Iranian Journal of Numerical Analysis and Optimization
Vol. ??, No. ??, ??, pp ??
https://doi.org/??
https://ijnao.um.ac.ir/

CC BY NC

Research Article

Stability analyses of block hybrid methods for numerical solutions of differential equations

O. Adedire* and U.W. Sirisena

Abstract

This research presents a comparative analysis of stability polynomials and stability functions for investigating the stability of derived block hybrid methods. The methods—an explicit (BHEM) and two implicit (BHIM) formulations—were obtained through continuous formulation using multistep collocation and matrix inversion techniques. A novel theoretical result establishes that for consistent block methods, stability polynomials and stability functions must yield congruent stability conclusions, with any discrepancy indicating potential derivation errors. It is demonstrated that the use of stability polynomials offers a straightforward determination of the angle α for methods that are $A(\alpha)$-stable. However, for investigating stronger stability properties such as L-stability and $L(\alpha)$-stability, the stability function is preferable due to its direct rational form, which allows immediate asymptotic analysis. Detailed experiments on block methods and block hybrid methods reveal the respective advantages and limitations of each stability analysis tool. The analysis conclusively answers the research question: The absence of poles in the left half-plane alone does not guarantee A-stability; both analyticity in the left half-plane and boundedness on the imaginary axis are required conditions. This

*Corresponding author

Received ??? ; revised ??? ; accepted ???

Oludare Adedire
Department of Mathematics, Faculty of Natural Sciences, University of Jos, Plateau State, Nigeria; and Department of Statistics, Federal College of Forestry Jos. e-mail: dharenss@gmail.com

Ullusihewage Wipuni Sirisena
Department of Mathematics, Faculty of Natural Sciences, University of Jos, Plateau State, Nigeria. e-mail: swipuni@yahoo.com

How to cite this article
Adedire, O. and Sirisena, U.W., Stability analyses of block hybrid methods for numerical solutions of differential equations. *Iran. J. Numer. Anal. Optim.*, ??; ??(??): ??-??. ??

research successfully extends stability analysis from the one-step domain to multi-step block formulations.

AMS subject classifications (2020): Primary 65C20; Secondary 65L07, 65L20, 65Y20.

Keywords: Block methods; Absolute stability regions; Congruence for consistent block methods; Stability polynomial; Stability function.

1 Introduction

The literature contains numerous numerical methods developed for solving differential equations (DEs) [2, 3, 11, 17, 24, 26, 28]. For such methods to be effective when solving ordinary differential equations (ODEs), it is essential to analyze both their stability and convergence properties. Convergence analysis primarily concerns the behavior of numerical solutions as the step size h approaches zero, while stability analysis examines the range of step sizes h that yield quantitatively or qualitatively correct results without undesirable error growth. A numerical method is convergent if it is both consistent and zero-stable [8, 15]. Consistency relates to the local truncation error of the method, whereas zero-stability ensures that errors remain bounded throughout the computation as the solution progresses [10, 20].

For nonstiff ODEs, establishing convergence may be sufficient. However, for stiff ODEs, convergence alone does not guarantee correct solutions; it is also necessary to determine the region of absolute stability to fully assess the method's performance. In the literature, two main approaches are used to investigate stability: The stability polynomial $\pi(r, z)$ and the stability function $L(z)$. The stability polynomial $\pi(r, z)$ is an algebraic equation in r and z, where $z = h\lambda$ (with h the step size and λ the eigenvalue of the test problem) and r represents the roots of the characteristic equation. For absolute stability, these roots must satisfy $|r_n(z)| < 1$ for $n = 1, 2, \ldots, k$ for a fixed z. The stability function $L(z)$, on the other hand, is a complex-valued function of $z = x + iy$ (with $x, y \in \mathbb{R}$) that represents the spectral radius of the amplification matrix derived from the block method. This amplification matrix can be constructed from the method's defining matrices.

The stability polynomial $\pi(r, z)$ is a tool for analyzing how errors propagate over time and how the choice of step size h influences the method's behavior through the parameter $z = h\lambda$. Proper analysis of the roots r of the stability polynomial helps determine the region of absolute stability, ensuring that the step size h remains within this region to maintain stability and prevent solution oscillations or blow-up. To derive the stability polynomial for a numerical method, one

must obtain its associated characteristic polynomials. In contrast, the stability function $L(z)$ for $z = x+iy$ is a complex-valued function representing the dominant eigenvalue of the amplification matrix [6, 18]. It is typically derived by applying the method to a test problem, and it quantifies how errors are amplified after one step. This function helps determine whether errors grow or decay over time: If $|L(z)| < 1$, then errors decay; if $|L(z)| = 1$, then the method is neutrally stable (errors neither grow nor decay); and if $|L(z)| > 1$, then the method is unstable [13].

A key distinction between the stability polynomial and the stability function lies in their error analysis procedures. The stability polynomial focuses on the locations of the roots r of the characteristic equation in the complex r-plane; for absolute stability, all roots must lie inside the unit disk ($|r| < 1$) when z belongs to the left half-plane. The stability function, however, is concerned with the behavior of $L(z)$ in the complex z-plane.

Akinfenwa, Jator, and Yao [4] investigated the stability of continuous block methods with step numbers $k = 2$ and $k = 3$ for solving stiff problems. They plotted stability regions and the zeros and poles of the stability functions $L(z)$ derived from these methods, concluding that the methods were A-stable based primarily on the absence of poles in the left half-plane. Although they obtained zeros and poles from the stability functions, they did not mention using the stability polynomial—though defined in the work—to plot the regions of absolute stability. Their study was limited to block backward differentiation formulas (BBDFs). Akinnukawe, Akinfenwa, and Okunuga [5] employed a similar approach, deriving the stability function $L(z)$ for a block numerical integrator and showing it to be A-stable and L-stable, but they presented the stability region plot without indicating the positions of the zeros and poles. Their work also focused exclusively on BBDF methods, without considering other classes of block numerical integrators. Other researchers, such as Ibrahim, Othman, and Suleiman [16], have also focused on BBDF methods, presenting stability regions without explicitly showing zeros and poles in the complex plane.

Ngwane and Jator [18] plotted zeros and poles from the stability function of a derived block method while determining its stability region; however, they did not attribute their plot to the stability polynomial. In the context of multistep block methods for ODEs, Chu and Hamilton [7] analyzed the stability of finite difference equations using vectorial representation. Their analysis did not explicitly define a stability function, but their elegant stability analysis of predictor-corrector methods provided insights into numerical schemes that, when applied to a test equation, produce a coefficient matrix whose spectral radius is less than unity—a condition for absolute stability of block methods. A similar procedure can be found in Celaya and Anza [6].

Prior studies on block methods [1, 16, 17, 18, 25] have typically used stability polynomials or stability functions in isolation, without reconciling the two approaches or addressing whether

their stability conclusions must coincide. What some researchers call the stability function in one variable z [2] is what others call the stability polynomial [23]. However, many discrepancies exist in the literature: The stability polynomial $\pi(r, z)$ is an algebraic equation in r and z; the stability function $L(z)$, on the other hand, is a complex-valued function of z only. While $z = h\lambda$ in $\pi(r, z)$, it is $z = x + iy$ in $L(z)$. The representation of z in both $\pi(r, z)$ and $L(z)$ is not exactly the same. While z is used in $\pi(r, z)$, the objective is to obtain eigenvalues $r_i(z)$ of the amplification matrix $D(z)$ (roots of $\det(rI - D(z)) = 0$); however, the use of z in the stability function $L(z)$ aims at obtaining zeros and poles of $L(z)$ and their positions in the complex plane. The motivation for this study is to investigate whether the stability polynomial and the stability function yield identical A-stability conclusions for consistent methods, thereby providing a novel theoretical verification mechanism. This paper's novelty is threefold: (i) proving that for consistent block methods both tools must yield congruent stability conclusions, with any discrepancy indicating derivation errors; (ii) demonstrating that the absence of poles in the left half-plane alone does not guarantee A-stability, thereby correcting a common misconception in the literature; and (iii) offering practical guidance on tool selection—$\pi(r, z)$ is preferable for determining $A(\alpha)$-stability angles, while $L(z)$ is better suited for finding zeros and poles, and for L-stability analysis (a much-desired stability property) due to its rational form. Unlike prior derivation-focused works, this study provides a unified comparative framework. For inconsistent methods (which are not convergent), the question of A-stability is not meaningful. Discrepancies between the two approaches for consistent methods serve as theoretical indicators of potential derivation errors or method inconsistencies. This study also addresses the research question: Does the absence of poles in the left half-plane of the complex plane guarantee that a block method is A-stable, as suggested in some literature? It is important to note that deriving block hybrid methods and applying them to solve stiff ODEs without a detailed analysis of the stability region risks selecting an inappropriate method for very stiff problems, leading to incorrect solutions. Determining stability regions with respect to a test problem allows for the selection of an appropriate step size that maintains both accuracy and efficiency while ensuring numerical stability for specific ODEs. Furthermore, some researchers have derived $A(\alpha)$-stable or almost A-stable block methods without specifying the angle α [16], which limits the applicability of such methods to stiff problems. For instance, Ibrahim, Othman, and Suleiman [16] developed an implicit $r = 3$ BBDF which they referred to as an almost A-stable method and, via a plot, showed $A(\alpha)$-stability without specifying the angle α. Knowing the value of α is crucial when considering these methods for solving stiff ODEs. A key theoretical contribution of this work is the demonstration that for consistent block methods, stability polynomials and stability functions must yield congruent stability conclusions; any discrepancy between them serves as a theoretical

indicator of potential derivation errors or method inconsistency, providing a novel verification mechanism.

In this paper, we investigate in detail the use of the stability polynomial and stability function for a block hybrid explicit method (BHEM) and two block hybrid implicit methods (BHIM). Further experiments are conducted to explore the strengths and weaknesses of both approaches for other types of block methods, showing the applicability of this research to both block hybrid and nonhybrid block methods. For illustration, we compare the accuracy of the derived block methods with MATLAB's `ode23s` solver. The remainder of this paper is organized as follows: Section 2 presents the derivation of the block hybrid methods. Section 3 analyzes the stability polynomials and stability functions of the new methods and includes their plots. Section 4 describes the implementation of the proposed methods, and Section 5 presents and discusses the results. Finally, section 6 provides the conclusions.

2 Derivation of the block methods

The methodology for deriving the block hybrid methods in this study follows the approach of Onumanyi, Sirisena, and Jator [22]. We derive three block hybrid methods from continuous k-step linear multistep methods. To begin, we define a polynomial of degree $p = t+m-1$, where $t > 0$ and $m > 0$, as

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y(x_{n+j}) + h \sum_{j=0}^{m-1} \beta_j(x) f(x_j, y(x_j)), \quad x_n \leqslant x \leqslant x_{n+k}, \tag{1}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients defined by

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i, \quad j \in \{0, \ldots, t-1\} \tag{2}$$

and

$$h\beta_j(x) = h \sum_{i=0}^{t+m-1} \beta_{j,i+1} x^i, \quad j \in \{0, \ldots, m\}. \tag{3}$$

In (1), the points x_j for $j = 0, \ldots, m-1$ are the m distinct collocation points, while the points x_{n+j} for $j = 0, \ldots, t-1$ (with $0 < t \leqslant k$) are the interpolation points.

The continuous coefficients $\alpha_j(x)$ and $\beta_j(x)$ can be obtained by inverting the multistep collocation matrix D, defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-1} \end{bmatrix}. \tag{4}$$

The dimension of D is $(t+m)\times(t+m)$. Let $C = D^{-1}$. Through simple matrix operations, we obtain

$$DC = I, \tag{5}$$

where I is the identity matrix of dimension $(t+m)$. For a more detailed explanation of multistep collocation using the matrix inversion technique, readers are referred to Onumanyi, Sirisena, and Jator [22]. We now proceed to derive a BHEM and two BHIMs.

