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Research Article

A modified school bullying model with a recovery compartment: Stability analysis and optimal control

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Abstract

This study proposes a compartmental dynamical model of school bullying transmission incorporating an explicit recovery mechanism through counseling and institutional intervention. The population is divided into susceptible, bully, exposed, violent, and recovered subgroups. The model's qualitative properties are established, including positivity, boundedness, equilibrium analysis, and stability conditions derived using the next-generation matrix approach. The bullying-free equilibrium is shown to be both locally and globally asymptotically stable whenever $\mathcal{R}_0 < 1$, and the endemic equilibrium is shown to exist and be locally asymptotically stable when $\mathcal{R}_0 > 1$ under explicit Routh–Hurwitz conditions. Sensitivity analysis reveals that the perpetrator-generation rate α is the primary driver of bullying persistence, whereas recovery-related interventions substantially reduce long-term prevalence. An optimal control framework based on Pontryagin's Maximum Principle is then formulated using disciplinary

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and victim-protection interventions. Numerical simulations show that combining both interventions yields the largest reductions in bullying and in overall cost, indicating that integrated institutional responses outperform isolated policies.

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1 Introduction

School bullying remains a major concern in education and public health because of its substantial impact on students' psychological well-being, academic engagement, and social development [2, 25]. Recent studies have shown that bullying involvement is strongly associated with anxiety, depression, social isolation, behavioral maladjustment, and long-term delinquent tendencies [7, 23]. Moreover, prevalence estimates vary considerably across demographic groups and school environments, suggesting that bullying is not merely an isolated behavioral incident but rather a complex social process embedded within peer interaction systems and social-ecological environments [6, 9, 21].

A growing body of evidence further indicates that bullying behavior evolves through mechanisms of peer influence and social contagion [6]. Adolescents are more likely to engage in bullying when they are surrounded by peers exhibiting similar aggressive behavior, and the probability of involvement increases according to the intensity of social interaction within friendship networks [9, 13]. From a social-ecological perspective, bullying dynamics are shaped not only by individual dispositions but also by repeated interactions, group reinforcement, classroom climate, and institutional responses [9]. These observations underscore the need for dynamic frameworks that describe how students transition between behavioral states over time.

Despite extensive empirical research on bullying prevalence, risk factors, and intervention outcomes, relatively limited attention has been given to analytical frameworks that explicitly capture long-term behavioral transitions and intervention-driven recovery processes. Most existing studies focus primarily on identifying correlates of bullying involvement. In contrast, much less attention has been devoted to understanding how bullying behavior propagates, persists, or diminishes dynamically within a school population.

Mathematical modeling provides a useful framework for investigating these issues. In particular, compartmental dynamical systems enable the analysis of threshold behavior, equilibrium stability, nonlinear interaction mechanisms, and intervention effectiveness that is difficult to observe directly from empirical studies alone [6, 18]. Such approaches have been widely used in epidemiological and social-dynamics research to examine behavioral propagation processes and optimal intervention policies [18, 16].

In the context of bullying dynamics, Crokidakis [8] proposed a compartmental model dividing the school population into susceptible, bully, exposed, and violent classes, thereby demonstrating that bullying behavior can be studied within a formal dynamical systems framework. Although this model successfully captures escalation pathways associated with bullying behavior, it does not explicitly incorporate recovery-related transitions resulting from counseling, psycho-social support, or school-based restorative intervention.

This limitation is important because recent intervention studies increasingly demonstrate that targeted school programs can reduce victimization and improve psychological outcomes among affected students [12]. Systematic reviews have shown that counseling-based and school-wide intervention strategies may substantially influence behavioral recovery and social reintegration [12, 10]. Consequently, excluding recovery dynamics may limit existing models' ability to investigate the persistence of intervention effects, relapse into susceptibility, and the long-term impact of restorative school policies on bullying prevalence.

Furthermore, although optimal control approaches have been widely applied to epidemic and behavioral-dynamics models [16], relatively few studies have examined how recovery-oriented interventions interact with the dynamics of bullying propagation and long-term control strategies. From a dynamical systems perspective, the inclusion of a recovery compartment introduces an additional recovery–susceptibility feedback mechanism that may alter the system's threshold structure and stability properties.

More recent work has extended compartmental bullying models with family education and memory effects [1], while related social-behavior models have combined next-generation-matrix stability analysis with Pontryagin-based optimal control [14, 22]. Our study differs by introducing an explicit recovery compartment with a recovery-to-susceptibility feedback and by comparing perpetrator- and victim-focused controls.

Motivated by these considerations, this study proposes a modified compartmental model of school bullying dynamics that explicitly incorporates a recovery compartment to represent counseling and school-based intervention outcomes. The model is analyzed qualitatively through positivity analysis, invariant-region analysis, equilibrium analysis, including the local and global stability of the bullying-free equilibrium and the local stability of the endemic equilibrium. In addition, an optimal control framework involving punishment-based intervention and victimization-prevention strategies is formulated using Pontryagin's Maximum Principle to determine efficient intervention policies for reducing bullying prevalence and minimizing implementation costs.

The remainder of this paper is organized as follows. Section 2 presents the mathematical formulation and modeling assumptions. Section 3 discusses the qualitative analysis of the uncontrolled model. Section 4 develops the optimal control formulation. Section 5 presents numerical simulations and discusses intervention strategies. Finally, Section 6 concludes the paper and outlines several directions for future research.

2 Mathematical formulation and the assumptions of the model

Let $S(t)$, $B(t)$, $E(t)$, $V(t)$, and $R(t)$ denote, respectively, susceptible students, active bullying perpetrators, nonretaliating victims, retaliating victims, and recovered students at time t. The total school population is

$$N(t) = S(t) + B(t) + E(t) + V(t) + R(t).$$

The recovered class represents students who are temporarily inactive in the bullying process following counseling or intervention. This compartment is introduced to distinguish students who have recovered from those who are merely susceptible to future involvement. The proposed model extends compartmental descriptions of school bullying dynamics by incorporating recovery and possible loss of intervention effects [8, 1, 3].

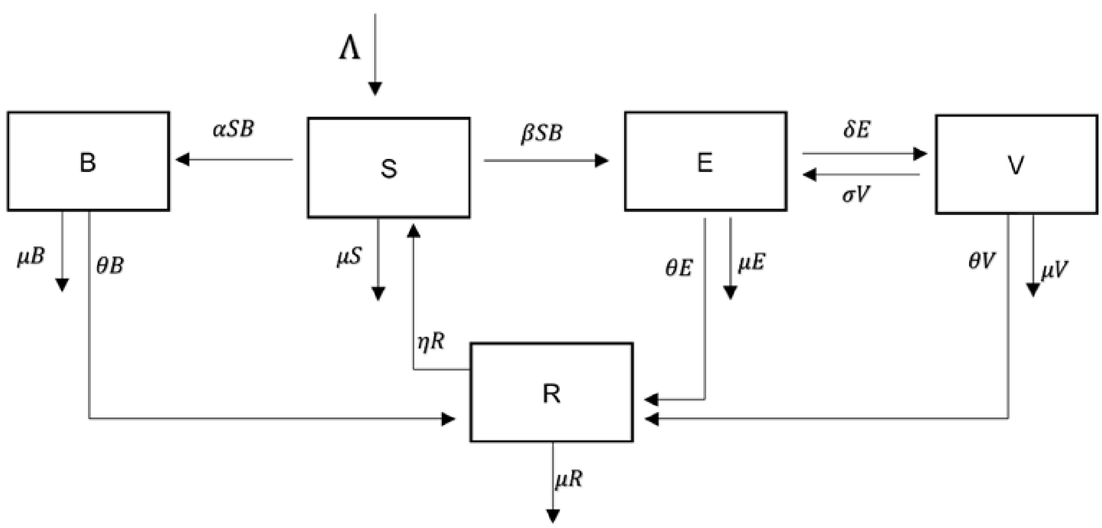

Figure 1: Transmission diagram of the bullying model with a recovery compartment.

The model is constructed under the following assumptions:

1. Students interact homogeneously within a single school environment. New students enter the system through the susceptible class at rate (Λ), while all students leave the system at the natural removal rate (μ).

2. Bullying-related transitions are generated by interactions between susceptible students and active perpetrators. Such interactions may lead susceptible students either to become perpetrators at rate αSB or nonretaliating victims at rate βSB.

3. Nonretaliating victims may develop aggressive responses at rate δ, while aggressively retaliating victims may de-escalate and return to the nonretaliating victim class at rate σ [8].

4. Active perpetrators, nonretaliating victims, and aggressively retaliating victims may recover through counseling or school-based intervention at the common rate θ [15].

5. Recovered students are assumed to be inactive in the bullying process, although the effect of intervention may weaken over time, causing them to return to the susceptible class at rate η.

