



Coupled fixed points for a class of noncyclic contractions in uniformly convex Banach spaces with error estimates

A. Safari-Hafshejani*,  and K. Amiri

Abstract

In this paper, we introduce a noncyclic contraction pair for an ordered pair of mappings. We then deduce the existence and uniqueness of coupled fixed points by associating a noncyclic contraction with them. In addition, we obtain both a priori and a posteriori error estimates for these points in uniformly convex Banach spaces. The main result is illustrated with an example, and as an application, a cyclic version of the main theorem is derived.

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*Corresponding author

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Akram Safari-Hafshejani

Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran.

e-mail: asafari@pnu.ac.ir

Khadije Amiri

Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran.

e-mail: amirikhadije@student.pnu.ac.ir

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1 Introduction and preliminaries

Let $(\mathcal{Z}, \varrho)$ be a complete metric space, and let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty closed subsets of $\mathcal{Z}$. A map $T : \mathcal{K} \cup \bar{\mathcal{K}} \rightarrow \mathcal{K} \cup \bar{\mathcal{K}}$ is termed a cyclic contraction map if it satisfies

- T is cyclic, that is, $T(\mathcal{K}) \subseteq \bar{\mathcal{K}}$ and $T(\bar{\mathcal{K}}) \subseteq \mathcal{K}$;
- there exists a constant $k \in (0, 1)$ such that for all $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$, the following inequality holds:

$$\varrho(Tx, T\bar{x}) \leq k\varrho(x, \bar{x}) + (1 - k)\varrho(\mathcal{K}, \bar{\mathcal{K}});$$

and is termed a noncyclic contraction map if it satisfies

- T is noncyclic, that is, $T(\mathcal{K}) \subseteq \mathcal{K}$ and $T(\bar{\mathcal{K}}) \subseteq \bar{\mathcal{K}}$;
- there exists a constant $k \in (0, 1)$ such that for all $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$, the following inequality holds:

$$\varrho(Tx, T\bar{x}) \leq k\varrho(x, \bar{x}) + (1 - k)\varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

where $\varrho(\mathcal{K}, \bar{\mathcal{K}}) := \inf\{\varrho(x, \bar{x}) : x \in \mathcal{K}, \bar{x} \in \bar{\mathcal{K}}\}$. A point $x^* \in \mathcal{K} \cup \bar{\mathcal{K}}$ is considered a best proximity point for a cyclic map T if $\varrho(x^*, Tx^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$ and is a fixed point for a cyclic or noncyclic map T if $Tx^* = x^*$ (see [9, 1] and their references).

The main theorems of [6, 7, 13] prove the existence of a fixed point and a best proximity point for two types of cyclic mappings. Additionally, they establish error estimates for the Picard iteration concerning these points.

Some generalizations of the concepts of a best proximity point and a fixed point are presented through the definitions in [11, 8, 3].

Definition 1 (see [11]). Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$. An ordered pair $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ is called a coupled best proximity point of Γ if

$$\varrho(x^*, \Gamma(x^*, x^{**})) = \varrho(\mathcal{K}, \bar{\mathcal{K}}) \quad \text{and} \quad \varrho(x^{**}, \Gamma(x^{**}, x^*)) = \varrho(\mathcal{K}, \bar{\mathcal{K}}).$$

Definition 2 (see [3]). Let $\mathcal{K}$ be nonempty subset of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$. An ordered pair $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ is called a coupled fixed point of Γ if $\Gamma(x^*, x^{**}) = x^*$ and $\Gamma(x^{**}, x^*) = x^{**}$.

Definition 3 (see [11]). Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K}$. For any initial values $x_0, y_0 \in \mathcal{K}$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{aligned} x_{2n+1} &:= \Gamma(x_{2n}, y_{2n}) \in \bar{\mathcal{K}}, & x_{2n+2} &:= \bar{\Gamma}(x_{2n+1}, y_{2n+1}) \in \mathcal{K}, \\ y_{2n+1} &:= \Gamma(y_{2n}, x_{2n}) \in \bar{\mathcal{K}}, & y_{2n+2} &:= \bar{\Gamma}(y_{2n+1}, x_{2n+1}) \in \mathcal{K}, \end{aligned}$$

for every $n \geq 0$. Also, we define $\Gamma' : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$ and $\bar{\Gamma}' : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K}$ by

$$\begin{aligned} \Gamma'(x, y) &= \Gamma(y, x), \text{ for any pairs } (x, y) \in \mathcal{K} \times \mathcal{K}, \\ \bar{\Gamma}'(x, y) &= \bar{\Gamma}(y, x), \text{ for any pairs } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}. \end{aligned}$$

Remark 1. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K}$. We define mappings $(\Gamma, \Gamma') : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ and $(\bar{\Gamma}, \bar{\Gamma}') : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K} \times \mathcal{K}$ by

$$(\Gamma, \Gamma')(x, y) = (\Gamma(x, y), \Gamma'(x, y)) \quad \text{for } (x, y) \in \mathcal{K} \times \mathcal{K}$$

and

$$(\bar{\Gamma}, \bar{\Gamma}')(x, y) = (\bar{\Gamma}(x, y), \bar{\Gamma}'(x, y)) \quad \text{for } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}.$$

Then for the sequences $\{x_n\}$ and $\{y_n\}$ defined in Definition 3, we have

$$\begin{aligned} (x_{2n+1}, y_{2n+1}) &= (\Gamma, \Gamma')(x_{2n}, y_{2n}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \\ (x_{2n+2}, y_{2n+2}) &= (\bar{\Gamma}, \bar{\Gamma}')(x_{2n+1}, y_{2n+1}) \in \mathcal{K} \times \mathcal{K}, \end{aligned}$$

for every $n \geq 0$. So

$$\begin{aligned} (x_1, y_1) &= (\Gamma, \Gamma')(x_0, y_0), \\ (x_2, y_2) &= (\bar{\Gamma}, \bar{\Gamma}')(x_1, y_1) = (\bar{\Gamma}, \bar{\Gamma}')((\Gamma, \Gamma')(x_0, y_0)) = ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))(x_0, y_0), \\ (x_3, y_3) &= (\Gamma, \Gamma')(x_2, y_2) = (\Gamma, \Gamma')((\bar{\Gamma}, \bar{\Gamma}')(x_1, y_1)) = ((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))(x_1, y_1), \\ (x_4, y_4) &= (\bar{\Gamma}, \bar{\Gamma}')(x_3, y_3) = (\bar{\Gamma}, \bar{\Gamma}')((\Gamma, \Gamma')(x_2, y_2)) = ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^2(x_0, y_0), \\ &\vdots \\ (x_{2n}, y_{2n}) &= ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n(x_0, y_0), \\ (x_{2n+1}, y_{2n+1}) &= ((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1), \end{aligned}$$

for every $n \in \mathbb{N} \cup \{0\}$.

