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Research Article

# Existence and Ulam–Hyers stability for a singular Caputo-fractional problem with integral boundary conditions

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**Abstract**

In this article, we study a singular fractional-order Caputo differential problem involving boundary conditions with linear operators. The analysis is carried out in an appropriate Banach space framework. First, by applying the Banach contraction principle, we establish a uniqueness result for the considered problem. Then, using Krasnosel'skii's fixed point theorem, we derive an existence theorem under suitable assumptions on the nonlinear term. In the second part of the paper, we investigate different types of stability for the obtained solutions, namely Ulam–Hyers stability and Ulam–Hyers–Rassias stability. Sufficient conditions ensuring these stability properties are presented and proved. The obtained theoretical results highlight the influence of the singular fractional operator and the associated boundary conditions. Finally, two illustrative examples are discussed in order to demonstrate the applicability and effectiveness of the established results.

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# 1 Introduction

In recent decades, fractional calculus (FC) has attracted the interest of many mathematicians and researchers. It gives the possibility to represent complex physical phenomena more accurately and efficiently than ordinary calculus. Models involving fractional differentiation highlight intermediate behaviors that cannot be done by models involving natural differentiation. Because of that, FC has, over time, become an essential tool in the description of complex dynamic systems with memory, nonlocal and hereditary properties. Compared to differential equations involving ordinary derivatives, fractional differential equations offer the possibility of better modeling in many areas such as viscolelastic materials, where the material's response depends on its entire deformation history (see [6, 7]), processes that exhibit memory effects like in biology (see [2, 24, 28]) and medicine (see [2, 4, 5, 13, 14, 20, 23]), in economy and finance see ( [8, 11, 19]), and others.

The use of fractional-order differential equations, particularly the Caputo equation, has significantly improved the understanding of boundary problems. This methodology effectively translates initial conditions, leading to important insights regarding the existence of solutions and their stability. To investigate these problems, various techniques are employed, including fixed-point theorems, Ulam–Hyers and Ulam–Hyers–Rassias stability concepts see [1, 3, 12, 15, 21, 22, 25, 27] and references therein.

Our starting point in the present work is the paper [22] in which, Rezapour and Shabibi considered the singular Caputo fractional integro-differential problem

$$\begin{cases} {}^{C}D^{\alpha}x(t) + f(t, x(t)) = 0, \quad t \in \,]0,1[\,, \quad n = [\alpha] + 1, \\ x^{(j)}(0) = 0, \quad 0 \leqslant j \neq 2 \leqslant n-1, \\ ax(1) = I^{p}x(1), \quad a \geqslant 1, \quad \alpha \geqslant 3, \quad p \geqslant 1, \quad 2\Gamma(p) \geqslant \Gamma(\alpha), \end{cases}$$

where $I^p$ is the Riemann–Liouville fractional integral of order $p$ and $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is singular at $t = 0$. They firstly proved that this problem is equivalent to an integral equation of the form:

$$y(t) = \int_0^1 G(t,s) f(s, y(s)) ds, \quad G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} \delta_{t,s} + t^2 \Phi(s), \tag{1}$$

where $\Phi(s)$ is a given function and $\delta_{t,s} = 1$ if $0 \leqslant s \leqslant t$ and 0 otherwise.

Using the representation (1), they gave in particular a sufficient uniqueness condition by means of the Banach contraction principle and a sufficient existence condition by using the compressive form of the Guo-Krasnosel'skii fixed point theorem. In 2020, Kheiryan and Rezapour [15] investigated the Ulam stability of this problem via its integral formulation. Formula (1) implies that the kernel $G(t,s)$ is a one-dimensional perturbation of the Green's function $g(t,s) := \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ of the equation ${}^C D^\alpha x(t) + f(t, x(t)) = 0$.

In the present work, we discuss the existence, the uniqueness, the Ulam–Hyers and the Ulam–Hyers–Rassias stability of solutions for the singular with two integral conditions boundary-value problem:

$${}^C D^\alpha y(\tau) + \psi(\tau, y(\tau)) = 0, \quad \tau \in [0,1], \ \alpha \geqslant 3, \tag{2}$$

$$y^{(j)}(0) = 0, \ j \in \{0, 1, \ldots, n-1\} \setminus \{j_1, j_2\}, \quad n = [\alpha] + 1, \tag{3}$$

$$A_k \left( y^{(q_k)}(1) \right) = \frac{1}{\Gamma(p_k)} \int_0^1 (1-\theta)^{p_k - 1} y(\theta) d\theta, \quad p_k \ \in \mathbb{N}^*, \ k = \overline{1,2}, \tag{4}$$

where $j_1, \ j_2, \ q_1, \ q_2, \ p_1, \ p_2$ are nonnegative integers such that

$$0 \leqslant q_1 \neq q_2 \leqslant j_1 \neq j_2 \leqslant [\alpha], \tag{5}$$

the function $\psi$ is defined on $[0,1] \times \mathbb{E} \longrightarrow \mathbb{E}$ with the following singularity:

$$\lim_{\tau \longrightarrow 0^+} \|\psi(\tau, \theta)\|_{\mathbb{E}} = +\infty, \quad \text{for all} \ \ \theta \in \mathbb{E} \tag{6}$$

and satisfying some conditions that will be specified later, $\mathbb{E}$ is a real Banach space, and $A_1$ and $A_2$ are two commuting bounded linear operators in $\mathbb{E}$. Compared to [22], the present work contains the following novelties: The function $\psi$ is Banach space valued, the number of integral conditions is greater, instead of the one-dimensional (axial) distortion represented by the scalar $a$ in [22], we have now two multi-dimensional (spatial) distortions represented by the operators $A_1$ and $A_2$.

The article is arranged as follows: In Section 2, we recall some preliminary concepts including basic definitions and properties. In Section 3, we first give the integral equivalent formulation of the problem proving in particular that the kernel is a two-dimensional perturbation of the Green's operator $\widetilde{G}(\tau, \theta) = \frac{-(\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)} I_{\mathbb{E}}$ of the equation ${}^C D^\alpha x(\tau) + \psi(\tau, x(\tau)) = 0$, $I_{\mathbb{E}}$ is the identity in $\mathbb{E}$. Using the integral formulation, we give a sufficient criterion for the uniqueness and existence of a solution by means of the Banach fixed point theorem and the Krasnosel'skii fixed point theorem successively. Some Ulam–Hyers and Ulam–Hyers–Rassias stability results are also obtained. In Section 4, two illustrative examples are given.

# 2 Preliminaries

In what follows, $\mathbb{E}$ is a real Banach space equipped with the norm $\|\cdot\|_{\mathbb{E}}$, $C(J,\mathbb{E})$ is the Banach space of continuous functions from $J=[0,1]$ into $\mathbb{E}$ with the norm

$$\|y\|_{C(J,\mathbb{E})} = \sup\left\{\|y(\tau)\|_{\mathbb{E}} : \ \tau \in J\right\}.$$

Moreover, $L^1(J,\mathbb{R})$ is the space of measurable functions $l : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|l\|_{L^1(J,\mathbb{R})} = \int_J |l(\theta)| d\theta.$$

Likewise, $L^1(J,\mathbb{E})$ is the space of measurable functions $l : J \longrightarrow \mathbb{E}$ that are Bochner integrable with the norm [9, 18]

$$\|l\|_{L^1(J,\mathbb{E})} = \int_J \|l(\theta)\|_{\mathbb{E}} d\theta.$$

Finally, $\mathcal{B}(\mathbb{E})$ is the Banach algebra of all bounded linear operators in $\mathbb{E}$, equipped with the operator norm

$$\|A\|_{\mathcal{B}(\mathbb{E})} = \inf\left\{M > 0 : \|A(x)\|_{\mathbb{E}} \leqslant M\,\|x\|_{\mathbb{E}}\,, \quad \text{for all} \ \ x \in \mathbb{E}\right\}.$$

The following definitions and lemmas are well known in the FC of real valued functions. They can be naturally extended to the Banach space valued case as follows.

**Definition 1.** [16] Let $\alpha > 0$ and let $[\alpha] := n-1$ be its integer part. The Riemann–Liouville fractional integral of order $\alpha$ of a function $l : J \longrightarrow \mathbb{E}$ is defined by

$$I_0^{\alpha} l(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^{\tau} (\tau-\theta)^{\alpha-1} l(\theta) d\theta,$$

provided that this integral exists.

**Definition 2.** [16] The right-side Caputo fractional derivative of order $\alpha$ of a function $l : J \longrightarrow \mathbb{E}$ is defined by

$$
{}^{C}D_0^{\alpha} l(\tau) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^{\tau} \frac{l^{(n)}(\theta)}{(\tau-\theta)^{\alpha+1-n}} d\theta & (n-1<\alpha<n), \\ \\ l^{(\alpha)}(\tau) & (\alpha = n). \end{cases}
$$

In the both previous definitions, $\Gamma(\cdot)$ is the Gamma function

$$
\Gamma(\varsigma) = \int_0^{\infty} e^{-u} u^{\varsigma-1} du, \quad \Gamma(\varsigma+1) = \varsigma\Gamma(\varsigma), \quad Re(\varsigma) > 0.
$$

**Lemma 1.** [16]

(i) If $g \in L^1(J, \mathbb{E}),\ \alpha > 0,\ \beta > 0$, then

$$
I_0^{\alpha} I_0^{\beta} g(\tau) = I_0^{\alpha+\beta} g(\tau).
$$

(ii) Let $g \in L^1(J, \mathbb{E}),\ \beta > \alpha > 0$; then

$$
{}^{C}D_0^{\alpha}\ I_0^{\beta} g(\tau) = I_0^{\beta-\alpha} g(\tau).
$$

(iii) If $\alpha > 0,\ \xi > -1$, then

$$
I_0^{\alpha}\left(\tau^{\xi} I_{\mathbb{E}}\right) = \frac{\Gamma(\xi+1)}{\Gamma(\alpha+\xi+1)} \tau^{\alpha+\xi} I_{\mathbb{E}}.
$$

(iv) If $\alpha > 0,\ \xi > -1$, then

$$
{}^{C}D_0^{\alpha}\left(\tau^{\xi} I_{\mathbb{E}}\right) = \frac{\Gamma(\xi+1)}{\Gamma(\alpha-\xi+1)} \tau^{\alpha-\xi} I_{\mathbb{E}},
$$

where $I_{\mathbb{E}}$ is the identity operator in $\mathbb{E}$.

**Lemma 2.** [16] Let $y \in C(J, \mathbb{E}),\ 0 < n-1 \leqslant \alpha < n$ and let $\phi \in L^1(J,\ \mathbb{E})$. Then, the general solution of the fractional differential equation

$$
{}^{C}D^{\alpha} y(\tau) + \phi(\tau) = 0,\ (0 \leqslant \tau \leqslant 1)
$$

is given by

$$
y(\tau) = \frac{-1}{\Gamma(\alpha)} \int_0^{\tau} (\tau-\theta)^{\alpha-1} \phi(\theta) d\theta + c_0 + c_1\tau + \cdots + c_{n-1}\tau^{n-1},
$$

where $c_0, c_1, \ldots, c_{n-1}$ are constants vectors in $\mathbb{E}$.

