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Research Article

Mathematical model on the effect of primary and secondary toxicants on a biological species: Occurrence of deformity

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Abstract

A mathematical model is proposed and analyzed to study the effect of primary and secondary toxicants on a biological species. Here, we consider a situation that after-effect of the primary and secondary toxicants a subclass of biological species shows deformity as stagnating the reproduction capability. We analyzed the model using stability analysis and Hopf-bifurcation theory. The analysis of model shows, primary and secondary toxicants badly affects the biological species. These toxicants decrease the growth rate of species as well as causing deformity in a subclass. For high emission of primary toxicant, system becomes unstable, total density and deformed subclass density of species start oscillating around their equilibrium levels. This study helps us to plan best courses of action to make a healthier environment for living things such as plant trees, properly dispose the wastes, use pesticides and insecticides sparingly, use eco-friendly vehicles, and so on.

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1 Introduction

The term environmental contaminants refer to those potentially objectionable substances (or toxicants) that exist in the environment at enough level to harmful effect on the health of each type of biological species (i.e., human beings, animals and plants). The environmental contaminants are the result of various activities of human populations, such as fuel combustion, industrialisation, chemical manufacturing, domestic sewage, and so on, and some natural factors, such as volcanic eruptions, forest fires which contain dangerous gases, chemicals, and so on. Some toxicants that are not directly emitted in the environment, although they are formed in the environment by chemical and physical processes from primary toxicants (those toxicants that are directly emitted into the environment) also cause to increase the contaminants levels in the environment. These toxicants are known as secondary toxicants and may be even more toxic and harmful than primary toxicants. To illustrate, some gaseous toxicants (or primary toxicants) such as Sulfur Dioxide (SO_2), Nitric Oxides (NO_X), Carbon Monoxide (CO), Volatile Organic Compounds (VOC) including Hydrocarbons (C_XH_Y) are discharged from particular sources, such as burning of fuels, industrial activities, volcanic eruptions, and so on. The product of these primary toxicants produce secondary toxicants, such as ground level ozone — produced by product of Nitric Oxides (NO_X), Hydrocarbons (C_XH_Y) and Sunlight — and acid rain which is formed when SO_2 and NO_X react with Water (H_2O) and Oxygen(O_2). There are many other toxicants that are the product of primary toxicants, sunlight, and water, for example, Peroxy Acetyl Nitrates (PAN), Formaldehydes, Aerosols, and so on (see Figure 1).

When these toxicants enter the food chain of biological species, a range of diseases may occur in biological species including cancer, genetic disorder, lesions, deformities, stagnated capability of reproduction, and so on [3, 5, 11, 12, 13, 14, 16, 19].

Many researchers, economists and other authorities are busy to monitor and assess the effect of toxicants on biological species. Some of them use mathematical models to highlight the various natures of biological species in the real life fields such as ecology, ecotoxicology, epidemiology, and so on. With the help of modeling, researchers are able to predict the growth of biological species in toxic environment and understand diseases and deformities in biological species in a more meaningful manner. In particular, researchers have proposed, analyzed, and simulated mathematical models to provide important insights for the effect of toxicants on biological species

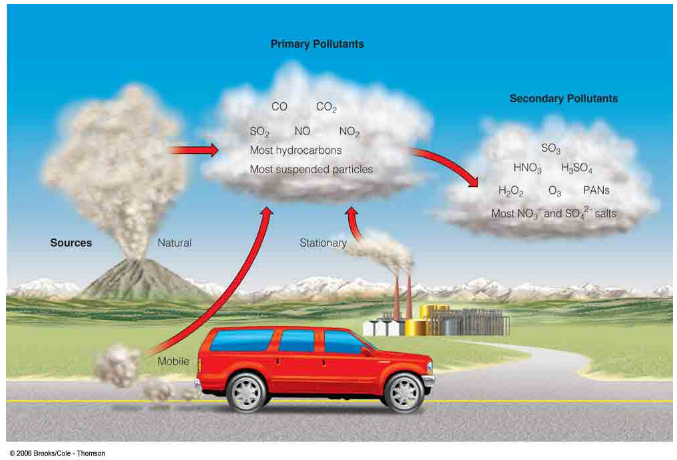

Figure 1: Primary and Secondary pollutants (or toxicants)
(Image Courtesy: sciencescene.com/Environmental%20Science/Graphics/1503.jpg)

[1, 4, 8, 17, 18]. Freedman and Shukla [4] have studied the effect of a single toxicant on biological species and predator: Prey case. They have shown that toxicant affected both the growth rate and carrying capacity. Shukla and Agrawal [17] have proposed a model by considering the case of allelopathy. In their model, the toxicant emitted into the environment by one species and decreased the growth rate and carrying capacity of other species. Shukla et al. [18] proposed and analyzed a model to study the growth of biomass resources (forests, agricultural, crops, etc.) with the effect of primary and secondary toxicants. They concluded that primary and secondary toxicants, both simultaneously affected the biomass resources. The biomass resources may extinct if the emission rate of primary and secondary toxicants are extremely high. Furthermore, a highly important observable fact when a subclass of biological species shows deformity and incapable in reproduction after the effect of a toxicant emitted from some external sources, was studied by Agrawal and Shukla [1]. Furthermore, Kumar et al. [8] proposed and analyzed a model with an assumption that toxicant emitted by biological species and a subclass of species is severely affected by this toxicant and suffer from deformity. They showed that the density of biological species decreased and density of the subclass of biological species increased when emission rate of toxicant increased. The system becomes unstable, and both densities (density of biological species and density of the subclass of biological species) show oscillations for higher emission rate of toxicant.

In this paper, we propose and analyze a mathematical model to study the effect of primary and secondary toxicant on a biological species of a subclass, which is severely affected and gets deformed. We also provide a numerical simulation of the model for the confirmation of our analytical results.

2 Mathematical model

We consider a biological species surviving in a polluted environment with population density $N(t)$ at time t. This polluted environment having primary and secondary toxicants with environmental concentrations $T(t)$ and $T_s(t)$ at time t. We assume that the primary toxicant is constantly discharged in the environment by some external sources. A part of this primary toxicant is converted into secondary toxicant. So, the rate at which primary toxicant is converted into secondary toxicant is directly proportional to the environmental concentration of primary toxicant. Let $U(t)$ and $U_s(t)$ be the concentrations of primary and secondary toxicants, respectively, taken up by the biological species $N(t)$ at time t. When these toxicants are being uptaken by the biological species, the growth rate of species $N(t)$ is decreasing as well as a subclass of species with population density $N_D(t)$ shows deformity as stagnating the reproduction capability. The remaining population density that is free from deformity is assumed as $N_A(t)$. Keeping these facts in mind, we propose the following mathematical model:

$$\begin{aligned}
\frac{dN_A}{dt} &= (b-d)N_A - (r_1U + r_2U_s)N_A - \frac{rN_AN}{K(T,T_s)},\\
\frac{dN_D}{dt} &= (r_1U + r_2U_s)N_A - \frac{rN_DN}{K(T,T_s)} - (\alpha + d)N_D,\\
\frac{dT}{dt} &= Q - \delta_1T - \gamma_1TN + \pi_1\nu_1NU - kT,\\
\frac{dT_s}{dt} &= \theta kT - \delta_2T_s - \gamma_2T_sN + \pi_2\nu_2NU_s,\\
\frac{dU}{dt} &= \gamma_1TN - \beta_1U - \nu_1NU,\\
\frac{dU_s}{dt} &= \gamma_2T_sN - \beta_2U_s - \nu_2NU_s.
\end{aligned} \tag{1}$$

$$\begin{aligned}
&N(t) = N_A(t) + N_D(t), \quad N_A(0), N_D(0), T(0), T_s(0) \geq 0, \quad U(0) \geq c_1T(0),\\
&U_s(0) \geq c_2T_s(0) \quad 0 < \theta < 1, \quad c_i(0) > 0, \quad 0 \leq \pi_i \leq 1, \quad \text{for all} \quad i = 1, 2.
\end{aligned}$$

