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Research Article

Comparative semi-analytical solutions of Caputo time-fractional Fokker–Planck equations via Laplace transform-based decomposition and iterative methods

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Abstract

In this study, the Laplace Adomian decomposition method and the Laplace variational iteration method are employed to obtain approximate analytical solutions for both linear and nonlinear time-fractional Fokker–Planck equations (TFFPEs). These methods combine the Laplace transform with the Adomian decomposition method and the variational iteration method, respectively. Fractional derivatives are considered in the Caputo sense. The proposed approaches are straightforward to implement and yield solutions without requiring linearization, discretization, or perturbation. Several examples are solved and presented both graphically and numerically to illustrate the accuracy and stability of the methods. A comparison with the exact solution for the integer-order case confirms that the approximate solutions are in excellent agreement with the exact results. Further comparisons with other methods indicate that the proposed approaches provide improved accuracy while reducing computational effort.

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All tabular and graphical results are generated using Python. Additionally, the stability and convergence of the proposed methods for TFFPEs are analyzed.

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1 Introduction

Fractional calculus is a branch of mathematics that deals with derivatives and integrals of arbitrary orders and plays an important role in the mathematical modeling of real-world phenomena such as anomalous diffusion, viscoelastic materials, signal and image processing, electromagnetic field modeling, electrochemical processes, fluid flow problems, reaction-diffusion systems, water wave dynamics, population growth studies, traffic flow analysis, fractional stochastic models, and diverse topics in medical sciences [37, 45, 3].

The Fokker–Planck equation is a statistical physics equation developed by Adriaan Fokker and Max Planck to describe Brownian motion and is commonly used to explain solute transport [39]. The Fokker–Planck equation (FPE) appears in various fields in natural science, including solid-state physics, astrophysics, chemical and biological systems, polymer dynamics, stochastic processes, and circuit theory [39]. Due to the wide range of applications of the FPE, the aim of this study is to obtain analytical and numerical solutions of the linear and nonlinear fractional FPEs.

The linear time-fractional Fokker–Planck equation (TFFPE), also known as the fractional forward Kolmogorov equation [39] has the form for the variables $\Im$ and $\wp$:

$$\frac{\partial^\zeta}{\partial \wp^\zeta} u(\Im, \wp) = -\frac{\partial}{\partial \Im}\left[P(\Im, \wp) u(\Im, \wp)\right] + \frac{\partial^2}{\partial \Im^2}\left[R(\Im, \wp) u(\Im, \wp)\right] \tag{1}$$

with the initial condition given by

$$u(\Im, 0) = f(\Im), \quad \Im \in \mathbb{R}. \tag{2}$$

Here, $\dfrac{\partial^\zeta}{\partial \wp^\zeta} u(\Im, \wp)$ represents the Caputo fractional derivative of order $\zeta \in (0,1)$ with respect to time. The function $u(\Im, \wp)$ is considered to be causal in both space and time. The terms $P(\Im, \wp)$ and $R(\Im, \wp) > 0$ denote the drift and diffusion coefficients, respectively, where both coefficients are assumed to depend only on the spatial variable $\Im$. A related fractional partial differential equation is the fractional backward Kolmogorov equation [39], which can be expressed as follows:

$$\frac{\partial^\zeta u}{\partial \wp^\zeta} = -\left[P(\Im,\wp)\frac{\partial}{\partial \Im} + R(\Im,\wp)\frac{\partial^2}{\partial \Im^2}\right]u(\Im,\wp). \tag{3}$$

A generalization of (1) to n variables $(\Im_1, \Im_2, \ldots, \Im_n)$ yields the following:

$$\frac{\partial^\zeta u}{\partial \wp^\zeta} = -\sum_{i=1}^{n}\frac{\partial}{\partial \Im_i}\left[P_i(\Im,\wp)u(\Im,\wp)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial \Im_i \partial \Im_j}\left[R_{ij}(\Im,\wp)u(\Im,\wp)\right] \tag{4}$$

with the initial condition given by

$$u(\Im,0) = f(\Im), \quad \Im = (\Im_1, \Im_2, \ldots, \Im_n) \in \mathbb{R}^n. \tag{5}$$

The nonlinear TFFPE, which extends the traditional FPE, appears in numerous areas of science and engineering, such as nonlinear hydrodynamics, plasma physics, surface physics, polymer dynamics, laser physics, engineering applications, biophysics, neuroscience, population modeling, pattern formation, and selected problems in psychology and marketing [9]. The nonlinear TFFPE can be formulated as follows:

$$\frac{\partial^\zeta u}{\partial \wp^\zeta} = -\sum_{i=1}^{n}\frac{\partial}{\partial \Im_i}\left[P_i(\Im,\wp,u)u(\Im,\wp)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial \Im_i \partial \Im_j}\left[R_{ij}(\Im,\wp,u)u(\Im,\wp)\right] \tag{6}$$

with the initial condition given by

$$u(\Im,0) = f(\Im), \quad \Im = (\Im_1, \Im_2, \ldots, \Im_n) \in \mathbb{R}^n, \tag{7}$$

where ζ is the order of the fractional derivative $(0 < \zeta \leq 1)$ and n is integer $(n \neq 0)$.

Recently, researchers have developed many methods for constructing approximate analytical solutions of TFFPEs, such as the new iterative method (NIM) [28], He's homotopy perturbation method (HPM) [34], new transform iterative method (NITM) and homotopy perturbation Elzaki transform method (HPETM) [30], Yang transform decomposition method (YTDM) [2], homotopy perturbation Sumudu transform method (HPSTM) [6], fractional power series method (FPSM) [11], fractional iteration method (FIM) [31], shifted Chebyshev collocation method (CPFK) and finite difference method (FDM) [10], homotopy analysis transform method (HATM) [43], asymptotic homotopy perturbation method (AHPM) [19], and finite volume methods (FVM) [48].

Although many analytical and numerical methods have been successfully applied to solve TFFPEs, the Laplace-based hybrid methods, namely the Laplace Adomian decomposition method (LADM) and the Laplace variational iteration method (LVIM), offer distinct advantages. These semi-analytical approaches provide highly accurate approximations that closely match exact solutions even with only a few terms, converge rapidly, and require significantly less computational effort compared to traditional iterative or numerical methods. Additionally, they can handle both linear and nonlinear problems in a unified framework, making them versatile and efficient for a wide range of fractional-order problems. Motivated by the need for fast, reliable, and accurate solutions of TFFPEs with complex drift and diffusion terms, this paper applies and compares LADM and LVIM, provides convergence and stability analysis, and validates the methods with numerical examples, and demonstrates their effectiveness for complex fractional systems.

In 1980, George Adomian introduced the Adomian decomposition method (ADM), which is a well-known methodology to solve linear and nonlinear differential equations of classical as well as fractional order that gives accurate solutions in a concurrent series form. However, ADM does not properly decompose the fractional order parts. So, it was modified with many integral transformations, such as Elzaki, Mohand, Adoodh, Sumudu, Natural, Kashuri-Fundo, Yang, and Shehu [44, 7, 22]. ADM also combined with the Laplace transform leads to a powerful method known as the LADM, which was introduced by Khuri [26]. In this method, we convert a differential equation to an algebraic equation, and the nonlinear terms are decomposed in terms of Adomian polynomials. LADM is more powerful than the standard ADM method [32].

Another well-known powerful method for solving differential equations is the variational iteration method (VIM), introduced by Ji-Huan He [15]. The combined form of the Laplace transform and the VIM is called the LVIM, introduced by Hammouch and Mekkaoui [12]. LVIM is more powerful than VIM. Furthermore, the results obtained using LVIM have been compared with LADM, showing that in some studies LVIM may be more accurate [33], but in our examples both methods produce identical results. Unlike LADM, there is no necessity for the distinct treatment of nonlinear terms.

Recent studies have shown the effectiveness of Laplace-based decomposition and iteration techniques in solving nonlinear and fractional PDEs. For example, LADM has been applied to the fractional Klein–Gordon [40], sine–Gordon [24], multi-dimensional Navier–Stokes [29], Chaffee-Infante equations [5], PHI-four equation [41], voltage of telegraph equation [23], Rotavirus model [1], SEIR epidemic model [8], smoking models [14], COVID-19 SDIQR epidemic models [42]. Likewise, LVIM has been applied to fractional diffusion and reaction-diffusion equations [13], heat-like equations [4], wave-like equations [21], Klein–Gordon and sine–Gordon [35], chemical kinetics and population dynamics [36], physical systems [25], microelectromechanical systems [47], transport phenomena [18], nonlinear PDEs [17] and many more. These applications highlight the wide utility of transform-based decomposition methods and motivate the present comparative study of LADM and LVIM for TFFPEs.

The rest of this paper is organized as follows. Section 2 presents the necessary mathematical preliminaries. Sections 3 and 4 describe the LADM and LVIM algorithms to solve TFFPE. Section 5 presents the convergence and error analysis of the proposed methods. Section 6 provides numerical examples and comparisons. Section 7 discusses the numerical and graphical results, and Section 8 concludes the paper.

2 Mathematical preliminaries

This section presents fundamental concepts and operators essential for our subsequent analysis of fractional differential equations.

Definition 1 (Caputo Fractional Derivative)**.** For a function $f \in C^n[0,\infty)$ and real number $\zeta > 0$, the Caputo fractional derivative of order ζ is defined as follows [37]:

$$ {}^{C}D^{\zeta}_{\Im} f(\Im) = \begin{cases} \dfrac{1}{\Gamma(n-\zeta)} \displaystyle\int_0^{\Im} (\Im - t)^{n-\zeta-1} f^{(n)}(t)\, dt, & n-1 < \zeta < n, \\ \dfrac{d^n}{d\Im^n} f(\Im), & \zeta = n, \end{cases} \tag{8} $$

where $n \in \mathbb{N}$ and $\Gamma(\cdot)$ represents the gamma function.

For power functions, the Caputo derivative exhibits the following behavior:

$$ {}^{C}D^{\zeta}_{\wp}\, \wp^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\zeta+1)} \wp^{\beta-\zeta}, & \beta > n-1, \\ 0, & \beta \le n-1, \end{cases} \tag{9} $$

where $n-1 < \zeta \le n$ and $\beta \in \mathbb{R}$.

Definition 2 (Laplace transform)**.** The Laplace transform of a function $f(\wp)$ is given by [37, 45]

$$ F(s) = \mathcal{L}\{f(\wp)\} = \int_0^{\infty} f(\wp)\, e^{-s\wp}\, d\wp, \tag{10} $$

where $F(s)$ denotes the Laplace transform of $f(\wp)$.

Definition 3 (Laplace transform of fractional derivatives)**.** For $m-1 < \zeta \le m$, where $m \in \mathbb{N}$, the Laplace transform of the Caputo fractional derivative satisfies [37]

$$\mathcal{L}\left\{{}^{C}D^{\zeta}f(\wp)\right\} = s^{\zeta}F(s) - \sum_{k=0}^{m-1} s^{\zeta-k-1}f^{(k)}(0), \tag{11}$$

where $F(s)$ denotes the Laplace transform of $f(\wp)$.