2.1 Continuous formulation of a two-step explicit method incorporating one off-grid collocation point

Let $k = 2$, let $t = 2$, and let $m = 2$, and let us incorporate the off-grid collocation point $x_{n+\frac{49}{25}}$. Then (1) becomes

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h\left[\beta_1(x)f_{n+1} + \beta_{\frac{49}{25}}(x)f_{n+\frac{49}{25}}\right], \tag{6}$$

and the matrix D from (4) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 \\ 0 & 1 & 2x_{n+\frac{49}{25}} & 3x_{n+\frac{49}{25}}^2 \end{bmatrix}. \tag{7}$$

Using computational software to compute $C = D^{-1}$ yields the continuous formulation for (6) as

$$\begin{aligned} y(x) = {} & \frac{(h-x+x_n)^2}{h^3}\left(h-\frac{25x}{61}+\frac{25x_n}{61}\right)y_n \\ & + \frac{147(x-x_n)}{61h^3}\left(h^2+\left(-\frac{37x}{49}+\frac{37x_n}{49}\right)h+\frac{25(x-x_n)^2}{147}\right)y_{n+1} \\ & - \frac{(-1825x+1825x_n+4753h)(x-x_n)(h-x+x_n)}{2928h^2}f_{n+1} \\ & + \frac{(625x-625x_n)(h-x+x_n)^2}{2928h^2}f_{n+\frac{49}{25}}. \end{aligned} \tag{8}$$

Evaluating (8) at $x = x_{n+1}$, $x = x_{n+2}$, and its derivative at $x = x_n$ yields

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{341}{144}hf_{n+1} - \frac{3125}{144}hf_{n+\frac{49}{25}} + \frac{61}{3}hf_{n+2}, \\ y_{n+\frac{49}{25}} &= \frac{6912}{38125}y_n + \frac{31213}{38125}y_{n+1} + \frac{28812}{38125}hf_{n+1} + \frac{588}{1525}hf_{n+\frac{49}{25}}, \\ y_{n+2} &= \frac{11}{61}y_n + \frac{50}{61}y_{n+1} + \frac{1103}{1464}hf_{n+1} + \frac{625}{1464}hf_{n+\frac{49}{25}}. \end{aligned}\right\} \tag{9}$$

2.2 Continuous formulation of a two-step implicit method incorporating one off-grid interpolation point

Let $k = 2$, $t = 2$, $m = 3$, and let $x = x_{n+\frac{125}{64}}$ be an off-grid interpolation point. Then (1) becomes

$$y(x) = \alpha_1(x)y_{n+1} + \alpha_{\frac{125}{64}}(x)y_{n+\frac{125}{64}} + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2}\right], \tag{10}$$

and the matrix D from (4) becomes

$$D = \begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+\frac{125}{64}} & x_{n+\frac{125}{64}}^2 & x_{n+\frac{125}{64}}^3 & x_{n+\frac{125}{64}}^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 \end{pmatrix}. \tag{11}$$

Using computational software to obtain $C = D^{-1}$ and substituting into (10) yields the continuous formulation

$$y(x) = \left[\left(-\frac{140625}{16636591h^4}\left(\frac{64x_n}{3}+h-\frac{64x}{3}\right)\right)\right.$$

$$
\begin{aligned}
&\times \left(h^2+\frac{8192}{375}(x-x_n)h-\frac{4096}{375}(x-x_n)^2\right)\\
&\times \left(\frac{64x_n}{125}+h-\frac{64x}{125}\right)\Bigg] y_{n+1}\\
&+\left[\frac{\left(16777216h^2+16777216(2x-2x_n)h-16777216(x-x_n)^2\right)(h-x+x_n)^2}{16636591h^4}\right] y_{n+\frac{125}{64}}\\
&+\left[-\frac{(h-x+x_n)^2(64x_n+125h-64x)(-70x+70x_n+143h)}{53652h^3}\right] f_n\\
&+\left[\left(-\frac{1093750(h-x+x_n)}{818193h^3}\left(\frac{64x_n}{125}+h-\frac{64x}{125}\right)\right)\right.\\
&\times \left.\left(h^2+\frac{189}{125}(x-x_n)h-\frac{8567}{8750}(x-x_n)^2\right)\right] f_{n+1}\\
&+\left[-\frac{(64x_n+125h-64x)(314x-314x_n+125h)(h-x+x_n)^2}{53652h^3}\right] f_{n+2}. \qquad (12)
\end{aligned}
$$

Evaluating (12) at $x = x_n$, $x = x_{n+2}$, and differentiating the continuous scheme (12) and evaluating it at $x = x_{n+\frac{125}{64}}$ yields

$$
\begin{aligned}
y_{n+1} =& -\frac{16636591}{140625}y_n+\frac{16777216}{140625}y_{n+\frac{125}{64}}-\frac{532103}{13500}hf_n-\frac{4270}{27}hf_{n+1}\\
&-\frac{3721}{108}hf_{n+2},\\
y_{n+\frac{125}{64}} =& y_{n+1}-\frac{15207727}{786432000}hf_n+\frac{1272155}{3145728}hf_{n+1}+\frac{272731}{96000}hf_{n+\frac{125}{64}}\\
&-\frac{4766601}{2097152}hf_{n+2},\\
y_{n+2} =& -\frac{140625}{16636591}y_{n+1}+\frac{16777216}{16636591}y_{n+\frac{125}{64}}+\frac{3}{17884}hf_n-\frac{942}{272731}hf_{n+1}\\
&+\frac{753}{17884}hf_{n+2}. \qquad (13)
\end{aligned}
$$

2.3 Continuous formulation of a three-step implicit method incorporating one off-grid collocation point

Let $k = 3$, let $t = 3$, let $m = 2$, and let $x_{n+\frac{72}{25}}$ be an off-grid collocation point. Then (1) becomes

$$
y(x) = \alpha_0(x)y_n+\alpha_1(x)y_{n+1}+\alpha_2(x)y_{n+2}+h\left[\beta_{\frac{72}{25}}(x)f_{n+\frac{72}{25}}+\beta_3(x)f_{n+3}\right], \qquad (14)
$$

and the matrix D from (4) becomes

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 \\ 0 & 1 & 2x_{n+\frac{72}{25}} & 3x_{n+\frac{72}{25}}^2 & 4x_{n+\frac{72}{25}}^3 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 \end{pmatrix}. \tag{15}$$

Using computational software to obtain $C = D^{-1}$ and substituting into (14) yields

$$\begin{aligned} y(x) =& \frac{1}{h^4}(x_n + h - x)\left(\frac{1}{2}x_n + h - \frac{1}{2}x\right) \\ &\times \left(h^2 + \left(-\frac{54033x}{93146} + \frac{54033x_n}{93146}\right)h + \frac{8075(x-x_n)^2}{93146}\right) y_n \\ &+ \frac{1}{46573h^4}(181440x - 181440x_n)\left(\frac{1}{2}x_n + h - \frac{1}{2}x\right) \\ &\times \left(h^2 + \left(-\frac{2879x}{5040} + \frac{2879x_n}{5040}\right)h + \frac{1535(x-x_n)^2}{18144}\right) y_{n+1} \\ &- \frac{(84564x - 84564x_n)(x_n + h - x)}{46573h^4} \\ &\times \left(h^2 + \left(-\frac{6215x}{12528} + \frac{6215x_n}{12528}\right)h + \frac{22625(x-x_n)^2}{338256}\right) y_{n+2} \\ &+ \frac{(15625x - 15625x_n)(x_n + 2h - x)(11x_n + 39h - 11x)(x_n + h - x)}{279438h^3} f_{n+\frac{72}{25}} \\ &- \frac{(x_n + 2h - x)(x - x_n)(75025x_n + 253296h - 75025x)(x_n + h - x)}{139719h^3} f_{n+3}. \end{aligned} \tag{16}$$

Evaluating (16) at $x = x_{n+\frac{72}{25}}$, $x = x_{n+3}$, and differentiating the continuous scheme (14) with evaluation at $x = x_{n+1}$ and $x = x_{n+2}$ yield

$$\begin{aligned} y_{n+1} =& -\frac{251}{776}y_n + \frac{1027}{776}y_{n+2} - \frac{46573}{36472}hf_{n+1} - \frac{109375}{54708}hf_{n+\frac{72}{25}} \\ &+ \frac{3793}{2328}hf_{n+3}, \\ y_{n+2} =& -\frac{395}{2837}y_n + \frac{3232}{2837}y_{n+1} + \frac{46573}{31207}hf_{n+2} - \frac{265625}{93621}hf_{n+\frac{72}{25}} \\ &+ \frac{18772}{8511}hf_{n+3}, \\ y_{n+\frac{72}{25}} =& \frac{29134501}{727703125}y_n - \frac{203233536}{727703125}y_{n+1} + \frac{180360432}{145540625}y_{n+2} \\ &+ \frac{2270664}{1164325}hf_{n+\frac{72}{25}} - \frac{923750784}{727703125}hf_{n+3} \\ y_{n+3} =& \frac{1861}{46573}y_n - \frac{12987}{46573}y_{n+1} + \frac{57699}{46573}y_{n+2} + \frac{93750}{46573}hf_{n+\frac{72}{25}} \\ &- \frac{56442}{46573}hf_{n+3}. \end{aligned} \tag{17}$$

Equations (9), (13), and (17) represent the derived block hybrid schemes. Their convergence analyses are presented in the next section.

3 Analyses of the derived block methods

This section presents convergence analyses and plots of the regions of absolute stability for the derived block methods.

3.1 Convergence analysis of the block method (9)

We employ the approach of Fatunla [12] by reformulating (9) as

$$M^{(1)}Y_{n+i} = M^{(0)}Y_n + hB^{(1)}f_{n+i}, \quad i \in \mathbb{R}, \tag{18}$$

where

$$M^{(1)} = \begin{pmatrix} 1 & 0\ 0 \\ -\frac{31213}{38125} & 1\ 0 \\ -\frac{50}{61} & 0\ 1 \end{pmatrix}, \quad M^{(0)} = \begin{pmatrix} 0\ 0 & 1 \\ 0\ 0 & \frac{6912}{38125} \\ 0\ 0 & \frac{11}{61} \end{pmatrix}, \tag{19}$$

and

$$B^{(1)} = \begin{pmatrix} \frac{341}{144} & -\frac{3125}{144} & \frac{61}{3} \\ \frac{28812}{38125} & \frac{588}{1525} & 0 \\ \frac{1103}{1464} & \frac{625}{1464} & 0 \end{pmatrix}. \tag{20}$$

The first characteristic polynomial for (9) is given by

$$\rho(\lambda) = \det(\lambda M^{(1)} - M^{(0)}) = |\lambda M^{(1)} - M^{(0)}| = 0.$$

Thus, we obtain

$$\rho(\lambda) = \left| \lambda \begin{pmatrix} 1 & 0\ 0 \\ -\frac{31213}{38125} & 1\ 0 \\ -\frac{50}{61} & 0\ 1 \end{pmatrix} - \begin{pmatrix} 0\ 0 & 1 \\ 0\ 0 & \frac{6912}{38125} \\ 0\ 0 & \frac{11}{61} \end{pmatrix} \right| = \lambda^2(\lambda - 1),$$

which implies

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0. \tag{21}$$

Since $|\lambda_i| \leqslant 1$ for $i = 1, 2, 3$, the block method (9) is zero-stable by Fatunla [11], with order $p = 3$ and error constants $\left(-\frac{83}{360}, -\frac{691488}{23828125}, -\frac{527}{18300}\right)$. Since (9) is zero-stable and has order $p > 1$, it is convergent by Henrici [15].

3.2 Convergence analysis of the block method (13)

We reformulate (13) as

$$M^{(1)}Y_{n+i} = M^{(0)}Y_n + h\left(B^{(1)}f_{n+i} + B^{(0)}f_n\right), \quad i \in \mathbb{R}, \tag{22}$$

where

$$M^{(1)} = \begin{pmatrix} 1 & -\frac{16777216}{140625} & 0 \\ -1 & 1 & 0 \\ \frac{140625}{16636591} & -\frac{16777216}{16636591} & 1 \end{pmatrix}, \quad M^{(0)} = \begin{pmatrix} 0 & 0 & -\frac{16636591}{140625} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

and

$$B^{(1)} = \begin{pmatrix} -\frac{4270}{27} & 0 & -\frac{3721}{108} \\ \frac{1272155}{3145728} & \frac{272731}{96000} & -\frac{4766601}{2097152} \\ -\frac{942}{272731} & 0 & \frac{753}{17884} \end{pmatrix}, \quad B^{(0)} = \begin{pmatrix} 0 & 0 & -\frac{532103}{13500} \\ 0 & 0 & -\frac{15207727}{786432000} \\ 0 & 0 & \frac{3}{17884} \end{pmatrix}. \tag{24}$$

The first characteristic polynomial for (13) is

$$\rho(\lambda) = \det(\lambda M^{(1)} - M^{(0)}) = |\lambda M^{(1)} - M^{(0)}| = 0. \tag{25}$$

Thus, we obtain

$$\begin{aligned} \rho(\lambda) &= \left| \lambda \begin{pmatrix} 1 & -\frac{16777216}{140625} & 0 \\ -1 & 1 & 0 \\ \frac{140625}{16636591} & -\frac{16777216}{16636591} & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -\frac{16636591}{140625} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| \\ &= -\frac{16636591}{140625}\lambda^3 + \frac{16636591}{140625}\lambda^2, \end{aligned}$$

which implies

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0. \tag{26}$$

Since $|\lambda_i| \leqslant 1$ for $i = 1, 2, 3$, the block method (13) is zero-stable by Fatunla [11], with order $p = 3$ and error constants

$$\left(\frac{271633}{207360}, \frac{6763352857}{1546188226560}, -\frac{439}{11445760}\right).$$

By Henrici [15], the block method (13) has order $p > 1$ and is zero-stable; therefore, it is convergent.

