represented as Fourier-cosine transforms of L^1 -functions.

5.1 On the Toeplitz plus Hankel-type integral equation

In this section, we consider the Toeplitz plus Hankel-type integral equation [34]

$$f_1(x_1) + \int_0^M [m_1(x_1 + y_1) + m_2(x_1 - y_1)] f_1(y_1) dy_1 = g_1(x_1),$$

$$0 \leq x_1, y_1 \leq M, \quad (24)$$

where g_1 is a given function, f_1 is an unknown function, and $m_1(x_1 + y_1) + m_2(x_1 - y_1)$ is the Toeplitz plus Hankel-type kernel. Equation (24) has practical applications in many fields such as scattering theory, fluid dynamics, linear filtering theory, problems in radiative wave transmission, and so on. However, up to now, a closed-form solution of (24) for the general case remains an open problem. Therefore, with the help of (5), we study a class of Toeplitz plus Hankel-type integral equations related to this polyconvolution operator.

We consider the following integral equation:

$$f_1(x_1) + \frac{1}{\sqrt{2\pi}} \int_0^\infty f_1(v_1) [g_1(x_1 + v_1) + g_1(|x_1 - v_1|)] dv_1$$

$$= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty [$$

$$\begin{aligned}
& \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \\
& \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] \\
& \times h_1(u_1)k_1(v_1)l_1(w_1) du_1 dv_1 dw_1, \quad x_1 > 0,
\end{aligned} \tag{25}$$

where g_1, h_1, k_1, l_1 are functions in $L_1(\mathbb{R}_+)$ and $f_1(x_1)$ is an unknown function in $L_1(\mathbb{R}_+)$.

Theorem 6. Assume that $1 + (F_c g_1)(y_1) \neq 0$, for all $y_1 > 0$. Then the integral equation (25) has a unique solution in $L_1(\mathbb{R}_+)$ in the form

$$f_1(x_1) = \underset{F_c, L}{\overset{\beta}{*}} (h_1, k_1, l_1)(x_1) - \underset{F_c, L}{\overset{\beta}{*}} \{ (h_1 \underset{F_c}{*} p_1), k_1, l_1 \}(x_1),$$

where $p_1 \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c p_1)(y_1) = \frac{(F_c g_1)(y_1)}{1 + (F_c g_1)(y_1)}.$$

Proof. Using (3) and (5), equation (25) can be rewritten in the form

$$f_1(x_1) + (f_1 \underset{F_c}{*} g_1) = \left[\underset{F_c, L}{\overset{\beta}{*}} (h_1, k_1, l_1) \right](x_1).$$

Because $f_1, g_1, \underset{F_c, L}{\overset{\beta}{*}} (h_1, k_1, l_1)(x_1)$ and $(f_1 \underset{F_c}{*} g_1)$ are functions in $L_1(\mathbb{R}_+)$, one can apply the factorization properties (4) and (6) to get

$$(F_c f_1)(y_1) + (F_c f_1)(y_1)(F_c g_1)(y_1) = \cos y_1 (F_c h_1)(y_1)(Lk_1)(y_1)(Ll_1)(y_1).$$

So

$$(F_c f_1)(y_1)[1 + (F_c g_1)(y_1)] = \cos y_1 (F_c h_1)(y_1)(Lk_1)(y_1)(Ll_1)(y_1).$$

From the condition $1 + (F_c g_1)(y_1) \neq 0$, for all $y_1 > 0$, we have

$$(F_c f_1)(y_1) = \cos y_1 (F_c h_1)(y_1)(Lk_1)(y_1)(Ll_1)(y_1) \left[1 - \frac{(F_c g_1)(y_1)}{1 + (F_c g_1)(y_1)} \right].$$

Due to the Wiener–Lévy theorem [1], there exists a function $p_1 \in L_1(\mathbb{R}_+)$ such that

$$(F_c p_1)(y_1) = \frac{(F_c g_1)(y_1)}{1 + (F_c g_1)(y_1)}.$$

Therefore, from (4) and (6), we get

$$(F_c f_1)(y_1) = F_c \left[\underset{F_c, L}{*}^{\beta} (h_1, k_1, l_1) \right](y_1) - F_c \left[\underset{F_c, L}{*}^{\beta} \{ (h_1 \underset{F_c}{*} p_1), k_1, l_1 \} \right](y_1).$$