We now refer to the definition of a cyclic contraction of an ordered pair defined by Sintunavarat and Kumam [11].

Definition 4 (see [11]). Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K}$. The ordered pair $(\Gamma, \bar{\Gamma})$ is said to be a cyclic contraction pair if there exist nonnegative numbers λ, μ such that $\lambda + \mu < 1$, and the following inequality holds:

$$\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) \leq \lambda \varrho(x, \bar{x}) + \mu \varrho(y, \bar{y}) + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

for all $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$.

Definition 5 (see [12]). Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. We say that the pair $(\mathcal{K}, \bar{\mathcal{K}})$ satisfies the *UC* property if the following condition holds: whenever sequences $\{x_n\}$ and $\{x'_n\}$ in $\mathcal{K}$ and a sequence $\{\bar{x}_n\}$ in $\bar{\mathcal{K}}$ satisfy

$$\lim_{n \rightarrow \infty} \varrho(x_n, \bar{x}_n) = \lim_{n \rightarrow \infty} \varrho(x'_n, \bar{x}_n) = \varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

then it follows that $\lim_{n \rightarrow \infty} \varrho(x_n, x'_n) = 0$.

Suzuki, Kikkawa, and Vetro [12] demonstrated that if $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, convex subsets of a uniformly convex Banach space $\mathcal{Z}$, then the pair $(\mathcal{K}, \bar{\mathcal{K}})$ possesses the *UC* property. They also established that if $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty subsets of a metric space $(\mathcal{Z}, \varrho)$ such that $\varrho(\mathcal{K}, \bar{\mathcal{K}}) = 0$, then the pair $(\mathcal{K}, \bar{\mathcal{K}})$ satisfies the *UC* property.

To obtain an error estimate for the sequence of successive iterations approximating the best proximity point, generated by a cyclic contraction, the modulus of convexity $\delta_{\mathcal{Z}}$ is used [13].

Definition 6 (see [2]). A Banach space $\mathcal{Z}$ is said to be uniformly convex if there exists a strictly increasing function $\delta_{\mathcal{Z}} : [0, 2] \rightarrow [0, 1]$ such that the following implication holds for all $x, x', \bar{x} \in \mathcal{Z}$, $\mathcal{R} > 0$ and $r \in [0, 2\mathcal{R}]$:

$$\left. \begin{array}{l} \|x - \bar{x}\| \leq \mathcal{R} \\ \|x' - \bar{x}\| \leq \mathcal{R} \\ \|x - x'\| \geq r \end{array} \right\} \Rightarrow \left\| \frac{x + x'}{2} - \bar{x} \right\| \leq \left(1 - \delta_{\mathcal{Z}}\left(\frac{r}{\mathcal{R}}\right)\right) \mathcal{R}. \quad (1)$$

In this paper, we demonstrate that a noncyclic contraction pair $(\Gamma, \bar{\Gamma})$ can be formulated as a noncyclic contraction within the space $\mathcal{Z}^2 = \mathcal{Z} \times \mathcal{Z}$ through the adoption of a suitable metric. This formulation facilitates the derivation of our main results concerning the existence and uniqueness of coupled fixed points for noncyclic contraction pairs, as well as the estimation of a priori and a posteriori error bounds.

2 Auxiliary results based on property UC

For a complete metric space $(\mathcal{Z}, \varrho)$, we define $\mathcal{Z}^2 = \mathcal{Z} \times \mathcal{Z}$ and $\varrho^* : \mathcal{Z}^2 \times \mathcal{Z}^2 \rightarrow \mathbb{R}^+$ such that $\varrho^*((x, y), (\bar{x}, \bar{y})) = \max\{\varrho(x, \bar{x}), \varrho(y, \bar{y})\}$. It can easily be seen that the completeness of the metric space $(\mathcal{Z}^2, \varrho^*)$ follows from the completeness of $(\mathcal{Z}, \varrho)$. On the normed space $(\mathcal{Z}, \|\cdot\|)$, we use $\|\cdot\|^*$ to denote the norm on $\mathcal{Z}^2$, defined as $\|(x, y)\|^* = \max\{\|x\|, \|y\|\}$, so we have $\|(x, y) - (\bar{x}, \bar{y})\|^* = \max\{\|x - \bar{x}\|, \|y - \bar{y}\|\}$.

Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of a metric space $(\mathcal{Z}, \varrho)$. Then

$$\begin{aligned}
 & \varrho^*(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}}) \\
 &= \inf\{\varrho^*((x, y), (\bar{x}, \bar{y})) : (x, y) \in \mathcal{K} \times \mathcal{K}, (\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}\} \\
 &\leq \inf\{\varrho^*((x, x), (\bar{x}, \bar{x})) : x \in \mathcal{K}, \bar{x} \in \bar{\mathcal{K}}\} \\
 &= \inf\{\varrho(x, \bar{x}) : (x, \bar{x}) \in \mathcal{K} \times \bar{\mathcal{K}}\} \\
 &= \varrho(\mathcal{K}, \bar{\mathcal{K}}) \\
 &= \inf\{\varrho(x, \bar{x}) : (x, \bar{x}) \in \mathcal{K} \times \bar{\mathcal{K}}\} \\
 &\leq \inf\{\max\{\varrho(x, \bar{x}), \varrho(y, \bar{y})\} : (x, \bar{x}) \in \mathcal{K} \times \bar{\mathcal{K}}, (y, \bar{y}) \in \mathcal{K} \times \bar{\mathcal{K}}\} \\
 &= \inf\{\varrho^*((x, y), (\bar{x}, \bar{y})) : (x, y) \in \mathcal{K} \times \mathcal{K}, (\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}\} \\
 &= \varrho^*(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}}).
 \end{aligned}$$