For parsimony, we assume a common recovery rate θ for perpetrators (B), nonretaliating victims (E) and retaliating victims (V). In practice these groups may respond differently to counseling and discipline, so θ should be interpreted as an average intervention effect rather than a group-specific rate. Allowing class-specific recovery rates $\theta_B, \theta_E, \theta_V$ is a straightforward extension; its effect on the threshold dynamics is discussed in Section 3.4.

The mathematical model derived from the transmission diagram in Figure 1 is given by the system of differential equations in (1). The parameters used in the proposed model are described in Table 1. Based on the assumptions stated above, we obtain the following nonlinear system of equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda + \eta R - \alpha SB - \beta SB - \mu S,\\
\frac{dB}{dt} &= \alpha SB - \theta B - \mu B,\\
\frac{dE}{dt} &= \beta SB + \sigma V - \delta E - \theta E - \mu E,\\
\frac{dV}{dt} &= \delta E - \sigma V - \theta V - \mu V,\\
\frac{dR}{dt} &= \theta B + \theta E + \theta V - \eta R - \mu R.
\end{aligned} \tag{1}$$

with the initial conditions given by $S(0) > 0, B(0) \geq 0, E(0) \geq 0, V(0) \geq 0$, and $R(0) \geq 0$.

Table 1: Parameters of the school bullying model (all rates are expressed per day)

Parameter	**Definition**
Λ	Recruitment rate into the school population
α	Transition rate from susceptible students to active bullying perpetrators.
β	Transition rate from susceptible students to nonretaliating victims.
δ	Progression rate from nonretaliating victims to aggressively retaliating victims.
σ	De-escalation rate from aggressive victims to nonretaliating victims.
θ	Recovery rate of individuals in B, E, and V due to counseling or intervention.
η	Return rate from recovered students to the susceptible class due to loss of intervention effects.
μ	Natural removal rate due to graduation or withdrawal.

3 Mathematical analysis of the model

Before analyzing the qualitative behavior of the system (1), we first establish that its solutions exist and are unique.

Lemma 1 (Existence and uniqueness)**.** The vector field f defined by the right-hand side of the system (1) is continuously differentiable, hence locally Lipschitz, on $\mathbb{R}^5_+$. By the Picard–Lindelöf theorem [19], for every nonnegative initial condition there exists a unique solution of (1) on a maximal forward interval. Together with the boundedness established in Lemma 2 below, this solution is defined for all $t \geq 0$.

3.1 Invariant region of the model

The following lemma establishes the boundedness of the model (1).

Lemma 2. If the initial conditions of the model (1) are within

$$\Omega = \left\{ (S, B, E, V, R) \in \mathbb{R}^5_+ : 0 \leq N(t) \leq \max \left\{ N(0), \frac{\Lambda}{\mu} \right\} \right\},$$

then all solutions of the model enter and remain in Ω.

Proof. Given the set $(S(t), B(t), E(t), V(t), R(t))$ for any solution of the system (1), and letting $N = S + B + E + V + R$, we have

$$\frac{dN}{dt} \leq \Lambda - \mu N.$$

Solving this linear differential equation yields

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu} \right) e^{-\mu t},$$

where $N_0 = N(0)$. Since $0 < e^{-\mu t} \leq 1$ for $t \geq 0$, it follows that $0 \leq N(t) \leq \max\{N(0), \frac{\Lambda}{\mu}\}$. Therefore Ω is positively invariant.

□

3.2 Positivity of the solutions

Since the model (1) tracks the populations of students, the solutions of the model (1) have to be nonnegative. Hence, the following theorem is proved to ensure that all solutions are nonnegative.

Theorem 1. For any nonnegative initial condition in $\mathbb{R}^5_+$, the solution of system (1) remains nonnegative for all $t \geq 0$.

Proof. By Lemma 1 the right-hand side of system (1) is locally Lipschitz, so a unique solution exists on a maximal forward interval. Suppose, for contradiction, that the solution leaves the nonnegative orthant $\mathbb{R}^5_+$, and set

$$t_1 = \inf\{\, t > 0 : \min\{S(t), B(t), E(t), V(t), R(t)\} = 0 \,\}.$$

Then $t_1 > 0$ (by continuity, since all components are strictly positive at $t = 0$ except possibly B, E, V, R which are ≥ 0), and all five components are nonnegative on $[0, t_1]$ with at least one equal to zero at t_1.

For the case $S(t_1) = 0$. On $[0, t_1]$, satisfies

$$\frac{dS}{dt} = \Lambda + \eta R - (\alpha + \beta) SB - \mu S \;\geq\; -\big[(\alpha + \beta) B(t) + \mu\big]\, S,$$

because $\Lambda + \eta R \geq 0$. Integrating this differential inequality gives

$$S(t_1) \;\geq\; S(0)\,\exp\Big(-\int_0^{t_1}\big[(\alpha+\beta)B(s)+\mu\big]\,ds\Big) \;>\; 0,$$

since $S(0) > 0$ and the exponential is strictly positive. This contradicts $S(t_1) = 0$.

The same argument applies to each equation, since each has the quasi-positive form $\frac{dx_i}{dt} \geq -c_i(t)\,x_i$ on the boundary:

$$\begin{aligned} \frac{dB}{dt} &\geq -(\theta+\mu)B, & \frac{dE}{dt} &\geq -(\delta+\theta+\mu)E, \\ \frac{dV}{dt} &\geq -(\sigma+\theta+\mu)V, & \frac{dR}{dt} &\geq -(\eta+\mu)R, \end{aligned}$$

where in each case the discarded terms (αSB; $\beta SB + \sigma V$; δE; $\theta B + \theta E + \theta V$) are nonnegative on $[0, t_1]$. Integrating yields $x_i(t_1) \geq x_i(0)\,e^{-\int_0^{t_1} c_i(s)\,ds} \geq 0$, and strictly positive whenever $x_i(0) > 0$, so no component can reach zero at t_1.

In every case we reach a contradiction. Hence the solution never leaves $\mathbb{R}^5_+$, i.e. $\mathbb{R}^5_+$ is positively invariant and all state variables remain nonnegative for all $t \geq 0$. □

3.3 Bullying free equilibrium point

The bullying-free equilibrium is the steady state in which bullying-related dynamics are absent in the school population. This occurs when there are no active bullies, no nonretaliating victims, and no violent responders, namely

$$B = E = V = 0.$$

In the absence of bullying involvement, no individuals enter the recovery class through intervention or counseling. Consequently, the recovered population also vanishes at equilibrium, that is, $R = 0$. The remaining population consists only of susceptible individuals. By setting the susceptible equation at steady state, one obtains

$$0 = \Lambda - \mu S,$$

which gives

$$S = \frac{\Lambda}{\mu}.$$

Thus, if the bullying-free equilibrium is denoted by $\mathcal{E}_0^* = (S_0, B_0, E_0, V_0, R_0)$, then

$$\mathcal{E}_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right). \tag{2}$$

3.4 Basic reproduction number

The basic reproduction number, denoted by $\mathcal{R}_0$, is defined as the expected number of perpetrators generated by a single active bully introduced into an otherwise bully-free school population during its active-bullying period [24]. It acts as a threshold: bulling cannot invade when $R_0 < 1$, whereas it can persist when $R_0 > 1$. We compute R_0 using the next-generation matrix method [24].

For the proposed model, recall the system (1) and its bullying-free equilibrium (2). To obtain $\mathcal{R}_0$, the next-generation matrix method is applied to the bullying-related compartments. Since the active bullying dynamics are represented by the compartments of bullies, nonretaliating victims, and violent victims, we take

$$x = (B, E, V)^T.$$

The terms responsible for generating new bullying-related cases are separated from the remaining transition terms. Thus, system (1), restricted to the variables (B, E, V), can be written as

$$\begin{aligned} \frac{dB}{dt} &= \alpha SB - (\theta + \mu)B, \\ \frac{dE}{dt} &= \beta SB + \sigma V - (\delta + \theta + \mu)E, \\ \frac{dV}{dt} &= \delta E - (\sigma + \theta + \mu)V. \end{aligned} \tag{3}$$

Following the next-generation matrix approach [24], we split the bullying-related subsystem into a new-appearance vector $\mathcal{F}$ and the transition vector $\mathcal{V}$, given by

$$\mathcal{F} = \begin{pmatrix} \alpha SB \\ \beta SB \\ 0 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} (\theta + \mu)B \\ (\delta + \theta + \mu)E - \sigma V \\ (\sigma + \theta + \mu)V - \delta E \end{pmatrix}. \tag{4}$$

The Jacobian matrices of $\mathbb{F}$ and $\mathbb{V}$ with respect to (B, E, V), evaluated at the bullying-free equilibrium $\mathcal{E}_0^*$, are

$$\mathbb{F} = \begin{pmatrix} \frac{\alpha\Lambda}{\mu} & 0 & 0 \\ \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbb{V} = \begin{pmatrix} \theta+\mu & 0 & 0 \\ 0 & \delta+\theta+\mu & -\sigma \\ 0 & -\delta & \sigma+\theta+\mu \end{pmatrix}. \tag{5}$$

The next-generation matrix is therefore $\mathbb{F}\mathbb{V}^{-1}$. Since the first row and first column of the transition matrix are decoupled from the remaining victim-response dynamics, the dominant eigenvalue of $\mathbb{F}\mathbb{V}^{-1}$ is obtained as

$$\mathcal{R}_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \frac{\alpha\Lambda}{\mu(\theta+\mu)}. \tag{6}$$

It is worth noting that β does not appear explicitly in $\mathcal{R}_0$, because the victimization pathway does not generate new perpetrators in the present model structure. Instead, β affects the distribution of bullying-related victims once the bully class has been established. Had the recovery rate been split into class-specific rates $\theta_B, \theta_E, \theta_V$, the basic reproduction number would read $R_0 = \alpha\Lambda/[\mu(\theta_B+\mu)]$, depending only on the perpetrator recovery rate θ_B. Hence the common-rate simplification does not affect the threshold $R_0 = 1$, and the qualitative results below remain valid; a full heterogeneous recovery analysis is left for future work.

