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Research Article

# Inverse eigenvalue problems for composite banded tridiagonal matrices: Symmetric and nonsymmetric cases

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**Abstract**

We study an inverse eigenvalue problem associated with a special class of structured matrices in both symmetric and nonsymmetric forms. In the symmetric case, the problem involves two sets of eigenpairs corresponding to the original matrix and its leading principal submatrix of order $n-1$. For the nonsymmetric case, the minimum and maximum eigenvalues of each leading principal submatrix, together with an eigenpair associated with the maximum eigenvalue of the original matrix, are prescribed. By establishing recurrence relations between successive principal submatrices, we derive the necessary and sufficient conditions for the solvability of the problems. Moreover, explicit reconstruction algorithms are developed, and several numerical results are presented to illustrate the effectiveness of the proposed methods.

**AMS subject classifications (2020):** Primary 05C50; Secondary 65F18.

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# 1 Introduction

The inverse eigenvalue problem concerns the reconstruction of a matrix (or a class of matrices) from prescribed spectral data such as eigenvalues or eigenpairs. Because spectral properties often reflect essential physical and geometric characteristics of systems, inverse eigenvalue problems arise in various applications, including vibration analysis, structural dynamics, control theory, and signal processing. For this reason, they have been extensively studied in both theoretical and numerical contexts [1, 2].

A large portion of the existing literature focuses on inverse eigenvalue problems for structured matrices such as symmetric, tridiagonal and banded classes

[8, 11, 10, 5, 9, 7]. In this paper, we consider a class of composite banded tridiagonal matrices, which generalize classical tridiagonal matrices by combining two different structural patterns within a single matrix. This composite structure leads to recurrence relations that differ from the standard ones, thereby making the associated inverse eigenvalue problems more challenging. However, to the best of our knowledge, inverse eigenvalue problems for composite banded tridiagonal matrices of the present type have not been investigated in the existing literature. The main contributions of this work can be summarized as follows. First, we formulate two inverse eigenvalue problems for a new class of composite banded tridiagonal matrices in both symmetric and nonsymmetric settings. Second, we derive necessary and sufficient conditions for solvability and establish existence and uniqueness results. Finally, we develop explicit reconstruction algorithms and validate the proposed methods through numerical examples.

This paper is organized as follows. Section 2 presents the matrix structures and formulates the corresponding inverse eigenvalue problems. The symmetric case is analyzed in Section 3. Section 4 addresses the nonsymmetric case. In Section 5, numerical examples are provided to illustrate the effectiveness of the proposed methods, and concluding remarks are given in Section 6.

# 2 Problem formulation and preliminaries

In this section, we introduce the class of composite banded tridiagonal matrices and formally formulate the associated inverse eigenvalue problems. We also present the notation and preliminary results that will be used in the subsequent analysis. We consider an $n \times n$ composite banded tridiagonal matrix of the following form. In the symmetric case, the matrix $A_n$ is given by

$$A_n = \begin{pmatrix} a_1 & b_1 & \dots & b_{m+1} & & & & \\ b_1 & a_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ b_{m+1} & & & a_{m+2} & b_{m+2} & & & \\ & & & b_{m+2} & a_{m+3} & b_{m+3} & & \\ & & & & b_{m+3} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & b_{n-1} \\ & & & & & & b_{n-1} & a_n \end{pmatrix}, \tag{1}$$

where $b_j \neq 0$ for $j = 1, \dots, n-1$.

In the nonsymmetric case, the matrix $B_n$ is defined as

$$B_n = \begin{pmatrix} a_1 & c_1 & \dots & c_{m+1} & & & & \\ b_1 & a_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ b_{m+1} & & & a_{m+2} & c_{m+2} & & & \\ & & & b_{m+2} & a_{m+3} & c_{m+3} & & \\ & & & & b_{m+3} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & c_{n-1} \\ & & & & & & b_{n-1} & a_n \end{pmatrix}, \tag{2}$$

where $b_j c_j \neq 0$ for $j = 1, \dots, n-1$.

The matrix $B_n$ of the form (2) is diagonally similar to the following symmetric matrix:

$$DB_nD^{-1} = \begin{pmatrix} a_1 & \sqrt{b_1c_1} & \dots & \sqrt{b_{m+1}c_{m+1}} & & & & \\ \sqrt{b_1c_1} & a_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ \sqrt{b_{m+1}c_{m+1}} & & & a_{m+2} & \sqrt{b_{m+2}c_{m+2}} & & & \\ & & & \sqrt{b_{m+2}c_{m+2}} & a_{m+3} & \sqrt{b_{m+3}c_{m+3}} & & \\ & & & & \sqrt{b_{m+3}c_{m+3}} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \sqrt{b_{n-1}c_{n-1}} \\ & & & & & & \sqrt{b_{n-1}c_{n-1}} & a_n \end{pmatrix}.$$

Also, $D = diag(1, d_2, \dots, d_n)$ such that

$$d_i = \begin{cases} \sqrt{\frac{c_{i-1}}{b_{i-1}}}, & i = 2, 3, \dots, m+1, \\ \sqrt{\prod_{j=m+1}^{i-1} \frac{c_j}{b_j}}, & i = m+2, \dots, n, \end{cases}$$

This implies that, under the transformation, the eigenvalues of the leading principal submatrices $B_j, j = 1, 2, \ldots, n$, of the matrix $B_n$ remain invariant. The term composite refers to the fact that the matrix structure changes after the $(m+1)$th row and column. Specifically, the leading part of the matrix exhibits a bordered diagonal pattern, while the remaining part follows a classical tridiagonal structure. This hybrid form generalizes standard tridiagonal matrices.

Based on the above matrix structures, we consider the following inverse eigenvalue problems.