So we obtain

$$\varrho^*(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}}) = \varrho(\mathcal{K}, \bar{\mathcal{K}}). \quad (2)$$

Definition 7. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of a Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ has the UC property and $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. If there exist $C > 0$ and $p > 0$ such that the following implication holds for all $x, x' \in \mathcal{K}$, $\bar{x} \in \bar{\mathcal{K}}$ and $\mathcal{R} \geq \varrho$:

$$\left. \begin{aligned} & \|x - \bar{x}\| \leq \mathcal{R}, \\ & \|x' - \bar{x}\| \leq \mathcal{R}, \end{aligned} \right\} \Rightarrow \|x - x'\| \leq \mathcal{R}^p \sqrt[p]{\frac{\mathcal{R} - \varrho}{\mathcal{R}C}}, \quad (3)$$

then, we say that $(\mathcal{K}, \bar{\mathcal{K}})$ has property UC with convexity approximation coefficients C and p .

Note that in the previous definition, since $(\mathcal{K}, \bar{\mathcal{K}})$ has the UC property, relation (3) clearly holds when $\mathcal{R} = \varrho$. Therefore, to prove that the definition is satisfied, it is enough to show that relation (3) holds for $\mathcal{R} > \varrho$.

Moreover, [10, Theorem 2] can be stated as follows for sets $\mathcal{K}$ and $\bar{\mathcal{K}}$ satisfying the property UC with convexity approximation coefficients C and p , using the same proof as provided by the original authors.

Theorem 1. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed subsets of a Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ and $(\bar{\mathcal{K}}, \mathcal{K})$ have the UC property and $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $T : \mathcal{K} \cup \bar{\mathcal{K}} \rightarrow \mathcal{K} \cup \bar{\mathcal{K}}$ be a noncyclic contraction map; that is,

$$\varrho(Tx, T\bar{x}) \leq k\varrho(x, \bar{x}) + (1 - k)\varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

for all $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$ and some $k \in [0, 1)$. Then

- (i) T has a unique optimal pair of fixed points $(x^*, \bar{x}^*) \in \mathcal{K} \times \bar{\mathcal{K}}$;
- (ii) for every $x_0 \in \mathcal{K}$ and $\bar{x}_0 \in \bar{\mathcal{K}}$ the sequences $\{x_n\} = \{T^n x_0\}$ and $\{\bar{x}_n\} = \{T^n \bar{x}_0\}$ converge to x^* and $\bar{x}^*$, respectively.

Furthermore, if there exist $C > 0$ and $p > 0$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ has the UC property with convexity approximation coefficients C and p , then

- (iii) the following priori error estimate holds

$$\|x^* - T^m x_0\| \leq \frac{\Delta_{x_0, \bar{x}_0}}{1 - \sqrt[p]{k}} \sqrt[p]{\frac{\Delta_{x_0, \bar{x}_0} - \varrho}{C\varrho}} (\sqrt[p]{k})^m;$$

- (iv) the following posteriori error estimate holds

$$\|T^n x_0 - x^*\| \leq \frac{\Delta_{x_n, \bar{x}_n}}{1 - \sqrt[p]{k}} \sqrt[p]{\frac{\Delta_{x_n, \bar{x}_n} - \varrho}{C\varrho}};$$

where $\Delta_{x, \bar{x}} := \max \{\|x - \bar{x}\|, \|Tx - \bar{x}\|\}$, for every $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$.

The following lemma is necessary to obtain the coupled version of Theorem 1 in the next section.

Lemma 1. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. If $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ for some $C > 0$, $p > 0$ and every $\epsilon \in (0, 2]$, then the pair $(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}})$ as subsets of the Banach space $(\mathcal{Z}^2, \|\cdot\|^*)$, has property UC with convexity approximation coefficients C and p .

Proof. It is easily proven that the pair $(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}})$ as subsets of the Banach space $(\mathcal{Z}^2, \|\cdot\|)$, has property *UC*. Since $(\mathcal{Z}, \|\cdot\|)$ is a uniformly convex Banach space, then there exists a strictly increasing function $\delta_{\mathcal{Z}} : [0, 2] \rightarrow [0, 1]$ such that the implication (1) holds for all $x, x', \bar{x} \in \mathcal{Z}$, $\mathcal{R} > 0$ and $r \in [0, 2\mathcal{R}]$. Let $(x, y), (x', y') \in \mathcal{K} \times \mathcal{K}$, $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, $\mathcal{R} > \varrho$ and

$$\begin{cases} \|(x, y) - (\bar{x}, \bar{y})\|^* \leq \mathcal{R}, \\ \|(x', y') - (\bar{x}, \bar{y})\|^* \leq \mathcal{R}. \end{cases}$$

So we obtain

$$\begin{cases} \|x - \bar{x}\| \leq \mathcal{R}, \\ \|x' - \bar{x}\| \leq \mathcal{R}, \end{cases} \quad \text{and} \quad \begin{cases} \|y - \bar{y}\| \leq \mathcal{R}, \\ \|y' - \bar{y}\| \leq \mathcal{R}. \end{cases}$$

Since $(\mathcal{Z}, \|\cdot\|)$ is a uniformly convex Banach space, then

$$\varrho \leq \left\| \frac{x + x'}{2} - \bar{x} \right\| \leq (1 - \delta_{\mathcal{Z}}(\frac{\|x - x'\|}{\mathcal{R}}))\mathcal{R}$$

and

$$\varrho \leq \left\| \frac{y + y'}{2} - \bar{y} \right\| \leq (1 - \delta_{\mathcal{Z}}(\frac{\|y - y'\|}{\mathcal{R}}))\mathcal{R}.$$