Definition 4 (Mittag-Leffler function)**.** The one-parameter Mittag-Leffler function $E_{\zeta}(\wp)$ for $\zeta > 0$ is defined through the series expansion [37]:

$$E_{\zeta}(\wp) = \sum_{k=0}^{\infty} \frac{\wp^{k}}{\Gamma(\zeta k+1)}, \tag{12}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

3 Laplace Adomian decomposition method

The LADM (Laplace Adomian decomposition method) has been successfully applied to a variety of fractional derivatives, including Riemann–Liouville, Caputo, Caputo–Fabrizio–Riemann, Caputo–Fabrizio–Caputo, Atangana–Baleanu–Riemann, and Atangana–Baleanu–Caputo types [20]. In this section, we focus on the implementation of the LADM algorithm for solving the nonlinear TFFPE.

Applying the Laplace transform to both sides of (6) for one variable case, we get

$$\mathcal{L}\left(\frac{\partial^{\zeta}u}{\partial\wp^{\zeta}}\right) = \mathcal{L}\left(-\frac{\partial}{\partial\Im}[P(\Im,\wp,u)u(\Im,\wp)] + \frac{\partial^{2}}{\partial\Im^{2}}[R(\Im,\wp,u)u(\Im,\wp)]\right). \tag{13}$$

Applying the Laplace transform property for fractional derivatives yields

$$s^{\zeta}\bar{u}(\Im,s) - s^{\zeta-1}u(\Im,0) = \mathcal{L}\left(-\frac{\partial}{\partial\Im}[P(\Im,\wp,u)u] + \frac{\partial^{2}}{\partial\Im^{2}}[R(\Im,\wp,u)u]\right). \tag{14}$$

Using the initial condition (7), we obtain

$$\bar{u}(\Im,s) = \frac{f(\Im)}{s} + \frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial\Im}[P(\Im,\wp,u)u] + \frac{\partial^{2}}{\partial\Im^{2}}[R(\Im,\wp,u)u]\right). \tag{15}$$

The standard Laplace decomposition method defines the solution $u(\Im,\wp)$ as an infinite series:

$$u(\Im,\wp) = \sum_{n=0}^{\infty} u_{n}(\Im,\wp). \tag{16}$$

The nonlinear terms $M(u) = P(\Im, \wp, u)u$ and $N(u) = R(\Im, \wp, u)u$ are decomposed using Adomian polynomials:

$$M(u) = \sum_{n=0}^{\infty} M_n, \quad N(u) = \sum_{n=0}^{\infty} N_n, \tag{17}$$

where the Adomian polynomials are computed as

$$M_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} M\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\Bigg|_{\lambda=0}, \tag{18}$$

$$N_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\Bigg|_{\lambda=0}. \tag{19}$$

Substituting these into (15) gives

$$\sum_{n=0}^{\infty} u_n(\Im, s) = \frac{f(\Im)}{s} + \frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial \Im}\sum_{n=0}^{\infty} M_n + \frac{\partial^2}{\partial \Im^2}\sum_{n=0}^{\infty} N_n\right). \tag{20}$$

The recursive solution scheme becomes

$$\begin{aligned} \bar{u}_0(\Im, s) &= \frac{f(\Im)}{s}, \\ \bar{u}_{n+1}(\Im, s) &= \frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial \Im} M_n + \frac{\partial^2}{\partial \Im^2} N_n\right), \quad n \geq 0. \end{aligned} \tag{21}$$

Applying the inverse Laplace transform, we obtain

$$u_0(\Im, \wp) = f(\Im), \tag{22}$$

$$u_{n+1}(\Im, \wp) = \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial \Im} M_n + \frac{\partial^2}{\partial \Im^2} N_n\right)\right], \quad n \geq 0. \tag{23}$$

The approximate solution is obtained by truncating the series,

$$u(\Im, \wp) \approx \sum_{n=0}^{k} u_n(\Im, \wp). \tag{24}$$

4 Laplace variational iteration method

Applying the Laplace transform to both sides of (6) and using the initial condition (7), we get

$$\bar{u}(\Im, s) = \frac{f(\Im)}{s} + \frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial \Im}[P(\Im, \wp, u)u] + \frac{\partial^2}{\partial \Im^2}[R(\Im, \wp, u)u]\right). \tag{25}$$

Applying the inverse Laplace transform yields

$$u(\Im,\wp)=f(\Im)+\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial\Im}[P(\Im,\wp,u)u]+\frac{\partial^2}{\partial\Im^2}[R(\Im,\wp,u)u]\right)\right]. \tag{26}$$

Differentiating with respect to $\wp$ gives

$$\frac{\partial u(\Im,\wp)}{\partial\wp}=\frac{\partial}{\partial\wp}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial\Im}[P(\Im,\wp,u)u]+\frac{\partial^2}{\partial\Im^2}[R(\Im,\wp,u)u]\right)\right]. \tag{27}$$

The correction functional is constructed as follows:

$$\begin{aligned}u_{n+1}(\Im,\wp)=u_n(\Im,\wp)-\int_0^{\wp}\lambda\Bigg[&\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon}\\&-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\bigg(\frac{1}{s^{\zeta}}\mathcal{L}\Big(-\frac{\partial}{\partial\Im}[P(\Im,\varepsilon,u_n)u_n]+\frac{\partial^2}{\partial\Im^2}[R(\Im,\varepsilon,u_n)u_n]\Big)\bigg)\Bigg]d\varepsilon.\end{aligned} \tag{28}$$

By the variational theory, the Lagrange multiplier λ for (28) can be obtained as follows:

$$1+\lambda|_{\varepsilon=\wp}=0 \quad\Rightarrow\quad \lambda=-1.$$

Substituting $\lambda=-1$ into (28) yields the iterative scheme:

$$\begin{aligned}u_{n+1}(\Im,\wp)=u_n(\Im,\wp)+\int_0^{\wp}\Bigg[&\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\bigg(\frac{1}{s^{\zeta}}\mathcal{L}\Big(-\frac{\partial}{\partial\Im}[P(\Im,\varepsilon,u_n)u_n]\\&+\frac{\partial^2}{\partial\Im^2}[R(\Im,\varepsilon,u_n)u_n]\Big)\bigg)\Bigg]d\varepsilon.\end{aligned} \tag{29}$$

The exact solution is given by

$$u(\Im,\wp)=\lim_{n\to\infty}u_n(\Im,\wp). \tag{30}$$

5 Convergence and error analysis

Definition 5. For a self map $\mathcal{T}:X\to X$ on a set X, a point $\Im$ is called a fixed point of $\mathcal{T}$ if

$$\mathcal{T}(\Im)=\Im. \tag{31}$$

Definition 6. Consider a metric space (Ξ, d). A self-map $\mathcal{T} : X \to X$ is said to be a contraction if there exists a constant γ with $0 \le \gamma < 1$ such that, for every pair $\Im, \wp \in X$ [27],

$$d\big(\mathcal{T}(\Im), \mathcal{T}(\wp)\big) \le \gamma\, d(\Im, \wp). \tag{32}$$

Theorem 1. Let (X, d) be a complete non-empty metric space. If $\mathcal{T} : X \to X$ is a contraction mapping, then $\mathcal{T}$ has specifically one fixed point.

Proof. See proof in [27]. □

Theorem 2. Consider a Banach space (X, d), and let $\mathcal{T} : X \to X$ be a self-mapping. Assume that there exist constants $\delta \ge 0$ and $0 \le \mu < 1$ such that, for all $x, y \in X$,

$$d\big(\mathcal{T}(x), \mathcal{T}(y)\big) \le \delta\, d(x, \mathcal{T}(x)) + \mu\, d(x, y). \tag{33}$$

If $\mathcal{T}$ has a fixed point q, then $\mathcal{T}$ is said to be Picard-$\mathcal{T}$ stable.

Proof. See proof in [38]. □

5.1 Laplace Adomian decomposition method

Theorem 3 (Picard-$\mathcal{T}$ stability of LADM for nonlinear TFFPEs)**.** Consider a Banach space $(X, \|\cdot\|)$ and a self-map $\mathcal{T} : X \to X$. Then, the iterative procedure (23) of the LADM is defined by

$$\mathcal{T}(u_n(\Im, \wp)) = u_{n+1}(\Im, \wp) = u(\Im, 0) + \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left(A_n(u)\right)\right], \tag{34}$$

where $A_n = -\frac{\partial}{\partial \Im} M_n + \frac{\partial^2}{\partial \Im^2} N_n$ is the Picard $\mathcal{T}$-stable if

1. $\|A_m(u) - A_n(u)\| \le \mu_1 \|u_m(\Im, \wp) - u_n(\Im, \wp)\|$ for some $\mu_1 \in \mathbb{R}^+$,
2. $\mu = \mu_1 \left\| \frac{\wp^\zeta}{\Gamma(\zeta+1)} \right\| < 1$.

Proof. First, we verify that the mapping $\mathcal{T}$ possesses a fixed point. Considering $n, m \in \mathbb{N}$, we obtain

$$\mathcal{T}(u_n(\Im, \wp)) = u(\Im, 0) + \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left(A_n(u)\right)\right], \tag{35}$$

$$\mathcal{T}(u_m(\Im,\wp)) = u(\Im,0) + \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(A_m(u)\right)\right]. \tag{36}$$

Subtracting (36) from (35), we obtain

$$\mathcal{T}(u_n(\Im,\wp)) - \mathcal{T}(u_m(\Im,\wp)) = \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(A_n(u) - A_m(u)\right)\right]. \tag{37}$$

Taking the norm on the both sides of (37) gives

$$\|\mathcal{T}(u_n(\Im,\wp)) - \mathcal{T}(u_m(\Im,\wp))\| = \left\|\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(A_n(u) - A_m(u)\right)\right]\right\|.$$

Using given assumption and the properties of the Laplace transform, we have

$$\begin{aligned}\|\mathcal{T}(u_n(\Im,\wp)) - \mathcal{T}(u_m(\Im,\wp))\| &\le \mu_1\|u_m(\Im,\wp) - u_n(\Im,\wp)\| \left\|\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left[1\right]\right]\right\| \\ &= \mu_1\|u_n(\Im,\wp) - u_m(\Im,\wp)\| \left\|\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right\| \\ &= \mu\|u_n(\Im,\wp) - u_m(\Im,\wp)\|,\end{aligned}$$

where $\mu = \mu_1 \left\|\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right\|$.

Thus, the self-map $\mathcal{T}$ has a fixed point and

$$\|\mathcal{T}(u_n(\Im,\wp)) - \mathcal{T}(u_m(\Im,\wp))\| \le \delta\|u_n(\Im,\wp) - u_m(\Im,\wp)\| + \mu\|u_n(\Im,\wp) - u_m(\Im,\wp)\|. \tag{38}$$

For $\delta = 0$, $\mu = \mu_1 \left\|\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right\| < 1$, the self-map $\mathcal{T}$ is a contraction mapping. Therefore, by the Banach fixed-point Theorem 2, the iterative scheme is Picard-$\mathcal{T}$ stable.