**Problem 1** (IEP-S (Symmetric Case))**.** The real numbers $\lambda_{\max}^{(n)}$, $\lambda_{\max}^{(n-1)}$ and the real vectors $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $x' = (x'_1, \ldots, x'_{n-1})^{\mathrm{T}} \in \mathbb{R}^{n-1}$ are given. Find an $n \times n$ symmetric matrix $A_n$ of the form (1) such that $\lambda_{\max}^{(n)}$ and $\lambda_{\max}^{(n-1)}$ are, respectively, the maximum eigenvalues of $A_n$ and its leading principal submatrix $A_{n-1}$, and $(\lambda_{\max}^{(n)}, x)$ and $(\lambda_{\max}^{(n-1)}, x')$ are eigenpairs of $A_n$ and $A_{n-1}$, respectively.

**Problem 2** (IEP-NS (Nonsymmetric Case))**.** The list of real numbers

$$\lambda_{\min}^{(j)},\ \lambda_{\max}^{(j)}, \quad j = 1, \ldots, n,$$

and the eigenpair $(\lambda_{\max}^{(n)}, x)$ are given. Find a matrix $B_n$ of the form (2) such that $B_n x = \lambda_{\max}^{(n)} x$, and for each $j = 1, \ldots, n$, $\lambda_{\min}^{(j)}$ and $\lambda_{\max}^{(j)}$ are, respectively, the minimum and maximum eigenvalues of the leading principal submatrix $B_j$ of $B_n$.

The considered inverse eigenvalue problems naturally arise in applications where partial spectral information of a structured system is available. For instance, in vibration analysis and structural dynamics, extremal eigenvalues of principal substructures may be measured experimentally, while only limited eigenvector information of the full system is accessible. In such situations, reconstructing the underlying matrix model from partial spectral data becomes essential. The composite banded structure considered here may represent systems composed of two interconnected subsystems with different internal configurations.

To prepare for the analysis in the subsequent sections, we first recall several known lemmas from the literature that will be used in the proofs of our results.

**Lemma 1.** [6] Let $\phi(\lambda)$ be a monic polynomial of degree $n$ with all real zeroes. If $\lambda_{\min}^{(n)}$ and $\lambda_{\max}^{(n)}$ are, respectively, the minimum and the maximum zero of $\phi(\lambda)$, then

i. If $x < \lambda_{\min}^{(n)}$, we have $(-1)^n \phi(x) > 0$.

ii. If $x > \lambda_{\max}^{(n)}$, we have $\phi(x) > 0$.

**Lemma 2.** [3] Let $A_n$ be an $n \times n$ matrix of the form (1) and let $\lambda_{\min}^{(j)}$, $\lambda_{\max}^{(j)}$ be the minimum and maximum eigenvalue of the leading principal submatrix $A_j$ of $A_n$, $j = 1, 2, \ldots, n$. Then

$$\lambda_{\min}^{(n)} < \lambda_{\min}^{(n-1)} < \cdots < \lambda_{\min}^{(2)} < \lambda_{\min}^{(1)} < \lambda_{\max}^{(2)} < \cdots < \lambda_{\max}^{(n)} \tag{3}$$

and

$$\lambda_{\min}^{(j)} < a_k < \lambda_{\max}^{(j)}, \quad k = 1, 2, \ldots, j; \quad j = 2, \ldots, n. \tag{4}$$

Let $\{\phi_j(\lambda) = \det(\lambda I_j - A_j)\}_{j=1}^{n}$ be the sequence of characteristic polynomials of $A_n$. We obtain the recurrence relation $\phi_j(\lambda)$ of the matrix $A_n$ in Lemma 3.

**Lemma 3.** [3] Let $A_n$ be an $n \times n$ matrix of the form (1). Then the sequence of $\{\phi_j(\lambda)\}_{j=1}^{n}$ satisfies the following recurrence relations:

i. $\phi_1(\lambda) = (\lambda - a_1)$,

ii. $\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - b_{j-1}^2 \prod_{k=2}^{j-1}(\lambda - a_k), \quad j = 2, \ldots, m+2,$

iii. $\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - b_{j-1}^2 \phi_{j-2}(\lambda), \quad j = m+3, \ldots, n,$

where $\phi_0(\lambda) = 1$.

# 3 Formulation and solution of the IEP-S (Symmetric Case)

In this section, we present the main theoretical results of the paper. By using the preliminary lemmas established in section 2, we derive solvability conditions and reconstruction formulas for the inverse eigenvalue problem in the symmetric case.

By Lemma 2, we now derive explicit formulas for the components of the eigenvector associated with the maximum eigenvalue of the matrix and its leading principal submatrix. These relations play a key role in linking the given spectral data to the unknown matrix entries.

**Lemma 4.** Let $x = (x_1, x_2, \ldots, x_n)^T$ and $x' = (x'_1, x'_2, \ldots, x'_{n-1})^T$ be eigenvectors of $A_n$ and $A_{n-1}$ corresponding to the eigenvalues $\lambda_{\max}^{(n)}$ and $\lambda_{\max}^{(n-1)}$, respectively. Then $x_1 \neq 0$, $x'_1 \neq 0$, and the components of these eigenvectors can be expressed in terms of their first components as follows:

$$x_j = \frac{-b_{j-1} x_1}{a_j - \lambda_{\max}^{(n)}}, \qquad j = 2, \ldots, m+1, \tag{5}$$

$$x_j = \frac{(-1)^j \phi_{j-1}(\lambda_{\max}^{(n)})\, x_1}{\displaystyle\prod_{i=m+1}^{j-1} b_i \prod_{k=2}^{m+1} (a_k - \lambda_{\max}^{(n)})}, \qquad j = m+2, \ldots, n, \tag{6}$$

$$x'_j = \frac{-b_{j-1} x'_1}{a_j - \lambda_{\max}^{(n-1)}}, \qquad j = 2, \ldots, m+1, \tag{7}$$

$$x'_j = \frac{(-1)^j \phi_{j-1}(\lambda_{\max}^{(n-1)})\, x'_1}{\displaystyle\prod_{i=m+1}^{j-1} b_i \prod_{k=2}^{m+1} (a_k - \lambda_{\max}^{(n-1)})}, \qquad j = m+2, \ldots, n-1. \tag{8}$$