It follows from the inequality $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ that $\delta_{\mathcal{Z}}^{-1}(\epsilon) \leq (\frac{\epsilon}{C})^{\frac{1}{p}}$ for every $\epsilon \in (0, 2]$. Since $0 < \frac{\mathcal{R} - \varrho}{\mathcal{R}} < 1$, then we get

$$\|x - x'\| \leq \delta_{\mathcal{Z}}^{-1}(\frac{\mathcal{R} - \varrho}{\mathcal{R}})\mathcal{R} \leq \mathcal{R} \sqrt[p]{\frac{\mathcal{R} - \varrho}{\mathcal{R}C}}$$

and

$$\|y - y'\| \leq \delta_{\mathcal{Z}}^{-1}(\frac{\mathcal{R} - \varrho}{\mathcal{R}})\mathcal{R} \leq \mathcal{R} \sqrt[p]{\frac{\mathcal{R} - \varrho}{\mathcal{R}C}}.$$

Therefore

$$\|(x, y) - (x', y')\|^* = \max\{\|x - x'\|, \|y - y'\|\} \leq \mathcal{R} \sqrt[p]{\frac{\mathcal{R} - \varrho}{\mathcal{R}C}}.$$

□

3 Coupled fixed points for noncyclic contraction pair

We define a noncyclic contraction for an ordered pair of mappings as follows.

Definition 8. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \bar{\mathcal{K}}$. The ordered pair $(\Gamma, \bar{\Gamma})$ is said to be a noncyclic contraction pair if there exist nonnegative numbers λ, μ such that $\lambda + \mu < 1$, and the following inequality holds:

$$\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) \leq \lambda \varrho(x, \bar{x}) + \mu \varrho(y, \bar{y}) + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

for all $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$.

For any initial values $x_0, y_0 \in \mathcal{K}$ and $\bar{x}_0, \bar{y}_0 \in \bar{\mathcal{K}}$, let the sequences $\{x_n\}$, $\{y_n\}$, $\{\bar{x}_n\}$, and $\{\bar{y}_n\}$ be defined by

$$\begin{aligned} x_{n+1} &:= \Gamma(x_n, y_n) \in \mathcal{K}, & \bar{x}_{n+1} &:= \bar{\Gamma}(\bar{x}_n, \bar{y}_n) \in \bar{\mathcal{K}}, \\ y_{n+1} &:= \Gamma(y_n, x_n) \in \mathcal{K}, & \bar{y}_{n+1} &:= \bar{\Gamma}(\bar{y}_n, \bar{x}_n) \in \bar{\mathcal{K}}, \end{aligned}$$

for every $n \geq 0$. From Remark 1,

$$\begin{aligned} (x_{n+1}, y_{n+1}) &= (\Gamma, \Gamma')(x_n, y_n) \in \mathcal{K} \times \mathcal{K}, \\ (\bar{x}_{n+1}, \bar{y}_{n+1}) &= (\bar{\Gamma}, \bar{\Gamma}')(\bar{x}_n, \bar{y}_n) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \end{aligned}$$

and

$$\begin{aligned} \{(x_n, y_n)\} &= \{(\Gamma, \Gamma')^n(x_0, y_0)\}, \\ \{(\bar{x}_n, \bar{y}_n)\} &= \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}, \end{aligned}$$

for every $n \geq 0$.

We continue this section by presenting two practical lemmas that will be useful in subsequent analyses.

Lemma 2. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty closed subsets of a complete metric space $(\mathcal{Z}, \varrho)$ and let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \bar{\mathcal{K}}$. Suppose that the ordered pair $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair. Then the noncyclic mapping $T : (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}}) \rightarrow (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}})$, defined by

$$T(x, y) = \begin{cases} (\Gamma, \Gamma')(x, y) & \text{if } (x, y) \in \mathcal{K} \times \mathcal{K}, \\ (\bar{\Gamma}, \bar{\Gamma}')(x, y) & \text{if } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \end{cases}$$

is a noncyclic contraction in the complete metric spaces $(\mathcal{Z}^2, \varrho^*)$ with constant $\lambda + \mu$.

Proof. Consider the fixed but arbitrary points $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. The ordered pair $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair, so we have

$$\begin{aligned} \varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) &\leq \lambda \varrho(x, \bar{x}) + \mu \varrho(y, \bar{y}) + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &\leq \lambda \max\{\varrho(x, \bar{x}), \varrho(y, \bar{y})\} + \mu \max\{\varrho(x, \bar{x}), \varrho(y, \bar{y})\} \\ &\quad + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &= \lambda \varrho^*((x, y), (\bar{x}, \bar{y})) + \mu \varrho^*((x, y), (\bar{x}, \bar{y})) \\ &\quad + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &= (\lambda + \mu) \varrho^*((x, y), (\bar{x}, \bar{y})) + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}). \end{aligned}$$

From the previous relation, for $(y, x) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{y}, \bar{x}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, we have

$$\begin{aligned} \varrho(\Gamma(y, x), \bar{\Gamma}(\bar{y}, \bar{x})) &\leq (\lambda + \mu) \varrho^*((y, x), (\bar{y}, \bar{x})) + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &= (\lambda + \mu) \varrho^*((x, y), (\bar{x}, \bar{y})) + (1 - (\lambda + \mu)) \varrho(\mathcal{K}, \bar{\mathcal{K}}). \end{aligned}$$

Based on the symbols introduced in Remark 1, we establish the following relationship:

$$\begin{aligned} \varrho^*((\Gamma, \Gamma')(x, y), (\bar{\Gamma}, \bar{\Gamma}')(\bar{x}, \bar{y})) \\ = \max\{\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})), \varrho(\Gamma(y, x), \bar{\Gamma}(\bar{y}, \bar{x}))\}. \end{aligned}$$

From (2) we have $\varrho^*(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}}) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$, combining these relations yield the inequality:

$$\begin{aligned} \varrho^*(T(x, y), T(\bar{x}, \bar{y})) &= \varrho^*((\Gamma, \Gamma')(x, y), (\bar{\Gamma}, \bar{\Gamma}')(\bar{x}, \bar{y})) \\ &\leq (\lambda + \mu) \varrho^*((x, y), (\bar{x}, \bar{y})) \\ &\quad + (1 - (\lambda + \mu)) \varrho^*(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}}). \end{aligned}$$

This demonstrates that T is a noncyclic contraction in the complete metric space $(\mathcal{Z}^2, \varrho^*)$. $\square$

Lemma 3. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty subsets of a metric space $(\mathcal{Z}, \varrho)$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ and $(\bar{\mathcal{K}}, \mathcal{K})$ have the UC property and $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \bar{\mathcal{K}}$ be mappings such that the ordered pair $(\Gamma, \bar{\Gamma})$ constitutes a noncyclic contraction pair. If Γ possesses a coupled fixed point $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ possesses a coupled fixed point $(\bar{x}^*, \bar{x}^{**}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, then it follows that $x^* = x^{**}$ and $\bar{x}^* = \bar{x}^{**}$ and $\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$.