□

Theorem 4 (Convergence of LADM)**.** Let $(X, \|\cdot\|)$ be a Banach space and let $\mathcal{T} : X \to X$ be the operator associated with the LADM iteration scheme (34). Then there exists a unique fixed point, and the sequence $\{\psi_n(\Im,\wp)\}_{n=0}^{\infty}$ generated by LADM with an initial value $u_0 \in X$ converges to this fixed point whenever $0 < \eta < 1$, $\|u_0\| < \infty$, and

$$\|u_{n+1}(\Im,\wp)\| \le \eta\|u_n(\Im,\wp)\|, \quad n \ge 0.$$

Proof. Take $u_0 \in X$, and define the sequence of partial sums

$$\psi_n = \sum_{k=0}^{n} u_k, \qquad n \ge 0.$$

From the assumption

$$\|\psi_{n+1} - \psi_n\| = \|u_{n+1}\| \leq \eta\|u_n\| \leq \eta^{n+1}\|u_0\|.$$

For $m > n$, applying the triangle inequality yields

$$\|\psi_m - \psi_n\| \leq \sum_{k=n+1}^{m} \|u_k\| \leq \sum_{k=n+1}^{m} \eta^k\|u_0\| \leq \frac{\eta^{n+1}}{1-\eta}\|u_0\|.$$

Since $0 < \eta < 1$, the right-hand side tends to zero as $n \to \infty$, so $\{\psi_n\}$ is Cauchy in X and hence convergent. Denote the limit by

$$\phi^* = \lim_{n\to\infty} \psi_n.$$

To establish uniqueness, suppose ψ is another fixed point. Then

$$\|\psi^* - \phi\| = \|\mathcal{T}\phi^* - \mathcal{T}\psi\| \leq \eta\|\phi^* - \psi\|.$$

As $0 < \eta < 1$, this forces $\phi^* = \psi$. Therefore, ψ^* is the unique fixed point of $\mathcal{T}$. $\square$

Theorem 5 (Error bound LADM)**.** Assume that $\sum_{k=0}^{n} u_k(\Im, \wp)$ is finite and that $u(\Im, \wp)$ is the corresponding series solution. Considering $\|u_{k+1}(\Im, \wp)\| \leq \eta\|u_k(\Im, \wp)\|$ for some $0 < \eta < 1$, then the maximum absolute error is given by

$$\|u(\Im, \wp) - \sum_{k=0}^{n} u_k(\Im, \wp)\| < \frac{\eta^{n+1}}{1-\eta}\|u_0(\Im, \wp)\|. \tag{39}$$

Proof. Let $\sum_{k=0}^{n} u_k$ be finite; then $\sum_{k=0}^{\infty} u_k < \infty$. We have

$$\begin{aligned}\|u - \sum_{k=0}^{n} u_k\| &= \|\sum_{k=n+1}^{\infty} u_k\| \leq \sum_{k=n+1}^{\infty} \|u_k\| \leq \sum_{k=n+1}^{\infty} \eta^k\|u_0\| \\ &= \eta^{n+1}\|u_0\| \sum_{k=0}^{\infty} \eta^k = \frac{\eta^{n+1}}{1-\eta}\|u_0\|.\end{aligned}$$

$\square$

5.2 Laplace variational iteration method

From the iterative formula (29), we can rewrite it as

$$u_{n+1}(\Im,\wp)=u_n(\Im,\wp)-\mathcal{T}[u_n], \tag{40}$$

where the LVIM operator $\mathcal{T}[u_n]$ is defined as follows:

$$\mathcal{T}[u]=\int_0^{\wp}\left[\frac{\partial u(\Im,\varepsilon)}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left(\frac{1}{s^{\zeta}}\mathcal{L}\Big(-\frac{\partial}{\partial\Im}[P(\Im,\varepsilon,u)u]+\frac{\partial^2}{\partial\Im^2}[R(\Im,\varepsilon,u)u]\Big)\right)\right]d\varepsilon. \tag{41}$$

Then the LVIM iteration can be written as

$$u_{n+1}(\Im,\wp)=u_n(\Im,\wp)-\mathcal{T}[u_n]. \tag{42}$$

Define the components ϑ_k, $k=0,1,2,\ldots$, through the recursive relation,

$$\begin{cases}\vartheta_0=u_0,\\ \vartheta_{k+1}=-\mathcal{T}[\vartheta_0+\vartheta_1+\cdots+\vartheta_k], \quad k=0,1,2,\ldots.\end{cases} \tag{43}$$

Then the series solution is given by

$$u(\Im,\wp)=\lim_{n\to\infty}u_n(\Im,\wp)=\sum_{k=0}^{\infty}\vartheta_k(\Im,\wp). \tag{44}$$

Theorem 6 (Convergence of LVIM)**.** Let $\mathcal{T}$, defined in (41), be an operator from a Banach space X to X. The series solution $u(\Im,\wp)=\sum_{k=0}^{\infty}\vartheta_k(\Im,\wp)$, defined in (44), converges if there exists $0<\eta<1$ such that

$$\|\vartheta_{k+1}\|\le\eta\|\vartheta_k\|$$

for all $k\in\mathbb{N}\cup\{0\}$.

Proof. From (43), we have $\vartheta_{k+1}=-\mathcal{T}[\sum_{i=0}^{k}\vartheta_i]$. The condition implies

$$\|\vartheta_{k+1}\|\le\eta\|\vartheta_k\|\le\eta^2\|\vartheta_{k-1}\|\le\cdots\le\eta^{k+1}\|\vartheta_0\|.$$

Define the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$ as

$$S_n=\sum_{k=0}^{n}\vartheta_k, \quad n=0,1,2,\ldots. \tag{45}$$

Consider the difference between consecutive partial sums:

$$\|S_{n+1} - S_n\| = \|\vartheta_{n+1}\| \leq \eta^{n+1}\|\vartheta_0\|. \tag{46}$$

For any $n, m \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned} \|S_n - S_m\| &= \left\| \sum_{k=m+1}^{n} (S_k - S_{k-1}) \right\| \leq \sum_{k=m+1}^{n} \|S_k - S_{k-1}\| \\ &\leq \sum_{k=m+1}^{n} \eta^k \|\vartheta_0\| = \eta^{m+1} \left(\frac{1 - \eta^{n-m}}{1 - \eta} \right) \|\vartheta_0\|. \end{aligned} \tag{47}$$

Since $0 < \eta < 1$, we obtain

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{48}$$

Hence, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space X, which implies that the series solution $u(\Im, \wp) = \sum_{k=0}^{\infty} \vartheta_k(\Im, \wp)$ converges. □

Theorem 7 (Error estimate for LVIM)**.** If the series solution $\sum_{k=0}^{\infty} \vartheta_k(\Im, \wp)$ converges to $u(\Im, \wp)$, and the truncated series $\sum_{k=0}^{j} \vartheta_k(\Im, \wp)$ is used as an approximation, then the maximum absolute error $\mathcal{E}_j(\Im, \wp)$ is estimated by

$$\mathcal{E}_j(\Im, \wp) = \left\| u(\Im, \wp) - \sum_{k=0}^{j} \vartheta_k(\Im, \wp) \right\| \leq \frac{\eta^{j+1}}{1 - \eta} \|\vartheta_0\|, \tag{49}$$

where $0 < \eta < 1$ and $\|\vartheta_{k+1}\| \leq \eta \|\vartheta_k\|$ for all $k \geq 0$.

Proof. From the convergence proof, for $n > j$, we have

$$\|S_n - S_j\| \leq \eta^{j+1} \left(\frac{1 - \eta^{n-j}}{1 - \eta} \right) \|\vartheta_0\|. \tag{50}$$

Taking the limit as $n \to \infty$, since $S_n \to u(\Im, \wp)$, we have

$$\|u(\Im, \wp) - S_j\| \leq \lim_{n\to\infty} \|S_n - S_j\| \leq \frac{\eta^{j+1}}{1 - \eta} \|\vartheta_0\|. \tag{51}$$

Thus,

$$\mathcal{E}_j(\Im, \wp) = \left\| u(\Im, \wp) - \sum_{k=0}^{j} \vartheta_k(\Im, \wp) \right\| \leq \frac{\eta^{j+1}}{1 - \eta} \|\vartheta_0\|. \tag{52}$$

□

6 Mathematical examples

In this section, we present the analytical solutions of several linear and nonlinear space-time fractional FPEs and display the solutions both numerically and graphically. Furthermore, we check the non-fractional case ($\zeta = 1$) to ensure consistency with the exact solution.

Example 1. Consider a linear TFFPE (1) with $P(\Im, \wp) = \Im, \quad R(\Im, \wp) = \frac{\Im^2}{2}$:

$$\frac{\partial^\zeta u}{\partial \wp^\zeta} = \left[-\frac{\partial}{\partial \Im}\Im + \frac{\partial^2}{\partial \Im^2}\frac{\Im^2}{2} \right] u(\Im, \wp), \tag{53}$$

subject to the initial condition

$$u(\Im, 0) = f(\Im) = \Im, \quad \Im \in \mathbb{R}. \tag{54}$$

(i) LADM solution

Applying the Laplace transform to both sides of (53), we obtain

$$s^\zeta \bar{u}(\Im, s) - s^{\zeta-1}u(\Im, 0) = \mathcal{L}\left[-\frac{\partial}{\partial \Im}(\Im u) + \frac{\partial^2}{\partial \Im^2}\left(\frac{\Im^2}{2}u\right) \right].$$

Using the initial condition (54), this becomes

$$\bar{u}(\Im, s) = \frac{\Im}{s} + \frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial \Im}(\Im u) + \frac{\partial^2}{\partial \Im^2}\left(\frac{\Im^2}{2}u\right) \right].$$

The recurrence relation is, therefore

$$\begin{aligned} \bar{u}_0(\Im, s) &= \frac{\Im}{s}, \\ \bar{u}_{n+1}(\Im, s) &= \frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial \Im}(\Im u_n) + \frac{\partial^2}{\partial \Im^2}\left(\tfrac{\Im^2}{2}u_n\right) \right], \quad n \geq 0. \end{aligned} \tag{55}$$

Applying the inverse Laplace transform, we have

$$u_0(\Im, \wp) = \mathcal{L}^{-1}\left(\frac{\Im}{s}\right) = \Im.$$

For $n = 0$ in (55):

$$u_1(\Im, \wp) = \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left(-\frac{\partial}{\partial \Im}\Im^2 + \frac{\partial^2}{\partial \Im^2}\frac{\Im^3}{2} \right) \right]$$

$$= \mathcal{L}^{-1}\left(\frac{\Im}{s^{\zeta+1}}\right) = \frac{\Im\,\wp^{\zeta}}{\Gamma(\zeta+1)}.$$

For $n = 1$:

$$\begin{aligned} u_2(\Im,\wp) &= \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\,\mathcal{L}\left(-\frac{\partial}{\partial\Im}\left(\frac{\Im^2\wp^{\zeta}}{\Gamma(\zeta+1)}\right)+\frac{\partial^2}{\partial\Im^2}\left(\frac{\Im^3\wp^{\zeta}}{2\Gamma(\zeta+1)}\right)\right)\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\,\mathcal{L}\left(\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}\right)\right] = \mathcal{L}^{-1}\left(\frac{\Im}{s^{2\zeta+1}}\right) = \frac{\Im\,\wp^{2\zeta}}{\Gamma(2\zeta+1)}. \end{aligned}$$