*Proof.* Let $(\lambda_{\max}^{(n)}, x)$ be an eigenpair of $A_n$. Then

$$A_n x = \lambda_{\max}^{(n)} x. \tag{9}$$

Comparing the first $(n-1)$ rows of (9) on both sides yields

$$(a_1 - \lambda_{\max}^{(n)}) x_1 + \sum_{j=2}^{m+2} b_{j-1} x_j = 0, \tag{10}$$

$$b_{j-1} x_1 + (a_j - \lambda_{\max}^{(n)}) x_j = 0, \qquad j = 2, \ldots, m+1, \tag{11}$$

$$b_{m+1} x_1 + (a_{m+2} - \lambda_{\max}^{(n)}) x_{m+2} + b_{m+2} x_{m+3} = 0, \tag{12}$$

$$b_{j-1} x_{j-1} + (a_j - \lambda_{\max}^{(n)}) x_j + b_j x_{j+1} = 0, \qquad j = m+3, \ldots, n-1. \tag{13}$$

By Lemma 2, $(a_j - \lambda_{\max}^{(n)}) \neq 0$ for $j = 2, \ldots, m+1$. Hence, from (11) we obtain

$$x_j = \frac{-b_{j-1} x_1}{a_j - \lambda_{\max}^{(n)}}, \qquad j = 2, \ldots, m+1,$$

which establishes (5).

Next, we determine the remaining components $x_{m+2}, \ldots, x_n$ recursively. From (10) we get

$$x_{m+2} = \frac{(\lambda_{\max}^{(n)} - a_1) x_1 - \sum_{i=2}^{m+1} b_{i-1} x_i}{b_{m+1}}.$$

Substituting the expressions for $x_2, \ldots, x_{m+1}$ from (5) yields

$$x_{m+2} = \frac{(-1)^{m+2} \phi_{m+1}(\lambda_{\max}^{(n)})\, x_1}{b_{m+1} \prod_{k=2}^{m+1} (a_k - \lambda_{\max}^{(n)})},$$

which verifies the base case of (6).

Assume now that (6) holds up to $j = n-1$. To prove it for $j = n$, consider (13) with $j = n-1$:

$$b_{n-2}x_{n-2} + (a_{n-1} - \lambda_{\max}^{(n)})x_{n-1} + b_{n-1}x_n = 0,$$

so that

$$x_n = \frac{-b_{n-2}x_{n-2} + (\lambda_{\max}^{(n)} - a_{n-1})x_{n-1}}{b_{n-1}}.$$

By the induction hypothesis,

$$x_{n-2} = \frac{(-1)^{n-2}\phi_{n-3}(\lambda_{\max}^{(n)})\, x_1}{\prod_{i=m+1}^{n-3} b_i \prod_{k=2}^{m+1}(a_k - \lambda_{\max}^{(n)})},$$

$$x_{n-1} = \frac{(-1)^{n-1}\phi_{n-2}(\lambda_{\max}^{(n)})\, x_1}{\prod_{i=m+1}^{n-2} b_i \prod_{k=2}^{m+1}(a_k - \lambda_{\max}^{(n)})}.$$

Substituting into the recurrence gives

$$x_n = \frac{(-1)^{n}\phi_{n-1}(\lambda_{\max}^{(n)})\, x_1}{\prod_{i=m+1}^{n-1} b_i \prod_{k=2}^{m+1}(a_k - \lambda_{\max}^{(n)})},$$

which proves (6).

Finally, since $x$ is an eigenvector, $x \neq 0$. If $x_1 = 0$, then by relations (5)–(6) all components vanish, which is a contradiction. Hence $x_1 \neq 0$.

The same reasoning applies to the eigenpair $(\lambda_{\max}^{(n-1)}, x')$ of $A_{n-1}$: Replacing $\lambda_{\max}^{(n)}$ with $\lambda_{\max}^{(n-1)}$ and noting that indices now run up to $n-1$, we obtain the relations (7)–(8). This completes the proof. □

With the component formulas of the eigenvectors established in Lemma 4, we are now able to formulate the main result of this section. The following theorem provides conditions under which the inverse eigenvalue problem admits a unique solution and shows how the unknown entries of the matrix can be determined from the eigenvalues and eigenvector components.

**Theorem 1.** Problem 1 has a unique solution if the following conditions are satisfied:

a: $x_j \neq 0$, for $j = 1, 2, \ldots, n$ and $x'_j \neq 0$, for $j = 1, 2, \ldots, n-1$.

b: $E_j = \begin{vmatrix} x_1 & x_{m+2} \\ x'_1 & x'_{m+2} \end{vmatrix} \neq 0.$

c: $F_j = \begin{vmatrix} x_j & x_{j+1} \\ x'_j & x'_{j+1} \end{vmatrix} \neq 0,\ j = m+2, \dots, n-2.$

*Proof.* Assume that $x_j \neq 0$ for $j = 1, 2, \dots, n$ and $x'_j \neq 0$ for $j = 1, 2, \dots, n-1$.