3.3 Convergence analysis of the block method (17)

We reformulate (17) as

$$M^{(1)} Y_{n+i} = M^{(0)} Y_n + hB^{(1)} f_{n+i}, \quad i \in \mathbb{R}, \tag{27}$$

where

$$M^{(1)} = \begin{pmatrix} 1 & -\frac{1027}{776} & 0 & 0 \\ -\frac{3232}{2837} & 1 & 0 & 0 \\ \frac{203233536}{727703125} & -\frac{180360432}{145540625} & 1 & 0 \\ \frac{12987}{46573} & -\frac{57699}{46573} & 0 & 1 \end{pmatrix}, \quad M^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{251}{776} \\ 0 & 0 & 0 & -\frac{395}{2837} \\ 0 & 0 & 0 & \frac{29134501}{727703125} \\ 0 & 0 & 0 & \frac{1861}{46573} \end{pmatrix} \tag{28}$$

and

$$B^{(1)} = \begin{pmatrix} -\frac{46573}{36472} & 0 & -\frac{109375}{54708} & \frac{3793}{2328} \\ 0 & \frac{46573}{31207} & -\frac{265625}{93621} & \frac{18772}{8511} \\ 0 & 0 & \frac{2270664}{1164325} & -\frac{923750784}{727703125} \\ 0 & 0 & \frac{93750}{46573} & -\frac{56442}{46573} \end{pmatrix}. \tag{29}$$

The first characteristic polynomial for (17) is

$$\rho(\lambda) = \det(\lambda M^{(1)} - M^{(0)}) = |\lambda M^{(1)} - M^{(0)}| = 0. \tag{30}$$

Thus, we obtain

$$\begin{aligned}\rho(\lambda) &= \left| \lambda \begin{pmatrix} 1 & -\frac{1027}{776} & 0 & 0 \\ -\frac{3232}{2837} & 1 & 0 & 0 \\ \frac{203233536}{727703125} & -\frac{180360432}{145540625} & 1 & 0 \\ \frac{12987}{46573} & -\frac{57699}{46573} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -\frac{251}{776} \\ 0 & 0 & 0 & -\frac{395}{2837} \\ 0 & 0 & 0 & \frac{29134501}{727703125} \\ 0 & 0 & 0 & \frac{1861}{46573} \end{pmatrix} \right| \\ &= -\frac{139719}{275189}\lambda^4 + \frac{139719}{275189}\lambda^3,\end{aligned}$$

which implies

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0. \tag{31}$$

Since $|\lambda_i| \leqslant 1$ for $i = 1, 2, 3, 4$, the block method (17) is zero-stable by Fatunla [11], with order $p = 4$ and error constants

$$\left(-\frac{863}{207360}, -\frac{20699}{425550}, \frac{38451126384}{2274072265625}, \frac{39267}{2328650}\right).$$

By Henrici [15], (17) has order $p > 1$ and is zero-stable; therefore, it is convergent.

3.4 Stability polynomial of the block method (9)

Definition 1. A block method among (9), (13), or (17) is absolutely stable with region of absolute stability S if all roots r_i of the stability polynomial

$$\pi(r, z) = \rho(r) - z\sigma(r) = 0 \tag{32}$$

satisfy $|r_i| < 1$ for $i = 1, 2, \ldots, k$ for a fixed $z = h\lambda$, where $\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials, respectively [4, 21].

Definition 2. A numerical integration scheme is $A(\alpha)$-stable for some $\alpha \in \left[0, \frac{\pi}{2}\right]$ if the wedge

$$S_\alpha = \{z : |\mathrm{Arg}(-z)| < \alpha, z \neq 0\} \tag{33}$$

lies within its region of absolute stability. The largest $\alpha_{\max}$ is called the angle of absolute stability or argument of stability [19].

For block method (9), the stability polynomial $\pi(r,z)$ from (32) is

$$\begin{aligned}\pi\,(r,z) &= M^{(1)}Y_{n+i} - M^{(0)}Y_n - hB^{(1)}f_{n+i} \quad i \in \mathbb{R},\\ &= M^{(1)}r^1 - M^{(0)}r^0 - zB^{(1)}r^1\\ &= M^{(1)}r - M^{(0)} - zB^{(1)}r\\ &= r(M^{(1)} - zB^{(1)}) - M^{(0)}. \end{aligned} \tag{34}$$

Using the values of $M^{(1)}$, $B^{(1)}$, and $M^{(0)}$ from (19) and (20) with computational software, we obtain

$$\pi(r,z) = \begin{pmatrix} r\left(1-\dfrac{341z}{144}\right) & \dfrac{3125rz}{144} & -\dfrac{61rz}{3}-1\\ r\left(-\dfrac{31213}{38125}-\dfrac{28812z}{38125}\right) & r\left(1-\dfrac{588z}{1525}\right) & -\dfrac{6912}{38125}\\ r\left(-\dfrac{50}{61}-\dfrac{1103z}{1464}\right) & -\dfrac{625rz}{1464} & r-\dfrac{11}{61}\end{pmatrix}. \tag{35}$$

The determinant of (35) gives the stability polynomial:

$$\pi(r,z) = r^3 - r^2 - \frac{124}{75}r^3z - \frac{26}{75}r^2z + \frac{197}{150}r^3z^2 - \frac{1}{150}r^2z^2 - \frac{49}{75}r^3z^3. \tag{36}$$

The plot of (36) using software is shown in Figure 1.

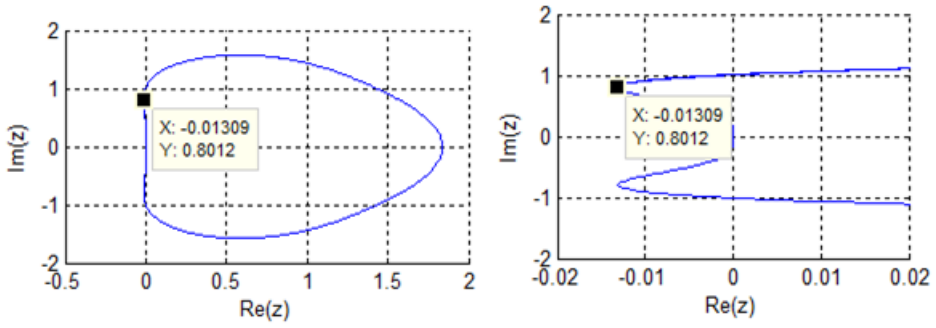

Figure 1: a) (Left) Plot of stability polynomial (36); b) (Right) Detail of (36) near the origin.

3.5 Stability polynomial of the block method (13)

For block method (13), the stability polynomial $\pi(r,z)$ from (32) is

$$\begin{aligned}\pi\left(r,z\right) &= M^{(1)}Y_{n+i} - M^{(0)}Y_n - hB^{(1)}f_{n+i} - hB^{(0)}f_n, \quad i \in \mathbb{R},\\ &= M^{(1)}r^1 - M^{(0)}r^0 - zB^{(1)}r^1 - zB^{(0)}r^0\\ &= M^{(1)}r - M^{(0)} - zB^{(1)}r - zB^{(0)}\\ &= r(M^{(1)} - zB^{(1)}) - M^{(0)} - zB^{(0)}. \end{aligned} \tag{37}$$

Using the values of $M^{(1)}$, $B^{(1)}$, $M^{(0)}$, and $B^{(0)}$ from (23) and (24) with computational software, we obtain

$$\pi(r,z) = \begin{pmatrix} r\left(1+\frac{4270\,z}{27}\right) & -\frac{16777216\,r}{140625} & \frac{3721\,rz}{108}+\frac{16636591}{140625}+\frac{532103\,z}{13500}\\ r\left(-1-\frac{1272155\,z}{3145728}\right) & r\left(1-\frac{272731\,z}{96000}\right) & \frac{4766601\,rz}{2097152}+\frac{15207727\,z}{786432000}\\ r\left(\frac{140625}{16636591}+\frac{942\,z}{272731}\right) & -\frac{16777216\,r}{16636591} & r\left(1-\frac{753\,z}{17884}\right)-\frac{3\,z}{17884}\end{pmatrix}. \tag{38}$$

The determinant of (38) gives the stability polynomial:

$$\begin{aligned}\pi(r,z) = &-\frac{16636591}{140625}r^3 + \frac{5273799347}{36000000}r^3z + \frac{216275683}{2400000}r^2z - \frac{8368205273}{108000000}r^3z^2\\ &+\frac{2279212967}{108000000}r^2z^2 + \frac{16636591}{864000}r^3z^3 + \frac{16636591}{36000000}r^2z^3 + \frac{16636591}{140625}r^2.\end{aligned} \tag{39}$$

The plot of (39) is shown in Figure 2.

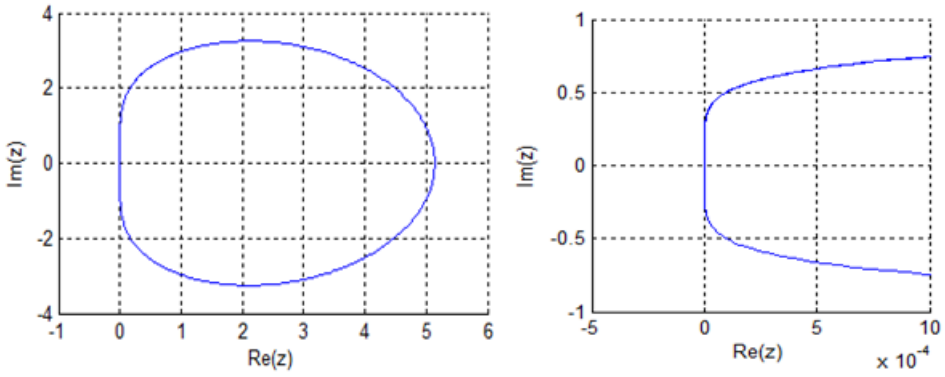

Figure 2: a) (Left) Plot of stability polynomial (39); b) (Right) Detail of (39) near the origin.

3.6 Stability polynomial of the block method (17)

For block method (17), the stability polynomial $\pi(r,z)$ from (32) is

$$\pi\left(r,z\right)=r\left(M^{(1)}-zB^{(1)}\right)-M^{(0)}. \tag{40}$$

Using the values of $M^{(1)}$, $M^{(0)}$, and $B^{(1)}$ from (28) and (29) with computational software, we obtain

$$\pi(r,z)=\begin{pmatrix} r\left(1+\frac{46573\,z}{36472}\right) & -\frac{1027\,r}{776} & \frac{109375\,rz}{54708} & -\frac{3793\,rz}{2328}+\frac{251}{776} \\ -\frac{3232\,r}{2837} & r\left(1-\frac{46573\,z}{31207}\right) & \frac{265625\,rz}{93621} & -\frac{18772\,rz}{8511}+\frac{395}{2837} \\ \frac{203233536\,r}{727703125} & -\frac{180360432\,r}{145540625} & r\left(1-\frac{2270664\,z}{1164325}\right) & \frac{923750784\,rz}{727703125}-\frac{29134501}{727703125} \\ \frac{12987\,r}{46573} & -\frac{57699\,r}{46573} & -\frac{93750\,rz}{46573} & r\left(1+\frac{56442\,z}{46573}\right)-\frac{1861}{46573} \end{pmatrix}. \tag{41}$$

The determinant of (41) gives the stability polynomial:

$$\begin{aligned}\pi(r,z)=&-\frac{139719}{275189}r^4+\frac{15508809}{13759450}r^4z+\frac{139719}{275189}r^3-\frac{32927111}{27518900}r^4z^2\\&+\frac{5449041}{13759450}r^3z+\frac{21935883}{27518900}r^4z^3+\frac{2747807}{27518900}r^3z^2\\&-\frac{2514942}{6879725}r^4z^4+\frac{139719}{27518900}r^3z^3.\end{aligned} \tag{42}$$

The plot of (42) is shown in Figure 3.