In the proposed model system (1), all the considered parameters are positive constants. The constants b, d, and r are the natural birth, death, and growth rate of biological species $N(t)$. The primary toxicant $T(t)$ is constantly emitted in the environment at the rate Q. The constant k represents the conversion rate at which primary toxicant is being converted into secondary toxicant. In the conversion process, it may happen that the entire part of primary toxicant (kT) is not fully converted into secondary toxicant, and other substances may be produced, which are harmless for biological species. For this kind of situation, we consider a constant $\theta(\leq 1)$

as a fraction of that part of primary toxicant, which is converted into secondary toxicant. For $\theta = 1$, the entire part of primary toxicant (kT) is converted into secondary toxicant. The primary and secondary toxicants are uptaken by the species at the rate γ_1 and γ_2, respectively. These primary and secondary toxicants are decreasing the population density at the rate r_1 and r_2, respectively. Also, α is the mortality rate of deformed population due to high toxicities in the environment. Moreover, δ_1 and δ_2 are the natural depletion rates of primary and secondary toxicants correspondingly. In addition, β_1 and β_2 are the natural depletion rates of uptake concentration of primary and secondary toxicants respectively. Since some members of species die out, the uptake concentrations $U(t), U_s(t)$ are depleted at the rates ν_1, ν_2 and fractions π_1, π_2 of these decay is re-entering into the environment, respectively. Also, $c_1, c_2 > 0$ are the proportionality constants to find out the initial uptake concentrations of primary and secondary toxicants. Furthermore, $K(T, T_s)$ represents the carrying capacity of the environment. It is a decreasing function of T and T_s, that is,

$$K(0,0) = K_0 > 0,\ K(T,T_s) > 0,\ \frac{\partial K(T,T_s)}{\partial T} < 0,\ \frac{\partial K(T,T_s)}{\partial T_s} < 0, \quad \text{for } T, T_s > 0, \tag{2}$$

where K_0 is the carrying capacity of the toxicants free environment.

To make our analysis easy, we reduce the model system (1) into following model system (3), which is free from $N_A(t)$ by using the fact that $N(t) = N_A(t) + N_D(t)$:

$$\begin{aligned}
\frac{dN}{dt} &= rN - \frac{rN^2}{K(T,T_s)} - (\alpha + b)N_D,\\
\frac{dN_D}{dt} &= (r_1 U + r_2 U_s)(N - N_D) - \frac{rN_D N}{K(T,T_s)} - (\alpha + d)N_D,\\
\frac{dT}{dt} &= Q - \delta_1 T - \gamma_1 TN + \pi_1 \nu_1 NU - kT,\\
\frac{dT_s}{dt} &= \theta kT - \delta_2 T_s - \gamma_2 T_s N + \pi_2 \nu_2 NU_s,\\
\frac{dU}{dt} &= \gamma_1 TN - \beta_1 U - \nu_1 NU,\\
\frac{dU_s}{dt} &= \gamma_2 T_s N - \beta_2 U_s - \nu_2 NU_s.
\end{aligned} \tag{3}$$

The region of attraction Ω that attracts all the positive solution of model system (3), is as follows:

$$\begin{aligned}
\Omega = \{(N, N_D, T, T_s, U, U_s) :\ & 0 \le N \le K_0,\\
& 0 \le N_D \le \frac{(r_1\delta_{m_2} + r_2\theta k)QK_0}{(r_1\delta_{m_2} + r_2\theta k)Q + (\alpha + d)\delta_{m_1}\delta_{m_2}},
\end{aligned}$$

$$0 \le T + U \le \frac{Q}{\delta_{m_1}},\ 0 \le T_s + U_s \le \frac{\theta k Q}{\delta_{m_1}\delta_{m_2}} \Bigg\},$$

where $\delta_{m_1} = \min(\delta_1 + k, \beta_1)$ and $\delta_{m_2} = \min(\delta_2, \beta_2)$.

The proof of region of attraction Ω is given in Appendix A.

3 Equilibrium points

After, equating the right-hand side of model system (3) is equal to zero, we obtain two nonnegative equilibrium points $E_1(0, 0, \frac{Q}{\delta_1+k}, \frac{\theta k Q}{\delta_2(\delta_1+k)}, 0, 0)$ and $E_2(N^*, N_D^*, T^*, T_s^*, U^*, U_s^*)$. Here, we omitted the existence of E_1 because it is obvious. The existence of E_2 is as follows:

The values $N^*, N_D^*, T^*, T_s^*, U^*$, and U_s^* are the positive solutions of the following nonlinear equations:

$$N = \frac{(r - r_1 U - r_2 U_s)K(T, T_s)}{r}, \tag{4a}$$

$$N_D = \frac{(r_1 U + r_2 U_s) N K(T, T_s)}{rN + (r_1 U + r_2 U_s + \alpha + d)K(T, T_s)}, \tag{4b}$$

$$T = \frac{Q(\beta_1 + \nu_1 N)}{f_1(N)} \qquad = g_1(N)\ (say), \tag{4c}$$

$$T_s = \frac{\theta k(\beta_2 + \nu_2 N)}{f_2(N)} \frac{Q(\beta_1 + \nu_1 N)}{f_1(N)} \qquad = g_2(N)\ (say), \tag{4d}$$

$$U = \frac{Q\gamma_1 N}{f_1(N)} \qquad = h_1(N)\ (say), \tag{4e}$$

$$U_s = \frac{\theta k \gamma_2 N}{f_2(N)} \frac{Q(\beta_1 + \nu_1 N)}{f_1(N)} \qquad = h_2(N)\ (say), \tag{4f}$$

where

$$f_1(N) = (\delta_1 + k)\beta_1 + \{\gamma_1\beta_1 + (\delta_1 + k)\nu_1\}N + \gamma_1\nu_1(1 - \pi_1)N^2, \tag{4g}$$

$$f_2(N) = \delta_2\beta_2 + (\gamma_2\beta_2 + \delta_2\nu_2)N + \gamma_2\nu_2(1 - \pi_2)N^2, \tag{4h}$$

From (4c) and (4d), we note that T is directly dependent on the parameter Q and T_s is directly proportional to the parameters Q, k, and θ. Since, carrying capacity $K(T, T_s)$ is a decreasing function of T and T_s. Hence, the carrying capacity of the environment decreases when the discharge rate of primary toxicant Q, the conversion rate of primary toxicant into secondary toxicant k or the fraction of that part of primary toxicant which is converted into secondary toxicant θ increase.

Let

$$F(N) = rN - (r - r_1 h_1(N) - r_2 h_2(N))K(g_1(N), g_2(N)). \tag{5}$$

At $N = 0$,

$$F(0) = -rK\left(\frac{Q}{(\delta_1 + k)}, \frac{\theta k Q}{(\delta_1 + k)\delta_2}\right) < 0. \tag{6}$$

At $N = K_0$,

$$F(K_0) = rK_0 - (r - r_1 h_1(K_0) - r_2 h_2(K_0))K(g_1(K_0), g_2(K_0)) > 0. \tag{7}$$

Equations (6) and (7) guarantee that $F(N) = 0$ has a solution in the interval $[0, K_0]$.

Also, the root N^* of $F(N) = 0$ is unique, if

$$\begin{aligned}\frac{dF}{dN} = \Bigg[r + K(g_1(N), g_2(N))\left\{r_1\frac{dh_1}{dN} + r_2\frac{dh_2}{dN}\right\}\\ -(r - r_1 h_1(N) - r_2 h_2(N))\left\{\frac{\partial K}{\partial T_1}\frac{dg_1}{dN} + \frac{\partial K}{\partial T_2}\frac{dg_2}{dN}\right\}\Bigg] > 0.\end{aligned} \tag{8}$$

Hence, $F(N) = 0$ must have a root N^* in the interval $[0, K_0]$ and the root N^* is unique if condition (8) holds. The values of other points can be found out using the value of N^* in (4).

4 Dynamical behavior

In this section, we show the dynamical behavior of model system (3) corresponding to the obtained equilibrium points E_1 and E_2.

4.1 Dynamical behavior corresponding to E_1

We evaluate the Jacobian matrices M_1(say) of model system (3) and its eigenvalues corresponding to the equilibrium point E_1.