3.5 The local stability of the bullying free-equilibrium

The following theorem establishes the local stability of the bullying-free equilibrium point.

Theorem 2. Given

$$\mathcal{R}_0 = \frac{\alpha\Lambda}{\mu(\theta+\mu)},$$

the bullying-free equilibrium point

$$\mathcal{E}_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right)$$

is locally asymptotically stable if $\mathcal{R}_0 < 1$ and becomes unstable if $\mathcal{R}_0 > 1$.

Proof. The local stability of the bullying-free equilibrium is determined by the eigenvalues of the Jacobian matrix of system (1) evaluated at $\mathcal{E}_0^*$.

The Jacobian matrix of system (1) is given by

$$J(S,B,E,V,R) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \eta \\ \alpha B & a_{22} & 0 & 0 & 0 \\ \beta B & \beta S & a_{33} & \sigma & 0 \\ 0 & 0 & \delta & a_{44} & 0 \\ 0 & \theta & \theta & \theta & a_{55} \end{pmatrix}.$$

with $a_{11} = -(\alpha+\beta)B - \mu, a_{12} = -(\alpha+\beta)S, a_{22} = \alpha S - (\theta+\mu), a_{33} = -(\delta+\theta+\mu), a_{44} = -(\sigma+\theta+\mu)$, and $a_{55} = -(\eta+\mu)$. Evaluating the Jacobian at the bullying-free equilibrium

$$\mathcal{E}_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right),$$

we obtain

$$J(\mathcal{E}_0^*) = \begin{pmatrix} -\mu & a_{12}^0 & 0 & 0 & \eta \\ 0 & a_{22}^0 & 0 & 0 & 0 \\ 0 & a_{32}^0 & a_{33}^0 & \sigma & 0 \\ 0 & 0 & \delta & a_{44}^0 & 0 \\ 0 & \theta & \theta & \theta & a_{55}^0 \end{pmatrix}.$$

with $a_{12}^0 = -\dfrac{(\alpha+\beta)\Lambda}{\mu}, a_{22}^0 = \dfrac{\alpha\Lambda}{\mu} - (\theta+\mu), a_{32}^0 = \dfrac{\beta\Lambda}{\mu}, a_{33}^0 = -(\delta+\theta+\mu), a_{44}^0 = -(\sigma+\theta+\mu)$, and $a_{55}^0 = -(\eta+\mu)$. Therefore, the characteristic polynomial can be written as

$$P(\lambda) = \left[\lambda - a_{22}^0\right](\lambda - a_{55}^0)(\lambda+\mu)\left[(\lambda - a_{33}^0)(\lambda - a_{44}^0) - \sigma\delta\right].$$

Equivalently,

$$\begin{aligned} P(\lambda) = {} & [\lambda + (\theta+\mu)(1-\mathcal{R}_0)](\lambda+\eta+\mu)(\lambda+\mu) \\ & \times \left[\lambda^2 + (\delta+\sigma+2\theta+2\mu)\lambda + (\theta+\mu)(\delta+\sigma+\theta+\mu)\right]. \end{aligned}$$

Hence, three eigenvalues are immediately identified as

$$\lambda_1 = -(\eta+\mu), \qquad \lambda_2 = -\mu, \qquad \lambda_3 = (\theta+\mu)(\mathcal{R}_0 - 1).$$

Since $\eta, \mu, \theta > 0$, it follows that $\lambda_1 < 0$ and $\lambda_2 < 0$. Moreover, $\lambda_3 < 0$ precisely when $\mathcal{R}_0 < 1$.

The remaining two eigenvalues are the roots of

$$\lambda^2 + (\delta + \sigma + 2\theta + 2\mu)\lambda + (\theta + \mu)(\delta + \sigma + \theta + \mu) = 0.$$

All coefficients of this quadratic polynomial are positive for nonnegative δ and σ, and positive θ and μ. Therefore, by the Routh–Hurwitz [17] criterion for a second-order polynomial, both roots have negative real parts.

Consequently, if $\mathcal{R}_0 < 1$, all eigenvalues of $J(\mathcal{E}_0^*)$ have negative real parts. By the linearization theorem [19], the bullying-free equilibrium $\mathcal{E}_0^*$ is locally asymptotically stable.

On the other hand, if $\mathcal{R}_0 > 1$, then

$$\lambda_3 = (\theta + \mu)(\mathcal{R}_0 - 1) > 0.$$

Thus, the Jacobian matrix has a positive eigenvalue, and the bullying-free equilibrium is unstable.

□

Having established local stability, we now strengthen this to a global result via the Castillo-Chávez approach.

3.6 Global stability of the bullying-free equilibrium

In this section, the global asymptotic stability of the bullying-free equilibrium is investigated using the Castillo-Chávez [4, 5] approach. Let

$$X = (S, R)^T \quad \text{and} \quad Y = (B, E, V)^T,$$

where X represents the bullying-free compartments and Y represents the bullying-related classes. The model can be rewritten in the form

$$\begin{cases} \dfrac{dX}{dt} = F(X, Y), \\ \dfrac{dY}{dt} = G(X, Y), \quad G(X, 0) = 0. \end{cases} \tag{7}$$

For the proposed model, one has

$$F(X, Y) = \begin{pmatrix} \Lambda + \eta R - (\alpha + \beta)SB - \mu S \\ \theta B + \theta E + \theta V - (\eta + \mu)R \end{pmatrix},$$

and

$$G(X,Y)=\begin{pmatrix}\alpha SB-(\theta+\mu)B\\ \beta SB+\sigma V-(\delta+\theta+\mu)E\\ \delta E-(\sigma+\theta+\mu)V\end{pmatrix}.$$

Theorem 3. Given

$$\mathcal{R}_0=\frac{\alpha\Lambda}{\mu(\theta+\mu)},$$

the bullying-free equilibrium point

$$\mathcal{E}_0^*=\left(\frac{\Lambda}{\mu},0,0,0,0\right)$$

is globally asymptotically stable in the feasible region Ω if $\mathcal{R}_0<1$.

Proof. Let the feasible region be

$$\Omega=\left\{(S,B,E,V,R)\in\mathbb{R}_+^5 : S+B+E+V+R\leq\frac{\Lambda}{\mu}\right\}.$$

Since

$$\frac{d}{dt}(S+B+E+V+R)=\Lambda-\mu(S+B+E+V+R),$$

the region Ω is positively invariant.

We apply the result of Castillo-Chávez, Feng, and Huang [4], write the system as $\dot{X}=F(X,Y)$, $\dot{Y}=G(X,Y)$ with $G(X,0)=0$, where $X=(S,R)^\top$ collects the bullying-free classes and $Y=(B,E,V)^\top$ collects the bullying-related classes. The equilibrium is globally asymptotically stable if

(H1) the reduced system $\dot{X}=F(X,0)$ has a globally asymptotically stable equilibrium $X^*=\left(\frac{\Lambda}{\mu},0\right)^\top$;

(H2) $\widehat{G}(X,Y)=AY-G(X,Y)\geq 0$ for all $(X,Y)\in\Omega$, where $A=D_YG(X^*,0)$.

First, consider the reduced system

$$\frac{dX}{dt}=F(X,0),$$

which gives

$$\begin{cases}\dfrac{dS}{dt}=\Lambda+\eta R-\mu S,\\ \dfrac{dR}{dt}=-(\eta+\mu)R.\end{cases}\tag{8}$$

It is clear that $R(t)\to 0\quad\text{as}\quad t\to\infty$. Consequently, the first equation in (8) implies

$$S(t) \to \frac{\Lambda}{\mu} \quad \text{as} \quad t \to \infty.$$

Thus, $X^* = (\Lambda/\mu, 0)^T$ is globally asymptotically stable for the reduced system.