Proof. Since Γ possesses a coupled fixed point $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ possesses a coupled fixed point $(\bar{x}^*, \bar{x}^{**}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, then

$$\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**}) = \varrho(\Gamma(x^*, x^{**}), \bar{\Gamma}(\bar{x}^*, \bar{x}^{**})) + \varrho(\Gamma(x^{**}, x^*), \bar{\Gamma}(\bar{x}^{**}, \bar{x}^*))$$

and

$$\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*) = \varrho(\Gamma(x^*, x^{**}), \bar{\Gamma}(\bar{x}^{**}, \bar{x}^*)) + \varrho(\Gamma(x^{**}, x^*), \bar{\Gamma}(\bar{x}^*, \bar{x}^{**})).$$

Given that $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair, we obtain the following inequalities from the above relations:

$$\begin{aligned} & \varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**}) \\ & \leq \lambda \varrho(x^*, \bar{x}^*) + \mu \varrho(x^{**}, \bar{x}^{**}) + (1 - (\lambda + \mu)) \varrho \\ & \quad + \lambda \varrho(x^{**}, \bar{x}^{**}) + \mu \varrho(x^*, \bar{x}^*) + (1 - (\lambda + \mu)) \varrho \\ & = (\lambda + \mu)(\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**})) + 2(1 - (\lambda + \mu)) \varrho \end{aligned}$$

and

$$\begin{aligned} & \varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*) \\ & \leq \lambda \varrho(x^*, \bar{x}^{**}) + \mu \varrho(x^{**}, \bar{x}^*) + (1 - (\lambda + \mu)) \varrho \\ & \quad + \lambda \varrho(x^{**}, \bar{x}^*) + \mu \varrho(x^*, \bar{x}^{**}) + (1 - (\lambda + \mu)) \varrho \\ & = (\lambda + \mu)(\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*)) + 2(1 - (\lambda + \mu)) \varrho. \end{aligned}$$

These imply that

$$(1 - (\lambda + \mu))(\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**})) \leq 2(1 - (\lambda + \mu)) \varrho$$

and

$$(1 - (\lambda + \mu))(\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*)) \leq 2(1 - (\lambda + \mu)) \varrho.$$

Hence

$$\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**}) \leq 2\varrho = 2\varrho(\mathcal{K}, \bar{\mathcal{K}})$$

and

$$\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*) \leq 2\varrho = 2\varrho(\mathcal{K}, \bar{\mathcal{K}}).$$

Since $\varrho(x^*, \bar{x}^*) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, $\varrho(x^{**}, \bar{x}^{**}) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, $\varrho(x^*, \bar{x}^{**}) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, and $\varrho(x^{**}, \bar{x}^*) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, then

$$\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}}) \quad (4)$$

and

$$\varrho(x^*, \bar{x}^{**}) = \varrho(x^{**}, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}}). \quad (5)$$

Since $(\mathcal{K}, \bar{\mathcal{K}})$ and $(\bar{\mathcal{K}}, \mathcal{K})$ satisfy the UC property, then from (4) and (5), we obtain $\bar{x}^* = \bar{x}^{**}$ and $x^* = x^{**}$, respectively. $\square$

We now have the necessary tools to present our main result.

Theorem 2. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{Z}, \|\cdot\|)$ with $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$, $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \bar{\mathcal{K}}$ such that the ordered pair $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair. Then

- (i) Γ has a unique coupled fixed point $(x^*, x^*) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^*) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. Moreover, $\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$;
- (ii) for every $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, the sequences $\{(x_n, y_n)\} = \{(\Gamma, \Gamma')^n(x_0, y_0)\}$ and $\{(\bar{x}_n, \bar{y}_n)\} = \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}$ converge to (x^*, x^*) and $(\bar{x}^*, \bar{x}^*)$, respectively.

Furthermore, if $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ for some $C > 0$, $p > 0$ and every $\epsilon \in (0, 2]$, then for every $n \geq 0$ we have

- (iii) the following priori error estimate holds

$$\max\{\|x^* - x_n\|, \|x^* - y_n\|\} \leq \frac{\Delta_0}{1 - \sqrt[p]{\lambda + \mu}} \sqrt[p]{\frac{\Delta_0 - \varrho}{C\varrho}} (\sqrt[p]{\lambda + \mu})^n;$$

- (iv) the following posteriori error estimate holds

$$\max\{\|x_n - x^*\|, \|y_n - x^*\|\} \leq \frac{\Delta_n}{1 - \sqrt[p]{\lambda + \mu}} \sqrt[p]{\frac{\Delta_n - \varrho}{C\varrho}},$$

where

$$\Delta_n = \max \{ \|(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*, \|(\Gamma, \Gamma')(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^* \},$$

or, equality

$$\Delta_n = \max \{ \|x_n - \bar{x}_n\|, \|y_n - \bar{y}_n\|, \|x_{n+1} - \bar{x}_n\|, \|y_{n+1} - \bar{y}_n\| \}.$$

Proof. Note that $\mathcal{K} \times \mathcal{K}$ and $\bar{\mathcal{K}} \times \bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of the Banach space $(\mathcal{Z}^2, \|\cdot\|^*)$ and that $T : (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}}) \rightarrow (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}})$ defined by

$$T(x, y) = \begin{cases} (\Gamma, \Gamma')(x, y) & \text{if } (x, y) \in \mathcal{K} \times \mathcal{K}, \\ (\bar{\Gamma}, \bar{\Gamma}')(x, y) & \text{if } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \end{cases}$$

from Lemma 2, is a noncyclic contraction with constant $\lambda + \mu$.