**Definition 3.** Let $(\mathcal{Q}, d)$ be a complet metric space. A map $H : \mathcal{Q} \longrightarrow \mathcal{Q}$ is called a contraction if there exists $0 \leqslant L < 1$ such that

$$d(Hy_1,\ Hy_2) \leqslant Ld(y_1,\ y_2), \quad \text{for all} \quad y_1,\ y_2 \in \mathcal{Q}.$$

**Lemma 3.** [26] (Banach contraction principle) Let $\mathcal{Q}$ be a complete metric space and let $H : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a contraction. Then, there exists a unique $y \in \mathcal{Q}$ such that $Hy = y$.

**Theorem 1** (Krasnosel'skii's fixed point theorem [17])**.** Let $\mathcal{S}$ be a closed, convex, and nonempty subset of a Banach space $\mathcal{L}$. Let $M,\ N$ be operators in $\mathcal{L}$ such that
(a) $My + Nz \in \mathcal{S}$ whenever $y, z \in \mathcal{S}$;
(b) $M$ is compact and continuous;
(c) $N$ is a contraction.
Then, there exists $x \in \mathcal{S}$ such that $Mx + Nx = x$.

The following lemma regroups some well-known facts about the bounded linear operators in Banach spaces [29].

**Lemma 4.** Let $\mathbb{E}$ be a Banach space.
1) If $A \in \mathcal{B}(\mathbb{E})$ is a bijection, then $A^{-1}$ is also an element of $\mathcal{B}(\mathbb{E})$.
2) If $A,\ B$ are two commuting elements of $\mathcal{B}(\mathbb{E})$ and $A$ is a bijection, then $A^{-1}$ and $B$ also commute.
3) If $A,\ B$ are two elements of $\mathcal{B}(\mathbb{E})$, then

$$\|AB\|_{\mathcal{B}(\mathbb{E})} \leqslant \|A\|_{\mathcal{B}(\mathbb{E})} \|B\|_{\mathcal{B}(\mathbb{E})}.$$

4) If $A,\ B \in \mathcal{B}(\mathbb{E})$ are such that $A$ is bijective and $\|B\|_{\mathcal{B}(\mathbb{E})} < \frac{1}{\|A^{-1}\|_{\mathcal{B}(\mathbb{E})}}$, then also $A + B$ is bijective.

# 3 Main results

## 3.1 Auxiliary results

We begin this section by introducing the following notations that will be used through the reminder of the article. Let $\mathcal{B}_0$ be a set of commuting elements of $\mathcal{B}(\mathbb{E})$ and let $j, p, q$ be nonnegative integers such that $q \leqslant j \leqslant \alpha - q - 1$. For each $A \in \mathcal{B}_0$, we set

$$w(A, j, p, q) := \frac{\Gamma(j+1)}{\Gamma(j-q+1)} A - \frac{\Gamma(j+1)}{\Gamma(j+p+1)} I_{\mathbb{E}}, \tag{7}$$

$$z(\theta, A, p, q) = \frac{(1-\theta)^{\alpha-q-1}}{\Gamma(\alpha-q)} A - \frac{(1-\theta)^{\alpha+p-1}}{\Gamma(\alpha+p)} I_{\mathbb{E}}, \quad \theta \in J, \tag{8}$$

where $I_{\mathbb{E}}$ is the identity in $\mathbb{E}$. We also set

$$\mathcal{W} = \left\{ w(A, j, p, q) : \ A \in \mathcal{B}_0, \ j, p, q \in \mathbb{Z}^+ \right\};$$

$$\mathcal{Z} = \left\{ z(\theta, A, p, q) : \ \theta \in J, \ A \in \mathcal{B}_0, \ p, q \in \mathbb{Z}^+ \right\}.$$

**Remark 1.** The following properties are verified:

1. Each of $\mathcal{W}$ and $\mathcal{Z}$ is a subset of commuting elements of $\mathcal{B}(\mathbb{E})$ with the following norm estimations:

$$\|w(A, j, p, q)\|_{\mathcal{B}(\mathbb{E})} \leqslant \frac{\Gamma(j+1)}{\Gamma(j-q+1)} \|A\|_{\mathcal{B}(\mathbb{E})} + \frac{\Gamma(j+1)}{\Gamma(j+p+1)}, \tag{9}$$

$$\|z(\theta, A, p, q)\|_{\mathcal{B}(\mathbb{E})} \leqslant \frac{\|A\|_{\mathcal{B}(\mathbb{E})}}{\Gamma(\alpha-q)} + \frac{1}{\Gamma(\alpha+p)}, \qquad \text{for all } \ \theta \in J. \tag{10}$$

2. Each element of $\mathcal{W}$ commutes with each element of $\mathcal{Z}$.

Suppose now that the nonnegative integers $j_1, \ j_2, \ p_1, \ p_2, \ q_1, \ q_2$ satisfy (5) and $\mathcal{B}_0 = \{A_1, \ A_2\}$, where $A_1, \ A_2$ are the commuting elements of $\mathcal{B}(\mathbb{E})$ appearing in the relation (4) and let

$$U = w(A_1, j_1, p_1, q_1) w(A_2, j_2, p_2, q_2) - w(A_1, j_2, p_1, q_1) w(A_2, j_1, p_2, q_2), \tag{11}$$

$$\begin{aligned} \Phi_1(\theta) = \ & w(A_2, j_2, p_2, q_2) z(\theta, A_1, p_1, q_1) \\ & - w(A_1, j_2, p_1, q_1) z(\theta, A_2, p_2, q_2), \quad \theta \in J, \end{aligned} \tag{12}$$

$$\begin{aligned} \Phi_2(\theta) = \ & w(A_1, j_1, p_1, q_1) z(\theta, A_2, p_2, q_2) \\ & - w(A_2, j_1, p_2, q_2) z(\theta, A_1, p_1, q_1), \quad \theta \in J. \end{aligned} \tag{13}$$

Using the commutativity of the operators, one can check that

$$w(A_1, j_1, p_1, q_1)\Phi_1(\theta) + w(A_1, j_2, p_1, q_1)\Phi_2(\theta) = Uz(\theta, A_1, p_1, q_1) \quad (\theta \in J) \tag{14}$$

and

$$w(A_2, j_1, p_2, q_2)\Phi_1(\theta) + w(A_2, j_2, p_2, q_2)\Phi_2(\theta) = Uz(\theta, A_2, p_2, q_2), \ (\theta \in J). \tag{15}$$

**Lemma 5.** Let $\phi \in L^1(J, \mathbb{E})$. If the operator $U$ is invertible, then the unique solution of the boundary value problem

$$({}^C D^\alpha y)(\tau) + \phi(\tau) = 0, \qquad 0 < \tau \leqslant 1, \ \ \alpha \geqslant 3 \tag{16}$$

with the conditions (3), (4), (5) is given by

$$y(\tau) = \int_0^1 G(\tau, \theta)\,(\phi(\theta))\, d\theta, \tag{17}$$

where

$$G(\tau, \theta) = \begin{cases} \frac{-(\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)} I_{\mathbb{E}} + \tau^{j_1} U^{-1}\Phi_1(\theta) + \tau^{j_2} U^{-1}\Phi_2(\theta), & 0 \leqslant \theta < \tau \leqslant 1, \\ \tau^{j_1} U^{-1}\Phi_1(\theta) + \tau^{j_2} U^{-1}\Phi_2(\theta), & 0 \leqslant \tau \leqslant \theta \leqslant 1. \end{cases} \tag{18}$$

*Proof.* Note firstly that

$$\Phi_1(\theta) \in \mathcal{B}(\mathbb{E}), \ \Phi_2(\theta) \in \mathcal{B}(\mathbb{E}), \ G(\tau, \theta) \in \mathcal{B}(\mathbb{E}), \quad \text{for all} \ \ (\theta, \ \tau) \in J^2.$$

According to Lemma 2, the solution of the problem (16) has the form

$$y(\tau) = \frac{-1}{\Gamma(\alpha)} \int_0^\tau (\tau - \theta)^{\alpha-1}\phi(\theta) d\theta + c_0 + \tau c_1 + \cdots + \tau^{n-1} c_{n-1},$$

where $c_0, c_1, \ldots, c_{n-1}$ are constant vectors in $\mathbb{E}$. Using the initial conditions at $\tau = 0$, we obtain that $c_k = 0$ for $k \notin \{j_1, j_2\}$ and then,

$$y(\tau) = \frac{-1}{\Gamma(\alpha)} \int_0^\tau (\tau - \theta)^{\alpha-1}\phi(\theta) d\theta + \tau^{j_1} c_{j_1} + \tau^{j_2} c_{j_2} \quad (\tau \in J).$$

The integral conditions

$$A_k \left( y^{(q_k)}(1) \right) = \frac{1}{\Gamma(p_k)} \int_0^1 (1-\theta)^{p_k - 1} y(\theta) d\theta, \quad k = \overline{1, 2}$$

give us

$$A_k\left(\frac{-1}{\Gamma(\alpha-q_k)}\int_0^1(1-\theta)^{\alpha-q_k-1}\phi(\theta)d\theta+\frac{\Gamma(j_1+1)}{\Gamma(j_1-q_k+1)}c_{j_1}+\frac{\Gamma(j_2+1)}{\Gamma(j_2-q_k+1)}c_{j_2}\right)$$

$$=\frac{-1}{\Gamma(\alpha+p_k)}\int_0^1(1-\theta)^{\alpha+p_k-1}\phi(\theta)d\theta+\frac{\Gamma(j_1+1)}{\Gamma(j_1+p_k+1)}c_{j_1}+\frac{\Gamma(j_2+1)}{\Gamma(j_2+p_k+1)}c_{j_2},$$

or equivalently,

$$\begin{aligned}&\left(\frac{\Gamma(j_1+1)}{\Gamma(j_1-q_k+1)}A_k-\frac{\Gamma(j_1+1)}{\Gamma(j_1+p_k+1)}I_{\mathbb{E}}\right)(c_{j_1})\\&+\left(\frac{\Gamma(j_2+1)}{\Gamma(j_2-q_k+1)}A_k-\frac{\Gamma(j_2+1)}{\Gamma(j_2+p_k+1)}I_{\mathbb{E}}\right)(c_{j_2})\\&=\int_0^1\left(\frac{(1-\theta)^{\alpha-q_k-1}}{\Gamma(\alpha-q_k)}A_k-\frac{(1-\theta)^{\alpha+p_k-1}}{\Gamma(\alpha+p_k)}I_{\mathbb{E}}\right)(\phi(\theta))d\theta.\end{aligned}$$

Taking in account formulas (7) and (8), we get finally that the constant vectors $c_{j_1}$ and $c_{j_2}$ satisfy the following linear system:

$$\begin{cases}w(A_1,j_1,p_1,q_1)(c_{j_1})+w(A_1,j_2,p_1,q_1)(c_{j_2})=\int_0^1 z(\theta,A_1,p_1,q_1)\,(\phi(\theta))\,d\theta,\\[2ex] w(A_2,j_1,p_2,q_2)(c_{j_1})+w(A_2,j_2,p_2,q_2)(c_{j_2})=\int_0^1 z(\theta,A_2,p_2,q_2)\,(\phi(\theta))\,d\theta.\end{cases}$$