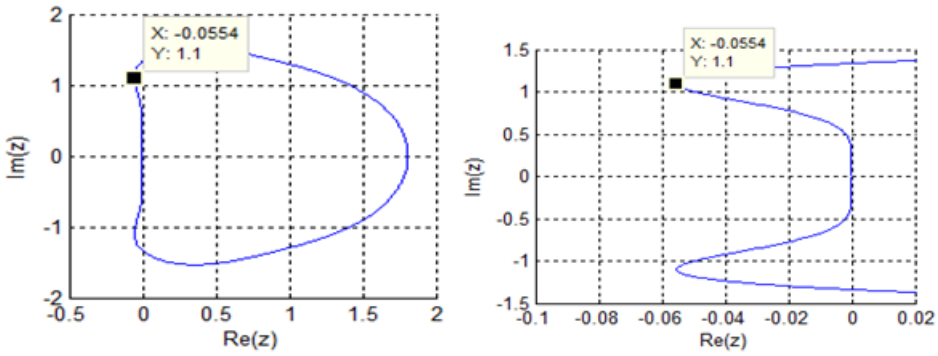

Figure 3: a) (Left) Plot of stability polynomial (42); b) (Right) Detail of (42) near the origin.

3.7 *L*-stability analysis via stability polynomials

Beyond A-stability, L-stability requires that all characteristic roots $r_i(z)$ of the stability polynomial satisfy $|r_i(z)| \to 0$ as $z \to -\infty$ along the real axis. For $L(\alpha)$-stability, this damping must occur within the wedge of $A(\alpha)$-stability. We analyze this asymptotic behavior for each method by examining the highest powers of z in their respective stability polynomials.

Block method (9)**:** The stability polynomial from (36) is

$$\pi(r,z) = r^3 - r^2 - \frac{124}{75}r^3z - \frac{26}{75}r^2z + \frac{197}{150}r^3z^2 - \frac{1}{150}r^2z^2 - \frac{49}{75}r^3z^3.$$

For large negative z, the dominant term is the highest power of z, which is z^3. Retaining only this term, we have

$$\pi(r,z) \approx -\frac{49}{75}r^3z^3 = 0.$$

This implies that all three roots satisfy $r \to 0$ as $z \to -\infty$. Since method (9) is $A(\alpha)$-stable with $\alpha = 89.06°$, it follows that it is $L(\alpha)$-stable within this wedge.

Block method (13)**:** The stability polynomial from (39) is

$$\begin{aligned}\pi(r,z) = &-\frac{16636591}{140625}r^3 + \frac{5273799347}{36000000}r^3z + \frac{216275683}{2400000}r^2z \\ &- \frac{8368205273}{108000000}r^3z^2 + \frac{2279212967}{108000000}r^2z^2 + \frac{16636591}{864000}r^3z^3 \\ &+ \frac{16636591}{36000000}r^2z^3 + \frac{16636591}{140625}r^2.\end{aligned}$$

The highest power of z is z^3. Isolating the z^3 terms, we have

$$\left(\frac{16636591}{864000}r^3 + \frac{16636591}{36000000}r^2\right)z^3 = 0.$$

Setting the coefficient to zero yields

$$r^2\left(\frac{1}{864000}r + \frac{1}{36000000}\right) = 0.$$

Thus, two roots approach $r = 0$, and the third root approaches $r = -\frac{864000}{36000000} = -0.024 \neq 0$ as $z \to -\infty$. Since not all roots tend to zero, method (13) is not L-stable, despite being A-stable.

Block method (17)**:** The stability polynomial from (42) is

$$\begin{aligned}\pi(r,z) = &-\frac{139719}{275189}r^4 + \frac{15508809}{13759450}r^4z + \frac{139719}{275189}r^3 - \frac{32927111}{27518900}r^4z^2 \\ &+ \frac{5449041}{13759450}r^3z + \frac{21935883}{27518900}r^4z^3 + \frac{2747807}{27518900}r^3z^2\end{aligned}$$

$$-\frac{2514942}{6879725}r^4z^4+\frac{139719}{27518900}r^3z^3.$$

The highest power of z is z^4. The dominant term is

$$-\frac{2514942}{6879725}r^4z^4=0,$$

which implies all four roots satisfy $r\to 0$ as $z\to-\infty$. Given that method (17) is $A(\alpha)$-stable with $\alpha=87.12^\circ$, it is $L(\alpha)$-stable within this wedge.

This analysis demonstrates that while stability polynomials are primarily used to determine A-stability regions, their asymptotic structure can also reveal L-stability properties by examining the behavior of roots as $z\to-\infty$.

3.8 Stability function of the block method (9)

Definition 3 (A-Stability using stability function)**.** Let $L(z)$ be the stability function defined by $L(z)=\frac{P(z)}{Q(z)}$, where $P(z)$ has degree k and $Q(z)$ has degree j. A method with stability function $L(z)$ is A-stable if and only if

$$|L(iy)|\leqslant 1,\quad \text{for all real } y, \tag{43}$$

and

$$L(z)\ \text{ is analytic for } \Re(z)<0. \tag{44}$$

Condition (43) alone, known as I-stability, ensures stability on the imaginary axis. It is equivalent to the condition that the polynomial

$$E(y)=|Q(iy)|^2-|P(iy)|^2=Q(iy)\,Q(-iy)-P(iy)\,P(-iy) \tag{45}$$

satisfies

$$E(y)\geq 0\quad \text{for all } y. \tag{46}$$

(see [14]).

Remark 1. The above definition from Hairer and Wanner [14] applies to one-step methods, where the stability function $L(z)$ completely characterizes stability. For block methods, $L(z)$ is

typically taken as the spectral radius $\rho(D(z))$ of the amplification matrix. Conditions (43) and (44) are necessary but not sufficient for A-stability. The insufficiency arises because $L(z)$ captures only the dominant eigenvalue; nondominant eigenvalues may violate the stability requirements even when the spectral radius satisfies the conditions. The necessary and sufficient conditions for A-stability of block methods are precisely those stated in Definition 4, which require that *all* eigenvalues of the amplification matrix satisfy the appropriate bounds (with the additional requirement that any eigenvalue on the unit circle be simple). Verifying these conditions is equivalent to checking that all roots of the stability polynomial $\pi(r, z) = 0$ lie in the closed unit disk for $\Re(z) \leq 0$, with strict inequality for $\Re(z) < 0$ and simplicity on the boundary. When nondominant eigenvalues are identically zero or have moduli orders of magnitude smaller than the dominant eigenvalue, conditions (43) and (44) may suffice for practical computations, but they do not constitute a rigorous mathematical proof of A-stability.

Definition 4 (Necessary and sufficient conditions for A-stability of block methods)**.** A consistent block method with amplification matrix $D(z)$ and eigenvalues $\{r_i(z)\}_{i=1}^{k}$ is A-stable if and only if

1. $|r_i(iy)| \leq 1$ for all real y and all $i = 1, \ldots, k$.
2. $|r_i(z)| < 1$ for all z with $\Re(z) < 0$ and all $i = 1, \ldots, k$.
3. any eigenvalue with $|r_i(iy)| = 1$ is simple (to avoid polynomial growth).

Theorem 1 (A-stability verification)**.** A block method is A-stable if and only if

1. All roots r of the stability polynomial $\pi(r, iy) = 0$ satisfy $|r| \leq 1$ for all real y.
2. All roots r of the stability polynomial $\pi(r, z) = 0$ satisfy $|r| < 1$ for all z with $\Re(z) < 0$.

Proof. Let the block method be applied to the test equation $y' = \lambda y$, with $z = h\lambda$. This yields an amplification matrix $D(z)$ such that the numerical solution advances as $Y_{n+1} = D(z)Y_n$. The stability polynomial is defined as $\pi(r, z) = \det(rI - D(z))$. Hence, the roots r of $\pi(r, z) = 0$ are exactly the eigenvalues of $D(z)$.

If the method is A-stable according to Definition 4, then by that definition, we have the following results:

- For every real y, each eigenvalue r of $D(iy)$ satisfies $|r| \leq 1$. This is precisely condition (1) of the theorem, because those eigenvalues are the roots of $\pi(r, iy) = 0$.

- For every z with $\Re(z) < 0$, each eigenvalue r of $D(z)$ satisfies $|r| < 1$. This is exactly condition (2) of the theorem, since those eigenvalues are the roots of $\pi(r, z) = 0$.

Conversely, assume conditions (1) and (2) hold. Then for every real y, all roots of $\pi(r, iy) = 0$ satisfy $|r| \leq 1$, which means all eigenvalues of $D(iy)$ satisfy $|r| \leq 1$. Similarly, for every z with $\Re(z) < 0$, all roots of $\pi(r, z) = 0$ satisfy $|r| < 1$, that is, all eigenvalues of $D(z)$ satisfy $|r| < 1$. Thus the method meets both requirements of Definition 4 and is therefore A-stable.

Hence, the two conditions are necessary and sufficient for A-stability. □

Remark 2. For a certain class of block methods where the amplification matrix $D(z)$ has rank 1 for all z, exactly one eigenvalue is nonzero and the remaining $k-1$ eigenvalues are identically zero. In such cases, the zero eigenvalues trivially satisfy any modulus bound satisfied by the nonzero eigenvalue. Consequently, verifying A-stability reduces to checking only the dominant eigenvalue, provided no poles exist in the left half-plane. However, this property is not general; some block methods may yield amplification matrices of full rank, requiring analysis of all eigenvalues.

To obtain the stability function for block method (9), we apply (18) to the test problem $y' = \lambda y$, $\lambda < 0$:

$$M^{(1)}Y_{n+1} = M^{(0)}Y_n + hB^{(1)}f_{n+1},$$

$$\begin{aligned}
M^{(1)}Y_{n+1} &= M^{(0)}Y_n + hB^{(1)}\lambda Y_{n+1},\\
M^{(1)}Y_{n+1} - h\lambda B^{(1)}Y_{n+1} &= M^{(0)}Y_n,\\
(M^{(1)} - h\lambda B^{(1)})Y_{n+1} &= M^{(0)}Y_n,\\
(M^{(1)} - zB^{(1)})Y_{n+1} &= M^{(0)}Y_n, \quad z = h\lambda,
\end{aligned}$$

$$Y_{n+1} = (M^{(1)} - zB^{(1)})^{-1}M^{(0)}Y_n, \tag{47}$$

$$Y_{n+1} = D(z)Y_n, \tag{48}$$

where $D(z) = (M^{(1)} - zB^{(1)})^{-1}M^{(0)}$ is a matrix with eigenvalues $\{d_1, d_2, \ldots, d_k\} = \{0, 0, \ldots, d_k\}$ such that the dominant eigenvalue d_k is the stability function $L(z) : \mathbb{C} \to \mathbb{C}$, a rational function with real coefficients. Using the values of $M^{(1)}$, $B^{(1)}$, and $M^{(0)}$ from (19) and (20) with Maple software, we obtain

$$L(z) = \frac{-z^2 - 52\,z - 150}{98\,z^3 - 197\,z^2 + 248\,z - 150}. \tag{49}$$

The plot of the stability function (49) is shown in Figure 4.

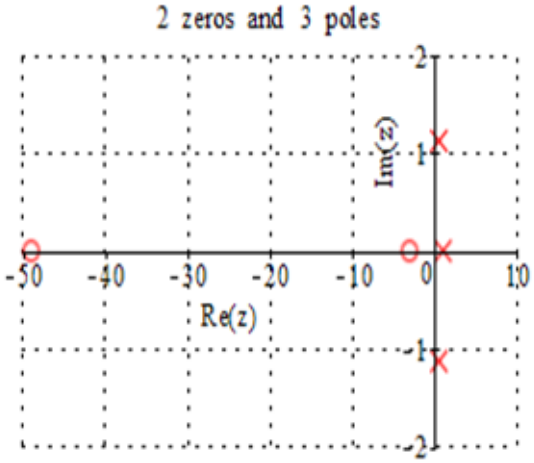

Figure 4: Plot of stability function (49).