Similarly,

$$u_3(\Im,\wp) = \frac{\Im\,\wp^{3\zeta}}{\Gamma(3\zeta+1)}.$$

Thus, the LADM approximate solution of (53) is

$$u(\Im,\wp) = \Im\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\dots\right) = \Im E_{\zeta}(\wp^{\zeta}). \tag{56}$$

(ii) LVIM solution

Applying the Laplace transform to both sides of (53) with initial condition (54), we obtain

$$\bar{u}(\Im,s) = \frac{\Im}{s}+\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}(\Im u)+\frac{\partial^2}{\partial\Im^2}\left(\frac{\Im^2}{2}u\right)\right]. \tag{57}$$

Applying the inverse Laplace transform yields

$$u(\Im,\wp) = \Im+\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left[-\frac{\partial}{\partial\Im}(\Im u)+\frac{\partial^2}{\partial\Im^2}\left(\frac{\Im^2}{2}u\right)\right]\right]. \tag{58}$$

Differentiating (58) with respect to $\wp$ gives

$$\frac{\partial u(\Im,\wp)}{\partial\wp} = \frac{\partial}{\partial\wp}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[\left(-\frac{\partial}{\partial\Im}\Im+\frac{\partial^2}{\partial\Im^2}\frac{\Im^2}{2}\right)u\right]\right]. \tag{59}$$

Now, from (29), the iterative formula is

$$\begin{aligned} u_{n+1}(\Im,\wp) &= u_n(\Im,\wp) \\ &\quad -\int_0^{\wp}\left[\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left(\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}(\Im u_n)+\frac{\partial^2}{\partial\Im^2}\left(\frac{\Im^2}{2}u_n\right)\right]\right)\right]d\varepsilon. \end{aligned} \tag{60}$$

The initial iteration:

$$u_0(\Im,\wp)=\Im.$$

For $n=0$:

$$\begin{aligned}u_1(\Im,\wp)&=\Im-\int_0^{\wp}\left[\frac{\partial\Im}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\Im^2+\frac{\partial^2}{\partial\Im^2}\frac{\Im^3}{2}\right]\right]\right]d\varepsilon\\&=\Im+\int_0^{\wp}\left[\frac{\Im\,\zeta\,\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}\right]d\varepsilon=\Im+\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}.\end{aligned}$$

For $n=1$:

$$\begin{aligned}u_2(\Im,\wp)&=\Im+\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}-\int_0^{\wp}\left[\frac{\partial}{\partial\varepsilon}\left(\Im+\frac{\Im\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)\right.\\&\quad\left.-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(\Im^2+\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)+\frac{\partial^2}{\partial\Im^2}\left(\frac{\Im^3}{2}\left(1+\frac{\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)\right)\right]\right]\right]d\varepsilon\\&=\Im+\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}+\int_0^{\wp}\left[\frac{2\Im\,\zeta\,\varepsilon^{2\zeta-1}}{\Gamma(2\zeta+1)}\right]d\varepsilon=\Im+\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\Im\wp^{2\zeta}}{\Gamma(2\zeta+1)}.\end{aligned}$$

For $n=2$:

$$u_3(\Im,\wp)=\Im+\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\Im\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\Im\wp^{3\zeta}}{\Gamma(3\zeta+1)}.$$

The LVIM solution is

$$u(\Im,\wp)=\Im\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\cdots\right)=\Im E_{\zeta}(\wp^{\zeta}).\tag{61}$$

Here, $E_{\zeta}(\wp^{\zeta})$ is the Mittag-Leffler function. The LADM and LVIM solutions at $\zeta=1$ converge to $u(\Im,\wp)=\Im e^{\wp}$, which is the exact solution for the nonfractional FPE [16]. Our LADM and LVIM solutions (56) and (61) match with the solutions obtained by NITM [30], HPETM [30], YTDM [2], and HPSTM [6].

Table 1: Comparative analysis of HPSTM [6], NITM [30], HPETM [30], YTDM [2], LADM, LVIM, and exact solution for Example 1.

$\wp$	$\Im$	HPSTM	NITM	HPETM	YTDM	LADM	LVIM	Exact
0.2	0.25	0.305333	0.305333	0.305333	0.305333	0.305333	0.305333	0.305351
0.2	0.50	0.610667	0.610667	0.610667	0.610667	0.610667	0.610667	0.610701
0.2	0.75	0.916000	0.916000	0.916000	0.916000	0.916000	0.916000	0.916052
0.2	1.00	1.221333	1.221333	1.221333	1.221333	1.221333	1.221333	1.221403
0.4	0.25	0.372667	0.372667	0.372667	0.372667	0.372667	0.372667	0.372956
0.4	0.50	0.745333	0.745333	0.745333	0.745333	0.745333	0.745333	0.745912
0.4	0.75	1.118000	1.118000	1.118000	1.118000	1.118000	1.118000	1.118869
0.4	1.00	1.490667	1.490667	1.490667	1.490667	1.490667	1.490667	1.491825
0.6	0.25	0.454000	0.454000	0.454000	0.454000	0.454000	0.454000	0.455530
0.6	0.50	0.908000	0.908000	0.908000	0.908000	0.908000	0.908000	0.911059
0.6	0.75	1.362000	1.362000	1.362000	1.362000	1.362000	1.362000	1.366589
0.6	1.00	1.816000	1.816000	1.816000	1.816000	1.816000	1.816000	1.822119

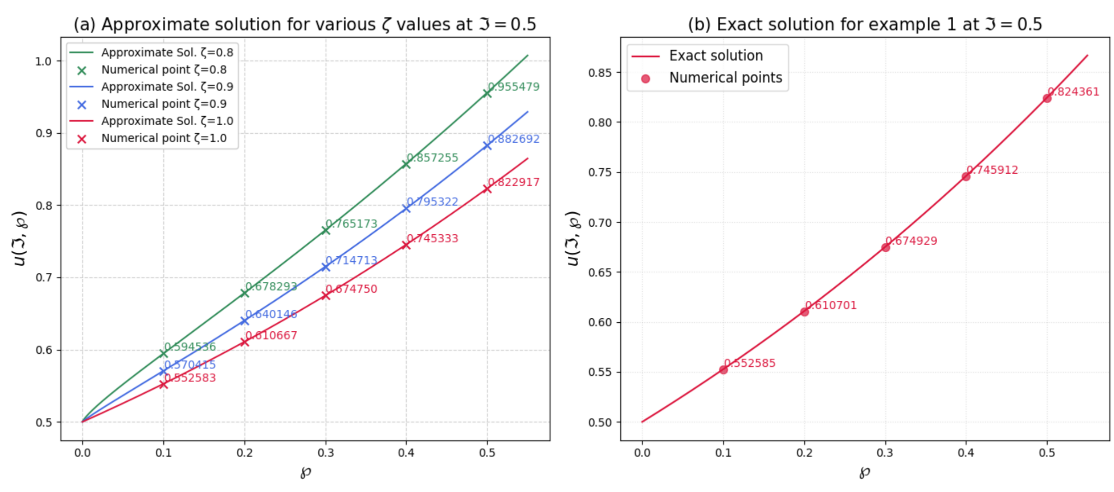

Figure 1: Two-dimensional graphical solutions of Example 1 at $\Im = 0.1$: (a) approximate solutions obtained by the LADM for $\zeta = 0.8, 0.9$ and 1.0, and (b) the corresponding exact solution.

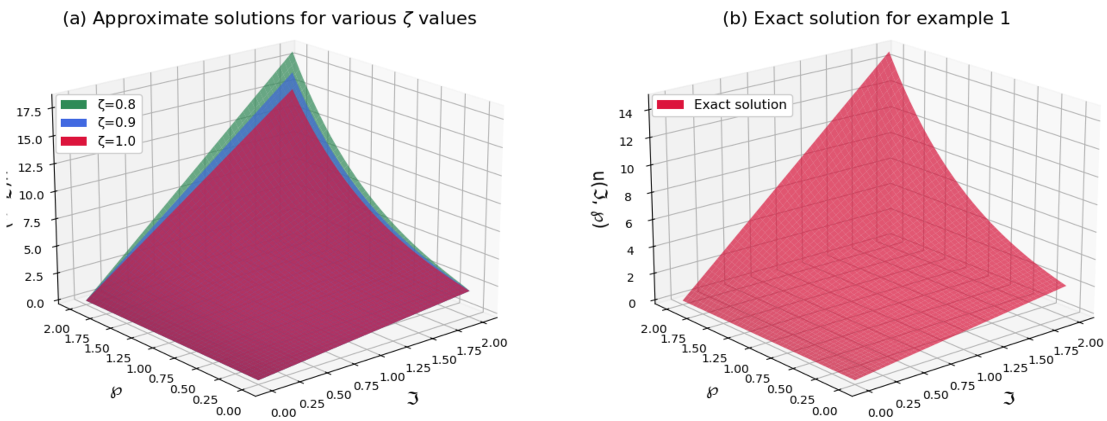

Figure 2: Three-dimensional graphical solutions of Example 1: (a) Approximate solutions for various values of ζ, and (b) Exact solution.

Example 2. Consider a backward Kolmogorov equation (3) with $P(\Im, \wp) = -(1 + \Im)$ and $R(\Im, \wp) = e^{\wp}\Im^2$:

$$\frac{\partial^\zeta}{\partial \wp^\zeta} u(\Im, \wp) = (1 + \Im)\frac{\partial}{\partial \Im} u(\Im, \wp) + e^{\wp}\Im^2 \frac{\partial^2}{\partial \Im^2} u(\Im, \wp), \tag{62}$$

subject to the initial condition:

$$u(\Im, 0) = 1 + \Im, \quad \Im \in \mathbb{R}. \tag{63}$$

(i) LADM solution

Applying the Laplace transform to both sides of (62) with initial condition (63), we obtain

$$\bar{u}(\Im, s) = \frac{1 + \Im}{s} + \frac{\mathcal{L}}{s^\zeta}\left[(1 + \Im)\frac{\partial u}{\partial \Im} + e^{\wp}\Im^2 \frac{\partial^2 u}{\partial \Im^2}\right]. \tag{64}$$

The recursive scheme is

$$\begin{aligned} \bar{u}_0(\Im, s) &= \frac{1 + \Im}{s} \\ \bar{u}_{n+1}(\Im, s) &= \frac{\mathcal{L}}{s^\zeta}\left[(1 + \Im)\frac{\partial}{\partial \Im} u_n(\Im, \wp) + e^{\wp}\Im^2 \frac{\partial^2}{\partial \Im^2} u_n(\Im, \wp)\right], \quad n \geq 0. \end{aligned} \tag{65}$$

Applying the inverse Laplace transform, we have

$$u_0(\Im, \wp) = \mathcal{L}^{-1}\left(\frac{1 + \Im}{s}\right) = 1 + \Im.$$

For $n = 0$:

$$\begin{aligned} u_1(\Im,\wp) &= \mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[(1+\Im)\frac{\partial}{\partial\Im}(1+\Im)+e^{\wp}\Im^2\frac{\partial^2}{\partial\Im^2}(1+\Im)\right]\right] \\ &= \mathcal{L}^{-1}\left[\frac{1+\Im}{s^{\zeta+1}}\right]=(1+\Im)\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}. \end{aligned}$$