By applying the operator $w(A_2,j_2,p_2,q_2)$ to the first equation and the operator $w(A_1,j_2,p_1,q_1)$ to the second equation and then subtracting, we get

$$\begin{aligned}U\left(c_{j_1}\right)&=\int_0^1\begin{pmatrix}w(A_2,j_2,p_2,q_2)z(\theta,A_1,p_1,q_1)\\ \\ -\,w(A_1,j_2,p_1,q_1)z(\theta,A_2,p_2,q_2)\end{pmatrix}(\phi(\theta))\,d\theta\\&=\int_0^1\Phi_1(\theta)\,(\phi(\theta))\,d\theta.\end{aligned}$$

Similarly, by applying the operator $w(A_2,j_1,p_2,q_2)$ to the first equation and the operator $w(A_1,j_1,p_1,q_1)$ to the second equation and then subtracting, we get

$$\begin{aligned}U\left(c_{j_2}\right)&=\int_0^1\begin{pmatrix}w(A_1,j_1,p_1,q_1)z(\theta,A_2,p_2,q_2)\\ \\ -\,w(A_2,j_1,p_2,q_2)z(\theta,A_1,p_1,q_1)\end{pmatrix}(\phi(\theta))\,d\theta\\&=\int_0^1\Phi_2(\theta)\,(\phi(\theta))\,d\theta.\end{aligned}$$

Since the operator $U$ is autonomous with respect to $\tau \in J$ and invertible by hypothesis, we have

$$(c_{j_1}, c_{j_2}) = \left( \int_0^1 U^{-1}\Phi_1(\theta)\left(\phi(\theta)\right) d\theta,\ \int_0^1 U^{-1}\Phi_2(\theta)\left(\phi(\theta)\right) d\theta \right).$$

Replacing in the expression of $y(\tau)$, we get finally

$$\begin{aligned} y(\tau) &= \frac{-1}{\Gamma(\alpha)} \int_0^\tau (\tau-\theta)^{\alpha-1}\phi(\theta)d\theta + \tau^{j_1}c_{j_1} + \tau^{j_2}c_{j_2} \\ &= \frac{-1}{\Gamma(\alpha)} \int_0^\tau (\tau-\theta)^{\alpha-1}\phi(\theta)d\theta + \tau^{j_1}\int_0^1 U^{-1}\Phi_1(\theta)\left(\phi(\theta)\right) d\theta \\ &\quad + \tau^{j_2}\int_0^1 U^{-1}\Phi_2(\theta)\left(\phi(\theta)\right) d\theta \\ &= \int_0^1 G(\tau,\theta)\left(\phi(\theta)\right) d\theta. \end{aligned}$$

Conversely, assume that the function $y(\tau),\ \tau \in J$ admits the representation given by (17), (18). Observing that

$$y(\tau) = -I_{0^+}^{\alpha}\phi(\tau) + \tau^{j_1}\int_0^1 U^{-1}\Phi_1(\theta)\left(\phi(\theta)\right) d\theta + \tau^{j_2}\int_0^1 U^{-1}\Phi_2(\theta)\left(\phi(\theta)\right) d\theta.$$

It is not difficult to check that $({}^{C}D^{\alpha}y)(\tau) + \phi(\tau) = 0,\ 0 < \tau \leqslant 1$ and that the initial conditions (3) are all satisfied. Let us prove the condition (4) for $k = 1$ (the case where $k = 2$ can be obtained by the same way). Direct calculations give

$$\begin{aligned} A_1\left(y^{(q_1)}(1)\right) &= \int_0^1 \frac{-(1-\theta)^{\alpha-q_1-1}}{\Gamma(\alpha-q_1)} A_1\left(\phi(\theta)\right) d\theta \\ &\quad + \int_0^1 \frac{\Gamma(j_1+1)}{\Gamma(j_1-q_1+1)} A_1U^{-1}\Phi_1(\theta)\left(\phi(\theta)\right) d\theta \\ &\quad + \int_0^1 \frac{\Gamma(j_2+1)}{\Gamma(j_2-q_1+1)} A_1U^{-1}\Phi_2(\theta)\left(\phi(\theta)\right) d\theta, \end{aligned}$$

$$\begin{aligned} I_{0^+}^{p_1}y(1) &= \int_0^1 \frac{-(1-\theta)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)} \phi(\theta)d\theta \\ &\quad + \int_0^1 \frac{\Gamma(j_1+1)}{\Gamma(p_1+j_1+1)} U^{-1}\Phi_1(\theta)\left(\phi(\theta)\right) d\theta \\ &\quad + \int_0^1 \frac{\Gamma(j_2+1)}{\Gamma(p_1+j_2+1)} U^{-1}\Phi_2(\theta)\left(\phi(\theta)\right) d\theta. \end{aligned}$$

Hence, we deduce that

$$
\begin{aligned}
I_{0+}^{p_1} y(1) - A_1\left(y^{(q_1)}(1)\right) = &\int_0^1 z(\theta, A_1, p_1, q_1)\left(\phi(\theta)\right) d\theta \\
&- \int_0^1 w(A_1, j_1, p_1, q_1) U^{-1}\Phi_1(\theta)\left(\phi(\theta)\right) d\theta \\
&- \int_0^1 w(A_1, j_2, p_1, q_1) U^{-1}\Phi_2(\theta)\left(\phi(\theta)\right) d\theta.
\end{aligned}
$$

By (14), $I_{0+}^{p_1} y(1) - A_1\left(y^{(q_1)}(1)\right) = 0$. □

In the best of our knowledge, the kernel $G(\tau, \theta)$ in the previous lemma is obtained for the first time. It is an uniformly bounded linear operator according to the norm-majoration (22) below. Moreover,

$$
G(0, \theta) = 0 = G(\tau, 1), \quad \text{for all} \ \ \tau,\ \theta \in [0, 1].
$$

Function $\widetilde{G}(\tau, \theta) = -\frac{(\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)} I_{\mathbb{E}}$ is such that ${}^{C}D^{\alpha}\left(\int_0^1 \widetilde{G}(\tau, \theta) h(\theta) d\theta\right) = h(\tau)$. Thus, it can be regarded as the Green's operator of the fractional differential equation ${}^{C}D^{\alpha}x(\tau) + h(\tau) = 0$.

If we set $\mathbb{E} = \mathbb{R},\ q_1 = q_2 := q,\ j_1 = j_2 := j,\ p_1 = p_2 := p$ and $A_1 = A_2 := a,\ a \geqslant 1$, then the problem (2)–(4) becomes

$$
{}^{C}D^{\alpha} y(\tau) + \psi(\tau, y(\tau)) = 0,\ \tau \in [0, 1]\,,\ \alpha \geqslant 3, \tag{19}
$$

$$
y^{(j)}(0) = 0,\ j \in \{0, 1, \ldots, n-1\} \setminus \{j\}\,,\ n = [\alpha] + 1, \tag{20}
$$

$$
a y^{(q)}(1) = \frac{1}{\Gamma(p)} \int_0^1 (1-\theta)^{p-1} y(\theta) d\theta,\ p\ \in \mathbb{N}^{*}. \tag{21}
$$

Setting in this new problem $q = 0,\ j = 2$, we retrieve the problem investigated in [22].

**Remark 2.** a) Since all the real numbers $\alpha - 1,\ \alpha - q_1 - 1,\ \alpha - q_2 - 1$ are positive (see the condition (5)), all the functions

$$
\theta \longmapsto (1-\theta)^{\alpha - q_1 - 1}, \quad \theta \longmapsto (1-\theta)^{\alpha - q_2 - 1}, \quad (\tau, \theta) \longmapsto (\tau - \theta)^{\alpha - 1}
$$

are continuous. This leads to the continuity of the operator-valued functions $\Phi_1(\theta),\ \Phi_2(\theta)$ from $J$ into $\mathcal{B}(\mathbb{E})$ and the continuity of the operator-valued function $G(\tau, \theta)$ from $J \times J$ into $\mathcal{B}(\mathbb{E})$.

b) Obviously,

$$
\sup_{\theta \in J}\left(\left\|U^{-1}\Phi_k(\theta)\right\|_{\mathcal{B}(\mathbb{E})}\right) \leqslant \left\|U^{-1}\right\|_{\mathcal{B}(\mathbb{E})}\left(\sup_{\theta \in J}\left\|\Phi_k(\theta)\right)\right\|_{\mathcal{B}(\mathbb{E})}\right) < +\infty, \quad k = \overline{1, 2},
$$

and by (18), we have

$$\delta=\sup_{\theta,\tau\in J}\left(\|G(\tau,\theta)\|_{\mathcal{B}(\mathbb{E})}\right)\leqslant\Delta:=\frac{1}{\Gamma(\alpha)}+\left\|U^{-1}\right\|_{\mathcal{B}(\mathbb{E})}\sum_{k=1}^{2}\sup_{\theta\in J}\left(\|\Phi_k(\theta))\|_{\mathcal{B}(\mathbb{E})}\right). \tag{22}$$

## 3.2 Existence of solutions

In the present and the following sections, $\psi$ is the function listed in the problem (2). We shall also suppose that there exist three real constants $k>0,\ m>0$, and $0<\sigma<1$ such that for all $\tau\in[0,\ 1]$,

$$\begin{aligned}\|\tau^{\sigma}\psi(\tau,y_1(\tau))-\tau^{\sigma}\psi(\tau,y_2(\tau))\|_{\mathbb{E}}\leqslant&k\|y_1-y_2\|_{C(J,\mathbb{E})},\\ &\text{for all}\ \ (y_1,y_2)\in(C(J,\mathbb{E}))^2,\end{aligned} \tag{23}$$

and

$$\sup_{\tau\in J}\|\tau^{\sigma}\psi(\tau,0)\|_{\mathbb{E}}\leqslant m. \tag{24}$$

**Lemma 6.** For the function $\psi$, the following holds:

$$\int_{\tau_0}^{\tau}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}d\theta\leqslant(k\|y\|_{C(J,\mathbb{E})}+m)\frac{\tau^{1-\sigma}-\tau_0^{1-\sigma}}{1-\sigma}$$

for all $(\tau,y)\in[0,\ 1]\times C(J,\mathbb{E})$. In particular,

$$\int_{0}^{\tau}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}d\theta=\lim_{\tau_0\longrightarrow0^+}\int_{\tau_0}^{\tau}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}d\theta\leqslant(k\|y\|_{C(J,\mathbb{E})}+m)\frac{\tau^{1-\sigma}}{1-\sigma}.$$

*Proof.* Indeed, if $0\leqslant\theta\leqslant1$ and $y\in C(J,\mathbb{E})$, then we get by using the conditions (23) and (24),

$$\begin{aligned}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}&\leqslant\|\psi(\theta,y(\theta))-\psi(\theta,0)\|_{\mathbb{E}}+\|\psi(\theta,0)\|_{\mathbb{E}}\\&=\theta^{-\sigma}\|\theta^{\sigma}\psi(\theta,y(\theta))-\theta^{\sigma}\psi(\theta,0)\|_{\mathbb{E}}+\theta^{-\sigma}\|\theta^{\sigma}\psi(\theta,0)\|_{\mathbb{E}}\\&\leqslant(k\|y\|_{C(J,\mathbb{E})}+m)\theta^{-\sigma}.\end{aligned}$$

Thus, for $0<\tau_0<\tau\leqslant1$ and $y\in C(J,\mathbb{E})$,

$$\int_{\tau_0}^{\tau}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}d\theta\leqslant(k\|y\|_{C(J,\mathbb{E})}+m)\int_{\tau_0}^{\tau}\theta^{-\sigma}d\theta$$

$$= (k\|y\|_{C(J,\mathbb{E})} + m)\frac{\tau^{1-\sigma} - \tau_0^{1-\sigma}}{1-\sigma}.$$

□

It follows from this lemma that the function $[0,\ 1] \ni \theta \longmapsto \psi(\theta, y(\theta))$ belongs to $L^1([0,\ 1], \mathbb{E})$ for every fixed in $C(J,\mathbb{E})$ function $y$. So, the fractional integral $I_0^\alpha \psi(\theta, y(\theta)),\ 0 \leqslant \tau \leqslant 1$ is well defined.