Next, let $A = D_Y G(X^*, 0)$. Then

$$A = \begin{pmatrix} \frac{\alpha\Lambda}{\mu} - (\theta + \mu) & 0 & 0 \\ \frac{\beta\Lambda}{\mu} & -(\delta + \theta + \mu) & \sigma \\ 0 & \delta & -(\sigma + \theta + \mu) \end{pmatrix}.$$

Furthermore, $G(X, Y)$ can be written as $G(X, Y) = AY - \widehat{G}(X, Y)$, where

$$\widehat{G}(X, Y) = \begin{pmatrix} \alpha \left(\frac{\Lambda}{\mu} - S \right) B \\ \beta \left(\frac{\Lambda}{\mu} - S \right) B \\ 0 \end{pmatrix}.$$

Since $(S, B, E, V, R) \in \Omega$, we have $0 \leq S \leq \frac{\Lambda}{\mu}$. Therefore, $\widehat{G}(X, Y) \geq 0$ for all $(X, Y) \in \Omega$. It remains to observe that, if $\mathcal{R}_0 < 1$, then

$$\frac{\alpha\Lambda}{\mu} - (\theta + \mu) = (\theta + \mu)(\mathcal{R}_0 - 1) < 0.$$

The remaining two eigenvalues associated with the (E, V)-subsystem are determined by

$$\lambda^2 + (\delta + \sigma + 2\theta + 2\mu)\lambda + (\theta + \mu)(\delta + \sigma + \theta + \mu) = 0,$$

whose coefficients are positive. Hence, both roots have negative real parts.

Thus, all conditions of the Castillo-Chávez and Song theorem [5] are satisfied. Therefore, the bullying-free equilibrium

$$\mathcal{E}_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0 \right)$$

is globally asymptotically stable in Ω whenever $\mathcal{R}_0 < 1$.

□

Theorem 3 implies that, when $\mathcal{R}_0 < 1$, bullying behavior cannot persist in the school population. In this case, any small or moderate introduction of active bullies eventually dies out, and the system converges to the bullying-free equilibrium. Biologically, this condition means that the average number of new bullies generated by one active bully during his or her active bullying period is less than unity. Therefore, the combined effects of counseling, intervention, and natural removal are sufficient to eliminate bullying dynamics from the school environment.

To simplify the notation, define the following composite parameters $a = \theta + \mu, b = \eta + \mu, c = \sigma + \theta + \mu, d = \delta + \sigma + \theta + \mu, g = \eta + \theta + \mu$, and $\Delta = \alpha\Lambda - \mu(\theta + \mu)$. These quantities are introduced only as shorthand notations for combinations of the original model parameters. The endemic equilibrium $\mathcal{E}_1^* = (S^*, B^*, E^*, V^*, R^*)$ is derived by equating the right-hand terms of the system (1) to zero, which gives

$$\mathcal{E}_1^* = \left(\frac{a}{\alpha}, \frac{b\Delta}{\mu(\alpha+\beta)g}, \frac{\beta cb\Delta}{\alpha\mu(\alpha+\beta)gd}, \frac{\beta\delta b\Delta}{\alpha\mu(\alpha+\beta)gd}, \frac{\theta\Delta}{\alpha\mu g}\right).$$

Since all parameters are assumed to be positive, the endemic equilibrium is feasible if and only if $\Delta > 0$ or equivalently $\alpha\Lambda - \mu(\theta + \mu) > 0$. Using the basic reproduction number $\mathcal{R}_0 = \frac{\alpha\Lambda}{\mu(\theta+\mu)}$, the above condition can be written as $\mathcal{R}_0 > 1$. Therefore, the endemic equilibrium exists in the positive feasible region whenever $\mathcal{R}_0 > 1$.

Proposition 1. System (1) admits a unique endemic equilibrium

$$\mathcal{E}_1^* = \left(\frac{a}{\alpha}, \frac{b\Delta}{\mu(\alpha+\beta)g}, \frac{\beta cb\Delta}{\alpha\mu(\alpha+\beta)gd}, \frac{\beta\delta b\Delta}{\alpha\mu(\alpha+\beta)gd}, \frac{\theta\Delta}{\alpha\mu g}\right)$$

with all components strictly positive if and only if $\mathcal{R}_0 > 1$, or equivalently $\Delta > 0$.

3.7 Local stability of the endemic equilibrium

The existence of the endemic equilibrium indicates the persistence of bullying in the population. Therefore, it is important to examine whether solutions of the system remain close to this equilibrium under small perturbations. This can be achieved by studying the eigenvalues of the Jacobian matrix evaluated at $\mathcal{E}_1^*$. The local stability condition is then derived from the characteristic polynomial associated with $J(\mathcal{E}_1^*)$. The result is summarized in the following theorem.

Theorem 4. Assume that all parameters are positive and that the endemic equilibrium

$$\mathcal{E}_1^* = \left(\frac{a}{\alpha}, \frac{b\Delta}{\mu(\alpha+\beta)g}, \frac{\beta cb\Delta}{\alpha\mu(\alpha+\beta)gd}, \frac{\beta\delta b\Delta}{\alpha\mu(\alpha+\beta)gd}, \frac{\theta\Delta}{\alpha\mu g}\right)$$

exists. Then the endemic equilibrium $\mathcal{E}_1^*$ is locally asymptotically stable if the following Routh–Hurwitz [17] conditions hold: $\psi_1 > 0, \psi_2 > 0, \psi_3 > 0$, and $\psi_1\psi_2 > \psi_3$, where

$$\psi_1 = p_1 + \eta + \mu, \quad \psi_2 = p_0 + (\eta+\mu)p_1, \quad \text{and} \quad \psi_3 = (\eta+\mu)p_0 - \eta\theta\frac{b\Delta}{\mu g},$$

with

$$p_1 = \frac{b\Delta}{\mu g} + 2\mu + \theta - a, \quad \text{and} \quad p_0 = \left(\frac{b\Delta}{\mu g} + \mu\right)(\theta+\mu) - a\mu.$$

Proof. The Jacobian matrix evaluated at the endemic equilibrium $\mathcal{E}_1^*$ is given by

$$J(\mathcal{E}_1^*) = \begin{pmatrix} a_{11}^1 & a_{12}^1 & 0 & 0 & \eta \\ \alpha B^* & a_{22}^1 & 0 & 0 & 0 \\ \beta B^* & \beta S^* & a_{33}^1 & \sigma & 0 \\ 0 & 0 & \delta & a_{44}^1 & 0 \\ 0 & \theta & \theta & \theta & a_{55}^1 \end{pmatrix},$$

where

$$\begin{aligned} &a_{11}^1 = -(\alpha+\beta)B^* - \mu, \quad a_{12}^1 = -(\alpha+\beta)S^*, \\ &a_{22}^1 = \alpha S^* - (\theta+\mu), \quad a_{33}^1 = -(\delta+\theta+\mu), \\ &a_{44}^1 = -(\sigma+\theta+\mu), \quad a_{55}^1 = -(\eta+\mu). \end{aligned}$$

The characteristic equation is

$$\det(\lambda I - J(\mathcal{E}_1^*)) = 0.$$

A direct computation gives

$$\det(\lambda I - J(\mathcal{E}_1^*)) = (\lambda+\theta+\mu)(\lambda+\delta+\sigma+\theta+\mu)P_3(\lambda),$$

where

$$P_3(\lambda) = (\lambda+\eta+\mu)\left(\lambda^2 + p_1\lambda + p_0\right) - \eta\theta\frac{b\Delta}{\mu g}.$$

Here

$$p_1 = \frac{b\Delta}{\mu g} + 2\mu + \theta - a, \quad \text{and} \quad p_0 = \left(\frac{b\Delta}{\mu g} + \mu\right)(\theta+\mu) - a\mu.$$

Hence, two eigenvalues are $\lambda_1 = -(\theta+\mu), \lambda_2 = -(\delta+\sigma+\theta+\mu)$. Since all parameters are positive, both eigenvalues are negative. It remains to analyze the cubic polynomial $P_3(\lambda)$. Expanding $P_3(\lambda)$ gives

$$P_3(\lambda) = \lambda^3 + \psi_1\lambda^2 + \psi_2\lambda + \psi_3,$$

where

$$\psi_1 = p_1 + \eta + \mu, \quad \psi_2 = p_0 + (\eta+\mu)p_1, \quad \psi_3 = (\eta+\mu)p_0 - \eta\theta\frac{b\Delta}{\mu g}.$$

By the Routh–Hurwitz criterion [17], all roots of the cubic polynomial $P_3(\lambda)$ have negative real parts if and only if $\psi_1 > 0, \psi_2 > 0, \psi_3 > 0$, and $\psi_1\psi_2 > \psi_3$. Therefore, under these conditions, all eigenvalues of $J(\mathcal{E}_1^*)$ have negative real parts. Hence, the endemic equilibrium $\mathcal{E}_1^*$ is locally asymptotically stable.