Since $(\lambda_{\max}^{(n)}, x)$ and $(\lambda_{\max}^{(n-1)}, x')$ are eigenpairs of the matrices $A_n$ and $A_{n-1}$, respectively, it follows that for $j = 2, \dots, m+1$ the following relations hold:

$$\begin{cases} b_{j-1}x_1 + (a_j - \lambda_{\max}^{(n)})x_j = 0, \\ b_{j-1}x'_1 + (a_j - \lambda_{\max}^{(n-1)})x'_j = 0. \end{cases}$$

Let $D_j$ denote the determinant of the coefficient matrix of the above system. Then

$$D_j = x_1 x'_j - x'_1 x_j.$$

If $D_j \neq 0$, the system admits a unique solution for $a_j$ and $b_{j-1}$, given by

$$b_{j-1} = \frac{(\lambda_{\max}^{(n)} - \lambda_{\max}^{(n-1)})x'_j x_j}{x_1 x'_j - x'_1 x_j},$$
$$a_j = \lambda_{\max}^{(n)} - \frac{b_{j-1}x_1}{x_j}.$$

According to Lemma 4, we have

$$D_j = x_1 \frac{-b_{j-1}x'_1}{(a_j - \lambda_{\max}^{(n-1)})} - x'_1 \frac{-b_{j-1}x_1}{(a_j - \lambda_{\max}^{(n)})} = \frac{-x_1 x'_1 b_{j-1}(\lambda_{\max}^{(n)} - \lambda_{\max}^{(n-1)})}{(a_j - \lambda_{\max}^{(n-1)})(a_j - \lambda_{\max}^{(n)})}.$$

By Lemma 2 and relation (3), it follows that

$$(\lambda_{\max}^{(n)} - \lambda_{\max}^{(n-1)}) \neq 0,$$

and thus $D_j \neq 0$. Since $(\lambda_{\max}^{(n)} - \lambda_{\max}^{(n-1)})x'_j x_j \neq 0$, we conclude that $b_{j-1} \neq 0$.

To determine $a_1$ and $b_{m+1}$, consider

$$\begin{cases} (a_1 - \lambda_{\max}^{(n)})x_1 + \sum_{j=2}^{m+2} b_{j-1}x_j = 0, \\ (a_1 - \lambda_{\max}^{(n-1)})x'_1 + \sum_{j=2}^{m+2} b_{j-1}x'_j = 0. \end{cases}$$

Since $b_j$ for $j = 1, \dots, m$ are known, this system can be solved uniquely for $b_{m+1}$ and $a_1$ (as $E_j \neq 0$) as follows:

$$b_{m+1} = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{j=1}^{m+1} x_j x'_j}{E_j},$$

$$a_1 = \lambda_{\max}^{(n)} - \frac{\sum_{j=2}^{m+2} b_{j-1} x_j}{x_1}.$$

From Lemma 4, we obtain

$$x_j x'_j = \frac{b_{j-1}^2 x_1 x'_1}{(a_j - \lambda_{\max}^{(n)})(a_j - \lambda_{\max}^{(n-1)})}.$$

Moreover, by Lemma 2, relation (4), we have

$$\frac{b_{j-1}^2}{(a_j - \lambda_{\max}^{(n)})(a_j - \lambda_{\max}^{(n-1)})} > 0.$$

Therefore, the signs of $x_j x'_j$ and $x_1 x'_1$ coincide. Hence, $\sum_{j=1}^{m+1} x_j x'_j \neq 0$, and thus $b_{m+1} \neq 0$. To determine the coefficients $a_{m+2}$ and $b_{m+2}$, consider the linear system

$$\begin{cases} b_{m+1} x_1 + (a_{m+2} - \lambda_{\max}^{(n)}) x_{m+2} + b_{m+2} x_{m+3} = 0, \\ b_{m+1} x'_1 + (a_{m+2} - \lambda_{\max}^{(n-1)}) x'_{m+2} + b_{m+2} x'_{m+3} = 0. \end{cases}$$

Since $F_{m+2} \neq 0$ by condition (c), the system has a unique solution. Therefore

$$b_{m+2} = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{j=1}^{m+2} x_j x'_j}{F_{m+2}},$$

$$a_{m+2} = \lambda_{\max}^{(n)} - \frac{b_{m+1} x_1 + b_{m+2} x_{m+3}}{x_{m+2}}.$$

Thus, both coefficients are uniquely determined once $b_{m+1}$ is known.

By (13) for $j = m+3, \dots, n-2$ we have

$$\begin{cases} b_{j-1} x_{j-1} + (a_j - \lambda_{\max}^{(n)}) x_j + b_j x_{j+1} = 0, \\ b_{j-1} x'_{j-1} + (a_j - \lambda_{\max}^{(n-1)}) x'_j + b_j x'_{j+1} = 0. \end{cases}$$

by condition (c), the system has a unique solution.

$$b_j = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{i=1}^{j} x_i x'_i}{F_j},$$

$$a_j = \lambda_{\max}^{(n)} - \frac{b_{j-1}x_{j-1} + b_j x_{j+1}}{x_j}.$$

Finally, to determine $a_{n-1}$, note that

$$b_{n-2}x'_{n-2} + (a_{n-1} - \lambda_{\max}^{(n-1)})x'_{n-1} = 0,$$

which implies

$$a_{n-1} = \lambda_{\max}^{(n-1)} - \frac{b_{n-2}x'_{n-2}}{x'_{n-1}}.$$

Using (13), we obtain

$$b_{n-1} = \frac{-(b_{n-2}x_{n-2} + (a_{n-1} - \lambda_{\max}^{(n)})x_{n-1})}{x_n} = \frac{(\lambda_{\max}^{(n)} - \lambda_{\max}^{(n-1)})\sum_{i=1}^{n-1} x_i x'_i}{x_n x'_{n-1}},$$

and consequently,

$$a_n = \lambda_{\max}^{(n)} - \frac{b_{n-1}x_{n-1}}{x_n}.$$

□

Theorem 1 shows that the solvability of Problem 1 depends only on the nonvanishing of certain eigenvector components and determinant conditions involving the given spectral data. These conditions ensure that each step of the reconstruction process is well defined and that the unknown matrix entries can be uniquely determined. In particular, the theorem guarantees that the prescribed eigenpairs contain sufficient information to recover the entire matrix.