**Theorem 2.** Let the operator $U$ be invertible, let the constants $\sigma$ and $k$ satisfy formulas (23) and (24). If $0 < k < \frac{1-\sigma}{\delta}$, then the problem (2)–(4) admits a unique solution $y \in C(J,\mathbb{E})$. Moreover,

$$\|y\|_{C(J,\mathbb{E})} \leqslant \frac{m\delta}{1-\sigma-k\delta}.$$

*Proof.* Recall firstly that according to (22),

$$\delta = \sup_{\theta,\tau\in J}\left(\|G(\tau,\theta)\|_{\mathcal{B}(\mathbb{E})}\right) < \infty.$$

Consider in $C(J,\mathbb{E})$ the self mapping $\Psi$, defined by

$$y \longmapsto \Psi y(\tau) = \int_0^1 G(\tau,\theta)\psi(\theta, y(\theta))d\theta. \tag{25}$$

Therefore, $\Psi$ is well defined and according to Lemma 5, the fixed points of $\Psi$ are the solutions of the problem (2)–(4). So, to prove the theorem, we need to prove that under the given conditions, the operator $\Psi$ has a unique fixed point. For instance, let $y_1,\ y_2$ be two elements of $C(J,\mathbb{E})$ and $0 \leqslant \tau \leqslant 1$. Then

$$\begin{aligned}
\|\Psi y_1(\tau) - \Psi y_2(\tau)\|_{\mathbb{E}} &= \left\|\int_0^1 G(\tau,\theta)\psi(\theta,y_1(\theta))d\theta - \int_0^1 G(\tau,\theta)\psi(\theta,y_2(\theta))d\theta\right\|_{\mathbb{E}} \\
&\leqslant \int_0^1 \|G(\tau,\theta)\|_{\mathcal{B}(\mathbb{E})}\|\psi(\theta,y_1(\theta)) - \psi(\theta,y_2(\theta))\|_{\mathbb{E}}d\theta \\
&\leqslant \delta\int_0^1 \theta^{-\sigma}\|\theta^\sigma\psi(\theta,y_1(\theta)) - \theta^\sigma\psi(\theta,y_2(\theta))\|_{\mathbb{E}}d\theta \\
&\leqslant k\delta\|y_1-y_2\|_{C(J,\mathbb{E})}\int_0^1 \theta^{-\sigma}d\theta = \frac{k\delta}{1-\sigma}\|y_1-y_2\|_{C(J,\mathbb{E})}.
\end{aligned}$$

Consequently,

$$\|\Psi y_1 - \Psi y_2\|_{C(J,\mathbb{E})} \leqslant \frac{k\delta}{1-\sigma}\|y_1-y_2\|_{C(J,\mathbb{E})}.$$

Since $0 < \frac{k\delta}{1-\sigma} < 1$, then by the Banach contraction principle (Lemma 3), the operator $\Psi$ has a unique fixed point $y$ in $C(J,\mathbb{E})$, which is the unique solution of the problem (2)–(4). On the other hand, taking in account Lemma 6, we have

$$\|y\|_{C(J,\mathbb{E})} = \|\Psi y\|_{C(J,\mathbb{E})} \leqslant \frac{k\|y\|_{C(J,\mathbb{E})} + m}{1-\sigma}\delta = \frac{k\delta}{1-\sigma}\|y\|_{C(J,\mathbb{E})} + \frac{m\delta}{1-\sigma}.$$

Consequently,

$$\left(1 - \frac{k\delta}{1-\sigma}\right)\|y\|_{C(J,\mathbb{E})} \leqslant \frac{m\delta}{1-\sigma},$$

or equivalently,

$$\|y\|_{C(J,\mathbb{E})} \leqslant \frac{m\delta}{1-\sigma-k\delta}.$$

□

Next, we give an existence result.

**Theorem 3.** Let the operator $U$ be invertible, and assume that (23) is verified with the constants $k,\ \sigma$ such that

$$0 < k < \frac{1-\sigma}{\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sum_{i=1}^{2}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)}.$$

Assume also that there exists a function $\mu \in C(J,\mathbb{R}^+)$ such that

$$\tau^{\sigma}\|\psi(\tau,u)\|_{\mathbb{E}} \leqslant \mu(\tau), \quad \text{for all} \quad \tau \in J, \quad \text{for all} \quad u \in \mathbb{E}. \tag{26}$$

Then, the problem (2)–(4) has at least one solution.

*Proof.* For the proof, we shall use the Krasnosel'skii's theorem (Theorem 1). For instance, consider the maps $M,\ N$ defined in $C(J,\mathbb{E})$ by

$$\begin{aligned} My(\tau) &= \frac{-1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\theta)^{\alpha-1}\psi(\theta,y(\theta))d\theta,\\ Ny(\tau) &= \sum_{i=1}^{2}\tau^{j_i}\int_0^1 U^{-1}\Phi_i(\theta)\psi(\theta,y(\theta))d\theta. \end{aligned}$$

Observe that $\Psi = M + N$, and let

$$R \geqslant \|\mu\|_{C(J,\mathbb{R}^+)}\frac{1+\Gamma(\alpha)\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sum_{i=1}^{2}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)}{\Gamma(\alpha)(1-\sigma)},$$

$$\mathcal{Q}_R = \left\{y \in C(J,\mathbb{E}) : \ \|y\|_{C(J,\mathbb{E})} \leqslant R\right\}.$$

Hence $\mathcal{Q}_R$ is a closed, bounded and convex nonempty subset of $C(J,\mathbb{E})$, the proof will be given in several steps.

**Step 1:** *$My_1+Ny_2\in\mathcal{Q}_R$, for $y_1,\ y_2\in\mathcal{Q}_R$.*

For $\tau\in J$ and $y_1,y_2\in\mathcal{Q}_R$, we have

$$\begin{aligned}
&\|My_1(\tau)+Ny_2(\tau)\|_{\mathbb{E}}\\
&\quad\leqslant \|My_1(\tau)\|_{\mathbb{E}}+\|Ny_2(\tau)\|_{\mathbb{E}}\\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^\tau \|(\tau-\theta)^{\alpha-1}\psi(\theta,y_1(\theta))\|_{\mathbb{E}}d\theta\\
&\qquad+\sum_{i=1}^{2}\tau^{j_i}\int_0^1 \|U^{-1}\Phi_i(\theta)\psi(\theta,y_2(\theta))\|_{\mathbb{E}}d\theta\\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^\tau \|\psi(\theta,y_1(\theta))\|_{\mathbb{E}}d\theta\\
&\qquad+\sum_{i=1}^{2}\int_0^1 \|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)\|\psi(\theta,y_2(\theta))\|_{\mathbb{E}}d\theta\\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^\tau \mu(\theta)\theta^{-\sigma}d\theta\\
&\qquad+\sum_{i=1}^{2}\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)\int_0^1 \mu(\theta)\theta^{-\sigma}d\theta\\
&\quad\leqslant \|\mu\|_{C(J,\mathbb{R}^+)}\left(\frac{1}{\Gamma(\alpha)}+\sum_{i=1}^{2}\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)\right)\int_0^1\theta^{-\sigma}d\theta\\
&\quad= \|\mu\|_{C(J,\mathbb{R}^+)}\frac{1+\Gamma(\alpha)\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sum_{i=1}^{2}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)}{\Gamma(\alpha)(1-\sigma)}\\
&\quad\leqslant R.
\end{aligned}$$

Thus, $y_1,y_2\in\mathcal{Q}_R\Longrightarrow My_1+Ny_2\in\mathcal{Q}_R$.

**Step 2:** *$N$ is a contraction.*

For $y_1,y_2\in C(J,\mathbb{E})$ and $\tau\in J=[0,\ 1]$, we have

$$\begin{aligned}
&\|Ny_1(\tau)-Ny_2(\tau)\|_{\mathbb{E}}\\
&\quad\leqslant \sum_{i=1}^{2}\tau^{j_i}\int_0^1 \|U^{-1}\Phi_i(\theta)\left(\psi(\theta,y_1(\theta))-\psi(\theta,y_2(\theta))\right)\|_{\mathbb{E}}d\theta\\
&\quad\leqslant \|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\left(\sum_{i=1}^{2}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)\right)\\
&\qquad\times\int_0^1\theta^{-\sigma}\theta^{\sigma}\|\psi(\theta,y_1(\theta))-\psi(\theta,y_2(\theta))\|_{\mathbb{E}}d\theta
\end{aligned}$$

$$\leqslant k\|y_1-y_2\|_{C(J,\mathbb{E})}\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\left(\sum_{i=1}^{2}\sup_{\xi\in J}\left(\|\Phi_i(\xi)\|_{\mathcal{B}(\mathbb{E})}\right)\right)\int_0^1\theta^{-\sigma}d\theta$$
$$= k\frac{\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sum_{i=1}^{2}\|\Phi_i\|_{C(J,\mathbb{E})}}{1-\sigma}\|y_1-y_2\|_{C(J,\mathbb{E})}.$$

By hypothesis, $0<k\frac{\|U^{-1}\|_{\mathcal{B}(\mathbb{E})}\sum_{i=1}^{2}\|\Phi_i\|_{C(J,\mathbb{E})}}{1-\sigma}<1$. Thus, $N$ is a contraction.

**Step 3:** *$M$ is compact and continuous.*

The proof will be given in three Claims

**Claim 1:** *$M$ is continuous.* Let $y_1$ and $y_2$ be two arbitrary elements of $C(J,\mathbb{E})$ and let $\tau\in J$. Then

$$\begin{aligned}\|My_1(\tau)-My_2(\tau)\|_{\mathbb{E}} &\leqslant \left\|\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-\theta)^{\alpha-1}(\psi(\theta,y_1(\theta))-\psi(\theta,y_2(\theta)))d\theta\right\|_{\mathbb{E}}\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1\theta^{-\sigma}\theta^{\sigma}\|\psi(\theta,y_1(\theta))-\psi(\theta,y_2(\theta))\|_{\mathbb{E}}d\theta\\ &\leqslant \frac{k}{\Gamma(\alpha)}\|y_1-y_2\|_{C(J,\mathbb{E})}\int_0^1\theta^{-\sigma}d\theta\\ &\leqslant \frac{k}{(1-\sigma)\Gamma(\alpha)}\|y_1-y_2\|_{C(J,\mathbb{E})}.\end{aligned}$$

Hence,

$$y_1,\ y_2\in C(J,\mathbb{E})\Longrightarrow\|My_1-My_2\|_{C(J,\mathbb{E})}\leqslant\frac{k}{(1-\sigma)\Gamma(\alpha)}\|y_1-y_2\|_{C(J,\mathbb{E})}.$$

The last relation expresses that the operator $M$ is $\dfrac{k}{(1-\sigma)\Gamma(\alpha)}$-Lypschitzian and so, is continuous.