(i) From Theorem 1(i), there exists a unique point $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$, which is a fixed point for the map T that is $(\Gamma, \Gamma')(x^*, x^{**}) = (x^*, x^{**})$. This implies that

$$\Gamma(x^*, x^{**}) = x^* \quad \text{and} \quad \Gamma(x^{**}, x^*) = x^{**}.$$

Conversely, since every coupled fixed point of Γ is a fixed point of T , it follows that (x^*, x^{**}) is the unique coupled fixed point of Γ in $\mathcal{K} \times \mathcal{K}$. Similarly, we can show that $\bar{\Gamma}$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^{**}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. Lemma 3 requires that $\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$, $x^* = x^{**}$ and $\bar{x}^* = \bar{x}^{**}$.

(ii) From Theorem 1(ii), for every $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ the sequences $\{(x_n, y_n)\} = \{T^n(x_0, y_0)\} = \{(\Gamma, \Gamma')^n(x_0, y_0)\}$ and $\{(\bar{x}_n, \bar{y}_n)\} = \{T^n(\bar{x}_0, \bar{y}_0)\} = \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}$, converge to (x^*, x^*) and $(\bar{x}^*, \bar{x}^*)$, respectively.

From Theorem 1, it follows that for every $n \geq 0$, we have

$$\begin{aligned} & \Delta_{(x_n, y_n), (\bar{x}_n, \bar{y}_n)} \\ &= \max \{ \|(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*, \|T(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^* \} \\ &= \max \{ \|(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*, \|(\Gamma, \Gamma')(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^* \} \\ &= \max \{ \|x_n - \bar{x}_n\|, \|y_n - \bar{y}_n\|, \|\Gamma(x_n, y_n) - \bar{x}_n\|, \|\Gamma(y_n, x_n) - \bar{y}_n\| \} \\ &= \max \{ \|x_n - \bar{x}_n\|, \|y_n - \bar{y}_n\|, \|x_{n+1} - \bar{x}_n\|, \|y_{n+1} - \bar{y}_n\| \} = \Delta_n. \end{aligned}$$

In particular

$$\Delta_{(x_0, y_0), (\bar{x}_0, \bar{y}_0)} = \max \{ \|x_0 - \bar{x}_0\|, \|y_0 - \bar{y}_0\|, \|x_1 - \bar{x}_0\|, \|y_1 - \bar{y}_0\| \} = \Delta_0.$$

On the other hand,

$$\|(x^*, x^*) - T^n(x_0, y_0)\|^* = \max\{\|x^* - x_n\|, \|x^* - y_n\|\}.$$

Moreover, T is a noncyclic contraction with constant $\lambda + \mu$. These observations, together with Theorem 1(iii)-(iv), complete the proof of parts (iii) and (iv). $\square$

We illustrate Theorem 2 with an example.

Example 1. Consider the real number space $\mathbb{R}$, equipped with the standard absolute value norm, $|\cdot|$. Then, we have $\delta_{\mathbb{R}}(\epsilon) = \frac{\epsilon}{2}$ (see [4]). So $C = \frac{1}{2}$ and $p = 1$.

Define the sets $\mathcal{K}$ to be the closed interval $[1, 2]$ and $\bar{\mathcal{K}}$ as the closed interval $[-2, -1]$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \bar{\mathcal{K}}$ be defined as follows:

$$\Gamma(x, y) := \frac{x + 2y + 5}{8} \quad \text{and} \quad \bar{\Gamma}(\bar{x}, \bar{y}) := \frac{\bar{x} + 2\bar{y} - 5}{8}.$$

It is easy to observe that $x = x^*, y = x^{**}, \bar{x} = \bar{x}^*, \bar{y} = \bar{x}^{**}$ in $\mathcal{K} \cup \bar{\mathcal{K}}$ constitute a solution of the system

$$\begin{cases} 7x - 2y = 5, \\ 7y - 2x = 5, \\ 7\bar{x} - 2\bar{y} = -5, \\ 7\bar{y} - 2\bar{x} = -5, \end{cases}$$

if and only if

$$\Gamma(x^*, x^{**}) = x^*, \Gamma(x^{**}, x^*) = x^{**}, \bar{\Gamma}(\bar{x}^*, \bar{x}^{**}) = \bar{x}^* \quad \text{and} \quad \bar{\Gamma}(\bar{x}^{**}, \bar{x}^*) = \bar{x}^{**}.$$

Note that

$$\begin{aligned} \varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) &= \left| \frac{x + 2y + 5}{8} - \frac{\bar{x} + 2\bar{y} - 5}{8} \right| \\ &= \left| \frac{1}{8}(x - \bar{x}) + \frac{2}{8}(y - \bar{y}) + \frac{5}{8} \times 2 \right| \\ &\leq \frac{1}{8}|x - \bar{x}| + \frac{2}{8}|y - \bar{y}| + \left(1 - \frac{3}{8}\right) \times \varrho(\mathcal{K}, \bar{\mathcal{K}}). \end{aligned}$$

Hence $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair. From Theorem 2(i), Γ has a unique coupled fixed point $(x^*, x^*) = (1, 1) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^*) = (-1, -1) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. For all $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ the sequences $\{(x_n, y_n)\} = \{(\Gamma, \Gamma')^n(x_0, y_0)\}$ and $\{(\bar{x}_n, \bar{y}_n)\} = \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}$ converge to $(1, 1)$ and $(-1, -1)$, respectively. See Table 1.