# 4 Formulation and solution of the IEP-NS (Nonsymmetric case)

In this section, we study the inverse eigenvalue problem for the nonsymmetric matrix $B_n$ of the form (2). To establish the main result, we first present some preliminary relations concerning the characteristic polynomials of the principal submatrices of $B_n$. These relations form the basis for deriving expressions of the eigenvectors associated with the dominant eigenvalue, which are essential for the subsequent analysis.

**Lemma 5.** Let $B_n$ be an $n \times n$ matrix of the form (2). Then the sequence of $\{\phi_j(\lambda)\}_{j=1}^{n}$ satisfies the following recurrence relations:

i. $\phi_1(\lambda) = (\lambda - a_1)$,

ii. $\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - b_{j-1}c_{j-1}\prod_{k=2}^{j-1}(\lambda - a_k), \quad j = 2, \ldots, m+2,$

iii. $\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - b_{j-1}c_{j-1}\phi_{j-2}(\lambda), \quad j = m+3, \ldots, n,$

where $\phi_0(\lambda) = 1$.

*Proof.* The relations can be verified by expanding the determinant. □

**Lemma 6.** Let $x = (x_1, x_2, \ldots, x_n)^T$ denote an eigenvector of the matrix $B_n$ corresponding to the eigenvalue $\lambda_{\max}^{(n)}$. Then $x_1 \neq 0$, and the components of this eigenvector can be expressed as

$$x_j = \frac{-b_{j-1}x_1}{a_j - \lambda_{\max}^{(n)}}, \qquad j = 2, \ldots, m+1, \tag{14}$$

$$x_j = \frac{(-1)^j \phi_{j-1}(\lambda_{\max}^{(n)})\, x_1}{\displaystyle\prod_{i=m+1}^{j-1} c_i \prod_{k=2}^{m+1}(a_k - \lambda_{\max}^{(n)})}, \qquad j = m+2, \ldots, n. \tag{15}$$

*Proof.* The proof follows the same reasoning as in Lemma 4, and is therefore omitted for brevity. □

To proceed toward the reconstruction of the matrix elements, we make use of the following result from [3], which establishes an interlacing property and explicit formulas for the entries of a symmetric matrix. This theorem will play a key role in proving our main result for the nonsymmetric case.

**Theorem 2.** [3] Let

$$\sigma = (\lambda_{\min}^{(1)}, \lambda_{\min}^{(2)}, \lambda_{\max}^{(2)}, \ldots, \lambda_{\min}^{(n)}, \lambda_{\max}^{(n)}),$$

be a list of real numbers. Then the list $\sigma$ can be interlacingly realized by an $n \times n$ symmetric matrix $A_n$ of the form (1) if and only if

$$\lambda_{\min}^{(n)} < \lambda_{\min}^{(n-1)} < \cdots < \lambda_{\min}^{(2)} < \lambda_{\min}^{(1)} < \lambda_{\max}^{(2)} < \cdots < \lambda_{\max}^{(n)}.$$

The entries of an $n \times n$ symmetric matrix $A$ of the form (1) as follows:

$$a_1 = \lambda_{\min}^{(1)},$$

$$a_j = \frac{\lambda_{\min}^{(j)}\, \phi_{j-1}(\lambda_{\min}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\max}^{(j)} - a_i) - \lambda_{\max}^{(j)}\, \phi_{j-1}(\lambda_{\max}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\min}^{(j)} - a_i)}{\phi_{j-1}(\lambda_{\min}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\max}^{(j)} - a_i) - \phi_{j-1}(\lambda_{\max}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\min}^{(j)} - a_i)},$$

$$b_{j-1}^2 = \frac{(\lambda_{\max}^{(j)} - \lambda_{\min}^{(j)})\, \phi_{j-1}(\lambda_{\min}^{(j)})\, \phi_{j-1}(\lambda_{\max}^{(j)})}{\phi_{j-1}(\lambda_{\min}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\max}^{(j)} - a_i) - \phi_{j-1}(\lambda_{\max}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\min}^{(j)} - a_i)},$$

for any $j = 2, \ldots, m$ and

$$a_j = \frac{\lambda_{\min}^{(j)} \phi_{j-1}(\lambda_{\min}^{(j)}) \phi_{j-2}(\lambda_{\max}^{(j)}) - \lambda_{\max}^{(j)} \phi_{j-1}(\lambda_{\max}^{(j)}) \phi_{j-2}(\lambda_{\min}^{(j)})}{\phi_{j-1}(\lambda_{\min}^{(j)})\, \phi_{j-2}(\lambda_{\max}^{(j)}) - \phi_{j-1}(\lambda_{\max}^{(j)})\, \phi_{j-2}(\lambda_{\min}^{(j)})},$$

$$b_{j-1}^2 = \frac{(\lambda_{\max}^{(j)} - \lambda_{\min}^{(j)}) \phi_{j-1}(\lambda_{\min}^{(j)})\, \phi_{j-1}(\lambda_{\max}^{(j)})}{\phi_{j-1}(\lambda_{\min}^{(j)})\, \phi_{j-2}(\lambda_{\max}^{(j)}) - \phi_{j-1}(\lambda_{\max}^{(j)})\, \phi_{j-2}(\lambda_{\min}^{(j)})},$$

for $j = m+1, \ldots, n$.