3.9 Stability function of the block method (13)

For the stability function of block method (13), we apply (22) to the test problem $y' = \lambda y$, $\lambda < 0$:

$$M^{(1)}Y_{n+1} = M^{(0)}Y_n + hB^{(1)}f_{n+1} + hB^{(0)}f_n,$$

$$\begin{aligned}
M^{(1)}Y_{n+1} &= M^{(0)}Y_n + hB^{(1)}\lambda Y_{n+1} + hB^{(0)}\lambda Y_n,\\
M^{(1)}Y_{n+1} - h\lambda B^{(1)}Y_{n+1} &= M^{(0)}Y_n + h\lambda B^{(0)}Y_n,\\
(M^{(1)} - h\lambda B^{(1)})Y_{n+1} &= (M^{(0)} + h\lambda B^{(0)})Y_n,\\
(M^{(1)} - zB^{(1)})Y_{n+1} &= (M^{(0)} + zB^{(0)})Y_n, \quad z = h\lambda,
\end{aligned}$$

$$Y_{n+1} = (M^{(1)} - zB^{(1)})^{-1}(M^{(0)} + zB^{(0)})Y_n, \tag{50}$$

$$Y_{n+1} = D(z)Y_n, \tag{51}$$

where $D(z) = (M^{(1)} - zB^{(1)})^{-1}(M^{(0)} + zB^{(0)})$. Using the values of $M^{(1)}$, $B^{(1)}$, $M^{(0)}$, and $B^{(0)}$ from (23) and (24) with computational software, we obtain

$$L(z) = \frac{-3\,z^3 - 137\,z^2 - 585\,z - 768}{125\,z^3 - 503\,z^2 + 951\,z - 768}. \tag{52}$$

The plot of the stability function (52) is shown in Figure 5.

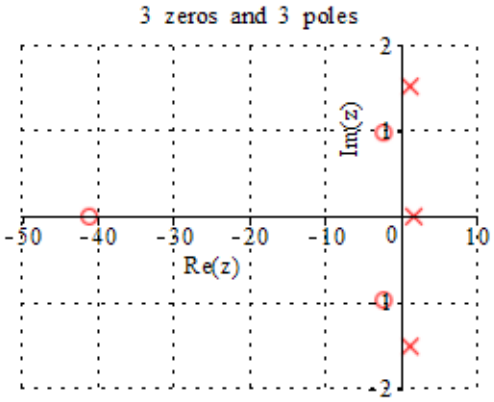

Figure 5: Plot of stability function (52).

3.10 Stability function of the block method (17)

To obtain the stability function for block method (17), we apply (27) to the test problem $y' = \lambda y$, $\lambda < 0$, which gives

$$Y_{n+1} = D(z)Y_n, \tag{53}$$

where $D(z) = (M^{(1)} - zB^{(1)})^{-1}M^{(0)}$. Using the values of $M^{(1)}$, $B^{(1)}$, and $M^{(0)}$ from (28) and (29) with computational software, we obtain

$$L(z) = \frac{3\,z^3 + 59\,z^2 + 234\,z + 300}{216\,z^4 - 471\,z^3 + 707\,z^2 - 666\,z + 300}. \tag{54}$$

The plot of the stability function (54) is shown in Figure 6.

Remark 3. This analysis provides a definitive answer to the research question posed in Section 1: "Does the absence of poles in the left half-plane of the complex plane guarantee that a block

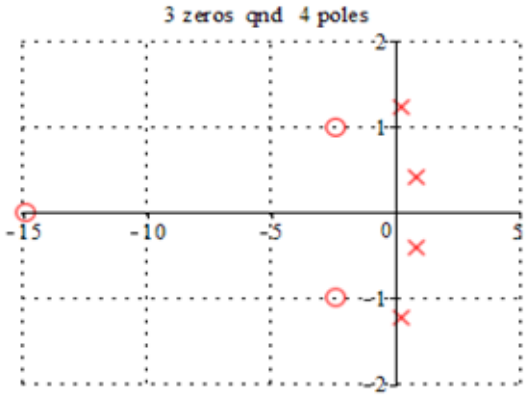

Figure 6: Plot of stability function (54).

method is A-stable, as suggested in some literature?" The answer is no. While the absence of poles in the left half-plane is necessary for A-stability, it is not sufficient. Condition (1) of Definition 4 (boundedness on the imaginary axis) must also be satisfied. Method (9) exemplifies this: it has no poles in the left half-plane but violates condition (1) of Definition 4, rendering it not A-stable. Similarly, method (17) does not satisfy condition (1) of Definition 4. Only method (13) satisfies all required conditions and is A-stable. Therefore, all conditions must be verified when using stability functions to assess A-stability.

3.11 *L*-stability analysis via stability function

Beyond the A-stability and $A(\alpha)$-stability analyzed above, L-stability is a stronger property that ensures the numerical method damps out stiff components rapidly. A method is L-stable if it is A-stable and its stability function satisfies

$$\lim_{\substack{z\to-\infty \\ \Re(z)<0}} |L(z)| = 0. \tag{55}$$

For methods that are $A(\alpha)$-stable, if the condition (55) holds within the wedge $|\arg(-z)| < \alpha$, the method is called $L(\alpha)$-stable.

Using the stability functions derived in the previous sections, we analyze the L-stability properties of the three block hybrid methods.

Block method (9):

The stability function given by (49) is

$$L(z) = \frac{-z^2 - 52z - 150}{98z^3 - 197z^2 + 248z - 150}.$$

The numerator has degree 2 and the denominator has degree 3. As $z \to -\infty$ along the negative real axis, the leading term behavior yields

$$L(z) \sim \frac{-z^2}{98z^3} = -\frac{1}{98z} \to 0.$$

Since method (9) is $A(\alpha)$-stable with $\alpha = 89.06°$ (as determined in Section 5), it follows that it is $L(\alpha)$-stable within the same wedge.

Block method (13):

The stability function in (52) is

$$L(z) = \frac{-3z^3 - 137z^2 - 585z - 768}{125z^3 - 503z^2 + 951z - 768}.$$

Here, the numerator and the denominator are both of degree 3. The limit as $z \to -\infty$ is the ratio of the leading coefficients:

$$\lim_{z \to -\infty} L(z) = \frac{-3}{125} = -0.024 \neq 0.$$

Although method (13) is A-stable (see Figure 2 and the analysis in Section 5), it does not satisfy condition (55). Hence, method (13) is not L-stable.

Block method (17):

The stability function in (54) is

$$L(z) = \frac{3z^3 + 59z^2 + 234z + 300}{216z^4 - 471z^3 + 707z^2 - 666z + 300}.$$

The degree of the numerator is 3 and that of the denominator is 4. As $z \to -\infty$,

$$L(z) \sim \frac{3z^3}{216z^4} = \frac{1}{72z} \to 0.$$

Given that method (17) is $A(\alpha)$-stable with $\alpha = 87.12°$, it is $L(\alpha)$-stable within this wedge.

This analysis demonstrates that the stability function not only facilitates the assessment of A-stability but also provides a straightforward means to check for L-stability by examining the asymptotic behavior as $z \to -\infty$. The stability function's rational form allows direct asymptotic analysis, making it preferable over stability polynomials for L-stability assessment, as the latter requires solving for root behavior as $z \to -\infty$ from $\pi(r, z) = 0$.

3.12 Experiments

In this subsection, three test problems are presented for experimental purposes. These problems are original constructions, not previously reported in the literature, and are designed to further examine the conditions for effectively applying stability polynomials and stability functions. Due to space constraints and to avoid repetition, the detailed derivation steps are omitted; however, the derivation procedure follows the one used to obtain block methods (9), (13), and (17). Interested readers may replicate the procedure to ascertain other properties of the test problems. The three experiments illustrate distinct stability characteristics and serve to validate the theoretical framework presented in the preceding sections.

3.12.1 Experiment 1

$$\begin{cases} f_{n+1} = \frac{984}{475\,h}\left(-y_{n+1} - \frac{227}{1312}y_n + \frac{1539}{1312}y_{n+\frac{4}{3}} - \frac{49}{984}hf_n - \frac{17}{492}hf_{n+2} + \frac{1}{328}hf_{n+3}\right), \\ f_{n+\frac{4}{3}} = -\frac{1200}{227\,h}\left(-y_{n+\frac{4}{3}} + y_{n+1} - \frac{23}{19440}\,hf_n + \frac{167}{1080}hf_{n+1} - \frac{4}{405}hf_{n+2} + \frac{29}{48600}hf_{n+3}\right), \\ y_{n+2} = -\frac{664}{227}y_{n+1} + \frac{891}{227}y_{n+\frac{4}{3}} + \frac{2}{227}hf_n - \frac{419}{681}hf_{n+1} + \frac{208}{681}hf_{n+2} - \frac{5}{681}hf_{n+3}\right), \\ y_{n+3} = \frac{875}{227}y_{n+1} - \frac{648}{227}y_{n+\frac{4}{3}} - \frac{25}{681}hf_n + \frac{800}{681}hf_{n+1} + \frac{1025}{681}hf_{n+2} + \frac{70}{227}hf_{n+3}\right). \end{cases} \tag{56}$$

3.12.2 Experiment 2

$$\begin{cases} y_{n+1} = y_n - \dfrac{9}{24}hf_n - \dfrac{19}{24}hf_{n+1} + \dfrac{5}{24}hf_{n+2} - \dfrac{1}{24}hf_{n+3}, \\ y_{n+2} = y_{n+1} - \dfrac{1}{24}hf_n + \dfrac{13}{24}hf_{n+1} + \dfrac{13}{24}hf_{n+2} - \dfrac{1}{24}hf_{n+3}, \\ y_{n+3} = y_{n+2} + \dfrac{8}{24}hf_{n+1} + \dfrac{32}{24}hf_{n+2} + \dfrac{8}{24}hf_{n+3}. \end{cases} \tag{57}$$

3.12.3 Experiment 3

$$\begin{cases} y_{n+1} = y_{n+2} + \dfrac{1}{1080}hf_n - \dfrac{7}{40}hf_{n+1} - \dfrac{7}{40}hf_{n+2} + \dfrac{53}{1080}hf_{n+3}, \\ y_{n+2} = y_n + \dfrac{41}{135}hf_n + \dfrac{8}{5}hf_{n+1} + \dfrac{3}{5}hf_{n+2} - \dfrac{4}{135}hf_{n+3}, \\ y_{n+3} = y_{n+1} - \dfrac{31}{1080}hf_n + \dfrac{17}{40}hf_{n+1} + \dfrac{57}{40}hf_{n+2} + \dfrac{329}{1080}hf_{n+3}. \end{cases} \tag{58}$$

3.13 Numerical simulation

In this subsection, the block hybrid schemes (9), (13), and (17) are implemented for the following problems, respectively:

3.13.1 Problem 1

$$\begin{cases} y_1' = -0.1y_1 - 199.9y_2, \quad y_1(0) = 5, \\ y_2' = -200y_2, \quad y_2(0) = 2. \end{cases} \tag{59}$$

3.13.2 Problem 2

$$\begin{cases} y_1' = -(2+\varepsilon^{-1})y_1(x) + \varepsilon^{-1}\left(y_2(x)\right)^2, \quad y_1(0) = 1, \\ y_2' = y_1(x) - y_2(x) - \left(y_2(x)\right)^2, \quad y_2(0) = 1. \end{cases} \tag{60}$$

3.13.3 Problem 3

$$\begin{cases} y_1' = 998y_1 + 1998y_2, \quad y_1(0) = 1, \\ y_2' = -999y_1 - 1999y_2, \quad y_2(0) = 0. \end{cases} \tag{61}$$

4 Implementation of the proposed block hybrid methods

This section provides a detailed description of how to implement the three derived block hybrid methods: The explicit method BHEM (9), the implicit method BHIM (13), and the implicit method BHIM (17). The implementation follows the standard block-by-block advancement strategy. For all three methods, the solution is advanced in blocks of fixed step size h. At each block step, values at r future points are computed simultaneously (for method (9), $r = 3$; for (13), $r = 4$; for (17), $r = 4$). Once the block unknowns are computed, the block window slides forward, and the process repeats.