Table 1: Convergence of the sequences in Example 1

	x_n	y_n	$\bar{x}_n$	$\bar{y}_n$
$n = 0$	1.35	1.45	-1.8	-1.35
$n = 1$	1.15625	1.14375	-1.1875	-1.24375
$n = 2$	1.05546875	1.05703125	-1.084375	-1.07734375
$n = 3$	1.0211914063	1.0209960938	-1.0298828125	-1.0307617188
$n = 4$	1.0078979492	1.0079223633	-1.0114257813	-1.0113159180
$n = 5$	1.0029678345	1.0029647827	-1.0042572021	-1.0042709351
$n = 6$	1.0011121750	1.0011125565	-1.0015998840	-1.0015981674
$n = 7$	1.0004171610	1.0004171133	-1.0005995274	-1.0005997419
$n = 8$	1.0001564234	1.0001564294	-1.0002248764	-1.0002248496
$n = 9$	1.0000586603	1.0000586595	-1.0000843219	-1.0000843253
$n = 10$	1.0000219974	1.0000219975	-1.0000316216	-1.0000316211

A priori error estimate holds

$$\begin{aligned} \max\{\|x^* - x_n\|, \|x^* - y_n\|\} &\leq \frac{\Delta_0}{1 - \sqrt[p]{\lambda + \mu}} \sqrt[p]{\frac{\Delta_0 - \varrho}{C\varrho}} (\sqrt[p]{\lambda + \mu})^n \\ &= 5.796 \times \left(\frac{3}{8}\right)^n. \end{aligned}$$

Therefore after ten iterations, the derived error bound guarantees that the priori error estimate is less than 0.0004.

A posteriori error estimate holds

$$\max\{\|x_n - x^*\|, \|y_n - x^*\|\} \leq \frac{\Delta_n}{1 - \sqrt[p]{\lambda + \mu}} \sqrt[p]{\frac{\Delta_n - \varrho}{C\varrho}} = \frac{8}{5} \Delta_n (\Delta_n - 2).$$

Therefore after ten iterations, the derived error bound guarantees that the posteriori error estimate is less than 0.0002.

As a consequence of Theorem 2, we obtain its cyclic version. The statements of this theorem are similar to [5, Theorem 3], but with new error bounds. The subsequent lemma will help us in the proof.

Lemma 4. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$, $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K}$ such that the ordered pair $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair with nonnegative constants λ and μ in which $\lambda + \mu < 1$. Then

$$(i) \quad (\bar{\Gamma} \circ (\Gamma, \Gamma'))' = \bar{\Gamma}' \circ (\Gamma, \Gamma') \text{ and } (\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))' = \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}');$$

(ii) $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$.

Proof. From the definitions of Γ and $\bar{\Gamma}$, the composition $\bar{\Gamma} \circ (\Gamma, \Gamma')$ defines a mapping from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$, while $\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}')$ defines a mapping from $\bar{\mathcal{K}} \times \bar{\mathcal{K}}$ to $\bar{\mathcal{K}}$.

(i) For all $(x, y) \in \mathcal{K} \times \mathcal{K}$, we have

$$\begin{aligned} (\bar{\Gamma} \circ (\Gamma, \Gamma'))'(x, y) &= (\bar{\Gamma} \circ (\Gamma, \Gamma'))(y, x) \\ &= \bar{\Gamma}(\Gamma(y, x), \Gamma(x, y)) \\ &= \bar{\Gamma}'(\Gamma(x, y), \Gamma'(x, y)) \\ &= (\bar{\Gamma}' \circ (\Gamma, \Gamma'))(x, y). \end{aligned}$$

So $(\bar{\Gamma} \circ (\Gamma, \Gamma'))' = \bar{\Gamma}' \circ (\Gamma, \Gamma')$. Similarly we can show that $(\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))' = \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}')$.

(ii) The ordered pair $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair with constants λ and μ , so for all $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, we have

$$\begin{aligned} &\varrho((\bar{\Gamma} \circ (\Gamma, \Gamma'))(x, y), (\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))(\bar{x}, \bar{y})) \\ &= \varrho((\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))(\bar{x}, \bar{y}), (\bar{\Gamma} \circ (\Gamma, \Gamma'))(x, y)) \\ &\leq \lambda\varrho(\bar{\Gamma}(\bar{x}, \bar{y}), \Gamma(x, y)) + \mu\varrho(\bar{\Gamma}(\bar{y}, \bar{x}), \Gamma(y, x)) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &= \lambda\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) + \mu\varrho(\Gamma(y, x), \bar{\Gamma}(\bar{y}, \bar{x})) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &\leq \lambda^2\varrho(x, \bar{x}) + \lambda\mu\varrho(y, \bar{y}) + \lambda(1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) + \mu\lambda\varrho(y, \bar{y}) \\ &\quad + \mu^2\varrho(x, \bar{x}) + \mu(1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &= (\lambda^2 + \mu^2)\varrho(x, \bar{x}) + 2\lambda\mu\varrho(y, \bar{y}) + (1 - (\lambda + \mu)^2)\varrho(\mathcal{K}, \bar{\mathcal{K}}). \end{aligned}$$

Hence $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$. $\square$

Corollary 1. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \rightarrow \bar{\mathcal{K}}$, $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \rightarrow \mathcal{K}$ such that the ordered pair $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair. Then

(i) Γ has a unique coupled best proximity point $(x^*, x^*) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ has a unique coupled best proximity point $(\bar{x}^*, \bar{x}^*) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, that is,

$$\|x^* - \Gamma(x^*, x^*)\| = \varrho \quad \text{and} \quad \|\bar{x}^* - \bar{\Gamma}(\bar{x}^*, \bar{x}^*)\| = \varrho.$$

Moreover,

$$\Gamma(x^*, x^*) = \bar{x}^*, \quad \bar{\Gamma}(\bar{x}^*, \bar{x}^*) = x^*, \quad \|x^* - \bar{x}^*\| = \varrho;$$

- (ii) For any arbitrary point (x_0, y_0) , let the sequences $\{x_n\}$ and $\{y_n\}$ defined as in Definition 3. Then

$$\lim_{n \rightarrow \infty} x_{2n} = \lim_{n \rightarrow \infty} y_{2n} = x^*, \quad \lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} y_{2n+1} = \bar{x}^*.$$

Furthermore, if $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ for some $C > 0$, $p > 0$ and every $\epsilon \in (0, 2]$, then