Thus

$$f_1(x_1) = \underset{F_c, L}{*}^{\beta} (h_1, k_1, l_1)(x_1) - \underset{F_c, L}{*}^{\beta} \{ (h_1 \underset{F_c}{*} p_1), k_1, l_1 \}(x_1) \in L_1(\mathbb{R}_+).$$

□

5.2 The integral equation of polyconvolution type

We consider the following integral equation:

$$\begin{aligned} f_1(x_1) + \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty & \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \\ & \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] \\ & \times f_1(u_1) g_1(v_1) h_1(w_1) du_1 dv_1 dw_1 = p_1(x_1), \quad x_1 > 0, \end{aligned} \quad (26)$$

where g_1, h_1, p_1 are functions in $L_1(\mathbb{R}_+)$ and $f_1(x_1)$ is an unknown function in $L_1(\mathbb{R}_+)$.

Theorem 7. Assume that $1 + \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1) \neq 0$, for all $y_1 > 0$. Then the integral equation (26) has a unique solution in $L_1(\mathbb{R}_+)$ in the form

$$f_1(x_1) = p_1(x_1) - \left(p_1 \underset{F_c}{*} q_1 \right)(x_1),$$

where $q_1(x_1) \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c q_1)(y_1) = \frac{F_c(g_1 \underset{F_c, L}{*}^{\beta} h_1)(y_1)}{1 + F_c(g_1 \underset{F_c, L}{*}^{\beta} h_1)(y_1)}.$$

Proof. Using (5), equation (26) can be rewritten in the following form:

$$f_1(x_1) + \left[\underset{F_c, L}{*}^{\beta} (f_1, g_1, h_1) \right](x_1) = p_1(x_1).$$

Because $f_1, \underset{F_c, L}{*}^{\beta} (f_1, g_1, h_1)$ and p_1 are functions in $L_1(\mathbb{R}_+)$, applying the Fourier cosine integral transform, we obtain

$$(F_c f_1)(y_1) + \cos y_1 (F_c f_1)(y_1)(Lg_1)(y_1)(Lh_1)(y_1) = (F_c p_1)(y_1).$$

Under the condition $1 + \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1) \neq 0$, for all $y_1 > 0$, we have

$$(F_c f_1)(y_1) = (F_c p_1)(y_1) \left[1 - \frac{\cos y_1 (Lg_1)(y_1)(Lh_1)(y_1)}{1 + \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1)} \right].$$

The generalized convolution operator with the weight function $\beta(y_1) = \cos y_1$ for two functions g_1, h_1 is denoted by $(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(x_1)$ and defined by

$$\begin{aligned} (g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(x_1) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty \left[\frac{u_1 + v_1}{(u_1 + v_1)^2 + (x_1 + 1)^2} \right. \\ &\quad \left. + \frac{u_1 + v_1}{(u_1 + v_1)^2 + (x_1 - 1)^2} \right] g_1(u_1) h_1(v_1) du_1 dv_1. \end{aligned}$$

We prove that if g_1, h_1 are arbitrary functions in $L_1(\mathbb{R}_+)$, then $(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(x_1)$ belongs to $L_1(\mathbb{R}_+)$ and satisfies the factorization property

$$F_c(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(y_1) = \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1), \quad \text{for all } y_1 > 0.$$

Therefore, we have

$$(F_c f_1)(y_1) = (F_c p_1)(y_1) \left[1 - \frac{F_c(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(y_1)}{1 + F_c(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(y_1)} \right].$$

Due to the Wiener–Lévy theorem [1], there exists a function $q_1 \in L_1(\mathbb{R}_+)$ such that

$$(F_c q_1)(y_1) = \frac{F_c(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(y_1)}{1 + F_c(g_1 \stackrel{\beta}{*}_{F_c, L} h_1)(y_1)}.$$