4.1 Implementing the explicit method BHEM (9)

Method (9) is explicit, requiring no iteration. Its matrix form is

$$M^{(1)}Y_{n+1} = M^{(0)}Y_n + hB^{(1)}F_{n+1},$$

where $Y_{n+1} = [y_{n+1}, y_{n+2}, y_{n+3}]^T$ and $F_{n+1} = [f_{n+1}, f_{n+2}, f_{n+3}]^T$. Because $M^{(1)}$ is lower triangular with ones on the diagonal (see (19)), the solution can be obtained by forward substitution without inverting the matrix.

4.2 Implementing the implicit methods BHIM (13) and BHIM (17)

Implicit methods require solving a system of nonlinear equations at each block. For a system of ODEs $y' = f(t, y)$ with d components, the block unknowns $Y_{n+1} \in \mathbb{R}^{rd}$ are obtained from

$$(M^{(1)} - zB^{(1)})Y_{n+1} = (M^{(0)} + zB^{(0)})Y_n, \quad \text{for method (13)},$$

$$(M^{(1)} - zB^{(1)})Y_{n+1} = M^{(0)}Y_n, \quad \text{for method (17)},$$

We write the block methods as $G(Y_{n+1}) = 0$, where G is the residual. For methods (13) and (17), the residuals for the jth component at each block point are defined by (22) and (27). We assemble the rd-dimensional unknown vector $\mathbf{Y} = [y_{n+1,1}, \ldots, y_{n+1,d}, y_{n+2,1}, \ldots, y_{n+r,d}]^T$. The Jacobian matrix $J = \partial G/\partial \mathbf{Y}$ is formed using the Jacobian of f, denoted $J_f = \partial f/\partial y$. The iteration used is

$$J(\mathbf{Y}^{(m)})\,\Delta\mathbf{Y}^{(m)} = -G(\mathbf{Y}^{(m)}), \quad \mathbf{Y}^{(m+1)} = \mathbf{Y}^{(m)} + \Delta\mathbf{Y}^{(m)}.$$

The iteration stops when $\|\Delta\mathbf{Y}^{(m)}\|_\infty < \text{tol}$.

5 Results and discussion

The stability polynomial $\pi(r, z)$ of (36) focuses on the positions of the characteristic roots r in the complex plane. Figure 1 demonstrates that BHEM (9) is $A(\alpha)$-stable. In contrast, the stability function $L(z)$ of (49) examines the complex parameter z and the positions of its zeros and poles, as shown in Figure 4 for the same block method. Notably, the stability functions (49), (52), and (54) for methods (9), (13), and (17) involve only z, whereas the stability polynomials (36), (39), and (42) explicitly contain the roots r. Consequently, Figures 1, 2, and 3 display stability regions based on the roots r of the characteristic equations, while Figures 4, 5, and 6 illustrate stability properties through the complex variable z for the stability functions.

Figure 4 indicates that the stability function $L(z)$ for block method (9) satisfies condition (44) of Definition 3, being analytic for all $\Re(z) < 0$ because all poles of $L(z)$ in (49) lie in the right half-plane. However, the mere absence of left-half-plane poles does not guarantee A-stability. From (43) and (49), restated here as,

$$|L(iy)| \leqslant 1, \quad \text{for all real } y;$$

and

$$L(z) = \frac{-z^2 - 52\,z - 150}{98\,z^3 - 197\,z^2 + 248\,z - 150}.$$

A complex number $z = x + iy$ on the imaginary axis has $z = 0 + iy$, that is, $z = iy$. Taking $y = -0.9$ and applying condition (43) to (49) gives

$$|L(-0.9i)| = \left| \frac{-(-0.9i)^2 - 52\,(-0.9i) - 150}{98\,(-0.9i)^3 - 197\,(-0.9i)^2 + 248\,(-0.9i) - 150} \right|.$$

Using the evalf function of Maple 2022.0 for numerical evaluation gives

$$|L(-0.9i)| = 1.028270409890 > 1,$$

which violates condition (43) of Definition 3 because the value obtained at $y = -0.9$ is 1.028270409890, which is not less than or equal to 1, as required. However, it can be noted that the choice of $y = -0.9$ was based on choosing from the lower interval of the imaginary axis based on common patterns of stability plots. If there exists at least one y in such an interval for which (43) is not satisfied, then condition (43) is violated. This is because $|L(iy)|$ must be less than or equal to 1 for all real y. More comprehensively, applying Theorem 1 to method (9) shows that it fails both conditions for A-stability. First, on the imaginary axis, numerical investigation reveals that for $y = -0.9$, at least one root r of $\pi(r, iy) = 0$ satisfies $|r| > 1$. Second, in the left half-plane, the stability region shown in Figure 1 does not entirely cover $\{z : \Re(z) < 0\}$; there exist z with negative real part for which some root r of $\pi(r, z) = 0$ has $|r| \geq 1$. Therefore, method (9) is not A-stable. Nevertheless, verification confirms it is $A(\alpha)$-stable with $\alpha = 89.06°$, meaning absolute stability holds within the wedge $|\arg(-z)| < 89.06°$.

The stability polynomial $\pi(r, z)$ for block method (13), given by (39), is plotted in Figure 2. The region of absolute stability, corresponding to the exterior of the enclosed shape, encompasses the entire left half-plane, indicating that method (13) is A-stable. Correspondingly, the stability function $L(z)$ in (52) satisfies $|L(z)| \leqslant 1$ and is analytic for $\Re(z) < 0$, as shown in Figure 5. All poles of $L(z)$ lie in the right half-plane, and all zeros lie in the left half-plane. Since no pole occurs in the left half-plane, $L(z)$ is analytic for $\Re(z) < 0$, satisfying condition (44) of Definition 3. Moreover, $|L(iy)| \leqslant 1$ for all real y, fulfilling condition (43) of Definition 3. Applying Theorem 1 confirms A-stability: first, on the imaginary axis, all roots of $\pi(r, iy) = 0$ satisfy $|r| \leq 1$ for every real y; second, in the left half-plane, the stability region covers $\{z : \Re(z) < 0\}$ entirely, and for every such z, all roots of $\pi(r, z) = 0$ satisfy $|r| < 1$. Consequently, method (13) is A-stable.

The stability polynomial $\pi(r, z)$ for block method (17), given by (42), is plotted in Figure 3. The region of absolute stability (the exterior of the enclosed portion) covers part of the left half-plane but not all of it, indicating that method (17) is $A(\alpha)$-stable. Using (33) yields $\alpha = 87.12°$. The stability function $L(z)$ in (54), plotted in Figure 6, shows all poles in the right half-plane, so $L(z)$ is analytic for $\Re(z) < 0$, satisfying condition (44) of Definition 3. However, for $y = -0.98$, we compute $|L(iy)| = 1.204625225 > 1$, violating condition (43) of Definition 3. Equivalently, $E(y) = -34356.13668$ for (54), which contradicts condition (46). Applying Theorem 1 confirms that method (17) is not A-stable: first, on the imaginary axis, for $y = -0.98$ at least one root of

$\pi(r, iy) = 0$ has modulus exceeding unity; second, in the left half-plane, the stability region does not cover the entire set $\{z : \Re(z) < 0\}$. Hence, method (17) is not A-stable but is $A(\alpha)$-stable with $\alpha = 87.12°$, meaning absolute stability holds within the wedge $|\arg(-z)| < 87.12°$.

Simple algebraic simplification of Experiment 1 converts it to the form:

$$\begin{aligned}
y_{n+1} =& -\frac{227}{1312}y_n + \frac{1539}{1312}y_{n+\frac{4}{3}} - \frac{49}{984}hf_n - \frac{475}{984}hf_{n+1} - \frac{17}{492}hf_{n+2} \\
&+ \frac{1}{328}hf_{n+3}, \\
y_{n+\frac{4}{3}} =& y_{n+1} - \frac{23}{19440}hf_n + \frac{167}{1080}hf_{n+1} + \frac{227}{1200}hf_{n+\frac{4}{3}} - \frac{4}{405}hf_{n+2} \\
&+ \frac{29}{48600}hf_{n+3}, \\
y_{n+2} =& -\frac{664}{227}y_{n+1} + \frac{891}{227}y_{n+\frac{4}{3}} + \frac{2}{227}hf_n - \frac{419}{681}hf_{n+1} + \frac{208}{681}hf_{n+2} \\
&- \frac{5}{681}hf_{n+3}, \\
y_{n+3} =& \frac{875}{227}y_{n+1} - \frac{648}{227}y_{n+\frac{4}{3}} - \frac{25}{681}hf_n + \frac{800}{681}hf_{n+1} + \frac{1025}{681}hf_{n+2} \\
&+ \frac{70}{227}hf_{n+3}.
\end{aligned} \tag{62}$$

The stability polynomial obtained using (32) is

$$\begin{aligned}
\pi(r, z) =& -\frac{227\,r^4}{1312} + \frac{227\,r^3}{1312} + \frac{227\,r^3z^4}{15744} + \frac{2497\,r^4z}{9840} - \frac{4313\,r^4z^2}{26240} \\
&+ \frac{7037\,r^4z^3}{118080} - \frac{227\,r^4z^4}{19680} + \frac{9223\,r^3z}{31488} + \frac{4767\,r^3z^2}{26240} + \frac{46183\,r^3z^3}{472320}.
\end{aligned} \tag{63}$$

Applying the same procedure used to obtain (49), (52), and (54) yields the stability function for Experiment 1:

$$L(z) = \frac{6810\,z^4 + 46183\,z^3 + 85806\,z^2 + 138345\,z + 81720}{5448\,z^4 - 28148\,z^3 + 77634\,z^2 - 119856\,z + 81720}. \tag{64}$$

Plots of (63) and (64) are shown in Figure 7. Although no poles of the stability function $L(z)$ in (64) exist in the left half-plane, condition (43), or equivalently condition (46), is not satisfied. For instance, consider $y = -3$; following a similar approach to that used earlier, it can be shown that $|L(iy)| = 1.928808042740$ or $E(y) = -520155100224$, contradicting the requirements of (43) or (46). Thus, the block method in Experiment 1 is not A-stable because it violates condition (1) of Definition 4. While methods that are not A-stable may be explored for $A(\alpha)$-stability with sufficiently large α [29], stability functions $L(z)$ provide limited information on $A(\alpha)$-stability, whereas stability polynomials $\pi(r, z)$ offer more insight. The stability polynomial $\pi(r, z)$ in (63)

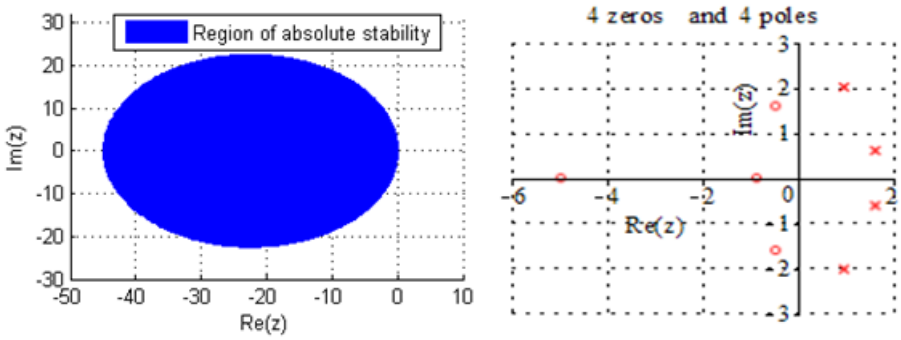

Figure 7: a) (Left) Plot of stability polynomial (63); b) (Right) Plot of stability function (64).

indicates that the block method of Experiment 1 has a very restricted region of absolute stability, as shown in its interval of absolute stability for the enclosed region on the left-hand side of the complex plane in Figure 7a. Placing wedges on the bounded region will not yield an infinite region on the left-hand side because it has a finite interval of absolute stability $(-45 < h\lambda < 0)$. Hence, the block method in Experiment 1 is neither A-stable nor $A(\alpha)$-stable but stable only in the interval $(-45 < h\lambda < 0)$. The block method in Experiment 1 does not possess an infinite interval of absolute stability like the block method (9), whose plot is shown in Figure 1; therefore, Experiment 1 should not be considered for stiff DEs. Using such methods for the solution of stiff DEs will, according to Dahlquist [9], impose prohibitive restrictions on the step size h.