□

3.8 Sensitivity analysis

The normalized forward sensitivity index is used to measure the relative change in the basic reproduction number $\mathcal{R}_0$ with respect to changes in each model parameter. The normalized forward sensitivity index of $\mathcal{R}_0$ with respect to a parameter p is defined as

$$\Upsilon_p^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0}.$$

Here, p denotes any parameter involved in the expression of $\mathcal{R}_0$. Since the basic reproduction number is given by

$$\mathcal{R}_0 = \frac{\alpha\Lambda}{\mu(\theta+\mu)},$$

the parameters that directly influence $\mathcal{R}_0$ are α, Λ, μ, and θ.

$$\begin{aligned}
\Upsilon_\alpha^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \alpha} \times \frac{\alpha}{\mathcal{R}_0} = 1\\
\Upsilon_\Lambda^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \Lambda} \times \frac{\Lambda}{\mathcal{R}_0} = 1\\
\Upsilon_\theta^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \theta} \times \frac{\theta}{\mathcal{R}_0} = -\frac{\theta}{\theta+\mu} < 0\\
\Upsilon_\mu^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \mu} \times \frac{\mu}{\mathcal{R}_0} = -\frac{\theta+2\mu}{\theta+\mu} < 0.
\end{aligned}$$

Therefore, the sensitivity indices of $\mathcal{R}_0$ are summarized in Table 2.

The results show that the parameters α and Λ have positive sensitivity indices. This indicates that an increase in either α or Λ will increase the value of $\mathcal{R}_0$. In particular, a 1% increase in α or Λ results in a 1% increase in $\mathcal{R}_0$, provided that the other parameters remain constant.

On the other hand, the parameters θ and μ have negative sensitivity indices. This implies that increasing either θ or μ will reduce the value of $\mathcal{R}_0$. Hence, θ and μ play an important role

Table 2: Sensitivity indices of $\mathcal{R}_0$ with respect to the model parameters

Parameter	Sensitivity index
α	1
Λ	1
θ	$-\dfrac{\theta}{\theta+\mu} < 0$
μ	$-\dfrac{\theta+2\mu}{\theta+\mu} < 0$

in reducing bullying transmission, since larger values of these parameters contribute to a lower basic reproduction number.

To further illustrate the influence of each parameter on the basic reproduction number, we present the variation of $\mathcal{R}_0$ with respect to the parameters Λ, α, θ, and μ. The threshold value $\mathcal{R}_0 = 1$ separates the nonendemic region from the endemic region. In particular, the bullying dynamics tend to disappear when $\mathcal{R}_0 < 1$, whereas they persist in the population when $\mathcal{R}_0 > 1$.

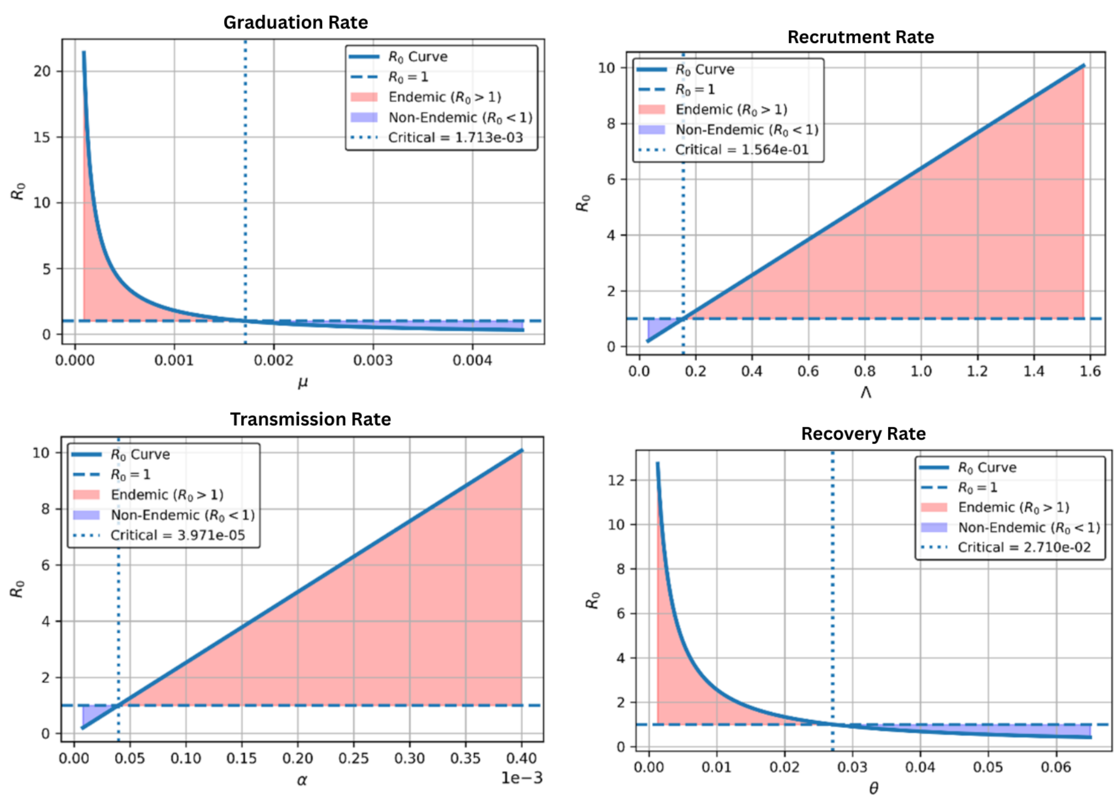

Figure 2: Graphical sensitivity analysis of $\mathcal{R}_0$ with respect to key model parameters.

Figure 2 shows that $\mathcal{R}_0$ increases as Λ and α increase. This agrees with the positive sensitivity indices of these two parameters. Biologically, a larger recruitment rate or transmission rate increases the potential for bullying spread in the population.

In contrast, $\mathcal{R}_0$ decreases as θ and μ increase. This is consistent with their negative sensitivity indices. Hence, increasing the values of θ and μ can contribute to reducing the basic reproduction number below the threshold value $\mathcal{R}_0 = 1$. The critical values shown in the figure indicate the parameter values at which the system changes from the endemic region to the nonendemic region, or conversely.

Notably, the parameters β, δ, σ, and η do not appear in $\mathcal{R}_0$. This structural property implies that interventions directed exclusively at the victimization pathway (without addressing perpetrator formation) are insufficient to drive the system below the persistence threshold. Because $\mathcal{R}_0$ is insensitive to the victimization pathway, we next ask how intervention effort should be allocated over time, which motivates the optimal-control formulation in Section 4.

4 An optimal control model

In this section, the mathematical model of bullying in (1) is further extended by incorporating optimal control theory as a strategic approach to reduce the spread of bullying behavior in the school environment. The application of control theory is intended to determine effective intervention strategies that minimize the number of perpetrators and victims while increasing the number of students who remain safe or recover from the effects of bullying.

From the perspective of applied mathematics, the control problem is formulated by introducing two time-dependent control variables, namely $u_1(t)$ and $u_2(t)$. The control variable $u_1(t)$ represents a punishment-based intervention, such as sanctions or disciplinary actions given to bullying perpetrators. Meanwhile, the control variable $u_2(t)$ represents a victimization-prevention intervention, such as teacher supervision, peer support, and anti-bullying programs designed to reduce the risk of susceptible students becoming nonretaliating victims.

Therefore, the controlled mathematical model with the implementation of $u_1(t)$ and $u_2(t)$ is given as follows:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda + \eta R - (1-u_1)\alpha SB - (1-u_2)\beta SB - \mu S, \\
\frac{dB}{dt} &= (1-u_1)\alpha SB - \theta B - \mu B, \\
\frac{dE}{dt} &= (1-u_2)\beta SB + \sigma V - \delta E - \theta E - \mu E, \\
\frac{dV}{dt} &= \delta E - \sigma V - \theta V - \mu V, \\
\frac{dR}{dt} &= \theta B + \theta E + \theta V - \eta R - \mu R.
\end{aligned} \tag{9}
$$

The objective of the optimal control problem is to minimize the numbers of bullying perpetrators and students at risk while also minimizing the implementation costs of the control strategies. Therefore, the objective functional is defined as

$$
J(u_1,u_2) = \int_0^{t_f} \left[A_1 B(t) + A_2 E(t) + \frac{1}{2} C_1 u_1^2(t) + \frac{1}{2} C_2 u_2^2(t) \right] dt, \tag{10}
$$

where t_f is the final time, A_1 and A_2 are positive weight constants associated with bullying perpetrators and students at risk, respectively, while C_1 and C_2 are positive weight constants representing the costs of implementing the control strategies $u_1(t)$ and $u_2(t)$. Next, we investigate the existence of an optimal control for the above-mentioned problem using the result of Fleming and Rishel [11].

Theorem 5 (Existence of an optimal control)**.** There exists an optimal control pair $u^* = (u_1^*, u_2^*) \in \mathcal{U}$ such that

$$
J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in\mathcal{U}} J(u_1,u_2), \tag{11}
$$

where the admissible control set $\mathcal{U}$ is defined by

$$
\mathcal{U} = \left\{ (u_1,u_2) \mid u_i(t) \text{ is measurable on } [0,t_f], \quad 0 \le u_i(t) \le 1, \quad i = 1,2 \right\}.
$$

The optimal control problem is subject to the controlled system (9) with nonnegative initial conditions.