- (iii) the following priori error estimate holds

$$\max\{\|x^* - x_{2m}\|, \|x^* - y_{2m}\|\} \leq \Delta_0 \sqrt[p]{\frac{\Delta_0 - \varrho}{C\varrho}} \frac{(\sqrt[p]{\lambda + \mu})^{2m}}{1 - \sqrt[p]{(\lambda + \mu)^2}};$$

- (iv) the following posteriori error estimate holds

$$\max\{\|x^* - x_{2n}\|, \|x^* - y_{2n}\|\} \leq \frac{\Delta_n}{1 - \sqrt[p]{(\lambda + \mu)^2}} \sqrt[p]{\frac{\Delta_n - \varrho}{C\varrho}},$$

where

$$\Delta_0 = \max\{\|(x_0, y_0) - (x_1, y_1)\|^*, \|(x_2, y_2) - (x_1, y_1)\|^*\}$$

and

$$\Delta_n = \max\{\|(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^*, \|(x_{2n+2}, y_{2n+2}) - (x_{2n+1}, y_{2n+1})\|^*\}.$$

Proof. (i) From Lemma 4, $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair. So, from Theorem 2(i), $\bar{\Gamma} \circ (\Gamma, \Gamma')$ has a unique coupled fixed point $(x^*, x^*) \in \mathcal{K} \times \mathcal{K}$ and $\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}')$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^*) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ such that $\|x^* - \bar{x}^*\| = \varrho(\mathcal{K}, \bar{\mathcal{K}})$. On the other hand, $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair, so we have

$$\begin{aligned} \|\Gamma(x^*, x^*) - x^*\| &= \|\Gamma(x^*, x^*) - (\bar{\Gamma} \circ (\Gamma, \Gamma'))(x^*, x^*)\| \\ &\leq (\lambda + \mu)\|x^* - \Gamma(x^*, x^*)\| + (1 - (\lambda + \mu))\varrho. \end{aligned}$$

This implies $\|\Gamma(x^*, x^*) - x^*\| = \varrho(\mathcal{K}, \bar{\mathcal{K}})$. Since $\|x^* - \bar{x}^*\| = \varrho(\mathcal{K}, \bar{\mathcal{K}})$, then from Property *UC* of $(\bar{\mathcal{K}}, \mathcal{K})$, we get $\Gamma(x^*, x^*) = \bar{x}^*$. The proof of the other cases is similar.

- (ii) From Theorem 2(ii), for every $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) := (\Gamma, \Gamma')(x_0, y_0) = (x_1, y_1) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ the sequences

$$\{(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^n(x_0, y_0)\}$$

and

$$\{(\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1)\}$$

converge to (x^*, x^*) and $(\bar{x}^*, \bar{x}^*)$, respectively. Indeed from Remark 1, we know

$$\begin{aligned} \{(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^n(x_0, y_0)\} &= \{((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n(x_0, y_0)\} \\ &= \{(x_{2n}, y_{2n})\} \end{aligned}$$

and

$$\begin{aligned} \{(\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1)\} &= \{((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1)\} \\ &= \{(x_{2n+1}, y_{2n+1})\}, \end{aligned}$$

which completes the proof of (ii).

(iii) In Theorem 2(iii), let $(\bar{x}_0, \bar{y}_0) := (\Gamma, \Gamma')(x_0, y_0) = (x_1, y_1)$. Then

$$\begin{aligned} \Delta_0 &= \max \{ \| (x_0, y_0) - (x_1, y_1) \|, \\ &\quad \| (\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))(x_0, y_0) - (x_1, y_1) \| \} \\ &= \max \{ \| (x_0, y_0) - (x_1, y_1) \|, \| ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))(x_0, y_0) - (x_1, y_1) \| \} \\ &= \max \{ \| (x_0, y_0) - (x_1, y_1) \|, \| (x_2, y_2) - (x_1, y_1) \| \}. \end{aligned}$$

Since from Lemma 4, $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$ and $\lambda^2 + \mu^2 + 2\lambda\mu = (\lambda + \mu)^2$, then the following priori error estimate holds

$$\begin{aligned} &\max \{ \| x^* - x_{2m} \|, \| x^* - y_{2m} \| \} \\ &= \| (x^*, x^*) - ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^m(x_0, y_0) \| \\ &= \| (x^*, x^*) - (\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^m(x_0, y_0) \| \\ &\leq \frac{\Delta_0}{1 - \sqrt{(\lambda + \mu)^2}} \sqrt[m]{\frac{\Delta_0 - \varrho}{C\varrho}} (\sqrt{\lambda + \mu})^{2m}. \end{aligned}$$

(iv) In Theorem 2(iv), let $(\bar{x}_0, \bar{y}_0) := (\Gamma, \Gamma')(x_0, y_0) = (x_1, y_1)$. Then

$$\begin{aligned} \Delta_n &= \Delta_{(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^n(x_0, y_0), (\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1)} \\ &= \Delta_{((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n(x_0, y_0), ((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1)} \\ &= \Delta_{(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})} \\ &= \max \{ \| (x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1}) \| \}, \end{aligned}$$

$$\begin{aligned}
& \|(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^* \} \\
& = \max \left\{ \|(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^*, \right. \\
& \quad \left. \|((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^* \right\} \\
& = \max \left\{ \|(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^*, \right. \\
& \quad \left. \|(x_{2n+2}, y_{2n+2}) - (x_{2n+1}, y_{2n+1})\|^* \right\}.
\end{aligned}$$

Since from Lemma 4, $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$ and $\lambda^2 + \mu^2 + 2\lambda\mu = (\lambda + \mu)^2$, then the following posteriori error estimate holds

$$\begin{aligned}
\max\{\|x^* - x_{2n}\|, \|x^* - y_{2n}\|\} & = \|((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n(x_0, y_0) - (x^*, x^*)\|^* \\
& \leq \frac{\Delta_n}{1 - \sqrt[p]{(\lambda + \mu)^2}} \cdot \sqrt[p]{\frac{\Delta_n - \varrho}{C_\varrho}}.
\end{aligned}$$

□

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Conflicts of Interest

All authors jointly worked on the results and they read and approved the final manuscript, and also declare that they have no competing interests.

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