We obtain the Jacobian matrix M_1 as follows:

$$M_1 = \begin{bmatrix} r & -(\alpha+b) & 0 & 0 & 0 & 0 \\ 0 & -(\alpha+d) & 0 & 0 & 0 & 0 \\ -\dfrac{\gamma_1 Q}{(\delta_1+k)} & 0 & -(\delta_1+k) & 0 & 0 & 0 \\ -\dfrac{\gamma_2 Q\theta k}{(\delta_1+k)\delta_2} & 0 & \theta k & -\delta_2 & 0 & 0 \\ \dfrac{\gamma_1 Q}{(\delta_1+k)} & 0 & 0 & 0 & -\beta_1 & 0 \\ \dfrac{\gamma_2 Q\theta k}{(\delta_1+k)\delta_2} & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}.$$

Here, the eigenvalues of the Jacobian matrix M_1 are $r, -(\alpha+d), -(\delta_1+k), -\delta_2, -\beta_1$, and $-\beta_2$, which shows that E_1 is a saddle point locally unstable manifold in $N-$ direction and locally stable manifold in $N_D - T - T_s - U - U_s$ space.

4.2 Dynamical behavior corresponding to E_2

Now, we linearize the model system (3) corresponding to the equilibrium point $E_2 = (N^*, N_D^*, T^*, T_s^*, U^*, U_s^*)$ by taking the following transformation:

$$\begin{aligned} N &= N^* + n, \quad N_D = N_D^* + n_d, \quad T = T^* + \tau, \\ T_s &= T_s^* + \tau_s, \quad U = U^* + u, \quad U_s = U_s^* + u_s. \end{aligned}$$

Here, n, n_d, τ, τ_s, u, and u_s are taken as small perturbations around E_2.

After using the above transformation, the model system (3) can be written in the terms of n, n_d, τ, τ_s, u, and u_s as follows:

$$\dot{X} = M_2 X + N_2, \tag{9}$$

where

$$X = \begin{bmatrix} n \\ n_d \\ \tau \\ \tau_s \\ u \\ u_s \end{bmatrix}, \quad M_2 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & 0 & m_{33} & 0 & m_{35} & 0 \\ m_{41} & 0 & m_{43} & m_{44} & 0 & m_{46} \\ m_{51} & 0 & m_{53} & 0 & m_{55} & 0 \\ m_{61} & 0 & 0 & m_{64} & 0 & m_{66} \end{bmatrix}, \quad N_2 = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{bmatrix}.$$

Moreover,

$$
\begin{aligned}
&m_{11} = -r\left\{\frac{2N^*}{K(T^*,T_s^*)} - 1\right\}, \qquad m_{12} = -(\alpha+b),\\
&m_{13} = \frac{rN^{*2}}{K^2(T^*,T_s^*)}\left[\frac{\partial K}{\partial T}\right]_{E_2}, \qquad m_{14} = \frac{rN^{*2}}{K^2(T^*,T_s^*)}\left[\frac{\partial K}{\partial T_s}\right]_{E_2},\\
&m_{21} = r_1U^* + r_2U_s^* - \frac{rN_D^*}{K(T^*,T_s^*)}, \qquad m_{22} = -(r_1U^* + r_2U_s^*)\frac{N^*}{N_D^*},\\
&m_{23} = \frac{rN^*N_D^*}{K^2(T^*,T_s^*)}\left[\frac{\partial K}{\partial T}\right]_{E_2}, \qquad m_{24} = \frac{rN^*N_D^*}{K^2(T^*,T_s^*)}\left[\frac{\partial K}{\partial T_s}\right]_{E_2},\\
&m_{25} = r_1(N^* - N_D^*), \qquad m_{26} = r_2(N^* - N_D^*),\\
&m_{31} = [-\gamma_1T^* + \pi_1\nu_1U^*], \qquad m_{33} = -\frac{(Q + \pi_1\nu_1N^*U^*)}{T^*},\\
&m_{35} = \pi_1\nu_1N^*, \qquad m_{41} = [-\gamma_2T_s^* + \pi_2\nu_2U_s^*], \qquad m_{43} = \theta k\\
&m_{44} = -(\delta_2 + \gamma_2N^*), \qquad m_{46} = \pi_2\nu_2N^*, \qquad m_{51} = \gamma_1T^* - \nu_1U^*,\\
&m_{53} = \gamma_1N^*, \qquad m_{55} = -(\beta_1 + \nu_1N^*), \qquad m_{61} = \gamma_2T_s^* - \nu_2U_s^*,\\
&m_{64} = \gamma_2N^*, \qquad m_{66} = -(\beta_2 + \nu_2N^*),
\end{aligned}
$$

$$
\begin{aligned}
n_1 =& -\frac{r}{K(T^*,T_s^*)}.n^2 + \frac{r(2N^*+n)}{K^2(T^*,T_s^*)}\left[\left[\frac{\partial K}{\partial T}\right]_{E_2}.\tau + \left[\frac{\partial K}{\partial T_s}\right]_{E_2}.\tau_s\right].n,\\
n_2 =&(r_1.u + r_2.u_s).(n - n_d) - \frac{r}{K(T^*,T_s^*)}.n.n_d\\
&+ \frac{r(N^*.n_d + N_D^*.n + n.n_d)}{K^2(T^*,T_s^*)}\left[\left[\frac{\partial K}{\partial T}\right]_{E_2}.\tau + \left[\frac{\partial K}{\partial T_s}\right]_{E_2}.\tau_s\right],\\
n_3 =& -\gamma_1.n.\tau + \pi_1\nu_1.n.u, \;\; n_4 = -\gamma_2.n.\tau_s + \pi_2\nu_2.n.u_s,\\
n_5 =&\gamma_1.n.\tau - \nu_1.n.u, \;\; n_6 = \gamma_2.n.\tau_s - \nu_2.n.u_s.
\end{aligned}
$$

In (9), M_2X and N_2 show the linear and nonlinear parts of the model system (3), respectively, and M_2 is a Jacobian matrix corresponding to the equilibrium point E_2. Thus, the characteristic equation of M_2 can be written as

$$p(x) = x^6 + c_1x^5 + c_2x^4 + c_3x^3 + c_4x^2 + c_5x + c_6. \tag{10}$$

For the coefficients c_1, c_2, c_3, c_4, c_5, and c_6, see **Appendix B**.

According to the Routh–Hurwitz criterion and characteristic equation (10), all the eigenvalues of Jacobian matrix M_2 are either negative or having negative real parts iff

$$c_j > 0, \quad (j = 1, 6), \tag{11a}$$

$$H_2 = c_1c_2 - c_3 > 0, \tag{11b}$$

$$H_3 = c_1c_2c_3 + c_1c_5 - c_3^2 - c_1^2c_4 > 0, \tag{11c}$$

$$H_4 = (c_1c_2 - c_3)(c_3c_4 - c_2c_5) + (c_1c_2 - c_3)c_1c_6 - (c_1c_4 - c_5)^2 > 0, \tag{11d}$$

$$\begin{aligned} H_5 = {} & c_3(c_1c_2 - c_3)(c_4c_5 - c_3c_6) + c_3c_5(c_2c_5 - c_1c_6) + c_1c_3c_6(c_1c_4 - 2c_5) \\ & - c_1(c_2c_5 - c_1c_6)^2 - c_5(c_1c_4 - c_5)^2 > 0. \end{aligned} \tag{11e}$$

So, we can state the following theorem for the locally asymptotically stability of the equilibrium point E_2.

Theorem 1. The equilibrium point E_2 of model system (3) is locally asymptotically stable, if conditions (11) are satisfied.

Hence, all the positive solutions of the model system (3) in the neighborhood of E_2, will become stable to E_2 if the conditions (11) are satisfied.

Existence of Hopf-bifurcation

In the model system (3), there is a possibility that Hopf-bifurcation [6, 9, 15] may exist corresponding to the equilibrium point E_2. So, we find the conditions for the existence of Hopf-bifurcation by treating Q (i.e., the discharge rate of primary toxicant, which is directly emitted in the environment from some sources) as a bifurcation parameter.

It is obvious that a Hopf-bifurcation may exist if all the eigenvalues of Jacobian matrix have negative real parts except a purely imaginary complex conjugate pair. In this case, the Jacobian matrix M_2 have six eigenvalues $x_j = R_j + iI_j$ $(j = 1, \ldots, 6)$ (say). So, the Hopf-bifurcation exist only when $R_1, R_2 = 0,\ I_1 = -I_2 \neq 0$ & $R_3, R_4, R_5, R_6 < 0$ at the critical value $Q = Q^*$(say).