Proof. To prove the existence of an optimal control, we follow the standard existence result in optimal control theory [11]. It is sufficient to verify the following conditions:

1. The control set $\mathcal{U}$ is nonempty, closed, and convex.

2. The state variables are nonnegative and bounded.

3. The right-hand side of the state system is bounded by a linear function of the state and control variables.

4. The integrand of the objective functional is convex with respect to the control variables on $\mathcal{U}$.

5. There exist constants $d_1 > 0$, $d_2 > 0$, and $c > 1$ such that the integrand satisfies a lower bound of the form
$$L(B, E, u_1, u_2) \geq d_1 \left(|u_1|^2 + |u_2|^2\right)^{c/2} - d_2.$$
where $L(B, E, u_1, u_2) = A_1 B + A_2 E + \frac{1}{2} C_1 u_1^2 + \frac{1}{2} C_2 u_2^2$ is the integrand of the objective functional (10).

First, the control set $\mathcal{U}$ is nonempty since the pair $(u_1, u_2) = (0, 0)$ belongs to $\mathcal{U}$. Moreover, since $0 \leq u_i(t) \leq 1$ for $i = 1, 2$, the set $\mathcal{U}$ is closed, bounded, and convex.

Next, for nonnegative initial conditions, the state variables remain nonnegative for all $t \in [0, t_f]$. Indeed, on the boundary of the nonnegative orthant, the right-hand side of the controlled system satisfies
$$\left.\frac{dS}{dt}\right|_{S=0} = \Lambda + \eta R \geq 0, \qquad \left.\frac{dB}{dt}\right|_{B=0} = 0,$$
$$\left.\frac{dE}{dt}\right|_{E=0} = (1 - u_2)\beta S B + \sigma V \geq 0, \qquad \left.\frac{dV}{dt}\right|_{V=0} = \delta E \geq 0,$$
and
$$\left.\frac{dR}{dt}\right|_{R=0} = \theta B + \theta E + \theta V \geq 0.$$
Thus, the nonnegative orthant is positively invariant.

Let
$$N(t) = S(t) + B(t) + E(t) + V(t) + R(t).$$
Adding all equations in the controlled system (9) gives
$$\frac{dN}{dt} = \Lambda - \mu N.$$
Hence,
$$N(t) \leq \max\left\{N(0), \frac{\Lambda}{\mu}\right\},$$
which implies that all state variables are bounded on $[0, t_f]$.

Furthermore, the right-hand side of the controlled system (9) is continuously differentiable with respect to the state variables and controls. Since the state variables are bounded and $0 \leq u_i(t) \leq 1$, the right-hand side of the system is bounded by a linear function of the state and control variables.

Recall the objective functional (10), whose integrand is

$$L(B, E, u_1, u_2) = A_1 B + A_2 E + \frac{1}{2} C_1 u_1^2 + \frac{1}{2} C_2 u_2^2.$$

Since $C_1 > 0$ and $C_2 > 0$, the function L is convex with respect to u_1 and u_2. Moreover, because $B(t) \geq 0$ and $E(t) \geq 0$, we have

$$L(B, E, u_1, u_2) \geq \frac{1}{2} C_1 u_1^2 + \frac{1}{2} C_2 u_2^2.$$

Therefore,

$$L(B, E, u_1, u_2) \geq d_1 \left(|u_1|^2 + |u_2|^2 \right) - d_2,$$

where

$$d_1 = \frac{1}{2} \min\{C_1, C_2\} \qquad \text{and} \qquad d_2 = 0.$$

Thus, all the required conditions for the existence of an optimal control are satisfied. Therefore, there exists an optimal control pair $(u_1^*, u_2^*) \in \mathcal{U}$ such that

$$J(u_1^*, u_2^*) = \min_{(u_1, u_2) \in \mathcal{U}} J(u_1, u_2).$$

This completes the proof. $\square$

To characterise the optimal controls, we apply Pontryagin's maximum principle [20]. The Hamiltonian $(\mathcal{H})$ of the optimal control problem (9)–(10) is as follows:

$$\begin{aligned} H = {} & A_1 B + A_2 E + \frac{1}{2} C_1 u_1^2 + \frac{1}{2} C_2 u_2^2 \\ & + \lambda_1 \left[\Lambda + \eta R - (1 - u_1) \alpha S B - (1 - u_2) \beta S B - \mu S \right] \\ & + \lambda_2 \left[(1 - u_1) \alpha S B - \theta B - \mu B \right] \\ & + \lambda_3 \left[(1 - u_2) \beta S B + \sigma V - \delta E - \theta E - \mu E \right] \\ & + \lambda_4 \left[\delta E - \sigma V - \theta V - \mu V \right] \\ & + \lambda_5 \left[\theta B + \theta E + \theta V - \eta R - \mu R \right]. \end{aligned}$$

where λ_i, $i = 1, 2, \ldots, 5$, are the adjoint variable functions to be determined.

Applying Pontryagin's maximum principle [20] to the Hamiltonian H yields the following adjoint system and optimality conditions. The adjoint variables $\lambda_i(t)$, $i = 1, \dots, 5$, satisfy $\dfrac{d\lambda_i}{dt} = -\dfrac{\partial H}{\partial x_i}$, $x_i \in \{S, B, E, V, R\}$, that is,

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial S} = \lambda_1 \left[(1-u_1)\alpha B + (1-u_2)\beta B + \mu\right] \\
&\qquad - \lambda_2 \left(1-u_1\right)\alpha B - \lambda_3(1-u_2)\beta B, \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial B} = -A_1 + \lambda_1 \left[(1-u_1)\alpha S + (1-u_2)\beta S\right] \\
&\qquad - \lambda_2 \left[(1-u_1)\alpha S - \theta - \mu\right] - \lambda_3(1-u_2)\beta S \\
&\qquad - \lambda_5\theta, \\
\frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial E} = -A_2 + \lambda_3 \left[\delta + \theta + \mu\right] - \lambda_4\delta - \lambda_5\theta, \\
\frac{d\lambda_4}{dt} &= -\frac{\partial H}{\partial V} = -\lambda_3\sigma + \lambda_4 \left[\sigma + \theta + \mu\right] - \lambda_5\theta, \\
\frac{d\lambda_5}{dt} &= -\frac{\partial H}{\partial R} = -\lambda_1\eta + \lambda_5 \left[\eta + \mu\right],
\end{aligned}
$$

together with the transversality conditions $\lambda_i(t_f) = 0, i = 1, 2, \dots, 5$. The optimality conditions follow from $\dfrac{\partial H}{\partial u_1} = 0$ and $\dfrac{\partial H}{\partial u_2} = 0$. Since

$$
\frac{\partial H}{\partial u_1} = C_1 u_1 + \alpha SB(\lambda_1 - \lambda_2), \qquad \frac{\partial H}{\partial u_2} = C_2 u_2 + \beta SB(\lambda_1 - \lambda_3),
$$

we obtain

$$
u_1 = \frac{\alpha SB(\lambda_2 - \lambda_1)}{C_1}, \qquad u_2 = \frac{\beta SB(\lambda_3 - \lambda_1)}{C_2}.
$$

Because the controls are bounded, $0 \leq u_i(t) \leq 1$, projecting these expressions onto the admissible set U gives the characterisation of the optimal controls

$$
\begin{aligned}
u_1^*(t) &= \min\Big\{1,\ \max\big\{0,\ \tfrac{\alpha S^* B^*(\lambda_2 - \lambda_1)}{C_1}\big\}\Big\}, \\
u_2^*(t) &= \min\Big\{1,\ \max\big\{0,\ \tfrac{\beta S^* B^*(\lambda_3 - \lambda_1)}{C_2}\big\}\Big\}.
\end{aligned}
$$

The optimality system is completed by the initial conditions for the state variables and the transversality conditions for the adjoint variables. The initial conditions are given by $S(0) = S_0, B(0) = B_0, E(0) = E_0, V(0) = V_0, R(0) = R_0$. Since the terminal states are free and there is no terminal cost in the objective functional, the transversality conditions are

$$
\lambda_i(t_f) = 0, \qquad i = 1, 2, \dots, 5.
$$

5 Numerical simulations and discussion

The optimality system was solved numerically using the *forward–backward sweep method* (FBSM) [16]. The state system was integrated forward in time using a fourth-order Runge–Kutta (RK4) scheme, while the adjoint system was integrated backward from the transversality conditions. All numerical computations and figures were produced in Python.

At each iteration, the controls were updated via the convex combination

$$u_i^{(k+1)} \;=\; \rho\,\hat{u}_i^{(k)} + (1-\rho)\,u_i^{(k)}, \qquad i = 1, 2,$$

where k indicates the iteration number, $\hat{u}_i^{(k)}$ denotes the control computed from the optimality conditions, and $\rho = 0.5$ is the relaxation parameter. Convergence was declared when

$$\max\Big\{\big\|u_1^{(k+1)} - u_1^{(k)}\big\|_2,\ \big\|u_2^{(k+1)} - u_2^{(k)}\big\|_2\Big\} < 10^{-6},$$

with a maximum of 200 iterations.