For $n = 1$:

$$\begin{aligned} u_2(\Im,\wp) &= \mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[(1+\Im)\frac{\partial}{\partial\Im}\left((1+\Im)\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right)+e^{\wp}\Im^2\frac{\partial^2}{\partial\Im^2}\left((1+\Im)\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right)\right]\right] \\ &= \mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[\frac{(1+\Im)\,\wp^{\zeta}}{\Gamma(\zeta+1)}\right]\right]=\mathcal{L}^{-1}\left[\frac{1+\Im}{s^{2\zeta+1}}\right]=(1+\Im)\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}. \end{aligned}$$

For $n = 2$:

$$u_3(\Im,\wp)=(1+\Im)\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}.$$

Thus, the LADM solution of (62) is given by

$$u(\Im,\wp)=(1+\Im)\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\cdots\right)=(1+\Im)E_{\zeta}(\wp^{\zeta}). \tag{66}$$

(ii) LVIM solution

Applying the Laplace transform and the inverse Laplace transform to both sides of (62) along with the initial condition, we get

$$u(\Im,\wp)=(1+\Im)+\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[(1+\Im)\frac{\partial u}{\partial\Im}+e^{\wp}\Im^2\frac{\partial^2 u}{\partial\Im^2}\right]\right]. \tag{67}$$

From (29) and (67), the iterative scheme becomes

$$\begin{aligned} u_{n+1}(\Im,\wp) &= u_n(\Im,\wp) \\ &\quad -\int_0^{\wp}\left[\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left(\frac{\mathcal{L}}{s^{\zeta}}\left[(1+\Im)\frac{\partial u_n(\Im,\varepsilon)}{\partial\Im}+e^{\wp}\Im^2\frac{\partial^2 u_n(\Im,\varepsilon)}{\partial\Im^2}\right]\right)\right]d\varepsilon. \end{aligned} \tag{68}$$

The initial iteration is

$$u_0(\Im,\wp)=1+\Im.$$

From (68), for $n = 0$, we have

$$
\begin{aligned}
u_1(\Im,\wp) &= (1+\Im) \\
&\quad - \int_0^{\wp}\left[\frac{\partial(1+\Im)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left((1+\Im)\frac{\partial(1+\Im)}{\partial\Im} + \left(e^{\wp}\Im^2\right)\frac{\partial^2(1+\Im)}{\partial\Im^2}\right)\right]\right]d\varepsilon \\
&= (1+\Im) + \int_0^{\wp}\left[\frac{(1+\Im)\,\zeta\,\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}\right]d\varepsilon = (1+\Im)\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right).
\end{aligned}
$$

Similarly, from (68), for $n=1$, we have

$$
\begin{aligned}
u_2(\Im,\wp) &= u_1(\Im,\wp) \\
&\quad - \int_0^{\wp}\left[\frac{\partial u_1(\Im,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left((1+\Im)\frac{\partial u_1(\Im,\varepsilon)}{\partial\Im} + \left(e^{\wp}\Im^2\right)\frac{\partial^2 u_1(\Im,\varepsilon)}{\partial\Im^2}\right)\right]\right]d\varepsilon \\
&= (1+\Im)\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right) + \int_0^{\wp}\left[(1+\Im)\cdot\frac{2\zeta\varepsilon^{2\zeta-1}}{\Gamma(2\zeta+1)}\right]d\varepsilon \\
&= (1+\Im)\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}\right).
\end{aligned}
$$

Similarly,

$$
u_3(\Im,\wp) = (1+\Im)\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}\right).
$$

The LVIM solution is

$$
u(\Im,\wp) = (1+\Im)\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\cdots\right) = (1+\Im)E_{\zeta}(\wp^{\zeta}). \tag{69}
$$

The LADM and LVIM solution at $\zeta=1$ converge to $u(\Im,\wp)=(1+\Im)e^{\wp}$, which is the exact solution for the nonfractional FPE [16]. Our LADM and LVIM solutions (66) and (69) match with the solutions obtained by NIM [28], NITM [30], HPETM [30], YTDM [2], and FPSM [11].

Table 2: Comparative analysis of FIM, LADM, LVIM, and exact solution for Example 2, including absolute errors for $\zeta = 1$.

$\wp$	$\Im$	FIM [31]	LADM	LVIM	Exact	FIM Er.	LADM Er.	LVIM Er.
0.2	0.1	1.3434667	1.3434667	1.3434667	1.3435430	0.0000764	0.0000764	0.0000764
0.3	0.1	1.4844500	1.4844500	1.4844500	1.4848447	0.0003947	0.0003947	0.0003947
0.4	0.1	1.6397333	1.6397333	1.6397333	1.6410072	0.0012738	0.0012738	0.0012738
0.2	0.6	1.9541333	1.9541333	1.9541333	1.9542444	0.0001111	0.0001111	0.0001111
0.3	0.6	2.1592000	2.1592000	2.1592000	2.1597741	0.0005741	0.0005741	0.0005741
0.4	0.6	2.3850667	2.3850667	2.3850667	2.3869195	0.0018529	0.0018529	0.0018529
0.2	0.9	2.3205333	2.3205333	2.3205333	2.3206652	0.0001319	0.0001319	0.0001319
0.3	0.9	2.5640500	2.5640500	2.5640500	2.5647317	0.0006817	0.0006817	0.0006817
0.4	0.9	2.8322667	2.8322667	2.8322667	2.8344669	0.0022003	0.0022003	0.0022003

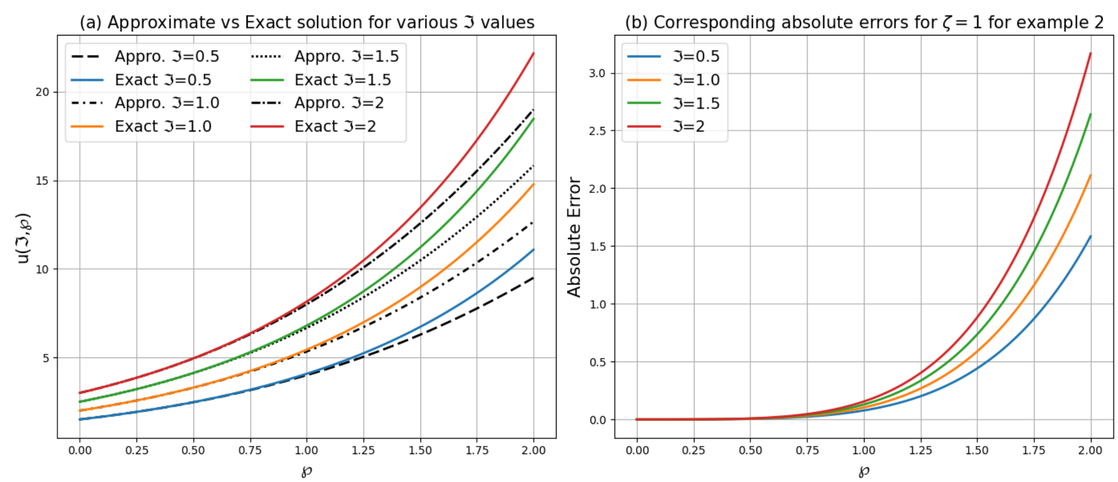

Figure 3: Two-dimensional graphical solutions of Example 2 for different values of $\Im$ when $\zeta = 1$ at : (a) Comparison of the approximate solutions with the exact solution, and (b) The corresponding absolute error .

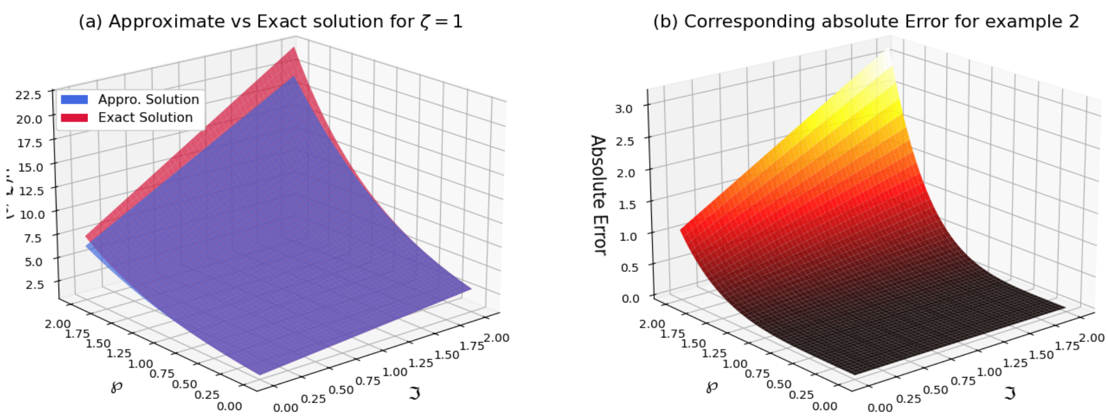

Figure 4: Three-dimensional graphical solutions of Example 2 for $\zeta = 1$: (a) Comparison of the approximate solutions with the exact solution, and (b) The corresponding absolute errors

Example 3. Consider a nonlinear TFFPE (6) with $P_1(\Im, \wp, u) = 4\frac{u}{\Im} - \frac{\Im}{3}$ and $R_{11}(\Im, \wp, u) = u$:

$$\frac{\partial^\zeta u}{\partial \wp^\zeta} = \left[-\frac{\partial}{\partial \Im}\left(4\frac{u^2}{\Im} - \frac{\Im u}{3}\right) + \frac{\partial^2}{\partial \Im^2}\left(u^2\right)\right], \tag{70}$$

subject to the initial condition:

$$u(\Im, 0) = \Im^2, \quad \Im \in \mathbb{R}. \tag{71}$$

(i) LADM solution

Applying the Laplace transform with initial condition, we have

$$\bar{u}(\Im, s) = \frac{\Im^2}{s} + \frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial \Im}\left(4\frac{u}{\Im} - \frac{\Im}{3}\right) u + \frac{\partial^2}{\partial \Im^2}(u^2)\right]. \tag{72}$$

Applying the inverse Laplace transform, we get

$$u(\Im, \wp) = \Im^2 + \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial \Im}\left(4\frac{u}{\Im} - \frac{\Im}{3}\right) u + \frac{\partial^2}{\partial \Im^2}(u^2)\right]\right]. \tag{73}$$

Now, applying the standard Laplace decomposition method, we get the iterative formula:

$$\sum_{n=0}^{\infty} u_n(\Im, \wp) = \Im^2 + \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial \Im}\left(\frac{4}{\Im}\sum_{n=0}^{\infty} N_n - \frac{\Im}{3}\sum_{n=0}^{\infty} u_n\right) + \frac{\partial^2}{\partial \Im^2}\sum_{n=0}^{\infty} N_n\right]\right], \tag{74}$$

where N_n are Adomian polynomials for u^2 and N_n are Adomian polynomials for $\frac{u^2}{\Im}$:

$$
\begin{aligned}
N_0(u) &= u_0^2,\\
N_1(u) &= 2u_0u_1,\\
N_2(u) &= u_1^2 + 2u_0u_2,\\
&\vdots
\end{aligned}
$$