According to the Liu's criterion [10], the model system (3) undergoes a Hopf-bifurcation, if the following conditions hold at the critical value $Q = Q^* > 0$:

$$[c_j]_{Q=Q^*} > 0 \quad \text{for } (j = 1, \ldots, 6), \tag{12a}$$

$$[H_2]_{Q=Q^*} = [c_1c_2 - c_3]_{Q=Q^*} > 0, \tag{12b}$$

$$[H_3]_{Q=Q^*} = [c_1c_2c_3 + c_1c_5 - c_3^2 - c_1^2c_4]_{Q=Q^*} > 0, \tag{12c}$$

$$\begin{aligned} [H_4]_{Q=Q^*} = {} & [(c_1c_2 - c_3)(c_3c_4 - c_2c_5) + (c_1c_2 - c_3)c_1c_6 - (c_1c_4 - c_5)^2]_{Q=Q^*} \\ & > 0, \end{aligned} \tag{12d}$$

$$\begin{aligned} [H_5]_{Q=Q^*} = {} & [c_3(c_1c_2 - c_3)(c_4c_5 - c_3c_6) + c_3c_5(c_2c_5 - c_1c_6) + c_1c_3c_6(c_1c_4 - 2c_5) \\ & - c_1(c_2c_5 - c_1c_6)^2 - c_5(c_1c_4 - c_5)^2\big]_{Q=Q^*} = 0, \end{aligned} \tag{12e}$$

$$\left[\frac{dR_j}{dQ}\right]_{Q=Q^*} \neq 0, \quad \text{for } j = 1, 2. \tag{12f}$$

With the help of *Existence of Hopf−bifurcation in a 6-dimensional system* [7], the last condition (12f) can be written in the terms of coefficients of characteristic equation (10) as follows:

$$\left[\frac{dR}{dQ}\right]_{Q=Q^*} = \left[\frac{1}{B}\frac{dH_5}{dQ}\right]_{Q=Q^*} \neq 0, \tag{13}$$

where H_5 is defined as (11e) and

$$B = 2\{c_1(c_1c_4 - c_5) - c_3(c_1c_2 - c_3)\}\begin{vmatrix} -c_1 & 1 & 1 & 0 & 0 \\ c_2 - 2A & -c_1 & 0 & 1 & 0 \\ c_1A - c_3 & c_2 - A & A & 0 & 1 \\ A^2 - c_2A + c_4 & c_1A - c_3 & 0 & A & 0 \\ 0 & A^2 - c_2A + c_4 & 0 & 0 & A \end{vmatrix}. \tag{14}$$

Also, $A = \dfrac{c_3c_5 + c_1^2c_6 - c_1c_2c_5}{c_1^2c_4 - c_1c_5 - c_1c_2c_3 + c_3^2}$.

So, we can state the following theorem for the existence of Hopf-bifurcation.

Theorem 2. The model (3) undergoes a Hopf-bifurcation corresponding to the equilibrium point E_2, if Q crosses the critical value $Q^* > 0$ such that

1. $[c_j]_{Q=Q^*} > 0 \quad \text{for } (j = 1, \ldots, 6)$,
2. $[H_2]_{Q=Q^*} = [c_1c_2 - c_3]_{Q=Q^*} > 0$,
3. $[H_3]_{Q=Q^*} = [c_1c_2c_3 + c_1c_5 - c_3^2 - c_1^2c_4]_{Q=Q^*} > 0$,
4. $[H_4]_{Q=Q^*} = [(c_1c_2 - c_3)(c_3c_4 - c_2c_5) + (c_1c_2 - c_3)c_1c_6 - (c_1c_4 - c_5)^2]_{Q=Q^*} > 0$,
5. $[H_5]_{Q=Q^*} = [c_3(c_1c_2 - c_3)(c_4c_5 - c_3c_6) + c_3c_5(c_2c_5 - c_1c_6)$ $+c_1c_3c_6(c_1c_4 - 2c_5) - c_1(c_2c_5 - c_1c_6)^2 - c_5(c_1c_4 - c_5)^2]_{Q=Q^*} = 0$,
6. $\left[\dfrac{dR}{dQ}\right]_{Q=Q^*} = \left[\dfrac{1}{B}\dfrac{dH_5}{dQ}\right]_{Q=Q^*} \neq 0$,

where B is defined as (14).

Hence, if Theorem 2 holds, there exist Hopf-bifurcation at the critical value Q^*. The model system (3) has bifurcating periodic solutions starting from Hopf-bifurcation. So, it is essential to know the nature of bifurcating periodic solutions.

Nature of bifurcating periodic solutions

The direction and stability of bifurcating periodic solutions can be examined by the sign of these important coefficients μ_2, β_2, τ_2 given by Hassard, Kazarinoff, and Wan [6]. To evaluate these coefficients μ_2, β_2, τ_2 for the parameter Q, we derive some explicit formulas with the help of normal form and center manifold theory [6, 9]. To transform the model system (3) in normal form, we assume all the eigenvalues of Jacobian matrix M_2 as $\pm iv$, $-J_1$, $-J_2$, $-J_3$ and $-J_4$.

Let $X = PY$; then the normal form is

$$\dot{Y} = JY + F, \quad Y = col.(y_1, y_2, y_3, y_4, y_5, y_6), \tag{15}$$

where

$$\begin{aligned} J =& P^{-1} M_2 P = \begin{bmatrix} 0 & -v & 0 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -J_4 \end{bmatrix} \\ \text{and} \quad F =& P^{-1} f = \begin{bmatrix} F_1(y_1, y_2, y_3, y_4, y_5, y_6) \\ F_2(y_1, y_2, y_3, y_4, y_5, y_6) \\ F_3(y_1, y_2, y_3, y_4, y_5, y_6) \\ F_4(y_1, y_2, y_3, y_4, y_5, y_6) \\ F_5(y_1, y_2, y_3, y_4, y_5, y_6) \\ F_6(y_1, y_2, y_3, y_4, y_5, y_6) \end{bmatrix}. \end{aligned} \tag{16}$$

Here P is a transformed matrix defined as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} \end{bmatrix},$$

where

$$P_{11}=m_{12}(L_3L_6-L_4L_7),\quad P_{12}=m_{12}(L_3L_7+L_4L_6),\quad P_{13}=m_{12}K_{31}K_{41},$$
$$P_{14}=m_{12}K_{32}K_{42},\quad P_{15}=m_{12}K_{33}K_{43},\quad P_{16}=m_{12}K_{34}K_{44},\quad P_{21}=-L_{13},$$
$$P_{22}=-L_{14},\quad P_{23}=K_{71},\quad P_{24}=K_{72},\quad P_{25}=K_{73},\quad P_{26}=K_{74},$$
$$P_{31}=m_{12}(L_1L_6+m_{31}vL_7),\quad P_{32}=m_{12}(L_1L_7-m_{31}vL_6)$$
$$P_{33}=m_{12}K_{11}K_{41},\quad P_{34}=m_{12}K_{12}K_{42},\quad P_{35}=m_{12}K_{13}K_{43},$$
$$P_{36}=m_{12}K_{14}K_{44},\quad P_{41}=m_{12}L_9,\quad P_{42}=m_{12}L_{10},\quad P_{43}=m_{12}K_{61},$$
$$P_{44}=m_{12}K_{62},\quad P_{45}=m_{12}K_{63},\quad P_{46}=m_{12}K_{64},$$
$$P_{51}=m_{12}(L_2L_6+m_{51}vL_7)\quad P_{52}=m_{12}(L_2L_7-m_{51}vL_6),$$
$$P_{53}=m_{12}K_{21}K_{41},\quad P_{54}=m_{12}K_{22}K_{42},\quad P_{55}=m_{12}K_{23}K_{43},$$
$$P_{56}=m_{12}K_{24}K_{44},\quad P_{61}=m_{12}L_{11},\quad P_{62}=m_{12}L_{12},\quad P_{63}=m_{12}K_{51},$$
$$P_{64}=m_{12}K_{52},\quad P_{65}=m_{12}K_{53},\quad P_{66}=m_{12}K_{54}.$$