**Claim 2:** *$M$ maps bounded sets into bounded sets in $C(J,\mathbb{E})$.*

It is enough to show that there exists a positive constant $r$ such that for each $y\in\mathcal{Q}_R$, one has $\|M(y)\|_{C(J,\mathbb{E})}\leqslant r$. In fact, we have

$$\begin{aligned}\|(My)(\tau)\|_{\mathbb{E}} &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1(\tau-\theta)^{\alpha-1}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}d\theta\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1\theta^{-\sigma}\theta^{\sigma}\|\psi(\theta,y(\theta))\|_{\mathbb{E}}d\theta\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1\theta^{-\sigma}\mu(\theta)d\theta\\ &\leqslant \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\int_0^1\theta^{-\sigma}d\theta.\end{aligned}$$

The operator $M$ is uniformly bounded in $\mathcal{Q}_R$ as

$$\|My\|_{C(J,\mathbb{E})} = \sup_{\tau\in J}(\|My(\tau)\|_{\mathbb{E}}) \leqslant r := \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{(1-\sigma)\Gamma(\alpha)}, \quad \text{for all } \ y \in \mathcal{Q}_R.$$

**Claim 3:** *$M$ maps each $\mathcal{Q}_R$ into an equicontinuous subset of $C(J,\mathbb{E})$.*
Let $\tau_1, \tau_2 \in J$ such that $\tau_1 < \tau_2$ and let $y \in \mathcal{Q}_R$. Then,

$$\begin{aligned}
&\|My(\tau_1) - My(\tau_2)\|_{\mathbb{E}} \\
&\quad = \frac{1}{\Gamma(\alpha)}\left\|\int_0^{\tau_2}(\tau_2-\theta)^{\alpha-1}\psi(\theta,y(\theta))d\theta - \int_0^{\tau_1}(\tau_1-\theta)^{\alpha-1}\psi(\theta,y(\theta))d\theta\right\|_{\mathbb{E}} \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)}\left\|\int_0^{\tau_1}\left((\tau_2-\theta)^{\alpha-1} - (\tau_1-\theta)^{\alpha-1}\right)\psi(\theta,y(\theta))d\theta\right\|_{\mathbb{E}} \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left\|\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\alpha-1}\psi(\theta,y(\theta))d\theta\right\|_{\mathbb{E}} \\
&\quad \leqslant \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\int_0^{\tau_1}\left((\tau_2-\theta)^{\alpha-1} - (\tau_1-\theta)^{\alpha-1}\right)\theta^{-\sigma}d\theta \\
&\qquad + \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\alpha-1}\theta^{-\sigma}d\theta \\
&\quad \leqslant \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\int_0^{\tau_1}\left((\tau_2-\theta)^{\alpha-1} - (\tau_1-\theta)^{\alpha-1}\right)\theta^{-\sigma}d\theta \\
&\qquad + \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\theta^{-\sigma}d\theta \\
&\quad = \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\int_0^{\tau_1}\left((\tau_2-\theta)^{\alpha-1} - (\tau_1-\theta)^{\alpha-1}\right)\theta^{-\sigma}d\theta \\
&\qquad + \|\mu\|_{C(J,\mathbb{R}^+)}\frac{\tau_2^{1-\sigma} - \tau_1^{1-\sigma}}{(1-\sigma)\Gamma(\alpha)} \\
&\quad = \frac{\|\mu\|_{C(J,\mathbb{R}^+)}}{\Gamma(\alpha)}\left(\int_0^{\tau_1}\left((\tau_2-\theta)^{\alpha-1} - (\tau_1-\theta)^{\alpha-1}\right)\theta^{-\sigma}d\theta\right. \\
&\qquad \left. + \frac{\tau_2^{1-\sigma} - \tau_1^{1-\sigma}}{(1-\sigma)}\right).
\end{aligned}$$

Clearly, $\lim_{\tau_1 \longrightarrow \tau_2}\|My(\tau_1) - My(\tau_2)\|_{\mathbb{E}}$ tends to zero independently on $y \in C(J,\mathbb{E})$. Thus, $M$ is equicontinuous and so, it is a relatively compact operator on $\mathcal{Q}_R$. As a consequence of Claims 1, 2, 3 and together with the Arzelá -Ascoli theorem, we conclude that $M$ is compact. According to Theorem 1, we deduce that $Mx + Nx$ has a fixed point $x \in \mathcal{Q}_R$, which is a solution of the problem (2)–(4). $\square$

## 3.3 Ulam Stability

In this section, we investigate different types of Ulam's stability of the fractional boundary value problem (2). For instance, we adopt the following definitions.

**Definition 4.** [10] The fractional boundary value problem (2) is said to be Ulam–Hyers stable if there exists a real number $C_\psi > 0$ depending only on $\psi$ and such that for every $\varepsilon > 0$ and every solution $y \in C(J, \mathbb{E})$ of the inequality

$$\|{}^C D_0^\alpha y(\tau) + \psi(\tau, y(\tau))\|_{\mathbb{E}} \leqslant \varepsilon, \ \tau \in [0,1], \tag{27}$$

with the conditions

$$\begin{cases} y^{(j)}(0) = 0, \quad j \in \{0,1,\dots,n-1\} \setminus \{j_1, j_2\}, \ n = [\alpha] + 1, \\ A_k \left(y^{(q_k)}(1)\right) = \dfrac{1}{\Gamma(p_k)} \int_0^1 (1-\theta)^{p_k - 1} y(\theta) d\theta, \ p_k \ \in \mathbb{N}^*, \ k = \overline{1,2}, \end{cases} \tag{28}$$

there exists a solution $y_0 \in C(J, \mathbb{E})$ of the fractional value problem (2) satisfying

$$\|y(\tau) - y_0(\tau)\|_{\mathbb{E}} \leqslant C_\psi \varepsilon, \ \tau \in [0,1].$$

**Definition 5.** [10] The fractional boundary value problem (2) is said to be generalized Ulam–Hyers stable if there exists a function $\omega \in C(J, \mathbb{R}^+)$ with $\omega(0) = 0$ such that for every $\varepsilon > 0$ and every solution $y \in C(J, \mathbb{E})$ of the inequality (27) with the conditions (28), there exists a solution $y_0 \in C(J, \mathbb{E})$ of the fractional value problem (2) satisfying

$$\|y(\tau) - y_0(\tau)\|_{\mathbb{E}} \leqslant \omega(\varepsilon), \ \tau \in [0,1].$$

**Definition 6.** [10] The fractional boundary value problem (2) is said to be Ulam–Hyers–Rassias stable with respect to the function $\varphi : [0,1] \longrightarrow \mathbb{R}^+$ if there exists a real number $C_{\psi,\varphi} > 0$ depending only on $\psi, \ \varphi$ and such that for every $\varepsilon > 0$ and every solution $y \in C(J, \mathbb{E})$ of the inequality

$$\|{}^C D_0^\alpha y(\tau) + \psi(\tau, y(\tau))\|_{\mathbb{E}} \leqslant \varepsilon\varphi(\tau), \ \tau \in [0,1], \tag{29}$$

with the conditions (28), there exists a solution $y_0 \in C(J, \mathbb{E})$ of the fractional value problem (2) satisfying

$$\|y(\tau) - y_0(\tau)\|_{\mathbb{E}} \leqslant C_{\psi,\varphi} \varepsilon\varphi(\tau), \ \tau \in [0,1].$$

**Remark 3.** 1) A function $y \in C(J, \mathbb{E})$ is a solution of the inequality (27) if and only if there exists a function $h_y : [0,1] \longrightarrow \mathbb{E}$ such that

$$^{C}D_0^{\alpha} y(\tau) + \psi(\tau, y(\tau)) = h_y(\tau) \text{ and } \|h_y(\tau)\|_{\mathbb{E}} \leqslant \varepsilon, \quad (0 \leqslant \tau \leqslant 1). \tag{30}$$

2) Ulam–Hyers stability $\Longrightarrow$ Generalized Ulam–Hyers stability.

**Theorem 4.** Let $\psi : [0,1] \times \mathbb{E} \longrightarrow \mathbb{E}$ be a function satisfying (6), (23), and (24), and suppose that the operator $U$ is invertible. If $0 < k < \frac{1-\sigma}{\delta}$, then the problem (2) is Ulam–Hyers stable.

*Proof.* Recall once again that according to (22),

$$\delta = \sup_{\theta, \tau \in J} \left( \|G(\tau, \theta)\|_{\mathcal{B}(\mathbb{E})} \right) < \infty.$$

It comes from Theorem 2 that under the assumptions, the problem (2) has a unique solution $y_0$. Let now $\varepsilon > 0$ and let $y \in C(J, \mathbb{E})$ be any solution of the inequality (27) with the conditions (28). The function

$$[0,1]^2 \ni (\tau, \theta) \longmapsto \widetilde{\psi_y}(\tau, \theta) = \psi(\tau, \theta) - h_y(\tau)$$

satisfies all the conditions (6), (23), and (24) with the majorating constant $m + \varepsilon$ instead of $m$. Moreover, $y$ is a solution of the problem

$$^{C}D^{\alpha} z(\tau) + \widetilde{\psi_y}(\tau, z(\tau)) = 0, \quad (0 \leqslant \tau \leqslant 1)$$

with the conditions (28). By Lemma 5

$$\begin{aligned} y(\tau) &= \int_0^1 G(\tau, \theta) \left( \psi(\theta, y(\theta)) - h_y(\theta) \right) d\theta \\ &= \int_0^1 G(\tau, \theta) \psi(\theta, y(\theta)) d\theta - \int_0^1 G(\tau, \theta) h_y(\theta) d\theta \quad (0 \leqslant \tau \leqslant 1). \end{aligned}$$

Consequently, for every $\tau \in [0,1]$,

$$\begin{aligned} &\|y(\tau) - y_0(\tau)\|_{\mathbb{E}} \\ &\quad \leqslant \left\| \int_0^1 G(\tau, \theta) \psi(\theta, y(\theta)) d\theta - \int_0^1 G(\tau, \theta) \psi(\theta, y_0(\theta)) d\theta \right\|_{\mathbb{E}} \\ &\qquad + \left\| \int_0^1 G(\tau, \theta) h_y(\theta) d\theta \right\|_{\mathbb{E}} \\ &\quad \leqslant \int_0^1 \|G(\tau, \theta)\|_{\mathcal{B}(\mathbb{E})} \|\psi(\theta, y(\theta)) - \psi(\theta, y_0(\theta))\|_{\mathbb{E}} d\theta \\ &\qquad + \int_0^1 \|G(\tau, \theta)\|_{\mathcal{B}(\mathbb{E})} \|h_y(\theta)\|_{\mathbb{E}} d\theta \\ &\quad \leqslant \delta \int_0^1 \theta^{-\sigma} \|\theta^{\sigma} \psi(\theta, y(\theta)) - \theta^{\sigma} \psi(\theta, y_0(\theta))\|_{\mathbb{E}} d\theta + \delta\varepsilon \end{aligned}$$

$$\leqslant \frac{\delta}{1-\sigma} k \|y - y_0\|_{C(J,\mathbb{E})} + \delta\varepsilon.$$

Finally,

$$\tau \in J \Longrightarrow \|y(\tau) - y_0(\tau)\|_{\mathbb{E}} \leqslant \|y - y_0\|_{C(J,\mathbb{E})} \leqslant C_\psi \varepsilon, \quad \text{with } C_\psi \geqslant \frac{(1-\sigma)\delta}{1-\sigma-k\delta}.$$

□

**Remark 4.** Setting $\omega(\varepsilon) = C_\psi \varepsilon$, it follows that $\omega(0) = 0$ and the problem (2) is generalized Ulam–Hyers stable.