The solution components are

$$
\begin{aligned}
u_0(\Im,\wp) &= \Im^2,\\
u_1(\Im,\wp) &= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial\Im}\left(4\frac{u_0^2}{\Im}-\frac{\Im u_0}{3}\right)+\frac{\partial^2}{\partial\Im^2}u_0^2\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial\Im}\left(4\Im^3-\frac{\Im^3}{3}\right)+\frac{\partial^2}{\partial\Im^2}\left(\Im^4\right)\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial\Im}\left(\frac{11\Im^3}{3}\right)+12\Im^2\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-11\Im^2+12\Im^2\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\cdot\frac{\Im^2}{s}\right]=\frac{\Im^2\wp^\zeta}{\Gamma(\zeta+1)},\\
u_2(\Im,\wp) &= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial\Im}\left(4\frac{2u_0u_1}{\Im}-\frac{\Im u_1}{3}\right)+\frac{\partial^2}{\partial\Im^2}(2u_0u_1)\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial\Im}\left(8\Im u_1-\frac{\Im u_1}{3}\right)+\frac{\partial^2}{\partial\Im^2}(2\Im^2u_1)\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[-\frac{\partial}{\partial\Im}\left(\frac{23}{3}\Im u_1\right)+4u_1+4\Im\frac{\partial u_1}{\partial\Im}+2\Im^2\frac{\partial^2u_1}{\partial\Im^2}\right]\right]\\
&= \mathcal{L}^{-1}\left[\frac{1}{s^\zeta}\mathcal{L}\left[\Im^2\left(\frac{\wp^\zeta}{\Gamma(\zeta+1)}\right)\right]\right]=\mathcal{L}^{-1}\left[\frac{\Im^2}{s^{2\zeta+1}}\right]=\frac{\Im^2\wp^{2\zeta}}{\Gamma(2\zeta+1)}\\
u_3(\Im,\wp) &= \frac{\Im^2\wp^{3\zeta}}{\Gamma(3\zeta+1)}.
\end{aligned}
$$

Thus, the LADM solution of (70) is given by

$$
u(\Im,\wp)=\Im^2\left(1+\frac{\wp^\zeta}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\cdots\right)=\Im^2E_\zeta(\wp^\zeta). \tag{75}
$$

(ii) LVIM solution

From (29) and (73), the iterative scheme becomes

$$u_{n+1}(\Im,\wp)=u_n(\Im,\wp) - \int_0^{\wp}\left[\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left(-\frac{\partial}{\partial\Im}\left(4\frac{u_n^2}{\Im}-\frac{\Im u_n}{3}\right)+\frac{\partial^2}{\partial\Im^2}u_n^2\right)\right]\right]d\varepsilon. \tag{76}$$

The initial iteration is

$$u_0(\Im,\wp)=\Im^2.$$

Then for $n=0$:

$$\begin{aligned}
u_1(\Im,\wp)&=\Im^2-\int_0^{\wp}\left[\frac{\partial\Im^2}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left(-\frac{\partial}{\partial\Im}\left(4\frac{\Im^4}{\Im}-\frac{\Im^3}{3}\right)+\frac{\partial^2\Im^4}{\partial\Im^2}\right)\right]\right]d\varepsilon\\
&=\Im^2-\int_0^{\wp}\left[0-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial\Im}\left(4\Im^3-\frac{\Im^3}{3}\right)+12\Im^2\right)\right]\right]d\varepsilon\\
&=\Im^2-\int_0^{\wp}\left[-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial\Im}\left(\frac{11\Im^3}{3}\right)+12\Im^2\right)\right]\right]d\varepsilon\\
&=\Im^2-\int_0^{\wp}\left[-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-11\Im^2+12\Im^2\right)\right]\right]d\varepsilon\\
&=\Im^2-\int_0^{\wp}\left[-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\cdot\frac{\Im^2}{s}\right]\right]d\varepsilon\\
&=\Im^2-\int_0^{\wp}\left[-\frac{\partial}{\partial\varepsilon}\left(\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)\right]d\varepsilon\\
&=\Im^2+\int_0^{\wp}\left[\frac{\Im^2\zeta\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}\right]d\varepsilon\\
&=\Im^2+\frac{\Im^2\wp^{\zeta}}{\Gamma(\zeta+1)}.
\end{aligned}$$

For $n=1$:

$$\begin{aligned}
u_2(\Im,\wp)&=u_1(\Im,\wp)-\int_0^{\wp}\left[\frac{\partial u_1(\Im,\varepsilon)}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left(-\frac{\partial}{\partial\Im}\left(4\frac{u_1^2}{\Im}-\frac{\Im u_1}{3}\right)+\frac{\partial^2}{\partial\Im^2}u_1^2\right)\right]\right]d\varepsilon\\
&=\Im^2+\frac{\Im^2\wp^{\zeta}}{\Gamma(\zeta+1)}-\int_0^{\wp}\left[\frac{\partial}{\partial\varepsilon}\left(\Im^2+\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)\right.\\
&\quad-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(-\frac{\partial}{\partial\Im}\left(\frac{4}{\Im}\left(\Im^2+\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)^2-\frac{\Im}{3}\left(\Im^2+\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)\right)\right.\right.\\
&\quad\left.\left.\left.+\frac{\partial^2}{\partial\Im^2}\left(\Im^2+\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)^2\right)\right]\right]d\varepsilon\\
&=\Im^2+\frac{\Im^2\wp^{\zeta}}{\Gamma(\zeta+1)}-\int_0^{\wp}\left[\frac{\Im^2\zeta\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left(\Im^2\left(1+\frac{\varepsilon^{\zeta}}{\Gamma(\zeta+1)}\right)\right)\right]\right]d\varepsilon\\
&=\Im^2+\frac{\Im^2\wp^{\zeta}}{\Gamma(\zeta+1)}-\int_0^{\wp}\left[\frac{\Im^2\zeta\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}-\frac{\partial}{\partial\varepsilon}\left(\frac{\Im^2\varepsilon^{\zeta}}{\Gamma(\zeta+1)}+\frac{\Im^2\varepsilon^{2\zeta}}{\Gamma(2\zeta+1)}\right)\right]d\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&= \Im^2 + \frac{\Im^2 \wp^\zeta}{\Gamma(\zeta+1)} - \int_0^\wp \left[\frac{\Im^2 \zeta \varepsilon^{\zeta-1}}{\Gamma(\zeta+1)} - \left(\frac{\Im^2 \zeta \varepsilon^{\zeta-1}}{\Gamma(\zeta+1)} + \frac{2\Im^2 \zeta \varepsilon^{2\zeta-1}}{\Gamma(2\zeta+1)} \right) \right] d\varepsilon \\
&= \Im^2 + \frac{\Im^2 \wp^\zeta}{\Gamma(\zeta+1)} + \int_0^\wp \left[\frac{2\Im^2 \zeta \varepsilon^{2\zeta-1}}{\Gamma(2\zeta+1)} \right] d\varepsilon \\
&= \Im^2 + \frac{\Im^2 \wp^\zeta}{\Gamma(\zeta+1)} + \frac{\Im^2 \wp^{2\zeta}}{\Gamma(2\zeta+1)}.
\end{aligned}
$$

Similarly,

$$
u_3(\Im, \wp) = \Im^2 \left(1 + \frac{\wp^\zeta}{\Gamma(\zeta+1)} + \frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)} + \frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)} \right).
$$

Hence, the LVIM solution is

$$
u(\Im, \wp) = \Im^2 \left(1 + \frac{\wp^\zeta}{\Gamma(\zeta+1)} + \frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)} + \frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)} + \dots \right) = \Im^2 E_\zeta(\wp^\zeta). \tag{77}
$$

The LADM and LVIM solution at $\zeta = 1$ converges to $u(\Im, \wp) = \Im^2 e^\wp$, which is the exact solution for the nonfractional FPE [16]. Our LADM and LVIM solutions (75) and (77) match with the solutions obtained by HPM [34], NITM [30], HPETM [30], YTDM [2], and HPSTM [6].

Table 3: Comparative analysis of FIM, LADM, LVIM, and exact solution for Example 3, including absolute errors for $\zeta = 1$.

$\Im$	$\wp$	FIM [31]	LADM	LVIM	Exact	FIM Er.	LADM Er.	LVIM Er.
0.1	0.2	0.0124081	0.0122133	0.0122133	0.0122140	0.0001941	0.0000007	0.0000007
0.1	0.3	0.0137434	0.0134950	0.0134950	0.0134986	0.0002452	0.0000036	0.0000036
0.1	0.4	0.0151706	0.0149182	0.0149182	0.0149182	0.0002524	0.0000001	0.0000001
0.6	0.2	0.4466918	0.4396800	0.4396800	0.4397050	0.0069868	0.0000250	0.0000250
0.6	0.3	0.4947610	0.4858200	0.4858200	0.4859492	0.0088118	0.0001292	0.0001292
0.6	0.4	0.5461427	0.5366400	0.5366400	0.5370569	0.0090858	0.0004169	0.0004169
0.9	0.2	1.0050566	0.9892800	0.9892800	0.9893362	0.0157204	0.0000562	0.0000562
0.9	0.3	1.1132122	1.0930950	1.0930950	1.0933856	0.0198265	0.0002906	0.0002906
0.9	0.4	1.2288211	1.2074400	1.2074400	1.2083780	0.0213781	0.0009380	0.0009380

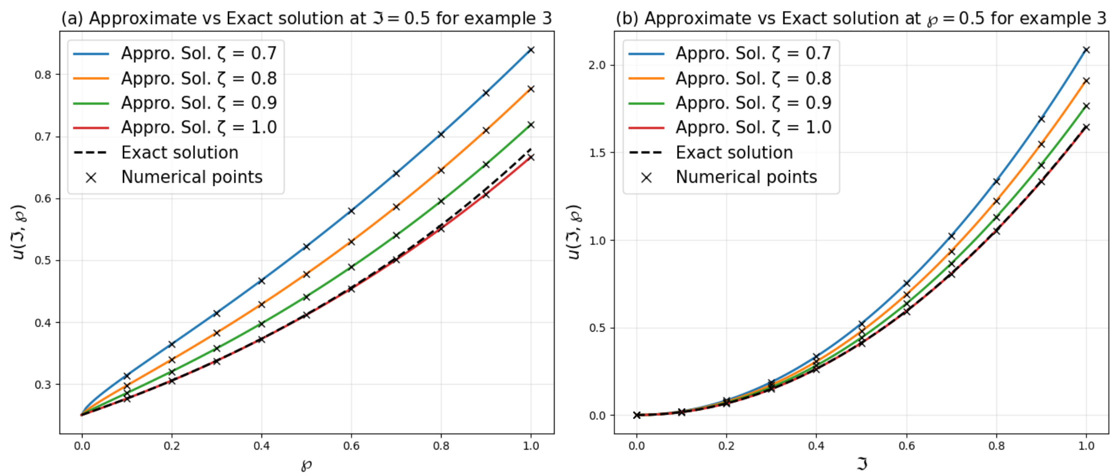

Figure 5: Two-dimensional graphical solutions of Example 3 for different values of ζ: (a) $u(\Im, \wp)$ versus $\wp$, and (b) $u(\Im, \wp)$ versus $\Im$.