Also,

$$L_1=m_{31}m_{55}-m_{35}m_{51},\quad L_2=m_{33}m_{51}-m_{31}m_{53},$$
$$L_3=m_{35}m_{53}-m_{33}m_{55}+v^2,\quad L_4=(m_{33}+m_{55})v,$$
$$L_5=m_{41}m_{66}-m_{46}m_{61},\quad L_6=m_{46}m_{64}-m_{44}m_{66}+v^2,$$
$$L_7=(m_{44}+m_{66})v,\quad L_8=m_{44}m_{61}-m_{41}m_{64},$$
$$L_9=L_3L_5+m_{43}m_{66}L_1+m_{41}vL_4-m_{43}m_{31}v^2,$$
$$L_{10}=L_4L_5-m_{41}vL_3-m_{43}vL_1-m_{31}m_{43}m_{66}v,$$
$$L_{11}=L_3L_8+m_{61}vL_4-m_{43}m_{64}L_1,$$
$$L_{12}=L_4L_8-m_{61}vL_3+m_{31}m_{43}m_{64}v,$$
$$L_{13}=m_{11}(L_3L_6-L_4L_7)+v(L_3L_7+L_4L_6)+m_{13}L_1+m_{14}L_9,$$
$$L_{14}=m_{11}(L_3L_7+L_4L_6)-v(L_3L_6-L_4L_7)+m_{14}L_{10}-m_{13}m_{31}v.$$

For $j=1,2,3,4$,

$$K_{1j}=m_{31}m_{55}-m_{35}m_{51}+m_{31}J_j,\quad K_{2j}=m_{33}m_{51}-m_{53}m_{31}+m_{51}J_j,$$
$$K_{3j}=m_{35}m_{53}-m_{33}m_{55}-(m_{33}+m_{55})J_j-J_j^2,$$
$$K_{4j}=m_{64}m_{46}-m_{44}m_{66}-(m_{44}+m_{66})J_j-J_j^2,$$
$$K_{5j}=(m_{61}m_{44}-m_{64}m_{41}+m_{61}J_j)K_{3j}-m_{64}m_{43}K_{1j},$$
$$K_{6j}=(m_{66}+J_j)(m_{41}K_{3j}+m_{43}K_{1j})-m_{61}m_{46}K_{3j},$$
$$K_{7j}=-\{(m_{11}+J_j)K_{3j}K_{4j}+m_{13}K_{1j}K_{4j}+m_{14}K_{6j}\},$$

and

$$f = \begin{bmatrix} f_1(y_1,y_2,y_3,y_4,y_5,y_6) \\ f_2(y_1,y_2,y_3,y_4,y_5,y_6) \\ f_3(y_1,y_2,y_3,y_4,y_5,y_6) \\ f_4(y_1,y_2,y_3,y_4,y_5,y_6) \\ f_5(y_1,y_2,y_3,y_4,y_5,y_6) \\ f_6(y_1,y_2,y_3,y_4,y_5,y_6) \end{bmatrix},$$

where

$$\begin{aligned} f_1(y_1,y_2,y_3,y_4,y_5,y_6) =& -\frac{r}{K(T^*,T_s^*)}.n^2 \\ &+ \frac{r(2N^*+n)}{K^2(T^*,T_s^*)}\left[\frac{\partial K}{\partial T}(T^*,T_s^*).\tau + \frac{\partial K}{\partial T_s}(T^*,T_s^*).\tau_s\right].n, \\ f_2(y_1,y_2,y_3,y_4,y_5,y_6) =& (r_1.u + r_2.u_s).(n-n_d) - \frac{r}{K(T^*,T_s^*)}.n.n_d \\ &+ \frac{r(N^*.n_d + N_D^*.n + n.n_d)}{K^2(T^*,T_s^*)} \\ &\times \left[\frac{\partial K}{\partial T}(T^*,T_s^*).\tau + \frac{\partial K}{\partial T_s}(T^*,T_s^*).\tau_s\right], \\ f_3(y_1,y_2,y_3,y_4,y_5,y_6) =& -\gamma_1.n.\tau + \pi_1\nu_1.n.u, \\ f_4(y_1,y_2,y_3,y_4,y_5,y_6) =& -\gamma_2.n.\tau_s + \pi_2\nu_2.n.u_s, \\ f_5(y_1,y_2,y_3,y_4,y_5,y_6) =& \gamma_1.n.\tau - \nu_1.n.u, \\ f_6(y_1,y_2,y_3,y_4,y_5,y_6) =& \gamma_2.n.\tau_s - \nu_2.n.u_s. \end{aligned}$$

Here,

$$\begin{aligned} n = \sum_{j=1}^{6} P_{1j}y_j,\ n_d = \sum_{j=1}^{6} P_{2j}y_j,\ \tau = \sum_{j=1}^{6} P_{3j}y_j, \\ \tau_s = \sum_{j=1}^{6} P_{4j}y_j,\ u = \sum_{j=1}^{6} P_{5j}y_j,\ u_s = \sum_{j=1}^{6} P_{6j}y_j. \end{aligned}$$

Now, we calculate the following quantities at critical value of parameter $Q = Q^*$ and $(y_1,y_2,y_3,y_4,y_5,y_6) = (0,0,0,0,0,0)$:

$$
\begin{aligned}
g_{11} =& \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2}+\frac{\partial^2 F_1}{\partial y_2^2}\right)+i\left(\frac{\partial^2 F_2}{\partial y_1^2}+\frac{\partial^2 F_2}{\partial y_2^2}\right)\right\},\\
g_{02} =& \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2}-\frac{\partial^2 F_1}{\partial y_2^2}-2\frac{\partial^2 F_2}{\partial y_1 \partial y_2}\right)+i\left(\frac{\partial^2 F_2}{\partial y_1^2}-\frac{\partial^2 F_2}{\partial y_2^2}+2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right\},\\
g_{20} =& \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2}-\frac{\partial^2 F_1}{\partial y_2^2}+2\frac{\partial^2 F_2}{\partial y_1 \partial y_2}\right)+i\left(\frac{\partial^2 F_2}{\partial y_1^2}-\frac{\partial^2 F_2}{\partial y_2^2}-2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right\},\\
g_{21} =& G_{21}+\sum_{j=1}^{4}(2G_{110}^{j}w_{11}^{j}+G_{101}^{j}w_{20}^{j}),
\end{aligned}
$$

where

$$
\begin{aligned}
G_{21} =& \frac{1}{8}\left\{\left(\frac{\partial^3 F_1}{\partial y_1^3}+\frac{\partial^3 F_2}{\partial y_2^3}+\frac{\partial^3 F_2}{\partial y_1^2 \partial y_2}+\frac{\partial^3 F_1}{\partial y_1 \partial y_2^2}\right)\right.\\
&\left. +i\left(\frac{\partial^3 F_2}{\partial y_1^3}-\frac{\partial^3 F_1}{\partial y_2^3}-\frac{\partial^3 F_1}{\partial y_1^2 \partial y_2}+\frac{\partial^3 F_2}{\partial y_1 \partial y_2^2}\right)\right\}.
\end{aligned}
$$

For $j = 1, 2, 3, 4$,

$$
\begin{aligned}
G_{110}^{j} =& \frac{1}{2}\left\{\left(\frac{\partial^2 F_1}{\partial y_1 \partial y_{(j+2)}}+\frac{\partial^2 F_2}{\partial y_2 \partial y_{(j+2)}}\right)+i\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_{(j+2)}}-\frac{\partial^2 F_1}{\partial y_2 \partial y_{(j+2)}}\right)\right\},\\
G_{101}^{j} =& \frac{1}{2}\left\{\left(\frac{\partial^2 F_1}{\partial y_1 \partial y_{(j+2)}}-\frac{\partial^2 F_2}{\partial y_2 \partial y_{(j+2)}}\right)+i\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_{(j+2)}}+\frac{\partial^2 F_1}{\partial y_2 \partial y_{(j+2)}}\right)\right\},\\
h_{11}^{j} =& \frac{1}{4}\left(\frac{\partial^2 F_{(j+2)}}{\partial y_1^2}+\frac{\partial^2 F_{(j+2)}}{\partial y_2^2}\right),\\
h_{20}^{j} =& \frac{1}{4}\left(\frac{\partial^2 F_{(j+2)}}{\partial y_1^2}-\frac{\partial^2 F_{(j+2)}}{\partial y_2^2}-2i\frac{\partial^2 F_{(j+2)}}{\partial y_1 \partial y_2}\right),\\
w_{11}^{k} =& \frac{h_{11}^{j}}{J_k}, \quad w_{20}^{j} = \frac{h_{20}^{j}}{J_j+2iv}.
\end{aligned}
$$

Now, we evaluate the coefficients μ_2, β_2 and τ_2 using the following formulas:

$$
\begin{aligned}
C_1(0) = \frac{i}{2v}\left[g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right]+\frac{1}{2}g_{21}, \quad \mu_2 = -\frac{\text{Re } C_1 0}{\text{Re } x_1'},\\
\beta_2 = 2\text{Re } C_1(0), \qquad \tau_2 = -\frac{(\text{Im } C_1(0)+\mu_2 \text{Im } x_1')}{v}.
\end{aligned}
\tag{17}
$$

By observing the sign of the coefficients μ_2, β_2 and τ_2, we can state following theorem for the nature of bifurcating periodic solutions corresponding to the parameter Q.