Next result concerns the Ulam–Hyers–Rassias stability.

**Theorem 5.** Let $\psi : [0,1] \times \mathbb{E} \longrightarrow \mathbb{E}$ be a function satisfying (6), (23), (24), let the operator $U$ be invertible and $0 < k < \frac{1-\sigma}{\delta}$. Then, the problem (2) is Ulam–Hyers–Rassias stable with respect to any Riemann integrable function $\varphi : [0,1] \longrightarrow \mathbb{R}^+$ for which there exists a constant $\delta_\varphi > 0$ such that

$$\text{for all } \ \tau \in [0,1] : \int_0^1 \varphi(\theta) d\theta \leqslant \delta_\varphi \varphi(\tau). \tag{31}$$

*Proof.* Let $\varepsilon > 0$ and let $\varphi : [0,1] \longrightarrow \mathbb{R}^+$ be a Riemann integrable function satisfying (31). Let also $y \in C(J,\mathbb{E})$ satisfy the inequality (29) with the conditions (28). As in the remark 3, this implies that there exists a function $h_y(\tau)$ satisfying $h_y(\tau) \leqslant \varepsilon\varphi(\tau)$ for all $\tau \in [0,1]$ and $y$ is a solution of the problem

$${}^C D_0^\alpha z(\tau) + \widetilde{\psi_y}(\tau, z(\tau)) = 0, \quad 0 \leqslant \tau \leqslant 1$$

with the conditions (28). The function $\widetilde{\psi_y}(\tau, z(\tau))$ obviously satisfies the conditions (6) and (23). Moreover, for $0 \leqslant \tau \leqslant 1$,

$$\begin{aligned} \|\tau^\sigma \widetilde{\psi_y}(\tau, 0)\|_{\mathbb{E}} &= \|\tau^\sigma(\psi(\tau,0) - h_y(\tau))\|_{\mathbb{E}} \leqslant \|\tau^\sigma \psi(\tau,0)\|_{\mathbb{E}} + \|\tau^\sigma h_y(\tau)\|_{\mathbb{E}} \\ &\leqslant m + \tau^\sigma \varepsilon\varphi(\tau) \leqslant m + \varepsilon \sup_{0\leqslant\tau\leqslant1} \varphi(\tau). \end{aligned}$$

So, the condition (24) is also satisfied by $\widetilde{\psi_y}$. According to Lemma 5, we have the representation

$$y(\tau) = \int_0^1 G(\tau,\theta)(\psi(\theta, y(\theta)) - h_y(\theta)) d\theta.$$

Therefore, if $y_0$ is the unique solution of problem (2)–(4), then for every $\tau \in [0,1]$,

$$\|y(\tau) - y_0(\tau)\|_{\mathbb{E}}$$

$$
\begin{aligned}
&\leqslant \int_0^1 \|G(\tau,\theta)\|_{\mathcal{B}(\mathbb{E})} \|\psi(\theta,y(\theta)) - \psi(\theta,y_0(\theta))\|_{\mathbb{E}} d\theta \\
&\quad + \int_0^1 \|G(\tau,\theta)\|_{\mathcal{B}(\mathbb{E})} \|h_y(\theta)\|_{\mathbb{E}} d\theta \\
&\leqslant \delta \int_0^1 \theta^{-\sigma} \|\theta^{\sigma}\psi(\theta,y(\theta)) - \theta^{\sigma}\psi(\theta,y_0(\theta))\|_{\mathbb{E}} d\theta \\
&\quad +\delta\varepsilon \int_0^1 \varphi(\theta) d\theta \\
&\leqslant \frac{\delta}{1-\sigma} k \|y-y_0\|_{C(J,\mathbb{E})} + \delta\varepsilon \int_0^1 \varphi(\theta) d\theta \\
&\leqslant \frac{\delta}{1-\sigma} k \|y-y_0\|_{C(J,\mathbb{E})} + \delta\varepsilon\delta_{\varphi}\varphi(\tau).
\end{aligned}
$$

Finally,

$$
\text{for all } \ \tau \in [0,\ 1]: \ \|y(\tau)-y_0(\tau)\|_{\mathbb{E}} \leqslant C_{\psi,\varphi}\varepsilon\varphi(\tau), \quad \text{with } \ C_{\psi,\varphi} \geqslant \frac{(1-\sigma)\delta_{\varphi}\delta}{1-\sigma-k\delta}.
$$

□

# 4 Examples

In this section, we treat two illustrative examples that differ only in the nature of the operators used. In the first example, we suppose that $A_1$ and $A_2$ are scalar; in the second example, we consider operators of multiplication by $\theta$ and $\theta^2$, respectively. Note firstly that for the operators $\Phi_1$ and $\Phi_2$ defined by (12) and (13), we have the following norm estimations:

$$
\begin{aligned}
\|\Phi_1\|_{\mathcal{B}(\mathbb{E})} &\leqslant \|w(A_2,j_2,p_2,q_2)\|_{\mathcal{B}(\mathbb{E})} \|z(\theta,A_1,p_1,q_1)\|_{\mathcal{B}(\mathbb{E})} \\
&\quad + \|w(A_1,j_2,p_1,q_1)\|_{\mathcal{B}(\mathbb{E})} \|z(\theta,A_2,p_2,q_2)\|_{\mathcal{B}(\mathbb{E})} \\
&\leqslant \left( \frac{\Gamma(j_2+1)}{\Gamma(j_2-q_2+1)} \|A_2\|_{\mathcal{B}(\mathbb{E})} + \frac{\Gamma(j_2+1)}{\Gamma(j_2+p_2+1)} \right) \\
&\quad \times \left( \frac{1}{\Gamma(\alpha-q_1)} \|A_1\|_{\mathcal{B}(\mathbb{E})} + \frac{1}{\Gamma(\alpha+p_1)} \right) \\
&\quad + \left( \frac{\Gamma(j_2+1)}{\Gamma(j_2-q_1+1)} \|A_1\|_{\mathcal{B}(\mathbb{E})} + \frac{\Gamma(j_2+1)}{\Gamma(j_2+p_1+1)} \right) \\
&\quad \times \left( \frac{1}{\Gamma(\alpha-q_2)} \|A_2\|_{\mathcal{B}(\mathbb{E})} + \frac{1}{\Gamma(\alpha+p_2)} \right)
\end{aligned}
$$

and similarly,

$$\begin{aligned}
\|\Phi_2\|_{\mathcal{B}(\mathbb{E})} \leqslant\ & \|w(A_1,j_1,p_1,q_1)\|_{\mathcal{B}(\mathbb{E})}\|z(\theta,A_2,p_2,q_2)\|_{\mathcal{B}(\mathbb{E})} \\
& +\|w(A_2,j_1,p_2,q_2)\|_{\mathcal{B}(\mathbb{E})}\|z(\theta,A_1,p_1,q_1)\|_{\mathcal{B}(\mathbb{E})} \\
\leqslant & \left(\frac{\Gamma(j_1+1)}{\Gamma(j_1-q_1+1)}\|A_1\|_{\mathcal{B}(\mathbb{E})}+\frac{\Gamma(j_1+1)}{\Gamma(j_1+p_1+1)}\right) \\
& \times\left(\frac{1}{\Gamma(\alpha-q_2)}\|A_2\|_{\mathcal{B}(\mathbb{E})}+\frac{1}{\Gamma(\alpha+p_2)}\right) \\
& +\left(\frac{\Gamma(j_1+1)}{\Gamma(j_1-q_2+1)}\|A_2\|_{\mathcal{B}(\mathbb{E})}+\frac{\Gamma(j_1+1)}{\Gamma(j_1+p_2+1)}\right) \\
& \times\left(\frac{1}{\Gamma(\alpha-q_1)}\|A_1\|_{\mathcal{B}(\mathbb{E})}+\frac{1}{\Gamma(\alpha+p_1)}\right).
\end{aligned}$$

Let $\mathbb{E}$ be a real Banach space and let $\psi : J\times\mathbb{E}\longrightarrow\mathbb{E}$ be given by

$$\psi(\tau,u)=\begin{cases}\beta\ln(e+\tau)\dfrac{7+\cos(\pi\tau)}{\sqrt{\tau}(1+\|u\|_{\mathbb{E}})}u_0, & \tau\in]0,1],\ u\in\mathbb{E},\\ \\ 0, & \tau=0,\ u\in\mathbb{E},\end{cases}$$

where $\beta>0$ and $u_0$ is a fixed element in $\mathbb{E}$.
The function $\psi$ satisfies the conditions

$$\lim_{\tau\longrightarrow 0^+}\|\psi(\tau,u)\|_{\mathbb{E}}=+\infty,\quad \text{for all }\ u\in\mathbb{E},$$

$$\begin{aligned}\|\sqrt{\tau}\psi(\tau,u)-\sqrt{\tau}\psi(\tau,v)\|_{\mathbb{E}}\leqslant 8\beta e\|u_0\|_{\mathbb{E}}\quad &\|u-v\|_{\mathbb{E}},\\ &\text{for all }\ \tau\in J,\quad \text{for all }\ (u,v)\in\mathbb{E}^2,\end{aligned}$$

and

$$\|\sqrt{\tau}\psi(\tau,u)\|_{\mathbb{E}}\leqslant\mu(\tau):=8\beta e\|u_0\|_{\mathbb{E}},\quad \text{for all }\ \tau\in J,\quad \text{for all }\ u\in\mathbb{E}.$$

For numerical simulation, we consider $\mathbb{E}=\mathbb{R}$, $u_0=1$, and $\beta=1$. The nonlinear function $\psi$ is implemented accordingly, and the singularity at $\tau=0$ is handled by setting $\psi(0,y)=0$.

Figure 1 highlights the amplification effect generated by increasing values of $\beta$.

Figure 2 emphasizes the strong growth generated by the singular term $\dfrac{1}{\sqrt{\tau}}$.

Figure 3 shows the damping effect produced by the term $(1+|y|)$ as $|y|$ increases.