The next step is to extend these relations to the nonsymmetric setting. For this purpose, we introduce a set of auxiliary quantities that simplify the notation and facilitate the proof of the main theorem.

$$\begin{aligned}
M_j &= \lambda_{\min}^{(j)} \phi_{j-1}(\lambda_{\min}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\max}^{(j)} - a_i), \\
N_j &= \lambda_{\max}^{(j)} \phi_{j-1}(\lambda_{\max}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\min}^{(j)} - a_i), \\
P_j &= (\lambda_{\max}^{(j)} - \lambda_{\min}^{(j)}) \phi_{j-1}(\lambda_{\min}^{(j)}) \phi_{j-1}(\lambda_{\max}^{(j)}), \\
D_j &= \phi_{j-1}(\lambda_{\min}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\max}^{(j)} - a_i) - \phi_{j-1}(\lambda_{\max}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{\min}^{(j)} - a_i), \\
S_j &= \lambda_{\min}^{(j)} \phi_{j-1}(\lambda_{\min}^{(j)}) \phi_{j-2}(\lambda_{\max}^{(j)}), \\
T_j &= \lambda_{\max}^{(j)} \phi_{j-1}(\lambda_{\max}^{(j)}) \phi_{j-2}(\lambda_{\min}^{(j)}), \\
D'_j &= \phi_{j-1}(\lambda_{\min}^{(j)}) \phi_{j-2}(\lambda_{\max}^{(j)}) - \phi_{j-1}(\lambda_{\max}^{(j)}) \phi_{j-2}(\lambda_{\min}^{(j)}).
\end{aligned}$$

We are now ready to state and prove the main theorem of this section, which establishes the existence and uniqueness of the solution to Problem 2 under the given spectral conditions.

**Theorem 3.** If the ordered spectral relation

$$\lambda_{\min}^{(n)} < \lambda_{\min}^{(n-1)} < \cdots < \lambda_{\min}^{(2)} < \lambda_{\min}^{(1)} < \lambda_{\max}^{(2)} < \cdots < \lambda_{\max}^{(n)}, \tag{16}$$

holds, and the component $x_j$ of the corresponding eigenvector is nonzero (i.e., $x_j \neq 0$) for all $j = 1, 2, \ldots, n$, then Problem 2 admits a unique solution.

*Proof.* To prove this theorem, we observe that Problem 2 is equivalent to solving the system of (17), (18), and (19) for the unknown parameters.

$$\phi_j(\lambda_{\max}^{(j)}) = 0, \quad j = 1, 2, \ldots, n, \tag{17}$$

$$\phi_j(\lambda_{\min}^{(j)}) = 0, \quad j = 1, 2, \ldots, n, \tag{18}$$

$$B_n x = \lambda_{\max}^{(n)} x. \tag{19}$$

It is immediate that $a_1 = \lambda_{\min}^{(1)}$. The following subsystem, defined by (17) and (18) for $j = 2, \ldots, m+2$.

$$\begin{cases} \phi_j(\lambda_{\max}^{(j)}) = a_j \phi_{j-1}(\lambda_{\max}^{(j)}) + b_{j-1}c_{j-1} \prod_{k=2}^{j-1} (\lambda_{\max}^{(j)} - a_k) \\ \qquad -\lambda_{\max}^{(j)} \phi_{j-1}(\lambda_{\max}^{(j)}) = 0, \\ \phi_j(\lambda_{\min}^{(j)}) = a_j \phi_{j-1}(\lambda_{\min}^{(j)}) + b_{j-1}c_{j-1} \prod_{k=2}^{j-1} (\lambda_{\min}^{(j)} - a_k) \\ \qquad -\lambda_{\min}^{(j)} \phi_{j-1}(\lambda_{\min}^{(j)}) = 0. \end{cases} \tag{20}$$

By Theorem 2 and (20), we can determine the coefficients $a_j$ and the product $b_{j-1}c_{j-1}$ as follows:

$$a_j = \frac{M_j - N_j}{D_j},$$

$$b_{j-1}c_{j-1} = \frac{P_j}{D_j}.$$

Using the eigenvalue condition $B_n x = \lambda_{\max}^{(n)} x$, and substituting into (11), we obtain

$$b_{j-1} = (\lambda_{\max}^{(n)} - a_j) \frac{x_j}{x_1}.$$

Substituting the obtained expressions for $a_j$ and $b_{j-1}$ back into the first equation of system (20) yields

$$c_{j-1} = \frac{(\lambda_{\max}^{(j)} - a_j)\phi_{j-1}(\lambda_{\max}^{(j)})}{b_{j-1} \prod_{k=2}^{j-1} (\lambda_{\max}^{(j)} - a_k))}.$$

Furthermore, from (16) and Lemma 1, we confirm the positivity condition:

$$b_{j-1}c_{j-1} = \frac{(-1)^{j-1} P_j}{(-1)^{j-1} D_j} > 0.$$

Next, using Lemma 5, the conditions in (17) and (18) can be reformulated as the following system for $j = m+3, \dots, n$:

$$\begin{cases} a_j \phi_{j-1}(\lambda_{\max}^{(j)}) + b_{j-1}c_{j-1}\phi_{j-2}(\lambda_{\max}^{(j)}) - \lambda_{\max}^{(j)}\phi_{j-1}(\lambda_{\max}^{(j)}) = 0, \\ a_j \phi_{j-1}(\lambda_{\min}^{(j)}) + b_{j-1}c_{j-1}\phi_{j-2}(\lambda_{\min}^{(j)}) - \lambda_{\min}^{(j)}\phi_{j-1}(\lambda_{\min}^{(j)}) = 0. \end{cases} \tag{21}$$