Theorem 3. If $\mu_2 > 0$ (or $\mu_2 < 0$), the model system (3) shows a supercritical (or subcritical) Hopf-bifurcation and the bifurcating periodic solutions exist for $Q > Q^*$ (or $Q < Q^*$), if $\beta_2 < 0$ (or $\beta_2 > 0$), the bifurcating periodic solutions are stable (or unstable), if $\tau_2 > 0$ (or $\tau_2 < 0$), the period of bifurcating solutions increases (or decreases).

5 Numerical simulation

To clarify the analytical results of model system (3), which are found in the previous sections, numerical simulation is given with the help of MATCONT [2]. For that, we assume the carrying capacity function as

$$K(T,T_s) = K_0 - \frac{b_{11}T}{1+b_{12}T} - \frac{b_{21}T_s}{1+b_{22}T_s}, \tag{18a}$$

and a set of parameters as

$$\begin{aligned}
&b = 0.2,\ d = 0.001,\ r_1 = 0.6,\ r_2 = 0.8,\ Q = 0.005, \alpha = 0.0008,\ \delta_1 = 0.08,\\
&\delta_2 = 0.005,\ \theta = 0.2,\ k = 0.15,\ \gamma_1 = 0.001,\ \gamma_2 = 0.0016,\ \pi_1 = 0.04,\\
&\pi_2 = 0.09,\ \nu_1 = 0.005,\ \nu_2 = 0.0004,\ \beta_1 = 0.025,\ \beta_2 = 0.02,\ K_0 = 10.0,\\
&b_{11} = 0.2,\ b_{12} = 1.0,\ b_{21} = 0.1,\ b_{22} = 2.0.
\end{aligned} \tag{18b}$$

For this set of the parameters, the equilibrium point $E_2(N^*, N_D^*, T^*, T_s^*, U^*, U_s^*)$ attains the value as $N^* = 9.10873,\ N_D^* = 0.79867,\ T^* = 0.02093,\ T_s^* = 0.03242,\ U^* = 0.00270,\ U_s^* = 0.01998$. From (8), $\frac{dF}{dN} = 0.19893 > 0$ shows that E_2 is unique. The local stability conditions (11) corresponding to E_2 are also satisfied.

In Figure 2, we show the changes in total density and density of deformed subclass of biological species with respect to the value of parameter θ upto 1 (keep the remaining parameters same as (18b)). Consequently, Figure 2 shows that when the fraction of the part of primary toxicant (kT), which is converted into secondary toxicant, increases, the total density decreases and the density of deformed subclass increases.

Figure 3 shows that the total density of species decreases and that the density of deformed subclass of species increases as conversion rate of primary toxicant into secondary toxicant k increase upto 1 (the other parameter same as (18b)).

Figure 4 shows the variation in total and deformed subclass densities corresponding to the discharge rate of primary toxicant Q. As the discharge rate Q increases, the total density of species N decreases and deformed subclass density N_D firstly increases then decreases with N. For large value $Q(= 0.200)$ both densities N and N_D start oscillating. Hence, there exists a Hopf-bifurcation corresponding to the parameter Q.

Figure 5 confirms that Hopf-bifurcation exists with respect to the parameter Q at critical value $Q^* = 0.13628$. This figure shows the real and imaginary parts of eigenvalues of Jacobian matrix M_2 for the parameter Q. The real parts of all eigenvalues (i.e., $R_i < 0$, $i = 1, \ldots, 6$) are negative for $Q < Q^*(= 0.13628)$. At $Q = Q^*$ two eigenvalues become purely imaginary (i.e.,

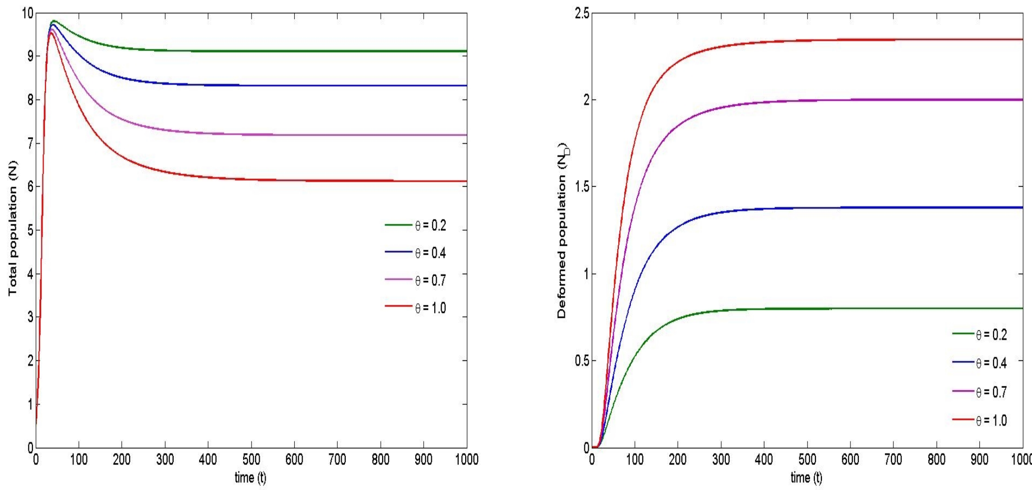

Figure 2: Time series graph for total and deformed subclass densities of species respect to the parameter θ.

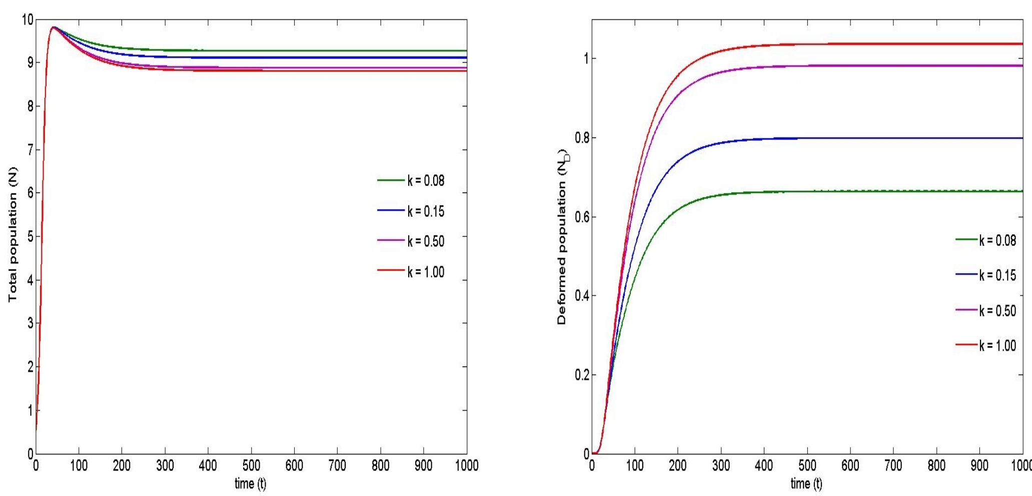

Figure 3: Time series graph for total and deformed subclass densities of species respect to the parameter k.

$R_1 = R_2 = 0$ and $I_1 = -I_2 \neq 0$) and other four eigenvalues have negative real parts (i.e., $R_i < 0$, $i = 3, \ldots, 6$).