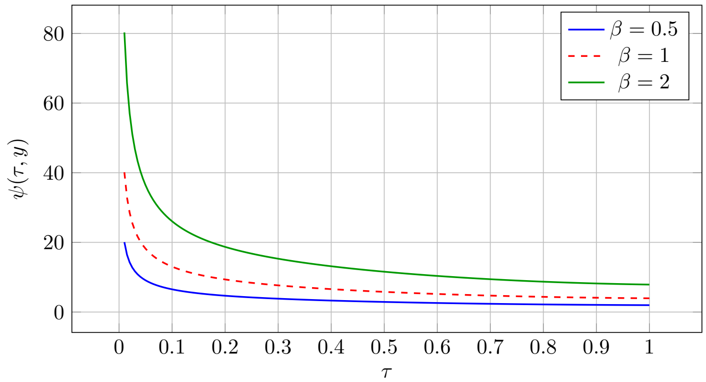


Figure 1: Influence of the parameter $\beta$ on the nonlinear function $\psi(\tau, y) = \beta \ln(e+\tau) \dfrac{7 + \cos(\pi\tau)}{\sqrt{\tau}(1 + |y|)}$ on the interval $[0, 1]$.

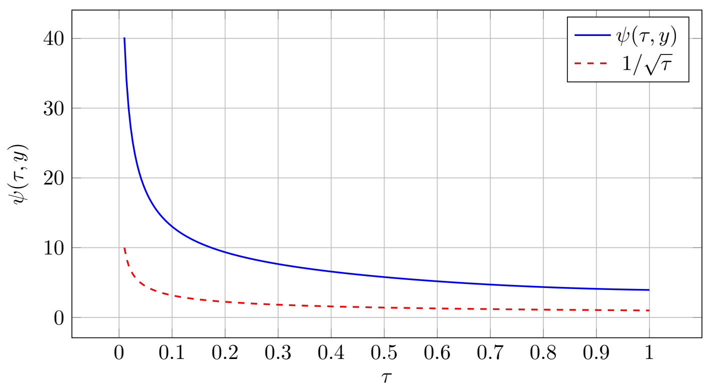


Figure 2: Singular dynamics of the nonlinear function $\psi(\tau, y)$ near $\tau = 0$ on the interval $[0, 1]$.

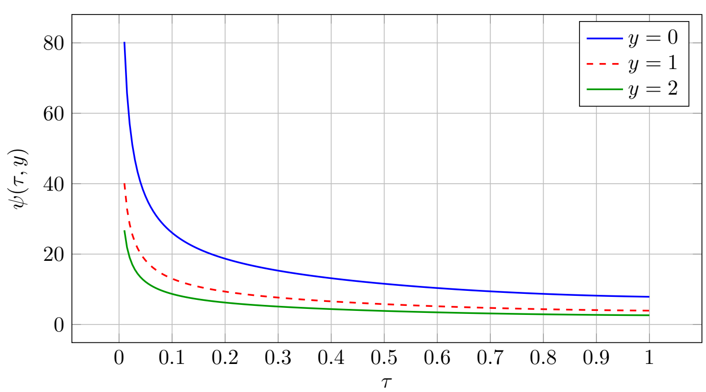


Figure 3: Effect of the state variable $y$ on the nonlinear function $\psi(\tau, y)$ on the interval $[0, 1]$.

**Example 1.** Consider now in the Banach space $C(J,\mathbb{E})$ the following Caputo-fractional problem:

$$(S1): \begin{cases} {}^{C}_{1}\partial^{\alpha}_{0} y(\tau) + \psi(\tau, y(\tau)) = 0, & 0 < \tau < 1, \alpha \geqslant 3, \\ \\ y^{(j)}(0) = 0, & j \notin \{3,4\}, \\ \\ a_1 y(1) = \int_0^1 (1-s) y(s) ds, & a_1 \in \mathbb{R}^*, \\ \\ a_2 y'(1) = \int_0^1 y(s) ds, & a_2 \in \mathbb{R}^*. \end{cases}$$

In this example, $A_1$ and $A_2$ are the scalar operators $a_1 I_{\mathbb{E}}$, $a_2 I_{\mathbb{E}}$, respectively, and

$$\sigma = \frac{1}{2},\ m = 8\beta e \|u_0\|_{\mathbb{E}},\ (j_1,j_2) = (3,4),\ (q_1,q_2) = (0,1),\ (p_1,p_2) = (2,1).$$

Consequently, the operator $U$ (formula (11)) is defined in $\mathbb{E}$ as follows:

$$Ux = \left( \frac{600 a_1 a_2 + 30 a_1 - 60 a_2 + 1}{600} \right) x, \quad x \in \mathbb{E}.$$

Moreover, $U$ is bounded with

$$\|U\|_{\mathcal{B}(\mathbb{E})} = \frac{|600 a_1 a_2 + 30 a_1 - 60 a_2 + 1|}{600}.$$

Also, $U$ is invertible if $600 a_1 a_2 + 30 a_1 - 60 a_2 + 1 \neq 0$; its inverse is then given by the formula

$$U^{-1} x = \left( \frac{600}{600 a_1 a_2 + 30 a_1 - 60 a_2 + 1} \right) x, \quad x \in \mathbb{E}$$

and $\left\|U^{-1}\right\|_{\mathcal{B}(\mathbb{E})} = \frac{1}{\|U\|_{\mathcal{B}(\mathbb{E})}}$.

Using the norm estimations given in the beginning of the section and the relation $\alpha\Gamma(\alpha) = \Gamma(\alpha+1),\ \alpha > 0$, we get

$$\begin{aligned} \|\Phi_1\|_{\mathcal{B}(\mathbb{E})} &\leqslant \Lambda_1 \\ &:= \frac{1}{30\Gamma(\alpha+2)} \begin{pmatrix} (30\alpha^3 + 120\alpha^2 + 90\alpha)|a_1 a_2| + (6\alpha^2 + 36\alpha + 30)|a_1| \\ +(\alpha^3 - \alpha + 120)|a_2| + (\alpha + 7) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \|\Phi_2\|_{\mathcal{B}(\mathbb{E})} &\leqslant \Lambda_2 \\ &:= \frac{1}{20\Gamma(\alpha+2)} \begin{pmatrix} (20\alpha^3 + 60\alpha^2 + 40\alpha)|a_1 a_2| + (5\alpha^2 + 25\alpha + 20)|a_1| \\ +(\alpha^3 - \alpha + 60)|a_2| + (\alpha + 6) \end{pmatrix} \end{aligned}$$

By (22), we can write

$$\delta \leqslant \Delta \leqslant \widetilde{\Delta} := \frac{1}{\Gamma(\alpha)} + \frac{1}{\|U\|_{\mathcal{B}(\mathbb{E})}}(\Lambda_1 + \Lambda_2). \tag{32}$$

Thus, the problem $(S1)$ has at least one solution if

$$k := 8\beta e\|u_0\|_{\mathbb{E}} < \frac{1-\sigma}{\widetilde{\Delta} - \frac{1}{\Gamma(\alpha)}} = \frac{\Gamma(\alpha)}{2\left(\Gamma(\alpha)\widetilde{\Delta} - 1\right)}.$$

If

$$k := 8\beta e\|u_0\|_{\mathbb{E}} < \frac{1-\sigma}{\widetilde{\Delta}} = \frac{1}{2\widetilde{\Delta}} < \frac{1}{2\delta}, \tag{33}$$

then the problem $(S1)$ admits a unique solution $y_0$ and it is Ulam–Hyers stable.

The function $\varphi(\tau) = 1 + \tau^2$, for all $\tau \in [0,1]$ is Riemann integrable and

$$\int_0^1 \varphi(\theta)d\theta = \frac{4}{3} = \delta_\varphi \leqslant \frac{4}{3}\varphi(\tau) \quad (0 \leqslant \tau \leqslant 1).$$

Here is the graphical representation of the function $\varphi(\tau) = 1 + \tau^2$ on $[0,1]$.

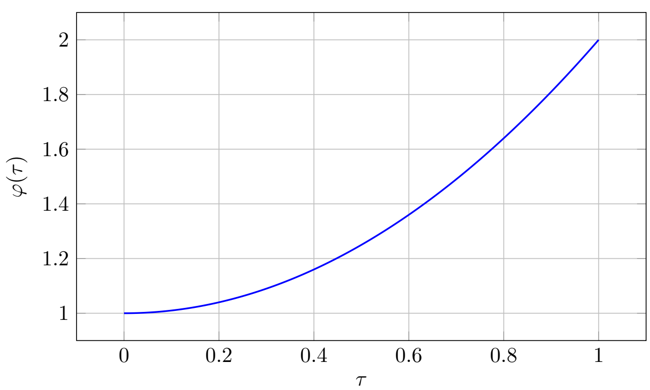


Figure 4: Plot of $\varphi(\tau) = 1 + \tau^2$.

Figure 4 confirms that the function is increasing.

Thus, for $k$ satisfying (33), the problem $(S1)$ is Ulam–Hyers–Rassias stable with respect to $\varphi$.

The following numerical simulations illustrate the behavior of the solution corresponding to Example 1 for several integer and noninteger fractional orders $\alpha \geqslant 3$. Figure 5 presents the two-dimensional evolution of the solution $y(\tau)$ with respect to the variable $\tau$, while Figure 6 provides a three-dimensional visualization showing the combined influence of both the fractional order $\alpha$ and the variable $\tau$ on the dynamics of the solution. These graphical representations clearly highlight the effect induced by the fractional operator as well as the singular behavior near $\tau = 0$.

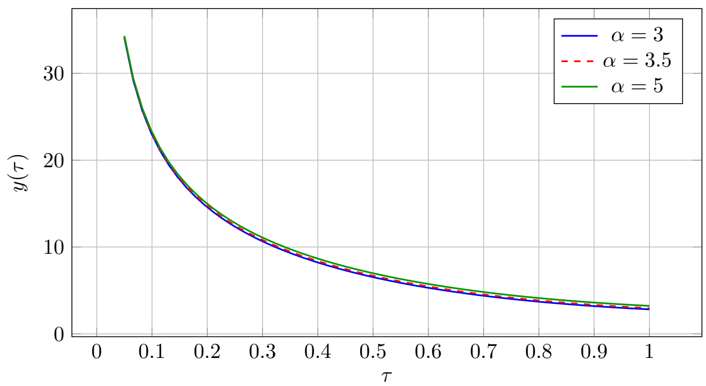


Figure 5: Numerical solution of Example 1 for integer and noninteger fractional orders $\alpha \geqslant 3$.

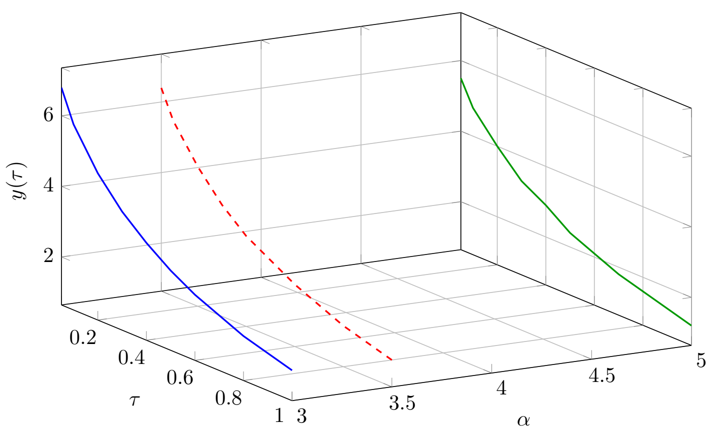


Figure 6: Three-dimensional numerical solution of Example 1 for integer and noninteger fractional orders $\alpha \geqslant 3$.