For Experiment 2, the stability polynomial is obtained as

$$\pi(r,z) = r^3 - r^2 - \frac{5}{24}r^3 z - \frac{3}{8}\,r^3 z^2 - \frac{43}{24}r^2 z + \frac{3}{8}\,r^2 z^2 + \frac{1}{4}\,r^3 z^3 + \frac{1}{4}\,r^2 z^3 \tag{65}$$

and the stability function is

$$L(z) = \frac{-6\,z^3 - 9\,z^2 + 43\,z + 24}{6\,z^3 - 9\,z^2 - 5\,z + 24}. \tag{66}$$

Plots of (65) and (66) are shown in Figure 8.

Figure 8b shows that the stability function $L(z)$ in (66) has a pole on the left-hand side of the complex plane; hence, condition (44) of Definition 3 is not satisfied. Therefore, the block method in Experiment 2 cannot be A-stable [26]. It can also be verified that conditions (43) and (44) are not met. The plot of the stability polynomial $\pi(r,z)$ in Figure 8a indicates a more restricted region with an interval of absolute stability $(-0.9 < h\lambda < 0)$. Hence, the block method in Experiment 2 is also neither A-stable nor $A(\alpha)$-stable but stable only in the interval

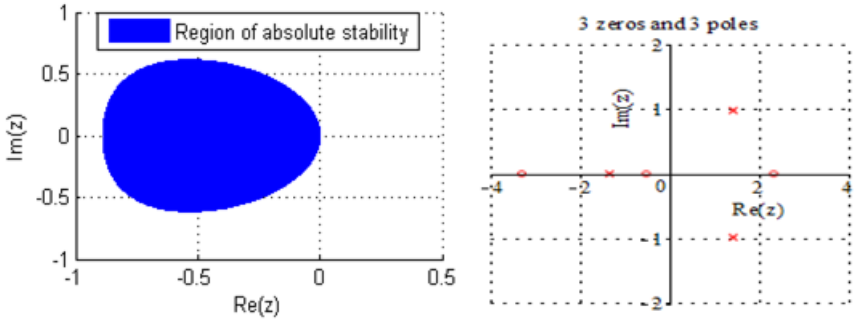

Figure 8: a) (Left) Plot of stability polynomial (65); b) (Right) Plot of stability function (66).

$(-0.9 < h\lambda < 0)$ and should not be considered for solving stiff DEs. However, it may be considered for nonstiff DEs, as discussed for Experiment 1.

For Experiment 3, the stability polynomial and stability function are

$$\begin{aligned}\pi(r,z) =& r^3 - r^2 - \frac{2537}{1080}r^3 z - \frac{2107}{1080}r^2 z + \frac{9181}{10800}r^3 z^2 - \frac{89}{120}r^2 z^2 \\ & - \frac{59}{400}r^3 z^3 - \frac{1}{20}\,r^2 z^3 \end{aligned} \tag{67}$$

and

$$L(z) = \frac{-540\,z^3 - 8010\,z^2 - 21070\,z - 10800}{1593\,z^3 - 9181\,z^2 + 25370\,z - 10800}. \tag{68}$$

Plots of (67) and (68) are shown in Figure 9.

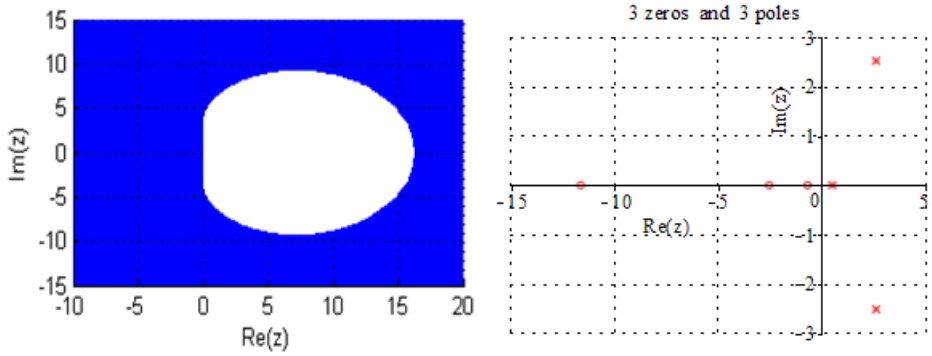

Figure 9: a) (Left) Plot of stability polynomial (67); b) (Right) Plot of stability function (68).

It can be verified that conditions (43) and (44) of Definition 3 are satisfied for the stability function $L(z)$ in (68), and the plot in Figure 9b shows no poles on the left-hand side and

$|L(z)| \leqslant 1$ for all $y \in \mathbb{R}$. Hence, the block method of Experiment 3 is A-stable. As confirmatory evidence, the plot of the stability polynomial $\pi(r, z)$ in (67) (Figure 9a) shows that the area outside the bounded region (blue) covers the entire left half-plane, implying an infinite interval $(-\infty < h\lambda < 0)$ of absolute stability. Thus, the block method in Experiment 3 is A-stable.

In summary, the stability function $L(z)$ can determine whether a numerical method is A-stable, but it may be limited in determining the angle α for $A(\alpha)$-stable methods. Nevertheless, stability functions provide a direct test for A-stability and are better suited for assessing L-stability properties because the stability function's rational form allows direct asymptotic analysis, making it preferable over stability polynomials for L-stability assessment, as the latter requires solving for root behavior as $z \to -\infty$ from $\pi(r, z) = 0$. Stability polynomials, however, can determine both A-stability and the angle α for $A(\alpha)$-stable methods. Additionally, methods that are neither A-stable nor $A(\alpha)$-stable can be shown to have a finite interval of absolute stability. A drawback of the stability polynomial $\pi(r, z)$ is that $A(\alpha)$-stable methods with very large α may have plots around the origin that appear A-stable. This can be addressed by adjusting the scale to see an amplified version of the plots, especially around the origin, as shown in Figures 1, 2, and 3. This enables clear visualization and valid conclusions. Stability polynomials and stability functions serve complementary theoretical roles. For consistent methods, the two approaches must yield congruent stability conclusions. A discrepancy between them serves as a theoretical indicator of potential derivation errors or method inconsistency.

Results from the numerical simulation of Problem 1 using BHEM (9) are shown in Figure 10.

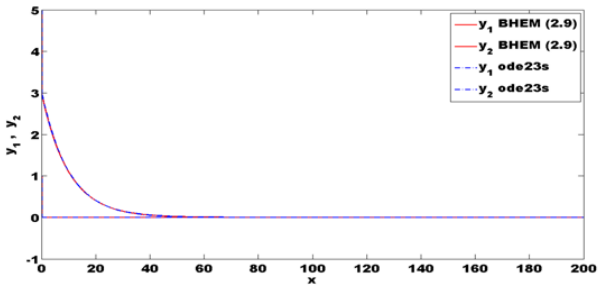

Figure 10: Numerical simulation of Problem 1 using BHEM (9).

Results from the numerical simulation of Problem 2 using the block hybrid implicit method BHIM (13) are shown in Figure 11.

The numerical simulation of Problem 3 using BHIM (17) yields the results shown in Figure 12.

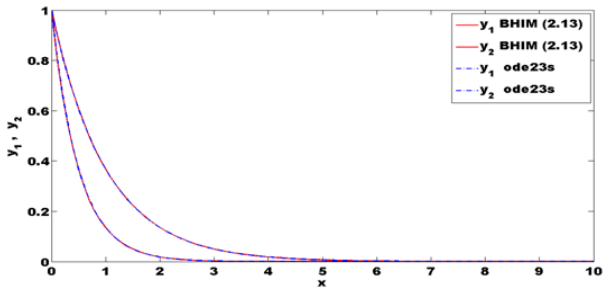

Figure 11: Numerical simulation of Problem 2 using BHIM (13).

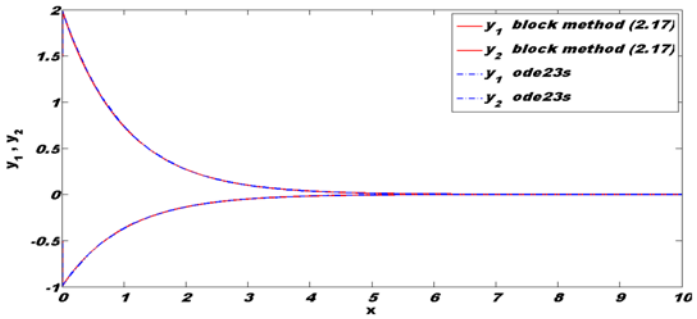

Figure 12: Numerical simulation of Problem 3 using BHIM (17).

Problem 1 is moderately stiff according to Fatunla [11], and Problem 2 is very stiff as $\varepsilon \to 0$. From Figures 10, 11, and 12, it can be seen that the block methods (9), (13), and (17) perform comparably to the built-in MATLAB `ode23s` solver for numerical solutions of Problems 1, 2, and 3.

To compare the accuracy of the results obtained from the block methods (9), (13), and (17) with exact solutions and the Adomian decomposition method reported in the literature, consider the following system:

$$\begin{cases} \dfrac{dx}{d\tau} = x + y, \quad x(0) = 0, \\ \dfrac{dy}{d\tau} = -x + y, \quad y(0) = 1, \end{cases} \tag{69}$$

whose exact solutions are given by

$$\begin{cases} x = e^{\tau} \sin \tau, \\ y = e^{\tau} \cos \tau. \end{cases}$$

The exact solutions from the system of ODEs (69) are compared with solutions from the block methods (9), (13), and (17) and the Adomian decomposition method described in the literature [27]. These comparisons are presented in Tables 1, 2, and 3.

Table 1: Comparison between the exact solutions (x and y), block method ($x_{eq(9)}$ and $y_{eq(9)}$) and Adomian decomposition method (x_ϕ and y_ϕ) solutions.

τ	Exact x	Exact y	$x_{eq(9)}$	$y_{eq(9)}$	x_ϕ [27]	y_ϕ [27]
0.50	0.79044	1.44689	0.79044	1.44689	0.79044	1.44688
1.00	2.28736	1.46869	2.28736	1.46869	2.28733	1.46868
1.50	4.47046	0.31702	4.47046	0.31702	4.47040	0.31701
2.00	6.71885	-3.07493	6.71885	-3.07493	6.71872	-3.07488
2.50	7.29088	-9.75993	7.29088	-9.75993	7.29069	-9.75970
3.00	2.83447	-19.88450	2.83447	-19.88453	2.83435	-19.88400
3.50	-11.61630	-31.01120	-11.61635	-31.01119	-11.61600	-31.01010
4.00	-41.32000	-35.68770	-41.32002	-35.68773	-41.31850	-35.68620
4.50	-87.99450	-18.97520	-87.99446	-18.97523	-87.99907	-18.97410

Table 2: Comparison between the exact solutions (x and y), block method ($x_{eq(13)}$ and $y_{eq(13)}$) and Adomian decomposition method (x_ϕ and y_ϕ) solutions.

τ	Exact x	Exact y	$x_{eq(13)}$	$y_{eq(13)}$	x_ϕ [27]	y_ϕ [27]
0.50	0.79044	1.44689	0.79043	1.44689	0.79044	1.44688
1.00	2.28736	1.46869	2.28733	1.46870	2.28733	1.46868
1.50	4.47046	0.31702	4.47042	0.31707	4.47040	0.31701
2.00	6.71885	-3.07493	6.71881	-3.07480	6.71872	-3.07488
2.50	7.29088	-9.75993	7.29095	-9.75965	7.29069	-9.75970
3.00	2.83447	-19.88450	2.83485	-19.88410	2.83435	-19.88400
3.50	-11.61630	-31.01120	-11.61531	-31.01082	-11.61600	-31.01010
4.00	-41.32000	-35.68770	-41.31797	-35.68805	-41.31850	-35.68620
4.50	-87.99450	-18.97520	-87.99141	-18.97758	-87.99907	-18.97410

Table 3: Comparison between the exact solutions (x and y), block method ($x_{eq(17)}$ and $y_{eq(17)}$) and Adomian decomposition method (x_ϕ and y_ϕ) solutions.