Therefore, from (4), we get

$$(F_c f_1)(y_1) = (F_c p_1)(y_1) - F_c(p_1 \stackrel{*}{F_c} q_1)(y_1).$$

Therefore

$$f_1(x_1) = p_1(x_1) - (p_1 \stackrel{*}{F_c} q_1)(x_1) \in L_1(\mathbb{R}_+).$$

□

5.3 The system of integral equations of polyconvolution-type

Consider the system of integral equations

$$\begin{aligned}
 & f_1(x_1) - \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty g_1(u_1) l_1(v_1) h_1(w_1) \\
 & \quad \times \left[\frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 + 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 + 1)^2} \right. \\
 & \quad \left. + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 + x_1 - 1)^2} + \frac{v_1 + w_1}{(v_1 + w_1)^2 + (u_1 - x_1 - 1)^2} \right] \\
 & \quad du_1 dv_1 dw_1 = p_1(x_1), \\
 & \frac{1}{\sqrt{2\pi}} \int_0^\infty f_1(u_1) [r_1(u_1 + x_1) + r_1(|u_1 - x_1|)] du_1 \\
 & \quad + g_1(x_1) = q_1(x_1), \quad x_1 > 0,
 \end{aligned} \tag{27}$$

where l_1, h_1, r_1, p_1, q_1 are functions in $L_1(\mathbb{R}_+)$ and f_1, g_1 are unknown functions.

Theorem 8. If the following condition is satisfied:

$$1 + F_c \left[\overset{\beta}{*}_{F_c, L} (r_1, l_1, h_1) \right] (y_1) \neq 0, \quad \text{for all } y_1 > 0,$$

then system (27) has a unique solution in $L_1(\mathbb{R}_+)$ given by

$$\begin{aligned}
 f_1(x_1) &= p_1(x_1) - \left[\overset{\beta}{*}_{F_c, L} (q_1, l_1, h_1) \right] (x_1) - (p_1 \overset{*}{*}_{F_c} k_1)(x_1) \\
 & \quad + \left\{ \left[\overset{\beta}{*}_{F_c, L} (q_1, l_1, h_1) \right] \overset{*}{*}_{F_c} k_1 \right\} (x_1); \\
 g_1(x_1) &= q_1(x_1) - (r_1 \overset{*}{*}_{F_c} p_1)(x_1) - (q_1 \overset{*}{*}_{F_c} k_1)(x_1) \\
 & \quad + [(r_1 \overset{*}{*}_{F_c} p_1) \overset{*}{*}_{F_c} k_1](x_1),
 \end{aligned}$$

where $k_1(x_1) \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c k_1)(y_1) = \frac{F_c \left[\overset{\beta}{*}_{F_c, L} (r_1, l_1, h_1) \right] (y_1)}{1 + F_c \left[\overset{\beta}{*}_{F_c, L} (r_1, l_1, h_1) \right] (y_1)}.$$

Proof. Using Definition 1 and (3), system (27) can be rewritten in the form

$$\begin{cases} f_1(x_1) - [\underset{F_c, L}{*}^{\beta} (g_1, l_1, h_1)](x_1) = p_1(x_1), \\ (f_1 \underset{F_c}{*} r_1)(x_1) + g_1(x_1) = q_1(x_1). \end{cases}$$

Applying the Fourier cosine integral transform to the system, we have

$$\begin{aligned} (F_c f_1)(y_1) - \cos y_1 (F_c g_1)(y_1) (L l_1)(y_1) (L h_1)(y_1) &= (F_c p_1)(y_1), \\ (F_c f_1)(y_1) (F_c r_1)(y_1) + (F_c g_1)(y_1) &= (F_c q_1)(y_1). \end{aligned}$$

The reciprocal of the determinant of this linear algebraic system has the form

$$\frac{1}{\Delta} = 1 - \frac{F_c [\underset{F_c, L}{*}^{\beta} (r_1, l_1, h_1)](y_1)}{1 + F_c [\underset{F_c, L}{*}^{\beta} (r_1, l_1, h_1)](y_1)}.$$