**Example 2.** Let now $\mathbb{E} = C(J, \mathbb{R})$ and consider in the Banach space $C(J, \mathbb{E})$ equipped with the norm

$$\|y\|_{C(J,\mathbb{E})} = \max_{\tau \in J} \|y(\tau, .)\|_{\mathbb{E}} = \max_{(\theta,\tau) \in J^2} |y(\tau, \theta)|$$

the following Caputo-fractional problem

$$(S2): \begin{cases} {}_{1}^{C}\partial_0^{\alpha} y(\tau,\theta) + \psi(\tau, y(\tau,\theta)) = 0, & 0<\tau<1,\ 0\leqslant\theta\leqslant 1, \alpha\geqslant 3,\\ y_{\tau}^{(j)}(r,\theta)=0, & j\notin\{3,4\},\ 0\leqslant\theta\leqslant 1,\\ \theta y(1,\theta) = \int_0^1 (1-s)y(s,\theta)ds, & 0\leqslant\theta\leqslant 1,\\ \theta^2 y_{\tau}^{(1)}(1,\theta) = \int_0^1 y(s,\theta)ds, & 0\leqslant\theta\leqslant 1, \end{cases}$$

where ${}_{1}^{C}\partial_0^{\alpha}$ is the Caputo partial derivative of order $\alpha$ with respect to the first argument $\tau$, $y_{\tau}^{(j)}(r,\theta) = \frac{d^j}{d\tau^j} y(\tau,\theta)\mid_{\tau=r}$ and the function $\psi$ is the same used in the first example.

We firstly prove that the problem $(S2)$ can be expressed under the form (2), (3), (4). Indeed, let $A_1$ and $A_2$ be the linear operators defined in $\mathbb{E}=C(J,\mathbb{R})$ by

$$(A_1 z)(\theta) = \theta z(\theta), \quad (A_2 z)(\theta) = \theta^2 z(\theta), \quad \text{for all} \ \ \theta\in J.$$

Now $A_1$ and $A_2$ commute and are bounded with $\|A_1\|_{\mathcal{B}(\mathbb{E})} = \|A_2\|_{\mathcal{B}(\mathbb{E})} = 1$. Using them, $(S2)$ can be rewritten in the desired form:

$$(S2): \begin{cases} {}_{1}^{C}\partial_0^{\alpha} y(\tau,\theta) + \psi(\tau, y(\tau,\theta)) = 0, & 0<\tau<1,\ 0\leqslant\theta\leqslant 1, \alpha\geqslant 3,\\ y_{\tau}^{(j)}(r,\theta)=0, & j\notin\{3,4\},\ 0\leqslant\theta\leqslant 1,\\ A_1 y(1,.)(\theta) = \int_0^1 (1-s)y(s,\theta)ds, & 0\leqslant\theta\leqslant 1\\ A_2 y_{\tau}^{(1)}(1,.)(\theta) = \int_0^1 y(s,\theta)ds, & 0\leqslant\theta\leqslant 1. \end{cases}$$

In the present case, the operator $U$ (formula (11)) is defined in $\mathbb{E}$ as follows:

$$Ux(\theta) = \left(\frac{600\theta^3 - 60\theta^2 + 30\theta + 1}{600}\right) x(\theta).$$

The function $h(\theta) = \dfrac{600\theta^3 - 60\theta^2 + 30\theta + 1}{600}$, $\theta\in J$ is continuous, increases and

$$h(0) = \frac{1}{600} < h(\theta) < h(1) = \frac{571}{600}, \quad 0<\theta<1.$$

Figure 7 illustrates the behavior of the function $h(\theta)$ on the interval $[0,1]$.

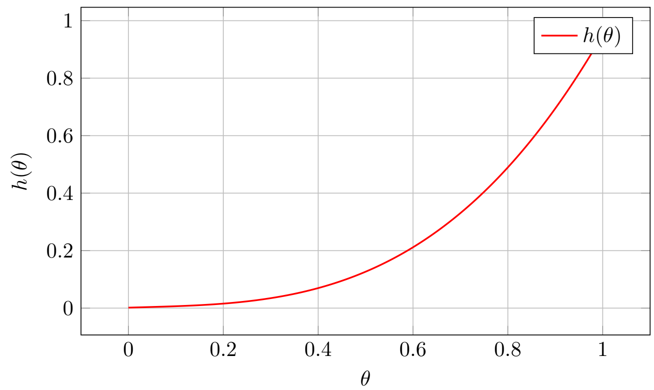


Figure 7: Plot of the function $h(\theta) = \dfrac{600\theta^3 - 60\theta^2 + 30\theta + 1}{600}$ on $[0,1]$. The graph confirms the monotone increasing behavior of the operator coefficient appearing in Example 2.

We can thus conclude that $U$ is bounded and $\|U\|_{\mathcal{B}(\mathbb{E})} = h(1) = \frac{571}{600}$. Moreover, $U$ is also invertible with

$$U^{-1}x(\theta) = \frac{1}{h(\theta)}x(\theta) = \frac{600}{600\theta^3 - 60\theta^2 + 30\theta + 1}x(\theta),$$

$$\left\|U^{-1}\right\|_{\mathcal{B}(\mathbb{E})} = \frac{1}{h(0)} = 600.$$

Since $\|A_1\|_{\mathcal{B}(\mathbb{E})} = \|A_2\|_{\mathcal{B}(\mathbb{E})} = 1$, then to estimate the corresponding $\|\Phi_1\|_{\mathcal{B}(\mathbb{E})}$ and $\|\Phi_2\|_{\mathcal{B}(\mathbb{E})}$, it suffices to set $|a_1| = |a_2| = 1$ in the quantities $\Lambda_1,\ \Lambda_2$ (see the first example). This gives us

$$\|\Phi_1\|_{\mathcal{B}(\mathbb{E})} \leqslant \frac{31\alpha^3 + 126\alpha^2 + 126\alpha + 157}{30\Gamma(\alpha+2)},$$
$$\|\Phi_2\|_{\mathcal{B}(\mathbb{E})} \leqslant \frac{21\alpha^3 + 65\alpha^2 + 65\alpha + 86}{20\Gamma(\alpha+2)}.$$

By (22), we get

$$\delta \leqslant \Delta \leqslant \widetilde{\Delta} = \frac{1250\alpha^3 + 4470\alpha^2 + 4471\alpha + 5720}{\Gamma(\alpha+2)}. \tag{34}$$

Thus, the problem $(S2)$ has at least one solution if

$$k := 8\beta e\|u_0\|_{\mathbb{E}} < \frac{1-\sigma}{\widetilde{\Delta} - \frac{1}{\Gamma(\alpha)}} = \frac{\Gamma(\alpha)}{2\left(\Gamma(\alpha)\widetilde{\Delta} - 1\right)}.$$

If

$$k := 8\beta e\|u_0\|_{\mathbb{E}} < \frac{1-\sigma}{\widetilde{\Delta}} = \frac{1}{2\widetilde{\Delta}} < \frac{1}{2\delta}, \tag{35}$$

then the problem $(S2)$ admits a unique solution $y_0$ and is Ulam–Hyers stable.

The function $\varphi(\tau) = 1 + \tau^2$, for all $\tau \in [0, 1]$ is Riemann integrable and

$$\int_0^1 \varphi(\theta)d\theta = \frac{4}{3} = \delta_\varphi \leqslant \frac{4}{3}\varphi(\tau), \ (0 \leqslant \tau \leqslant 1).$$

Thus, for $k$ satisfying (35), the problem $(S2)$ is Ulam–Hyers–Rassias stable with respect to $\varphi$. Figures 8 and 9 illustrate the numerical behavior of the solution of Example 2 for several integer and noninteger fractional orders $\alpha \geqslant 3$. They show the influence of the fractional order on the dynamics of the solution and highlight the singular behavior near $\tau = 0$.

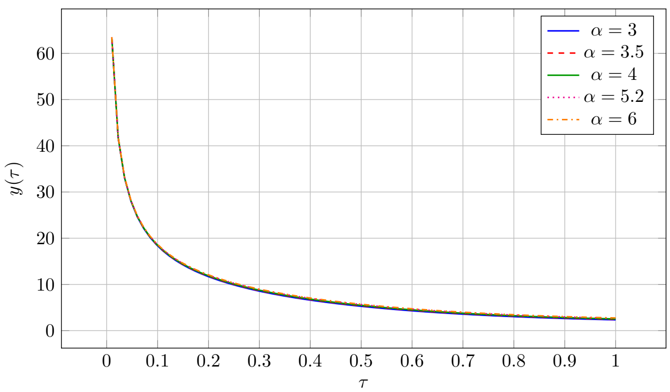


Figure 8: Numerical solution of example 2 for integer and noninteger fractional orders $\alpha \geqslant 3$.

Figure 8 illustrates the influence of the fractional order on the dynamics of the solution and highlights the effect of the singular fractional operator.

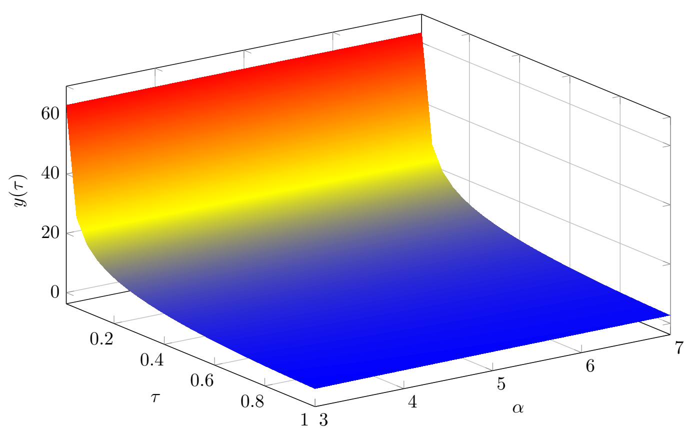


Figure 9: Three-dimensional numerical solution of Example 2 for integer and noninteger fractional orders $\alpha \geqslant 3$.

The surface illustrates the dependence of the solution $y(\tau)$ on both the variable $\tau$ and the fractional order $\alpha$. Figure 9 highlights the effect induced by the fractional operator and the singular behavior near $\tau = 0$.

## 5 Conclusion

This article investigated a class of singular Caputo-fractional boundary value problems that involve two integral boundary conditions connected through linear operators. It employs Banach's fixed point theorem to establish a uniqueness criterion and Krasnosel'skii's fixed point theorem to obtain an existence criterion. Furthermore, the authors presented two Ulam-type stability results for the problems under consideration and illustrated the theoretical results by a concrete example. The study remains largely theoretical, and the authors propose future work to assess practical applicability by utilizing real data and performing numerical simulations.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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