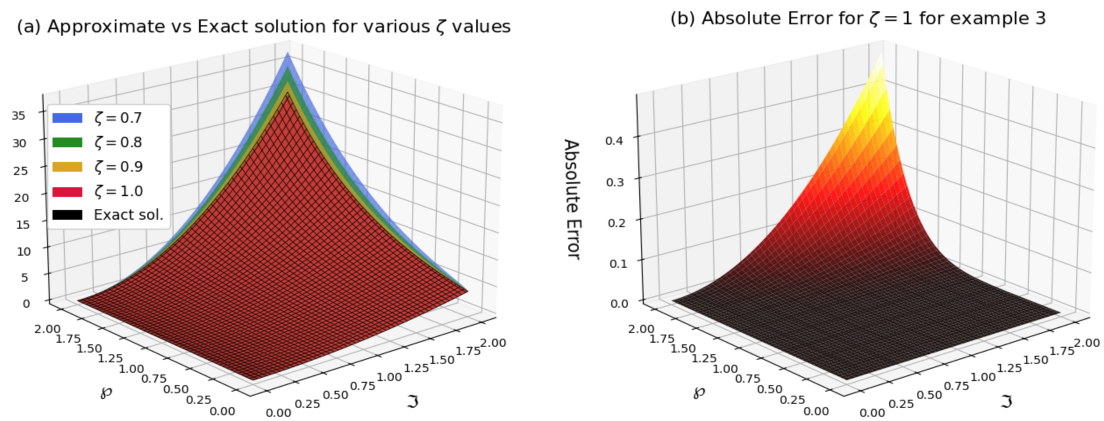

Figure 6: Three-dimensional graphical solutions of Example 3: (a) Comparison of the approximate solutions obtained by the LADM for different values of ζ with the exact solution, and (b) the absolute error surface for $\zeta = 1$.

Example 4. Consider another nonlinear TFFPE:

$$\frac{\partial^{\zeta} u}{\partial \wp^{\zeta}} = -\frac{\partial}{\partial \Im}\left(3u^2 - \frac{\Im u}{2}\right) + \frac{\partial^2}{\partial \Im^2}\left(\Im u^2\right), \tag{78}$$

subject to the initial condition:

$$u(\Im, 0) = \Im, \quad \Im \in \mathbb{R}. \tag{79}$$

(i) LADM solution

Applying the Laplace transform and the inverse Laplace transform to both sides of (78) along with the initial condition, we get

$$u(\Im,\wp)=\Im+\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(3u^{2}-\frac{\Im u}{2}\right)+\frac{\partial^{2}}{\partial\Im^{2}}\left(\Im u^{2}\right)\right]\right]. \tag{80}$$

Now, applying the standard Laplace decomposition method, we get the iterative formula:

$$\sum_{n=0}^{\infty}u_n(\Im,\wp)=\Im+\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(3\sum_{n=0}^{\infty}N_n-\frac{\Im}{2}\sum_{n=0}^{\infty}u_n\right)+\frac{\partial^{2}}{\partial\Im^{2}}\left(\Im\sum_{n=0}^{\infty}N_n\right)\right]\right], \tag{81}$$

where N_n are Adomian polynomials for u^2 [46]:

$$N_0(u)=u_0^2,\quad N_1(u)=2u_0u_1,\quad N_2(u)=u_1^2+2u_0u_2,\dots$$

The solution components are

$$\begin{aligned}
u_0(\Im,\wp)&=\Im,\\
u_1(\Im,\wp)&=\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(3u_0^2-\frac{\Im u_0}{2}\right)+\frac{\partial^{2}}{\partial\Im^{2}}\left(\Im u_0^2\right)\right]\right]\\
&=\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(\frac{5\Im^2}{2}\right)+\frac{\partial^{2}}{\partial\Im^{2}}\left(\Im^3\right)\right]\right]=\mathcal{L}^{-1}\left[\frac{\Im}{s^{\zeta+1}}\right]=\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)},\\
u_2(\Im,\wp)&=\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(6u_0u_1-\frac{\Im u_1}{2}\right)+\frac{\partial^{2}}{\partial\Im^{2}}\left(2\Im u_0u_1\right)\right]\right]\\
&=\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left(-\frac{\partial}{\partial\Im}\left(\frac{11\Im^2}{2}\cdot\frac{\wp^{\zeta}}{\Gamma\left(\zeta+1\right)}\right)+\frac{\partial^{2}}{\partial\Im^{2}}\left(2\Im^3\frac{\wp^{\zeta}}{\Gamma\left(\zeta+1\right)}\right)\right)\right]\\
&=\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\left[\frac{\Im\wp^{\zeta}}{\Gamma(\zeta+1)}\right]\right]=\mathcal{L}^{-1}\left[\frac{\Im}{s^{2\zeta+1}}\right]=\frac{\Im\wp^{2\zeta}}{\Gamma(2\zeta+1)}\\
u_3(\Im,\wp)&=\frac{\Im\wp^{3\zeta}}{\Gamma(3\zeta+1)}.
\end{aligned}$$

Thus, the LADM solution of (78) is given by

$$u(\Im,\wp)=\Im\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\cdots\right)=\Im E_{\zeta}(\wp^{\zeta}). \tag{82}$$

(ii) LADM solution

From (29) and (80), the iterative scheme becomes

$$u_{n+1}(\Im,\wp) = u_n(\Im,\wp) - \int_0^{\wp}\left[\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon}\right. \tag{83}$$

$$\left. - \frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\Big[-\frac{\partial}{\partial\Im}\left(3u_n^2(\Im,\varepsilon) - \frac{\Im\, u_n(\Im,\varepsilon)}{2}\right) + \frac{\partial^2}{\partial\Im^2}\left(\Im\, u_n^2(\Im,\varepsilon)\right)\Big]\right]\right] d\varepsilon. \tag{84}$$

The initial iterationis

$$u_0(\Im,\wp) = \Im.$$

For $n = 0$:

$$\begin{aligned} u_1(\Im,\wp) &= \Im - \int_0^{\wp}\left[\frac{\partial\Im}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{\mathcal{L}}{s^{\zeta}}\Big[-\frac{\partial}{\partial\Im}\left(3\Im^2 - \frac{\Im^2}{2}\right) + \frac{\partial^2}{\partial\Im^2}\left(\Im^3\right)\Big]\right]\right] d\varepsilon \\ &= \Im + \int_0^{\wp}\left[\frac{\Im\zeta\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}\right] d\varepsilon = \Im\left(1 + \frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right). \end{aligned}$$

For $n = 1$:

$$\begin{aligned} u_2(\Im,\wp) &= u_1(\Im,\wp) \\ &- \int_0^{\wp}\left[\frac{\partial u_n(\Im,\varepsilon)}{\partial\varepsilon} - \frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left(\frac{\mathcal{L}}{s^{\zeta}}\left[-\frac{\partial}{\partial\Im}\left(3u_1^2 - \frac{\Im u_1}{2}\right) + \frac{\partial^2}{\partial\Im^2}\left(\Im u_1^2\right)\right]\right)\right] d\varepsilon \\ &= \Im\left(1 + \frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right) + \int_0^{\wp}\left[\frac{\Im 2\zeta\varepsilon^{2\zeta-1}}{\Gamma(2\zeta+1)}\right] d\varepsilon \\ &= \Im\left(1 + \frac{\wp^{\zeta}}{\Gamma(\zeta+1)} + \frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}\right). \end{aligned}$$

For $n = 2$:

$$u_3(\Im,\wp) = \Im\left(1 + \frac{\wp^{\zeta}}{\Gamma(\zeta+1)} + \frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)} + \frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}\right).$$

The LVIM solution is

$$u(\Im,\wp) = \Im\left(1 + \frac{\wp^{\zeta}}{\Gamma(\zeta+1)} + \frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)} + \frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)} + \cdots\right) = \Im E_{\zeta}(\wp^{\zeta}). \tag{85}$$

The LADM and LVIM solution at $\zeta = 1$ converges to $u(\Im,\wp) = \Im e^{\wp}$, which is the exact solution for the nonfractional and our LADM and LVIM solutions (82) and (85) match with the solutions obtained by HPSTM [6].

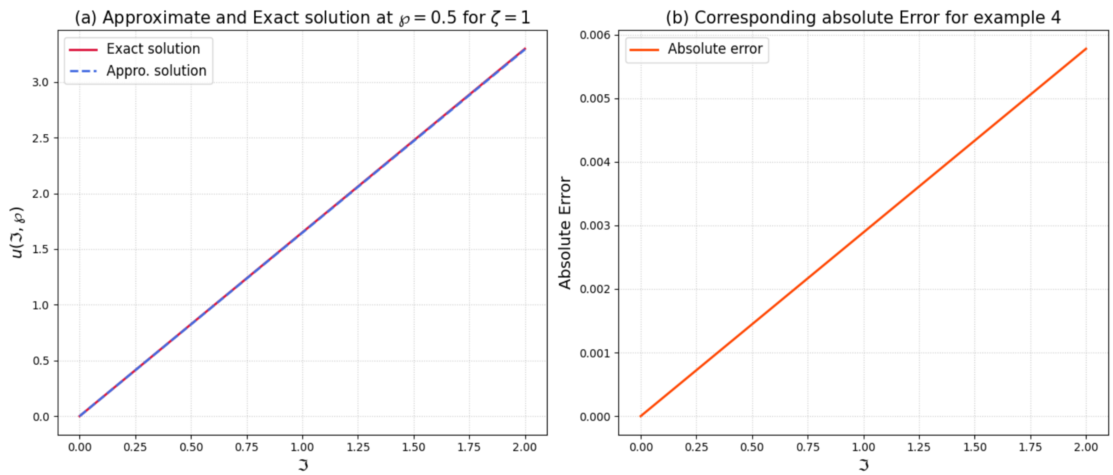

Figure 7: Two-dimensional graphical solutions of Example 4 for $\zeta = 1$: (a) Comparison of approximate and exact solutions, and (b) the corresponding absolute error.

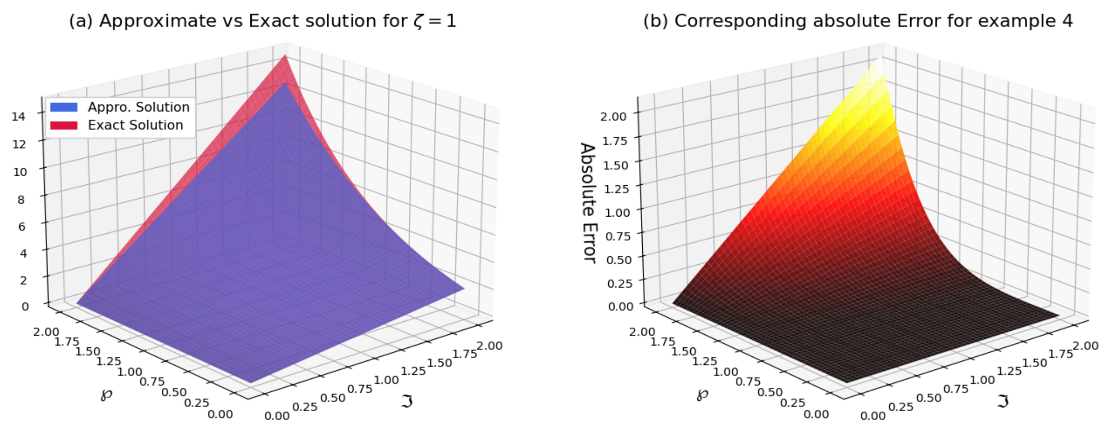

Figure 8: Three-dimensional graphical solutions of Example 4 at $\zeta = 1$: (a) Comparison of approximate and exact solutions, (b) the corresponding absolute error.