By Theorem 2 solving (21) for $a_j$ and the product $b_{j-1}c_{j-1}$ yields

$$\begin{aligned} a_j &= \frac{S_j - T_j}{D'_j}, \\ b_{j-1}c_{j-1} &= \frac{(\lambda_{\max}^{(j)} - \lambda_{\min}^{(j)})\phi_{j-1}(\lambda_{\min}^{(j)})\phi_{j-1}(\lambda_{\max}^{(j)})}{D'_j}. \end{aligned}$$

Furthermore, $b_{j-1}$ can be determined recursively from (15) as

$$c_{j-1} = \frac{(-1)^j \phi_{j-1}(\lambda_{\max}^{(n)})x_1}{\prod_{i=m+1}^{j-2} c_i \prod_{k=2}^{m+1}(a_k - \lambda_{\max}^{(n)})x_j}.$$

Finally, substituting the expressions for $a_j$ and $c_{j-1}$ into (17), we obtain

$$b_{j-1} = \frac{(\lambda_{\max}^{(j)} - a_j)\phi_{j-1}(\lambda_{\max}^{(j)})}{c_{j-1}\phi_{j-2}(\lambda_{\max}^{(j)})}.$$

The positivity of the resulting products follows from (16) and Lemma 1. Since all the required parameters are uniquely determined by the given spectral data and the components of the eigenvector, the solution to Problem 2 is unique. □

Theorem 3 establishes that the given extremal eigenvalues of the leading principal submatrices, together with a single eigenpair of the full matrix, uniquely determine the nonsymmetric matrix $B_n$. The interlacing condition plays a crucial role in ensuring the positivity of the reconstructed parameters and the consistency of the solution.

# 5 Algorithms and numerical examples

In this section, we present the solution algorithms for Problems 1 and 2, together with numerical examples.

**Algorithm 1** To solve Problem 1

**Require:** $\lambda_{\max}^{(n)}$, $\lambda_{\max}^{(n-1)}$, $\varepsilon$, $x = (x_1, \ldots, x_n)$, $x' = (x'_1, \ldots, x'_{n-1})$
**Ensure:** $A_n$
  **for** $j = 2$ **to** $m+1$ **do**
    $b_{j-1} = \frac{(\lambda_{\max}^{(n)} - \lambda_{\max}^{(n-1)}) x'_j x_j}{x_1 x'_j - x'_1 x_j}$.
    $a_j = \lambda_{\max}^{(n)} - \frac{b_{j-1} x_1}{x_j}$.
  **end for**
  **if** $|x_1 x'_{m+2} - x'_1 x_{m+2}| < \varepsilon$ **then**
    Problem cannot be solved by this algorithm.
  **end if**
  $b_{m+1} = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{j=1}^{m+1} x_j x'_j}{E_j}$
  $a_1 = \lambda_{\max}^{(n)} - \frac{\sum_{j=2}^{m+2} b_{j-1} x_j}{x_1}$.
  **if** $|x_{m+2} x'_{m+3} - x'_{m+2} x_{m+3}| < \varepsilon$ **then**
    Problem cannot be solved by this algorithm.
  **end if**
  $b_{m+2} = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{j=1}^{m+2} x_j x'_j}{F_j}$
  $a_{m+2} = \lambda_{\max}^{(n)} - \frac{b_{m+1} x_1 + b_{m+2} x_{m+3}}{x_{m+2}}$.
  **for** $j = m+3$ **to** $n-2$ **do**
    **if** $|x_j x'_{j+1} - x'_j x_{j+1}| < \varepsilon$ **then**
      Problem cannot be solved by this algorithm.
    **end if**
    $b_j = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{i=1}^{j} x_i x'_i}{F_j}$
    $a_j = \lambda_{\max}^{(n)} - \frac{b_{j-1} x_{j-1} + b_j x_{j+1}}{x_j}$.
  **end for**
  $a_{n-1} = \lambda_{\max}^{(n-1)} - \frac{b_{n-2} x'_{n-2}}{x'_{n-1}}$.
  $b_{n-1} = \frac{(\lambda_{\max}^{(n-1)} - \lambda_{\max}^{(n)}) \sum_{i=1}^{n-1} x_i x'_i}{x_n x'_{n-1}}$,
  $a_n = \lambda_{\max}^{(n)} - \frac{b_{n-1} x_{n-1}}{x_n}$.

**Example 1.** Consider two distinct real numbers $\lambda_{\max}^{(7)} = 9.8796$ and $\lambda_{\max}^{(6)} = 7.1115$, together with the real vectors

$$\mathbf{x} = (0.0453,\ 0.0066,\ 0.0046,\ 0.0084,\ 0.2260,\ 0.7525,\ 0.6168)^T,$$

and

$$\mathbf{x}' = (0.21946,\ 0.0534,\ 0.0343,\ 0.0935,\ 0.4707,\ 0.8470)^T.$$

Determine a $7 \times 7$ matrix of the form (1) such that $(\lambda_{\max}^{(7)}, \mathbf{x})$ forms an eigenpair of $A_7$ and $(\lambda_{\max}^{(6)}, \mathbf{x}')$ is an eigenpair of $A_6$.

**Solution:** Using Algorithm 1, we obtain the following unique matrix:

$$A_7 = \begin{bmatrix} 4 & 1 & 0.8 & 0.9 & 1.1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 5 & 0 & 0 & 0 \\ 1.1 & 0 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 5 \end{bmatrix}.$$

To verify the correctness of this result, the eigenvalues of $A_6$ and $A_7$ are recalculated from the above matrix. The obtained spectra are

$$\sigma(A_6) = \{\, \underline{7.1115},\ 5.7253,\ 4.2778,\ 2.6905,\ 2.0000,\ 1.1947\},$$
$$\sigma(A_7) = \{\, \underline{9.8796},\ 5.8562,\ 4.4413,\ 3.2988,\ 2.4780,\ 1.5593,\ 0.4865\}.$$

The underlined eigenvalues correspond to the prescribed maximum eigenvalues, confirming the consistency of the constructed matrix with the given spectral data.