Figure 6 shows the behavior of both densities N and N_D and also nature of bifurcating periodic solutions with respect to the discharge rate of primary toxicant Q. Both densities N and N_D become stable at equilibrium level for $Q < Q^*$. When Q crosses the critical value $Q^*(= 0.13628)$, the equilibrium point losses its stability and a supercritical Hopf-bifurcation oc-

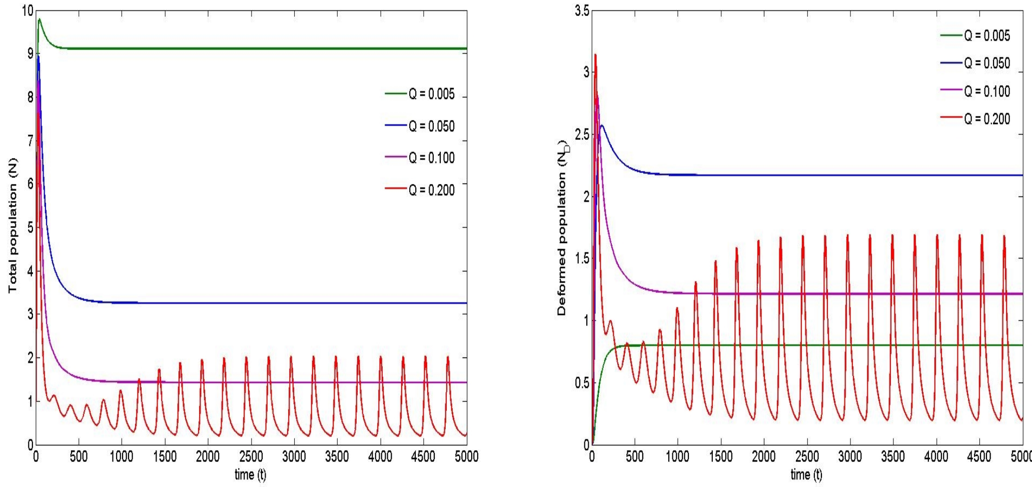

Figure 4: Time series graph for total and deformed population respect to the parameter Q.

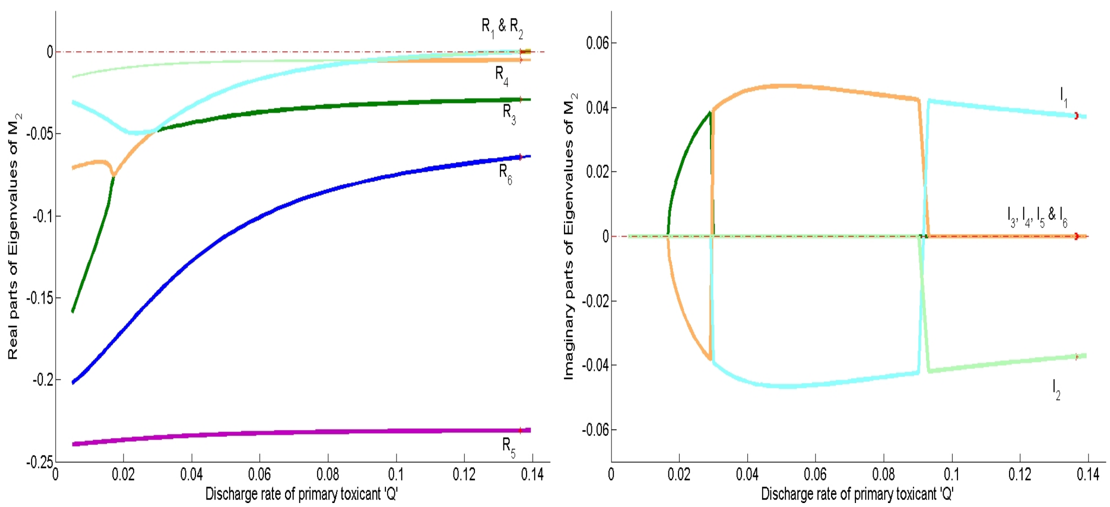

Figure 5: Real and Imaginary parts of eigenvalues of Jacobian matrix M_2 respect to the emission rate Q.

curs (since $\mu_2 = 1.8418 * 10^{-12} > 0$). Both densities start oscillating around their equilibrium level with stable bifurcating periodic orbits (since $\beta_2 = -1.2169 * 10^{-14} < 0$).

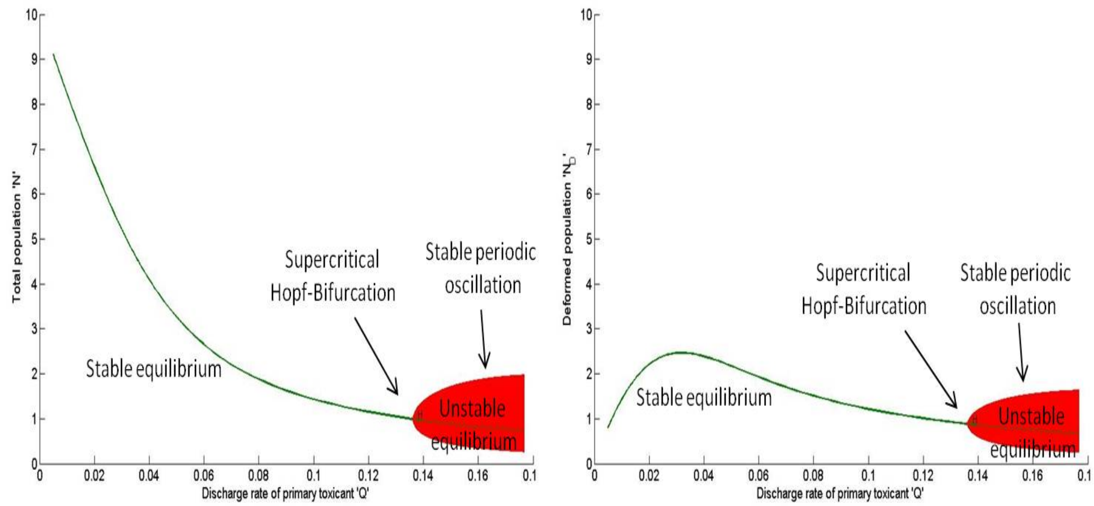

Figure 6: Dynamic behavior of densities of total and deformed populations for emission rate Q.

6 Conclusion

This paper is based on effect of primary and secondary toxicants on a biologically species. We have proposed and analyzed a mathematical model for the case in which after-effect of primary and secondary toxicants, a subclass of biological species shows deformity such as stagnating the reproduction capability. The stability analysis of model shows that total density decreases and deformed subclass density increases as the conversion rate of primary toxicant into secondary toxicant increases. If the discharge rate of primary toxicant into the environment increases, then total density of species decreases and deformed subclass density firstly increases then decreases with total density. If discharge rate crosses the critical value, then both densities start oscillations around their equilibrium and system shows a supercritical Hopf-bifurcation. All the bifurcating periodic solutions of model system are stable in nature.

A Proof of the region of attraction Ω

Proof. From the model system (3), we have

$$\frac{dN}{dt} \leq rN - \frac{rN^2}{K_0} = r\left(1 - \frac{N}{K_0}\right)N.$$

Thus, $\limsup_{t\to\infty} N(t) \leq K_0$. Also, we have

$$\begin{aligned}\frac{dT}{dt}+\frac{dU}{dt}&=Q-(\delta_1+k)T-\beta_1 U-(1-\pi_1)\nu_1 NU\\&\le Q-\delta_{m_1}(T+U),\end{aligned}$$

where $\delta_{m_1}=\min(\delta_1+k,\beta_1)$. Thus, $\limsup_{t\to\infty}(T(t)+U(t))\le \frac{Q}{\delta_{m_1}}$.
Similarly,

$$\begin{aligned}\frac{dT_s}{dt}+\frac{dU_s}{dt}&=\theta kT-\delta_2 T_s-\beta_2 U_s-(1-\pi_2)\nu_2 NU_s\\&\le\theta k\frac{Q}{\delta_{m_1}}-\delta_{m_2}(T_s+U_s),\end{aligned}$$

where $\delta_{m_2}=\min(\delta_2,\beta_2)$. Thus, $\limsup_{t\to\infty}(T_s(t)+U_s(t))\le\frac{\theta kQ}{\delta_{m_1}\delta_{m_2}}$.