Furthermore, according to the Wiener-Lévy theorem [1], there exists a function $k_1 \in L_1(\mathbb{R}_+)$ such that

$$(F_c k_1)(y_1) = \frac{F_c [\underset{F_c, L}{*}^{\beta} (r_1, l_1, h_1)](y_1)}{1 + F_c [\underset{F_c, L}{*}^{\beta} (r_1, l_1, h_1)](y_1)}.$$

Hence

$$\frac{1}{\Delta} = 1 - (F_c k_1)(y_1).$$

Therefore, using the factorization properties (4) and (6), we get

$$\begin{aligned} (F_c f_1)(y_1) &= (F_c p_1)(y_1) - F_c [\underset{F_c, L}{*}^{\beta} (q_1, l_1, h_1)](y_1) - F_c (p_1 \underset{F_c}{*} k_1)(y_1) \\ &\quad + F_c \left\{ [\underset{F_c, L}{*}^{\beta} (q_1, l_1, h_1)] \underset{F_c}{*} k_1 \right\}(y_1); \\ (F_c g_1)(y_1) &= (F_c q_1)(y_1) - F_c (r_1 \underset{F_c}{*} p_1)(y_1) - F_c (q_1 \underset{F_c}{*} k_1)(y_1) \\ &\quad + F_c [(r_1 \underset{F_c}{*} p_1) \underset{F_c}{*} k_1](y_1). \end{aligned}$$

It follows that

$$\begin{aligned} f_1(x_1) &= p_1(x_1) - [\underset{F_c, L}{*}^{\beta} (q_1, l_1, h_1)](x_1) - (p_1 \underset{F_c}{*} k_1)(x_1) \\ &\quad + \left\{ [\underset{F_c, L}{*}^{\beta} (q_1, l_1, h_1)] \underset{F_c}{*} k_1 \right\}(x_1) \in L_1(\mathbb{R}_+); \\ g_1(x_1) &= q_1(x_1) - (r_1 \underset{F_c}{*} p_1)(x_1) - (q_1 \underset{F_c}{*} k_1)(x_1) \\ &\quad + [(r_1 \underset{F_c}{*} p_1) \underset{F_c}{*} k_1](x_1) \in L_1(\mathbb{R}_+). \end{aligned}$$

□

5.4 On the integro-differential equation of Barbashin-type

The Barbashin integro-differential equation has applications in many fields such as mathematical physics, radiation propagation, and mathematical biology; see, for example, [2, 3]. The general form of Barbashin's equation is

$$\frac{\partial f_1(x_1, y_1)}{\partial x_1} = \theta(x_1, y_1)f_1(x_1, y_1) + \int_a^b K(x_1, y_1, t_1)f_1(x_1, t_1) dt_1 + \eta(x_1, y_1),$$

where $\theta, \eta : I \times [a, b] \rightarrow \mathbb{R}$ are given functions, $K : I \times [a, b]^2 \rightarrow \mathbb{R}$ is the kernel, and I is a closed set in $\mathbb{R}$ (see [2, 3]). In this subsection, we study the integro-differential equation of Barbashin-type

$$f_1(x_1) + \left(1 - \frac{d^2}{dx_1^2}\right) \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1) \right](x_1) = q_1(x_1), \quad (28)$$

where g_1, h_1, q_1 are given functions and f_1 is an unknown function.

Theorem 9. Suppose that g_1, h_1, q_1 belong to $L_2(\mathbb{R}_+)$ and are given functions satisfying the condition: $1 + (1 + y_1^2) \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1) \neq 0$;

$$\frac{(F_c q_1)(y_1)}{1 + (1 + y_1^2) \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1)} \in L_2(\mathbb{R}_+), \quad y_1 > 0, \quad (29)$$

and $x_1 \cdot \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1) \right](x_1), x_1^2 \cdot \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1) \right](x_1) \in L_2(\mathbb{R}_+)$, then (28) has a unique solution in $L_2(\mathbb{R}_+)$ which can be presented in the form

$$f_1(x_1) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(F_c q_1)(y_1) \cos(x_1 y_1)}{1 + (1 + y_1^2) \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1)} dy_1, \quad x_1 \in \mathbb{R}_+.$$