τ	Exact x	Exact y	$x_{eq(17)}$	$y_{eq(17)}$	x_ϕ [27]	y_ϕ [27]
0.50	0.79044	1.44689	0.79058	1.44690	0.79044	1.44688
1.00	2.28736	1.46869	2.28777	1.46851	2.28733	1.46868
1.50	4.47046	0.31702	4.47115	0.31613	4.47040	0.31701
2.00	6.71885	-3.07493	6.71924	-3.07738	6.71872	-3.07488
2.50	7.29088	-9.75993	7.28917	-9.76474	7.29069	-9.75970
3.00	2.83447	-19.88450	2.82694	-19.89126	2.83435	-19.88400
3.50	-11.61630	-31.01120	-11.63527	-31.01559	-11.61600	-31.01010
4.00	-41.32000	-35.68770	-41.35528	-35.67792	-41.31850	-35.68620
4.50	-87.99450	-18.97520	-88.04313	-18.92789	-87.99907	-18.97410

From the results in Tables 1, 2, and 3, it can be observed that the newly derived block methods (9), (13), and (17) perform competitively when compared with the exact solutions of system (69), and when compared and with the Adomian decomposition method.

5.1 CPU time performance of the proposed block methods

To assess the computational efficiency of the derived block hybrid methods, CPU time measurements were performed for Problems 1–3 described in Section 3.12. All codes were implemented in MATLAB R2018a on a system with an Intel Core i7-4900MQ processor, 32 GB RAM, and a 512 GB SSD, with a Newton iteration tolerance of 10^{-6}. The explicit method BHEM (9), applied to Problem 1, required 2.5788 seconds of CPU time, reflecting the higher cost associated with its stability constraints. In contrast, the implicit method BHIM (13) demonstrated remarkable efficiency for the moderately stiff Kaps Problem 2, completing the integration in just 0.1235 s of CPU time, a performance attributable to its A-stability, which permits larger step sizes. The implicit method BHIM (17), applied to the severely stiff linear system (Problem 3), required 2.3805 seconds of CPU time.

For system (69), CPU time measurements were performed for both short-range and long-range integration. The measured CPU times are summarized in Table 4.

Table 4: CPU times (in seconds) of the proposed block methods for (69).

Method	**Interval**	**CPU Time (s)**
BHEM (9)	$[0, 10]$	1.9762
BHIM (13)	$[0, 300]$	0.7544
BHIM (17)	$[0, 300]$	0.4779

The explicit method BHEM (9), which has weaker stability properties, was applied over $[0, 10]$ with $h = 0.001$, requiring 1.9762 s even over the short interval. In contrast, the implicit methods BHIM (13) and BHIM (17) were applied over the extended interval $[0, 300]$ with $h = 0.1$, requiring 0.7544 s and 0.4779 s, respectively. These results reveal a clear distinction in computational efficiency between the explicit and implicit formulations. The explicit method incurs a significantly higher cost due to the small step size necessary to maintain stability, making it suitable only for moderate integration ranges. The implicit methods, by contrast, leverage their superior stability properties—A-stability for BHIM (13) and $A(\alpha)$-stability with $L(\alpha)$-stability for BHIM (17)—to permit substantially larger step sizes, resulting in far lower CPU times despite integrating over a 30-fold longer interval. Notably, BHIM (17) is the most efficient among the proposed methods, completing the long-range integration in about half the time of BHIM (13). Overall, these findings confirm that the implicit block formulations, particularly BHIM (17), offer a compelling balance of accuracy and efficiency for long-time integration of problems with rapid exponential growth.

6 Conclusion

This study has compared the stability polynomial $\pi(r, z)$ and the stability function $L(z)$ for investigating the absolute stability regions of block hybrid methods. The analysis demonstrates that the stability polynomial focuses primarily on the characteristic roots r, while the stability function concentrates on the complex parameter z and the positions of its zeros and poles.

The research question posed in the introduction receives a clear answer: the absence of poles in the left half-plane is necessary but not sufficient for A-stability. Condition (1) of Definition 4 (boundedness on the imaginary axis) must also be satisfied, as illustrated by methods (9) and (17), which have no left-half-plane poles yet are not A-stable. This finding corrects a potential misconception in the literature that analyticity alone guarantees A-stability. The main contributions of the work are as follows: a novel theoretical result stating that for consistent block methods, the two stability tools must yield congruent stability conclusions, with any discrepancy serving as an indicator of potential derivation errors or method inconsistency; three practical rec-

ommendations, (i) when using stability polynomials, plots should be examined near the origin to accurately assess A-stability, (ii) when employing stability functions, both associated conditions must be verified, and (iii) a combined approach using both tools is advisable for comprehensive stability investigation; a demonstration that the stability function is preferable for analyzing stronger stability properties such as L-stability and $L(\alpha)$-stability, owing to its direct rational form that allows immediate asymptotic analysis; and a detailed stability analysis of three new block hybrid methods—one explicit (BHEM) and two implicit (BHIM)—using both stability polynomials and stability functions.

The use of stability polynomials offers a straightforward determination of the angle α for methods that are $A(\alpha)$-stable. A drawback is that an $A(\alpha)$-stable method with a large α may be mistakenly identified as A-stable if the plot is not examined at an appropriate scale near the origin.

For stability functions, when both prescribed conditions are satisfied, they provide a clear means to determine whether a block hybrid method is A-stable. Some researchers have used only one of these conditions to conclude A-stability; however, satisfying only one condition does not guarantee A-stability. Both conditions must be examined. A limitation of stability functions is their reduced effectiveness in determining the angle α for $A(\alpha)$-stable methods, as they focus primarily on the positions of zeros and poles. Nevertheless, stability functions are valuable for investigating stronger stability properties such as L-stability.

In light of these findings, we recommend that stability be examined using both stability polynomials and stability functions to ensure a comprehensive and reliable analysis. The two approaches are complementary: stability polynomials are particularly useful for identifying $A(\alpha)$-stability angles and finite stability intervals, while stability functions provide a direct test for A-stability and are better suited for assessing L-stability properties. For consistent methods, agreement between the two approaches validates the derivation, whereas discrepancies may signal errors or inconsistencies.

Acknowledgements

The contributions of Professor G.M. Kumleng of the Department of Mathematics, University of Jos, to the simulations of block hybrid methods are gratefully acknowledged.

Funding

The authors neither applied for nor received any external funding for this research.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Adedire, O., Sirisena, U.W. and Ndam, J.N., *Implementation of an A-stable block hybrid method for the solution of Bazykin ecological model and Trambouze reactor model*, New Trends Math. Sci., 8(2) (2020), 1–8.

[2] Akinfenwa, O.A., Abdulganiy, R.I., Akinnukawe, B.I. and Okunuga, S.A., *Seventh order hybrid block method for solution of first order stiff systems of initial value problems*, J. Egypt. Math. Soc., 28(34) (2020), 1–11, https://doi.org/10.1186/s42787-020-00095-3.

[3] Akinfenwa, O.A. and Jator, S.N., *Continuous block backward differentiation formula for solving stiff ordinary differential equations*, Comput. Math. Appl., 65 (2013), 996–1005, http://dx.doi.org/10.1016/j.camwa.2012.03.111.

[4] Akinfenwa, O.A., Jator, S.N. and Yao, N.M., *On the stability of continuous block backward differentiation formula for solving stiff ordinary differential equations*, J. Mod. Methods Numer. Methods, 3(2) (2012), 50–58, https://doi.org/10.1186/s42787-020-00095-3.

[5] Akinnukawe, B.I., Akinfenwa, O.A. and Okunuga, S.A., *L-stable block backward differentiation formula for parabolic partial differential equations*, Ain Shams Eng. J., 7 (2016), 867–972, http://dx.doi.org/10.1016/j.asej.2015.12.012.

[6] Celaya, E.A. and Anza, J.J., *BDF-α: A multistep method with numerical damping control*, J. Comput. Math., 1(3) (2013), 96–108, doi: 10.13189/ujcmj.2013.010305.

[7] Chu, M.T. and Hamilton, H., *Implicit parallel solution of ODEs by multi-block methods*, SIAM J. Sci. Stat. Comput., 8(3) (1987), 342–353.

[8] Dahlquist, G., *Convergence and stability in the numerical integration of ordinary differential equations*, Math. Scand., 4 (1956), 33–53, https://doi.org/10.7146/math.scand.a-10454.

[9] Dahlquist, G.G., *A special stability problem for linear multistep methods*, BIT Numer. Math., 3 (1963), 27–43, https://doi.org/10.1007/BF01963532.

[10] Fatunla, S.O., *Numerical methods for initial value problems in ordinary differential equations*, Academic Press, 1978.

[11] Fatunla, S.O., *A class of block methods for second order IVPs*, Int. J. Comput. Math., 55 (1995), 119–133, https://doi.org/10.1080/00207169508804368.

[12] Fatunla, S.O., *Block methods for second order ODEs*, Int. J. Comput. Math., 41 (1991), 55–63.

[13] Hairer, E., Lubich, C. and Wanner, G., *Geometric numerical integration: Structure-preserving algorithm for ordinary differential equations*, Springer Berlin, Heidelberg, 2006, https://doi.org/10.1007/3-540-30666-8.

[14] Hairer, E. and Wanner, G., *Solving ordinary differential equations II: stiff and differential algebraic problems (2nd ed.)*, Springer-verlag, Berlin, 1996, doi: 10.1007/978-3-642-05221-7.

[15] Henrici, P., *Discrete variable methods in ODEs*, John Wiley, 1962.

[16] Ibrahim, Z.B., Othman, K.I. and Suleiman, M., *Implicit r-point block backward differentiation formula for solving first-order stiff ODEs*, Appl. Math. Comput., 186 (2007), 558–565.

[17] Jator, S.N., Sahi, R., Akinyemi, M. and Nyonna, D., *Exponentially fitted block backward differentiation formulas for pricing options*, Cogent Econ. Finance, 9 (2021), 1875565, doi: 10.1080/23322039.2021.1875565.

[18] Ngwane, F.F. and Jator, S.N., *L-Stable block hybrid second derivative algorithm for parabolic partial differential equations*, Am. J. Comput. Math., 4 (2014), 87–92, http://dx.doi.org/10.4236/jasmi.2014.42008.

[19] Okuonghae, R.I. and Ikhile, M.N.O., *A class of hybrid linear multistep methods with $A(\alpha)$-stability properties for stiff IVPs in ODEs*, J. Numer. Math., 21(2) (2013), 157–172, doi: 10.1515/jnum-2013-0006.

[20] Okuonghae, R.I. and Ikhile, M.N.O., *A family of highly stable second derivative block methods for stiff IVPs in ODEs*, Numer. Anal. Appl., 7(1) (2014), 57–69, https://doi.org/10.1134/S1995423914010066.

[21] Okuonghae, R.I. and Ikhile, M.N.O., *Stiffly stable second derivative linear multistep methods with two hybrid points*, Numer. Anal. Appl., 8 (2015), 248–259, https://doi.org/10.1134/S1995423915030052.

[22] Onumanyi, P., Sirisena, U. W. and Jator, S. N., *Continuous finite difference approximations for solving differential equations*, Int. J. Comput. Math., 72(1) (1999), 15–27, doi: 10.1080/00207169908804831.

[23] Oyelami, O.A., Gideon, I.E., Mohammed, U., Akeem, D., Garba, J. and Elebor, C.C., *Optimization of two hybrid points in a one step method*, J. NAMP, 72 (2026), 11–22, https://doi.org/10.60787/jnamp.vol72no.658.

[24] Ramos, H. and Rufai, M., *A two-step hybrid block method with fourth derivatives for solving third-order boundary value problems*, J. Comput. Appl. Math., 404 (2021), 113419, doi: 10.1016/j.cam.2021.113419.

[25] Ramos, H. and Rufai, M.A., *A new one-step method with three intermediate points in a variable step-size mode for stiff differential systems*, J. Math. Chem., 61 (2023), 673–688, https://doi.org/10.1007/s10910-022-01427-7.

[26] Shateyi, S., *On the application of the block hybrid methods to solve linear and nonlinear first order differential equations*, Axioms, 12(2) (2023), 158–189, https://doi.org/10.3390/axioms12020158.

[27] Shawagfeh, N. and Kaya, D., *Comparing numerical methods for the solutions of systems of ordinary differential equations*, Appl. Math. Lett., 17(2) (2004), 323–328.

[28] Tassaddiq, A., Qureshi, S., Soomro, A., Hincal, E. and Shaikh, A.A., *A new continuous hybrid block method with one optimal intrastep point through interpolation and collocation*, Fixed Point Theory Algorithms Sci. Eng., 22 (2022), 1–17, https://doi.org/10.1186/s13663-022-00733-8.

[29] Widlund, O.B., A note on unconditionally stable linear multistep methods, BIT, 7 (1967), 65–70.