**Algorithm 2** To solve Problem 2

**Require:** Spectral data $\{\lambda_{\min}^{(j)}, \lambda_{\max}^{(j)}\}_{j=1}^{n}$, eigenvector $x = (x_1, \dots, x_n)$ with $x_j \neq 0$, integer $m$.

**for** $j = 2$ **to** $m+2$ **do**

compute auxiliary scalars $M_j, N_j, P_j, D_j$ from spectral data and $\{\phi_k\}$

$a_j \leftarrow \dfrac{M_j - N_j}{D_j}$.

$b_{j-1} \leftarrow (\lambda_{\max}^{(n)} - a_j)\dfrac{x_j}{x_1}$.

$c_{j-1} \leftarrow \dfrac{(\lambda_{\max}^{(j)} - a_j)\,\phi_{j-1}(\lambda_{\max}^{(j)})}{b_{j-1}\,\prod_{k=2}^{j-1}(\lambda_{\max}^{(j)} - a_k)}$.

**end for**

**for** $j = m+3$ **to** $n$ **do**

$a_j \leftarrow \dfrac{S_j - T_j}{D'_j}$.

$c_{j-1} \leftarrow \dfrac{(-1)^j\,\phi_{j-1}(\lambda_{\max}^{(n)})\,x_1}{\left(\prod_{i=m+1}^{j-2} c_i\right)\left(\prod_{k=2}^{m+1}(a_k - \lambda_{\max}^{(n)})\right)x_j}$.

$b_{j-1} \leftarrow \frac{(\lambda_{\max}^{(j)} - a_j)\phi_{j-1}(\lambda_{\max}^{(j)})}{c_{j-1}\phi_{j-2}(\lambda_{\max}^{(j)})}$.

**end for**

**Example 2.** Given the 15 real numbers shown in Table 1, which displays the minimum and maximum eigenvalues of the leading principal submatrices $B_j$, and the real vector

$$x = (-0.0664, -0.0109, -0.0198, -0.2552, -0.4714, -0.6657, -0.4745, -0.1985)^T,$$

assume that the conditions of Theorem 3 are satisfied.

Table 1: minimum and maximum eigenvalues of leading principal submatrices $B_j$.

| j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\lambda_{\min}^{(j)}$ | 4 | 2.3170 | 1.4499 | 0.5659 | 0.4374 | 0.3939 | 0.2575 | -1.4229 |
| $\lambda_{\max}^{(j)}$ | 4 | 8.1830 | 10.8934 | 14.3663 | 16.7496 | 18.5961 | 19.6192 | 19.9541 |

Find a matrix $B_8$ of the form (2) such that $\lambda_{\min}^{(j)}$ and $\lambda_{\max}^{(j)}$ are the minimum and maximum eigenvalues of $B_j$, $j = 1, 2, \ldots, 8$, moreover $(\lambda_{\max}^{(8)}, x)$ is its eigenpair.

**Solution**: By applying Algorithm 2, we obtain the following matrix

$$B_8 = \begin{bmatrix} 4.0 & 3.2 & 2.8 & 3.8 & 0 & 0 & 0 & 0 \\ 2.2 & 6.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.2 & 0 & 9.2 & 0 & 0 & 0 & 0 & 0 \\ 3.8 & 0 & 0 & 12.5 & 3.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.8 & 11 & 4.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.2 & 12 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.8 & 6 & 7.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 8 \end{bmatrix}.$$

The eigenvalues of the leading principal submatrices are listed in the Table 2. The underlined values represent the minimum and maximum eigenvalues of these leading principal submatrices.

Table 2: Eigenvalues of the leading principal submatrices of $B_8$

| **Submatrix** | **Eigenvalues** |
|---|---|
| $B_1$ | $\{\underline{4.0000}\}$ |
| $B_2$ | $\{\underline{2.3170}, \underline{8.1830}\}$ |
| $B_3$ | $\{\underline{1.4499}, 7.3567, \underline{10.8934}\}$ |
| $B_4$ | $\{\underline{0.5659}, 7.1812, 10.0865, \underline{14.3663}\}$ |
| $B_5$ | $\{\underline{0.4374}, 6.9207, 8.1816, 10.9108, \underline{16.7496}\}$ |
| $B_6$ | $\{\underline{0.3939}, 5.1089, 7.2774, 10.2381, 13.5856, \underline{18.5961}\}$ |
| $B_7$ | $\{\underline{0.2575}, 1.1955, 6.8389, 7.7846, 10.5835, 14.9208, \underline{19.6192}\}$ |
| $B_8$ | $\{\underline{-1.4229}, 0.4466, 5.7080, 7.2957, 10.0053, 11.6269, 15.5863, \underline{19.9541}\}$ |

The underlined eigenvalues are the minimum and maximum eigenvalues, and

$$B_8x = 19.9541x.$$

Although the presented examples are of moderate size, they illustrate all essential steps of the reconstruction procedure. Since the proposed algorithm is based on explicit recurrence relations and closed form expressions, its implementation does not depend on the matrix dimension in a structural sense. Therefore, the same reconstruction process applies to larger matrices without modification. In practice, the numerical computations remain stable whenever the solvability conditions of Theorem 1 and Theorem 3 are satisfied.

## 6 Conclusions

In this paper, we addressed two partially described inverse eigenvalue problems for a special class of structured matrices. By utilizing limited spectral data and establishing recurrence relations between successive principal submatrices, we derived necessary and sufficient solvability conditions and proposed explicit reconstruction algorithms. Numerical experiments confirmed the effectiveness of the developed methods, demonstrating that such inverse problems can be successfully solved even when only partial eigendata are available.

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## Conflicts of Interest

The author declares no conflicts of interest.

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