Proof. Applying the Fourier cosine integral transform to (28) and the factorization identity (6), and using the hypothesis that $x_1 \cdot \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1) \right](x_1), x_1^2 \cdot \left[{}_{F_c, L}^{\beta} (f_1, g_1, h_1) \right](x_1)$ belong to $L_2(\mathbb{R}_+)$, we get

$$(F_c f_1)(y_1) + (1 + y_1^2) \cos y_1 (F_c f_1)(y_1)(Lg_1)(y_1)(Lh_1)(y_1) = (F_c q_1)(y_1).$$

From the condition (29), we get

$$(F_c f_1)(y_1) = \frac{1}{1 + (1 + y_1^2) \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1)} (F_c q_1)(y_1).$$

Using the inverse transform in $L_2(\mathbb{R}_+)$, we get the solution of (28) in $L_2(\mathbb{R}_+)$ and express it as follows:

$$f_1(x_1) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(F_c q_1)(y_1) \cos(x_1 y_1)}{1 + (1 + y_1^2) \cos y_1 (Lg_1)(y_1)(Lh_1)(y_1)} dy_1, \quad x_1 \in \mathbb{R}_+.$$

□

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Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] Akhiezer, N.I. *Lectures on approximation theory*, Science Publishing House, Moscow, 1965. (In Russian.)
- [2] Appell, J.M., Kalitvin, A.S. and Zabrejko, P.P. *Partial integral operators and integro-differential equations*, Pure and Applied Mathematics, Marcel Dekker, New York, 2000.
- [3] Barbashin, E.A. *Condition for invariance of stability of solution of integro-differential equation*, Izv. Vyssh. Uchebn. Zaved. Mat. 1 (1957) 25–34. (In Russian.)
- [4] Castro, L.P., Guerra, R.C. and Tuan, N.M. *New convolutions weighted by Hermite functions and their applications*, Math. Inequal. Appl. 22 (2019) 719–745.
- [5] Castro, L.P., Guerra, R.C. and Tuan, N.M. *Convolutions and integral equations weighted by multi-dimensional Hermite functions*, Asian-Eur. J. Math. 15 (2022) Article ID 2250151, 35 pp.

- [6] Castro, L.P., Phuong, N.T.H. and Tuan, T. *Qualitative properties and applications of convolutions associated with Hartley and Fourier transforms*, Bol. Soc. Mat. Mex. (3) 32 (2026) Paper No. 54, 46 pp.
- [7] Castro, L.P. and Saitoh, S. *New convolutions and norm inequalities*, Math. Inequal. Appl. 15 (2012) 707–716.
- [8] Castro, L.P., Silva, A.S. and Tuan, N.M. *New convolutions with Hermite weight functions*, Bull. Iran. Math. Soc. 47 (2021) Suppl. 1, 365–379.
- [9] Castro, L.P., Tuan, N.M. and Tuan, T. *Polyconvolutions and their applications to Fredholm and Barbashin-type integral equations*, Quaest. Math. 49 (2026) 335–369.
- [10] Churchill, R.V. *Operational mathematics*, McGraw-Hill, New York, 1972.
- [11] Debnath, L. and Bhatta, D. *Integral transforms and their applications*, 2nd ed., Chapman & Hall/CRC Press, New York, 2006.
- [12] Gakhov, F.D. and Cherskii, Yu.I. *Equations of convolution type*, Nauka, Moscow, 1978. (In Russian.)
- [13] Kakichev, V.A. *On the convolution for integral transforms*, Izv. Vyssh. Uchebn. Zaved. Mat. 2 (1967) 48–57. (In Russian.)
- [14] Kakichev, V.A. *Polyconvolution*, TPTU, Taganrog, 1997. (In Russian.)
- [15] Kakichev, V.A. and Thao, N.X. *A method for constructing generalized integral convolutions*, Izv. Vyssh. Uchebn. Zaved. Mat. 42 (1998) 31–40. (In Russian.)
- [16] Khoa, N.M. *On the convolutions of Fourier-type transforms*, Acta Math. Vietnam. 36 (2011) 283–298.
- [17] Khoa, N.M. *On a polyconvolution with a weight function for Fourier cosine and Laplace transforms*, Acta Math. Vietnam. 41 (2016) 549–562.
- [18] Khoa, N.M. *On the polyconvolution of Hartley integral transforms H_1, H_1, H_2 and integral equation of polyconvolution type*, Far East J. Math. Sci. 132 (2021) 143–158.
- [19] Khoa, N.M. *On the polyconvolution operator with a trigonometric weight function for the Hartley integral transforms and applications*, Integral Transforms Spec. Funct. 34 (2023) 861–877.