The terminal time was set to $t_f = 200$ days (the approximate number of school days in one academic year), and the interval $[0, t_f]$ was discretized into $n = 4{,}000$ equally spaced grid points, giving a time step $\Delta t = t_f/(n-1)$. The parameter set in Table 3 yields

$$\mathcal{R}_0 = \frac{\alpha\Lambda}{\mu(\theta+\mu)} = \frac{(8\times10^{-5})(0.315)}{(9\times10^{-4})(0.013+9\times10^{-4})} \approx 2.014,$$

confirming that the baseline scenario corresponds to an endemic bullying equilibrium ($\mathcal{R}_0 > 1$). The values of δ and σ follow Crokidakis [8], and the remaining parameters are grounded in the school setting: μ gives an average schooling duration of about 5.6 years, Λ fixes the school size $\Lambda/\mu \approx 350$, θ reflects the effectiveness of anti-bullying programs [12], and α, β lie in the small mass-action range used in prior bullying models [8, 3, 1]. As these values are illustrative rather than calibrated, the robustness check below confirms the conclusions are insensitive to them.

The initial conditions were $S(0) = 275$, $B(0) = 20$, $E(0) = 40$, $V(0) = 15$, and $R(0) = 0$, and the weight constants in the objective functional were $C_1 = 5$, $C_2 = 50$, $A_1 = 2$, and $A_2 = 1$. The weights reflect features of the program. Taking $A_1 > A_2$ places greater priority on reducing the number of active perpetrators than on protecting students who are merely at risk, while $C_2 \gg C_1$ reflects that broad victimization-prevention measures are more costly to sustain than targeted disciplinary action against identified perpetrators. Such measures include school-wide supervision, peer support, and anti-bullying initiatives.

Table 3: Parameter values used in the numerical simulation.

Parameter	**Value**	**Source**
Λ	0.3150	Assumed
α	0.00008	Assumed
β	0.00020	Assumed
θ	0.0130	Assumed
η	0.0055	Assumed
μ	0.0009	Assumed
δ	0.0500	[8]
σ	0.0700	[8]

We stress, however, that the ineffectiveness of the u_2-only strategy is not an artifact of this cost ratio; it is structural. It follows from the form of R_0, which depends on the perpetrator-generation rate α but not on the victimization rate β. Lowering C_2 therefore reduces the cost of deploying u_2 while leaving R_0, and hence its capacity to suppress the bulling dynamics, untouched.

Three intervention strategies were investigated:

- **Strategy 1** (perpetrator-focused only): $u_1 \neq 0$, $u_2 = 0$.
- **Strategy 2** (victimization-prevention only): $u_1 = 0$, $u_2 \neq 0$.
- **Strategy 3** (combined intervention): $u_1 \neq 0$, $u_2 \neq 0$.

The control profiles are presented in Figure 3. The perpetrator-focused control $u_1(t)$ remains at its upper bound for approximately the first 150 days before declining monotonically to zero at t_f, indicating that intensive early intervention is essential during the most critical phase of the school year. Moreover, the profiles of u_1 under Strategies 1 and 3 are virtually identical, confirming that the presence of u_2 does not alter the optimal timing of perpetrator-focused measures.

The victimization-prevention control $u_2(t)$ exhibits a markedly different profile. Under Strategy 2 (applied alone), u_2 rises gradually from zero, peaks at day 140, and then decreases toward t_f. Under Strategy 3 (combined), u_2 starts at a high level ($u_2(0) \approx 0.85$) and decreases monotonically throughout the horizon. The higher initial intensity of u_2 in Strategy 3 reflects the synergistic interaction between the two controls: when u_1 suppresses perpetrator numbers early, the marginal benefit of victimization-prevention is greatest at the start of the intervention.

The objective functional values are compared in Figure 4 and Table 4. In the absence of control, the objective attains J_0 = 18,171.01. Strategy 1 reduces J to 7,951.98, a decrease of

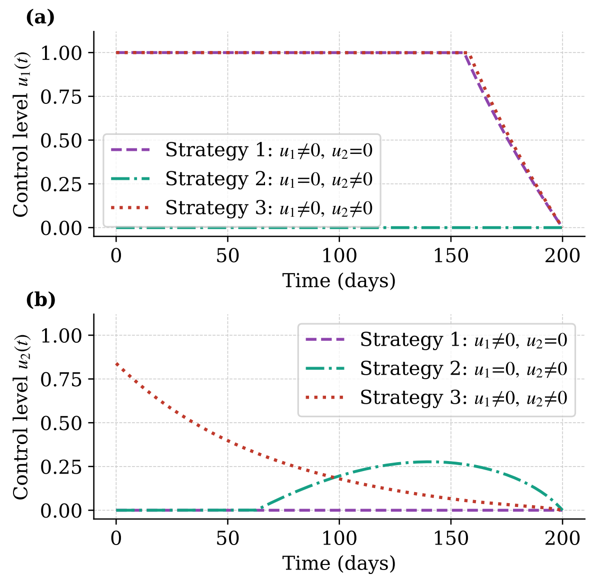

Figure 3: Optimal control profiles under the three intervention strategies: (a) perpetrator-focused control $u_1(t)$, and (b) victimization-prevention control $u_2(t)$. Strategy 2 sets $u_1 \equiv 0$ by construction; Strategy 1 sets $u_2 \equiv 0$.

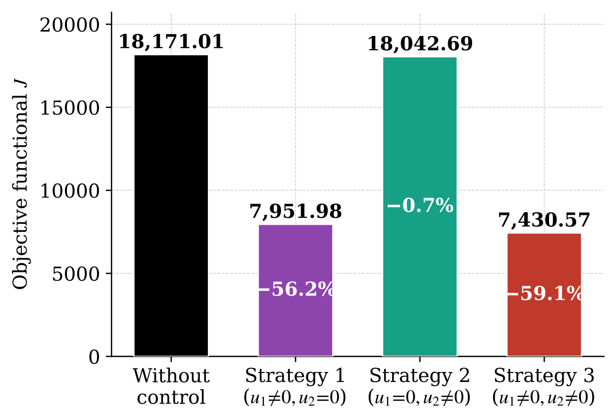

Figure 4: Comparison of the objective functional J across the uncontrolled baseline and the three intervention strategies. Percentage values inside each bar indicate the reduction relative to the uncontrolled value $J_0 =$ 18,171.01.

approximately 56.24% relative to the baseline. The combined Strategy 3 achieves a further reduction to $J = 7{,}430.57$, corresponding to an overall decrease of 59.11%. By contrast, Strategy 2 yields a negligible improvement of only 0.71%, with $J = 18{,}042.69$, confirming that victimization prevention alone is insufficient to reduce the bullying burden in a cost-effective manner.

These results are consistent with the structure of the basic reproduction number

$$\mathcal{R}_0 = \frac{\alpha\Lambda}{\mu(\theta+\mu)},$$

which depends directly on the perpetrator-generation pathway through the parameter α but is independent of the victimization rate β. Because $\mathcal{R}_0$ governs the persistence of bullying in the population, any intervention that does not target the pathway controlled by α cannot substantially alter the endemic equilibrium. The simulated population trajectories are displayed in Figures 5–6, and the terminal values are summarized in Table 4.