From the second equation of model (3), we have

$$\begin{aligned}\frac{dN_D}{dt}&=(r_1U+r_2U_s)(N-N_D)-\frac{rN_DN}{K(T,T_s)}-(\alpha+d)N_D\\&\le(r_1\delta_{m_2}+r_2\theta k)\frac{Q}{\delta_{m_1}\delta_{m_2}}(K_0-N_D)-(\alpha+d)N_D.\end{aligned}$$

Thus, $\limsup_{t\to\infty}N_D(t)\le\frac{(r_1\delta_{m_2}+r_2\theta k)QK_0}{[(r_1\delta_{m_2}+r_2\theta k)Q+(\alpha+d)\delta_{m_1}\delta_{m_2}]}$, proving the region of attraction Ω. $\square$

B Coefficients of characteristic equation (10)

$$\begin{aligned}c_1=&-(m_{11}+m_{22}+m_{33}+m_{44}+m_{55}+m_{66}),\\c_2=&(m_{11}+m_{22})(m_{55}+m_{66})+m_{22}(m_{33}+m_{44})\\&+(m_{33}+m_{55})(m_{44}+m_{66})\\&+\begin{vmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{vmatrix}+\begin{vmatrix}m_{11}&m_{13}\\m_{31}&m_{33}\end{vmatrix}+\begin{vmatrix}m_{11}&m_{14}\\m_{41}&m_{44}\end{vmatrix}+\begin{vmatrix}m_{33}&m_{35}\\m_{53}&m_{55}\end{vmatrix}+\begin{vmatrix}m_{44}&m_{46}\\m_{64}&m_{66}\end{vmatrix},\\c_3=&-\{m_{11}m_{55}m_{66}+m_{12}(m_{25}m_{51}+m_{26}m_{61})\\&+m_{22}(m_{33}+m_{55})(m_{44}+m_{66})\}+m_{13}\begin{vmatrix}m_{31}&m_{35}\\m_{51}&m_{55}\end{vmatrix}\\&+m_{14}\left\{\begin{vmatrix}m_{41}&m_{46}\\m_{61}&m_{66}\end{vmatrix}-\begin{vmatrix}m_{31}&m_{33}\\m_{41}&m_{43}\end{vmatrix}\right\}\\&-m_{31}\begin{vmatrix}m_{12}&m_{13}\\m_{22}&m_{23}\end{vmatrix}-m_{41}\begin{vmatrix}m_{12}&m_{14}\\m_{22}&m_{24}\end{vmatrix}\end{aligned}$$

$$
\begin{aligned}
&- m_{55}\begin{vmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{vmatrix} - (m_{44}+m_{66})\begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} \\
&- (m_{11}+m_{22}+m_{44}+m_{66})\begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix} \\
&- (m_{11}+m_{22}+m_{33}+m_{55})\begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \\
&- (m_{33}+m_{44}+m_{55}+m_{66})\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix},
\end{aligned}
$$

$$
\begin{aligned}
c_4 = {}& m_{12}\Bigg\{ m_{26}m_{61}(m_{33}+m_{55}) + m_{25}m_{51}(m_{44}+m_{66}) + m_{31}\begin{vmatrix} m_{23} & m_{25} \\ m_{53} & m_{55} \end{vmatrix} \\
&+ m_{41}\begin{vmatrix} m_{24} & m_{26} \\ m_{64} & m_{66} \end{vmatrix} - m_{51}\begin{vmatrix} m_{23} & m_{25} \\ m_{33} & m_{35} \end{vmatrix} - m_{61}\begin{vmatrix} m_{24} & m_{26} \\ m_{44} & m_{46} \end{vmatrix} \Bigg\} \\
&+ m_{31}(m_{44}+m_{66})\begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} + m_{41}(m_{33}+m_{55})\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} \\
&- m_{13}(m_{22}+m_{44}+m_{66})\begin{vmatrix} m_{31} & m_{35} \\ m_{51} & m_{55} \end{vmatrix} \\
&- m_{14}(m_{22}+m_{33}+m_{55})\begin{vmatrix} m_{41} & m_{46} \\ m_{61} & m_{66} \end{vmatrix} \\
&+ m_{43}\Bigg\{ m_{14}m_{31}m_{66} + m_{14}\begin{vmatrix} m_{31} & m_{35} \\ m_{51} & m_{55} \end{vmatrix} - m_{31}\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} \Bigg\} \\
&+ \Bigg\{ m_{11}m_{55} + m_{22}(m_{33}+m_{55}) + \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} \Bigg\}\begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \\
&+ \Bigg\{ (m_{33}+m_{55})(m_{44}+m_{66}) + \begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix} + \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \Bigg\}\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \\
&+ \Bigg\{ m_{11}m_{66} + m_{22}(m_{44}+m_{66}) + \begin{vmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{vmatrix} + \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \Bigg\}\begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix},
\end{aligned}
$$

$$
\begin{aligned}
c_5 = {}& m_{31}m_{43}\Bigg\{ m_{66}\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} - m_{12}m_{26}m_{64} \Bigg\} \\
&+ m_{43}\Bigg\{ \begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} - m_{14}m_{66} \Bigg\}\begin{vmatrix} m_{31} & m_{35} \\ m_{51} & m_{55} \end{vmatrix} \\
&+ (m_{33}+m_{55})\Bigg\{ m_{14}m_{22}\begin{vmatrix} m_{41} & m_{46} \\ m_{61} & m_{66} \end{vmatrix} + m_{61}m_{12}\begin{vmatrix} m_{24} & m_{26} \\ m_{44} & m_{46} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&- m_{12}m_{41}\begin{vmatrix} m_{24} & m_{26} \\ m_{64} & m_{66} \end{vmatrix} \Bigg\} \\
&+ (m_{44}+m_{66})\left\{ m_{12}m_{51}\begin{vmatrix} m_{23} & m_{25} \\ m_{33} & m_{35} \end{vmatrix} - m_{12}m_{31}\begin{vmatrix} m_{23} & m_{25} \\ m_{53} & m_{55} \end{vmatrix} \right\} \\
&+ m_{13}\left\{ m_{22}(m_{44}+m_{66}) + \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \right\} \begin{vmatrix} m_{31} & m_{35} \\ m_{51} & m_{55} \end{vmatrix} \\
&- \left\{ m_{12}m_{25}m_{51} + m_{31}\begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \right\} \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \\
&- \left\{ (m_{44}+m_{66})\begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix} + (m_{33}+m_{55})\begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \right\} \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \\
&- \Bigg\{ (m_{11}+m_{22})\begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} + m_{12}m_{26}m_{61} + m_{41}\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} \\
&- m_{14}\begin{vmatrix} m_{41} & m_{46} \\ m_{61} & m_{66} \end{vmatrix} \Bigg\} \begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
c_6 = {} & \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix} \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} + \begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} \begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix} \begin{vmatrix} m_{41} & m_{46} \\ m_{61} & m_{66} \end{vmatrix} \\
&+ \begin{vmatrix} m_{31} & m_{35} \\ m_{51} & m_{55} \end{vmatrix} \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} - m_{12}m_{26}\begin{vmatrix} m_{41} & m_{44} \\ m_{61} & m_{64} \end{vmatrix} \begin{vmatrix} m_{33} & m_{35} \\ m_{53} & m_{55} \end{vmatrix} \\
&- m_{12}m_{25}\begin{vmatrix} m_{31} & m_{33} \\ m_{51} & m_{53} \end{vmatrix} \begin{vmatrix} m_{44} & m_{46} \\ m_{64} & m_{66} \end{vmatrix} \\
&+ \left\{ m_{12}m_{26}m_{43}m_{64} - m_{43}m_{66}\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} \right\} \begin{vmatrix} m_{31} & m_{35} \\ m_{51} & m_{55} \end{vmatrix},
\end{aligned}
$$

where $\begin{vmatrix} m_1 & m_2 \\ m_3 & m_4 \end{vmatrix} = m_1m_4 - m_2m_3.$

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Conflicts of Interest

The authors declare no conflicts of interest.

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