- [20] Khoa, N.M. and Luong, D.X. *On the polyconvolution of Hartley integral transforms H_1, H_2, H_1 and integral equations*, Aust. J. Math. Anal. Appl. 16 (2019) 1–10.
- [21] Khoa, N.M. and Thang, T.V. *On the polyconvolution of Hartley integral transforms H_2 and integral equations*, J. Integral Equations Appl. 32 (2020) 171–180.
- [22] Khoa, N.M. and Thang, T.V. *On the polyconvolution with weight function $\gamma(y) = \cos y$ for Hartley integral transforms H_1, H_2, H_1 and integral equations*, Ukrainian Math. J. 75 (2023) 652–662.
- [23] Napalkov, V.V. *Convolution equations in multidimensional space*, Nauka, Moscow, 1982.
- [24] Saitoh, S. *Weighted L_p -norm inequalities in convolutions*, Survey on Classical Inequalities (T.M. Rassias, ed.), Mathematics and its Applications, Springer, Dordrecht, 2000, pp. 225–234.
- [25] Sneddon, I.N. *The use of integral transforms*, McGraw-Hill, New York, 1972.
- [26] Sogge, C.D. *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics 105, Cambridge University Press, Cambridge, 1993.
- [27] Thao, N.X. and Khoa, N.M. *On the convolution with a weight function for the cosine-Fourier integral transforms*, Acta Math. Vietnam. 29 (2004) 149–162.
- [28] Thao, N.X. and Khoa, N.M. *On the generalized convolution with a weight function for the Fourier sine and cosine transforms*, Integral Transforms Spec. Funct. 17 (2006) 673–685.
- [29] Thao, N.X., Khoa, N.M. and Anh, P.T.V. *On the polyconvolution for Hartley, Fourier cosine and Fourier sine transforms*, Integral Transforms Spec. Funct. 24 (2013) 517–531.
- [30] Thao, N.X., Tuan, T. and Huy, L.X. *The Fourier-Laplace generalized convolutions and applications to integral equations*, Vietnam J. Math. 41 (2013) 451–464.
- [31] Thao, N.X., Tuan, T. and Huy, L.X. *The generalized convolutions with a weight function for Laplace transform*, Nonlinear Funct. Anal. Appl. 19 (2014) 61–77.
- [32] Thao, N.X., Tuan, V.K. and Khoa, N.M. *A generalized convolution with a weight function for the Fourier cosine and sine transform*, Fract. Calc. Appl. Anal. 7 (2004) 323–337.
- [33] Titchmarsh, E.C. *Introduction to the theory of Fourier integrals*, 3rd ed., Chelsea Publishing Co., New York, 1986.

- [34] Tsitsiklis, J.N. and Levy, B.C. *Integral equations and resolvents of Toeplitz plus Hankel kernel*, Report No. LIDS-P 1170, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 1981.
- [35] Tuan, T. *New polyconvolution product for Fourier-cosine and Laplace integral operators and their applications*, Math. Methods Appl. Sci. 47 (2024) 1680–1700.
- [36] Tuan, V.K. and Saigo, M. *Convolution of Hankel transform and its applications to an integral involving Bessel function of first kind*, Int. J. Math. Math. Sci. 18 (1995) 545–550.
- [37] Yakubovich, S.B. and Britvina, L.E. *Convolution related to the Fourier and Kontorovich-Lebedev transforms revisited*, Integral Transforms Spec. Funct. 21 (2010) 255–276.