Example 5. Consider another TFFPE (4) with $A_1(\Im, \xi) = \Im$, $A_2(\Im, \xi) = 5\xi$, $B_{1,1}(\Im, \xi) = \Im^2$, $B_{1,2}(\Im, \xi) = 1$, $B_{2,1}(\Im, \xi) = 1$, $B_{2,2}(\Im, \xi) = \xi^2$, that is,

$$\frac{\partial^\zeta u}{\partial \wp^\zeta} = -\frac{\partial}{\partial \Im} \Im u - \frac{\partial}{\partial \xi} 5\xi u + \frac{\partial^2}{\partial \Im^2} \Im^2 u + \frac{\partial^2}{\partial \Im \partial \xi} u + \frac{\partial^2}{\partial \xi \partial \Im} u + \frac{\partial^2}{\partial \xi^2} \xi^2 u, \tag{86}$$

Subject to the initial condition:

$$u(\Im, \xi, 0) = \Im, \quad \Im, \xi \in \mathbb{R}. \tag{87}$$

(i) LADM solution

Applying the Laplace transform to both sides and using the initial condition, we obtain

$$u(\Im, \xi, s) = \frac{\Im}{s} + \frac{1}{s^{\zeta}}\mathcal{L}\left[-\frac{\partial}{\partial \Im}(\Im u) - \frac{\partial}{\partial \xi}(5\xi u) + \frac{\partial^2}{\partial \Im^2}(\Im^2 u) + \frac{\partial^2}{\partial \Im \partial \xi}u + \frac{\partial^2}{\partial \xi \partial \Im}u + \frac{\partial^2}{\partial \xi^2}(\xi^2 u)\right]. \tag{88}$$

The recurrence relation is

$$\begin{aligned} u_0(\Im, \xi, s) &= \frac{\Im}{s}, \\ u_{n+1}(\Im, \xi, s) &= \frac{1}{s^{\zeta}}\mathcal{L}\left[-\frac{\partial}{\partial \Im}(\Im u_n) - \frac{\partial}{\partial \xi}(5\xi u_n) + \frac{\partial^2}{\partial \Im^2}(\Im^2 u_n) + \frac{\partial^2}{\partial \Im \partial \xi}u_n + \frac{\partial^2}{\partial \xi \partial \Im}u_n + \frac{\partial^2}{\partial \xi^2}(\xi^2 u_n)\right]. \end{aligned} \tag{89}$$

Applying the inverse Laplace transform, we have

$$u_0(\Im, \xi, \wp) = \mathcal{L}^{-1}\left(\frac{\Im}{s}\right) = \Im.$$

For $n = 0$:

$$\begin{aligned} u_1(\Im, \xi, \wp) &= \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left[-\frac{\partial}{\partial \Im}(\Im^2) - \frac{\partial}{\partial \xi}(5\xi \Im) + \frac{\partial^2}{\partial \Im^2}(\Im^3) + \frac{\partial^2}{\partial \Im \partial \xi}\Im + \frac{\partial^2}{\partial \xi \partial \Im}\Im + \frac{\partial^2}{\partial \xi^2}(\xi^2 \Im)\right]\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left[\Im\right]\right] = \mathcal{L}^{-1}\left[\frac{\Im}{s^{\zeta+1}}\right] = \Im\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}. \end{aligned}$$

For $n = 1$:

$$u_2(\Im, \xi, \wp) = \mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left[\Im\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}\right]\right] = \mathcal{L}^{-1}\left[\frac{\Im}{s^{2\zeta+1}}\right] = \Im\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}.$$

Similarly,

$$u_3(\Im, \xi, \wp) = \Im\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}.$$

Thus, the LADM solution is

$$u(\Im, \xi, \wp) = \Im\left(1 + \frac{\wp^{\zeta}}{\Gamma(\zeta+1)} + \frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)} + \frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)} + \cdots\right) = \Im E_{\zeta}(\wp^{\zeta}). \tag{90}$$

(ii) LVIM solution

The iterative formula is:

$$u_{n+1}(\Im,\xi,\wp)=u_n(\Im,\xi,\wp)-\int_0^{\wp}\left[\frac{\partial u_n}{\partial\varepsilon}-\frac{\partial}{\partial\varepsilon}\mathcal{L}^{-1}\left[\frac{1}{s^{\zeta}}\mathcal{L}\left[-\frac{\partial}{\partial\Im}(\Im u_n)-\frac{\partial}{\partial\xi}(5\xi u_n)\right.\right.\right.$$
$$\left.\left.\left.+\frac{\partial^2}{\partial\Im^2}(\Im^2 u_n)+\frac{\partial^2}{\partial\Im\partial\xi}u_n+\frac{\partial^2}{\partial\xi\partial\Im}u_n+\frac{\partial^2}{\partial\xi^2}(\xi^2 u_n)\right]\right]\right]d\varepsilon. \tag{91}$$

The initial iteration is

$$u_0(\Im,\xi,\wp)=\Im.$$

For $n=0$:

$$u_1(\Im,\xi,\wp)=\Im+\int_0^{\wp}\left[\frac{\Im\zeta\varepsilon^{\zeta-1}}{\Gamma(\zeta+1)}\right]d\varepsilon=\Im+\Im\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}.$$

For $n=1$:

$$u_2(\Im,\xi,\wp)=\Im+\Im\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\int_0^{\wp}\left[\frac{2\Im\zeta\varepsilon^{2\zeta-1}}{\Gamma(2\zeta+1)}\right]d\varepsilon=\Im+\Im\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\Im\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}.$$

Similarly,

$$u_3(\Im,\xi,\wp)=\Im+\Im\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\Im\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\Im\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}.$$

Hence, the LVIM solution is

$$u(\Im,\xi,\wp)=\Im\left(1+\frac{\wp^{\zeta}}{\Gamma(\zeta+1)}+\frac{\wp^{2\zeta}}{\Gamma(2\zeta+1)}+\frac{\wp^{3\zeta}}{\Gamma(3\zeta+1)}+\cdots\right)=\Im E_{\zeta}(\wp^{\zeta}). \tag{92}$$

The LADM and LVIM solution at $\zeta=1$ converge to $u(\Im,\xi,\wp)=\Im e^{\wp}$, which is the exact solution for the nonfractional and our LADM and LVIM solutions (90) and (92) match with the solutions obtained by HPM [34].

7 Results and discussion

In Table 1, we present the numerical results of Example 1 for different values of $\wp$ and $\Im$, obtained using HPSTM [6], NITM [30], HPETM [30], YTDM [2], LADM, and LVIM. The results are compared with the exact solution when $\zeta=1$. It can be observed that LADM and LVIM produce identical results, which are in excellent agreement with the exact solution. Moreover, the consistency of LADM and LVIM with other recently developed methods (HPSTM, NITM, HPETM, and YTDM) further demonstrates the validity, accuracy, and robustness of the proposed approaches.

Tables 2 and 3 summarize the numerical results for Examples 2 (linear) and 3 (nonlinear) for different values of $\Im$ and $\wp$, comparing the FIM [31], LADM, LVIM, and exact solutions at $\zeta=1$. In both linear and nonlinear cases, LADM and LVIM produce nearly identical results that closely match the exact solutions, exhibiting minimal absolute errors. While FIM performs adequately for the linear problem, it shows comparatively larger deviations in the nonlinear case. These observations confirm that the proposed Laplace-based hybrid methods are highly accurate, reliable, and computationally efficient, achieving rapid convergence with fewer iterations and lower computational cost compared to traditional approaches. Furthermore, the accuracy of the approximate solutions can be systematically improved by including additional terms in the series expansions.

Since LADM and LVIM solutions yield the same results, we consider only the LADM solution in the graphs. Figures 1–6 illustrate the LADM solutions compared with the exact solutions for Examples 1–3.

In Figure 1, the two-dimensional LADM solutions for $\zeta = 0.8, 0.9, 1$ at $\Im = 0.5$ over $\wp \in [0, 0.5]$ are shown, with cross markers indicating discrete solution points, while graph (b) displays the exact solution at $\zeta = 1$ with circular markers. Excellent agreement between the approximate and exact solutions is observed. Figure 2 presents the corresponding three-dimensional surfaces over $\Im \in [0, 1]$ and $\wp \in [0, 1]$, further confirming this agreement.

For Example 2, Figures 3 and 4 show two-dimensional and three-dimensional comparisons at $\zeta = 1$ across different $\Im$ and $\wp$ values. The two-dimensional absolute error in Figure 3(b) and the three-dimensional error in Figure 4(b) remain very small, demonstrating that LADM converges quickly and accurately even with only the first four terms of the series.

Figures 5 and 6 illustrate the effect of different fractional orders $\zeta = 0.7, 0.8, 0.9, 1$ for Example 3. Two-dimensional plots show LADM solutions for fixed $\Im$ or $\wp$ compared with the exact solution at $\zeta = 1$, and three-dimensional plots display the solution surfaces and absolute errors. In all cases, LADM solutions closely match the exact solutions, confirming the accuracy and reliability of the proposed method.

Figures 7 and 8 present the graphical results for Example 4 at $\zeta = 1$. The two-dimensional plots compare the approximate and exact solutions along with the corresponding absolute error. The three-dimensional plots illustrate the solution surfaces and the absolute error distribution. The results show an excellent agreement between the approximate and exact solutions, demonstrating the accuracy and effectiveness of the proposed method.

8 Conclusion

In this study, the LADM and the LVIM were successfully applied to solve both linear and nonlinear TFFPEs (time-fractional Fokker–Planck equations), as well as related fractional differential equations. The proposed methods proved to be efficient and robust in obtaining analytical and numerical solutions, achieving excellent agreement with exact solutions after just a few iterations. Compared to existing approaches such as YTDM, NITM, HPETM, HPSTM, and FIM, LADM and LVIM require less computational effort while providing higher accuracy, particularly for nonlinear TFFPEs. Unlike traditional numerical techniques, these methods do not require discretization, linearization, or perturbation, making them highly practical for complex fractional systems. The present work can be extended to more sophisticated fractional models arising in physics, engineering, and finance, demonstrating the versatility of LADM and LVIM. Future research may explore higher-dimensional fractional models or the application of other fractional operators, including Atangana–Baleanu, Caputo–Fabrizio, and conformable derivatives.

Declarations

- **Data Availability**: All data presented in this study were generated using Python, and no external datasets were used. The numerical data and Python codes used for computations and plotting are available from the corresponding author upon reasonable request.
- **Conflict of Interest**: The authors declare no conflict of interest.
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