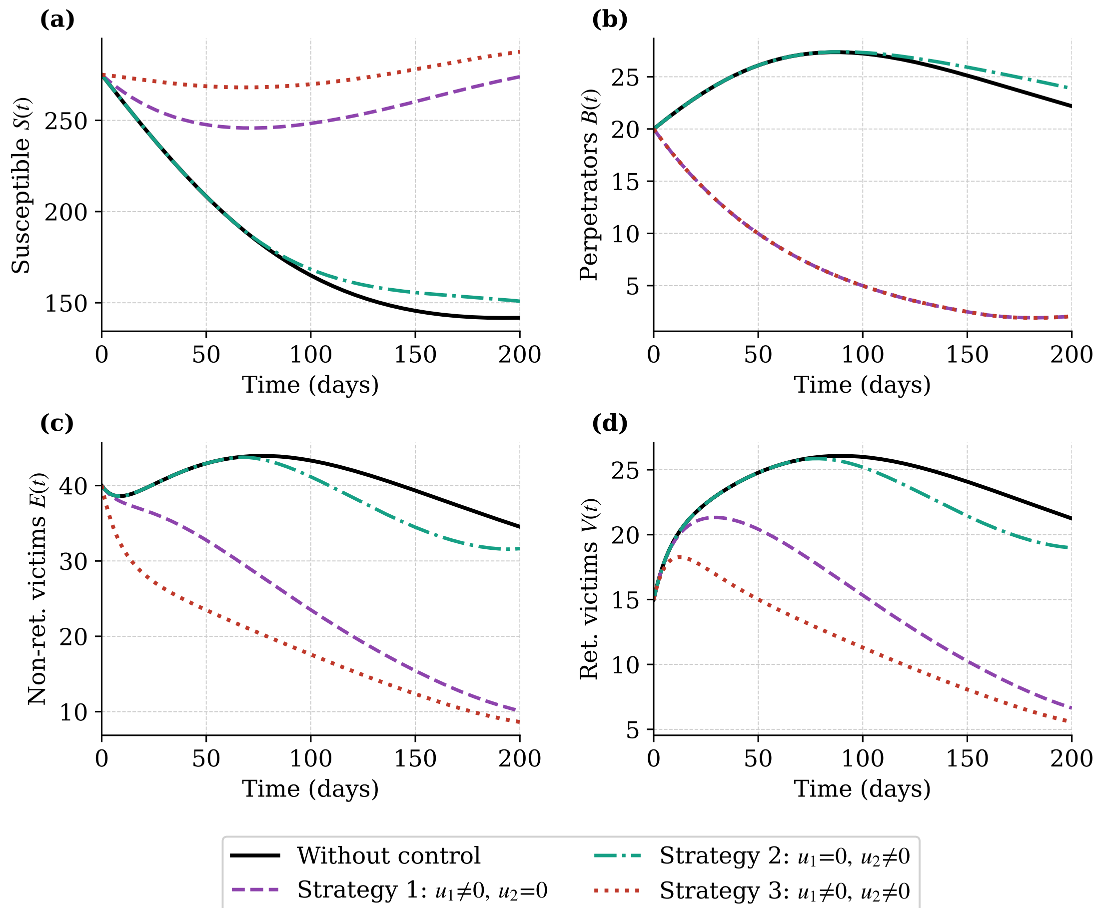

Figure 5: Simulated population trajectories under the three intervention strategies and the uncontrolled baseline: (a) susceptible students $S(t)$, (b) bullying perpetrators $B(t)$, (c) nonretaliating victims $E(t)$, and (d) retaliating victims $V(t)$.

Under the uncontrolled scenario and Strategy 2, the susceptible population declines steadily throughout the horizon as students are recruited into bullying-related compartments. Strategies 1 and 3, by suppressing perpetrator recruitment, preserve a substantially larger susceptible

Table 4: Terminal population values under each scenario. S_0: uncontrolled baseline; S_1, S_2, S_3: Strategies 1–3.

Compartment	S_0	S_1	S_2	S_3
$S(t_f)$	142	274	151	288
$B(t_f)$	22	2	24	2
$E(t_f)$	35	10	32	9
$V(t_f)$	21	7	19	6
$R(t_f)$	130	57	125	46
J	18,171.01	7,951.98	18,042.69	7,430.57
Reduction (%)	—	56.24	0.71	59.11

population: $S(t_f) = 274$ under Strategy 1 and $S(t_f) = 288$ under Strategy 3, compared with $S(t_f) = 142$ in the uncontrolled case.

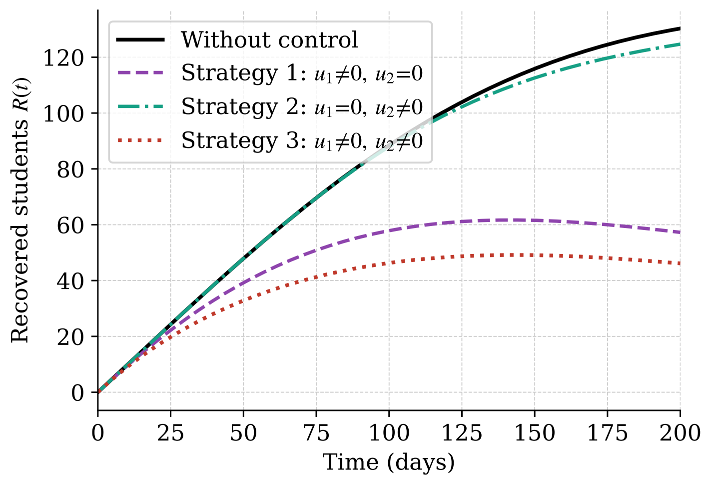

Figure 6: Recovered student population $R(t)$ under the three intervention strategies and the uncontrolled baseline. A large recovered population reflects greater cumulative exposure to bullying, not a favorable outcome.

The perpetrator population evolves most dramatically under the controls that include u_1. Under Strategies 1 and 3, $B(t)$ declines continuously from the initial value $B(0) = 20$ to terminal values of 2.065 and 2.070, respectively—reductions of more than 90%. Strategy 2 fails to reduce $B(t)$; indeed, the terminal perpetrator count (24) slightly *exceeds* the uncontrolled value (22), because diverting susceptibles away from victimization does not interrupt the self-reinforcing perpetrator-generation pathway [6].

Both victim compartments are reduced most effectively by Strategy 3. The terminal values $(E(t_f), V(t_f)) = (9, 6)$ under Strategy 3 represent reductions of 75.0% and 73.8%, respectively, relative to the uncontrolled case. Strategy 1 achieves comparable but slightly smaller reductions ($E(t_f) = 10$, $V(t_f) = 7$), while Strategy 2 produces only marginal improvements.

Counterintuitively, the uncontrolled scenario yields the largest recovered population ($R(t_f) = 130$). This reflects the model structure: recovery occurs only after students have already passed through bullying-related compartments. A large $R(t_f)$ therefore indicates a heavier cumulative burden of bullying exposure, not a favorable epidemiological outcome. Strategies 1 and 3 produce the smallest recovered populations (57 and 46, respectively) because fewer students enter the bullying process in the first place.

Finally, in Table 5, varying each assumed parameter $(\Lambda, \alpha, \beta, \theta, \eta, \mu)$ by $\pm 25\%$ and re-solving the optimality system leaves the ranking unchanged: Strategy 3 is always the best and Strategy 2 is always the worst. This reflects the structure of R_0, which depends on α but not β, rather than the precise parameter values.

Table 5: Robustness of the strategy ranking to $\pm 25\%$ variation in each assumed parameter. In every case Strategy 3 attains the lowest objective value and Strategy 2 the highest, so the ranking is preserved.

Parameter	Factor	J (S1)	J (S2)	J (S3)	Ranking preserved
Λ	0.75	7,914.1	17,598.2	7,402.4	yes
Λ	1.25	7,989.8	18,499.6	7,458.4	yes
α	0.75	7,940.3	14,445.5	7,419.5	yes
α	1.25	7,959.6	21,583.3	7,437.8	yes
β	0.75	7,359.4	17,963.0	7,046.3	yes
β	1.25	8,513.8	18,132.2	7,740.9	yes
θ	0.75	9,991.5	22,932.2	9,105.1	yes
θ	1.25	6,496.4	14,262.4	6,170.0	yes
η	0.75	7,934.4	17,719.8	7,418.9	yes
η	1.25	7,967.9	18,344.3	7,441.0	yes
μ	0.75	8,102.3	18,664.9	7,555.5	yes
μ	1.25	7,806.3	17,446.4	7,308.8	yes

6 Conclusion

This study developed and analyzed a compartmental mathematical model of school bullying dynamics incorporating a recovery compartment. The model was examined in two settings: without and with optimal control. For the uncontrolled model, the basic reproduction number $\mathcal{R}_0$ was derived as a threshold parameter governing the persistence of bullying. The bullying-free equilibrium was shown to be locally asymptotically stable when $\mathcal{R}_0 < 1$ and globally asymptotically stable under the same condition.

The model was extended by introducing two time-dependent control variables: $u_1(t)$, representing punishment-based intervention targeting perpetrator formation, and $u_2(t)$, representing victimization-prevention measures. The optimality system was derived via Pontryagin's Maximum Principle and solved numerically using a fourth-order Runge–Kutta scheme within the forward–backward sweep method.

Three intervention strategies were evaluated and compared against an uncontrolled baseline (J_0 = 18,171.01). Strategy 1 (perpetrator-focused only) reduced the objective functional by 56.24%, to J = 7,951.98, while Strategy 2 (victimization-prevention only) yielded a negligible reduction of 0.71%, with J = 18,042.69. Strategy 3 (combined intervention) achieved the greatest reduction of 59.11%, with J = 7,430.57. These results demonstrate that perpetrator-focused intervention is the primary driver of bullying suppression, consistent with the structure of $\mathcal{R}_0$, which depends on the perpetrator-generation rate α but not on the victimization rate β. The combined strategy, however, produced the lowest victim populations at the terminal time, confirming that supplementing perpetrator-focused measures with victimization-prevention support yields the most cost-effective outcome.

Several limitations suggest directions for future research. The model assumes a single recovery rate across the bullying-related classes; allowing class-specific recovery rates $\theta_B, \theta_E, \theta_V$ is a natural extension. In addition, calibrating the model against empirical bullying data would strengthen its applied relevance.

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Conflicts of Interest

The authors declares no conflicts of interest.

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