Iranian Journal of Numerical Analysis and Optimization
Vol. 16, No. 2, 2026, pp 539–562
https://doi.org/10.22067/ijnao.2025.93216.1634
https://ijnao.um.ac.ir/

Research Article

Local fractional Khalouta transform: Properties and applications

A. Khalouta*

Abstract

In the current paper, we propose a new hybrid method for solving local fractional differential equations. The proposed method is based on the coupling of the Khalouta transform with the local fractional derivative, and we call the local fractional Khalouta transform method. Also, we provide important results and properties of this method. For justification and verification of the present method, some applications of linear local fractional differential equations are presented. The results of the solved examples show the effectiveness of this method. Moreover, the present method can be applied to handle the solutions of various problems with local fractional derivatives.

AMS subject classifications (2020): 44A05, 44A20, 26A33, 34A08.

Keywords: Khalouta transform; Local fractional derivative; Local fractional differential equation; Local fractional Volterra integro-differential equation; Exact solution.

---

*Corresponding author

Received 28 April 2025; revised 30 November 2025; accepted 24 December 2025

Ali Khalouta
Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Setif 1 University-Ferhat ABBAS, Algeria.
e-mail: nadjibkh@yahoo.fr ; ali.khalouta@univ-setif.dz

How to cite this article
Khalouta, A., Local fractional Khalouta transform: Properties and applications. *Iran. J. Numer. Anal. Optim.*, 2026; 16(2): 539-562. https://doi.org/10.22067/ijnao.2025.93216.1634

1 Introduction

In recent years, fractional calculus has attracted increasing attention due to its applications in various fields. The history of this theory dates back to the 17th century, when in 1695 Leibniz [16] described the derivative of order $\theta = \frac{1}{2}$ in his letter to L'Hospital. Since then, the new theory has become very attractive to mathematicians as well as physicists, biologists, engineers, and economists. Fractional calculus has various applications in biophysics, quantum mechanics, wave theory, polymers, continuum mechanics, Lie theory, field theory, spectroscopy, group theory, and other applications in science and engineering [7, 18, 21, 22].

There are many concepts of fractional derivatives defined and developed in the literature, which are not compatible with each other, among them is, Riemann–Liouville fractional derivative [23], Caputo fractional derivative [15], conformable fractional derivative [1], Hadamard fractional derivative [14], Atangana–Baleanu fractional derivative [2], and local fractional derivative [25].

Today, fractional differential equations involving the local fractional derivative constitute an important area of modeling natural phenomena, and researchers are paying considerable attention to them due to their numerous applications in science and engineering. New developments in this field have emerged in disciplines such as mathematics, physics, chemistry, biology, medicine, economics, and engineering, highlighting the growing importance of this type of equation in the analysis of real-world problems. For this reason, researchers in mathematics and physics have focused on proposing various methods for solving local fractional differential equations [3, 4, 5, 9].

There are many computational methods for dealing with local fractional differential equations, such as the local fractional decomposition method [10], the local fractional variational iteration method [8], the local fractional differential transform method [6], the local fractional reduced differential transform method [17], the local fractional homotopy perturbation method [27], the local fractional homotopy analysis method [20], and others.

Recently, the author [12, 13] proposed a new integral transform called Khalouta transform, which is a generalization of many well known integral transforms such as Laplace–Carson transform, Sumudu transform, ZZ transform, ZMA transform, Elzaki transform, Aboodh transform, Natural transform, and Shehu transform. Thus, the main objective of the present paper is to propose a new hybrid method based on the coupling of the Khalouta transform with the local fractional derivative for solving local fractional differential equations.

The proposed method offers great advantages of straightforward applicability, computational efficiency and high accuracy. Its strength lies in its ability to combine two powerful concepts for obtaining exact solutions of linear local fractional differential equations. Unlike other methods, the proposed method requires neither linearization nor discretization, nor small or large parameters in the problem.

The structure of this paper is as follows. In Section 2, we give the basic definitions and properties of the local fractional calculus. In Section 3, we present important results regarding the local fractional Khalouta transform. In Section 4, we illustrate the relation between the local fractional Khalouta transform and other known local fractional integral transforms. In Section 5, we extend this new integral transform and apply it to solve linear differential equations using the local fractional derivative. Finally, in Section 6, we present our conclusions.

2 Local fractional calculus

This section gives the basic concepts of local fractional calculus, and in particular the local fractional derivative and the local fractional integral, which have been utilized in this work. For more details, see [25, 26].

Definition 1. If we have the relation

$$|u(t)-u(t_0)|<\varepsilon^{\theta}, \qquad 0<\theta\leq 1,$$

where there is $|t-t_0|<\delta$ for $\delta,\varepsilon\in\mathbb{R}_+^*$, then the function $u(t)$ is called local fractional continuous at $t=t_0$.

We say that the function $u(t)$ is local fractional continuous on the interval (a,b) if it is local fractional continuous at any point t_0 of (a,b), and we denote it by $u(t)\in C_\theta(a,b), 0<\theta\leq 1$.

Definition 2. Consider the function $u(t)\in C_\theta(a,b)$. The local fractional integral of $u(t)$ of order θ in the interval $[a,b]$ is defined by

$$\begin{aligned}{}_aI_b^{(\theta)}u(t) &= \frac{1}{\Gamma(1+\theta)}\int_a^b u(\tau)(d\tau)^{\theta}\\ &= \frac{1}{\Gamma(1+\theta)}\lim_{\Delta\tau\to 0}\sum_{i=0}^{N-1}u(\tau_i)(\Delta\tau_i)^{\theta},\end{aligned}$$

where $\Delta\tau_i=\tau_{i+1}-\tau_i$ with $\Delta\tau=\max\{\Delta\tau_0,\Delta\tau_1,\Delta\tau_2,\Delta\tau_3,\ldots\}$ and $[\tau_{i+1},\tau_i], i=0,\ldots,N-1, \tau_0=a, \tau_N=b$ is a partition of the interval $[a,b]$.

Definition 3. Consider the function $u(t)\in C_\theta(a,b)$. The local fractional derivative of $u(t)$ of order θ with $0<\theta\leq 1$ in the interval $[a,b]$ is defined by

$$D^{(\theta)}u(t) = \frac{d^{\theta}u(t_{0)}}{dt^{\theta}} = u^{(\theta)}(t) = \lim_{t\to t_0} \frac{\Delta^{\theta}\left(u(t)-u(t_0)\right)}{\left(t-t_0\right)^{\theta}},$$

where

$$\Delta^{\theta}\left(u(t)-u(t_0)\right) \simeq \Gamma(1+\theta)\left[u(t)-u(t_0)\right].$$

Moreover, the local fractional derivative of higher order is defined as

$$D^{(n\theta)}u(t) = \frac{d^{n\theta}u(t_{0)}}{dt^{n\theta}} = u^{(n\theta)}(t) = \underbrace{D^{(\theta)}D^{(\theta)}\dots D^{(\theta)}u(t)}_{n-times}.$$

Definition 4. The local fractional partial differential operator of order θ with $0<\theta\leq 1$ is given by

$$D_t^{(\theta)}u(x,t) = \frac{\partial^{\theta}u(x,t_{0)}}{\partial t^{\theta}} = u^{(\theta)}(x,t_0) = \lim_{t\to t_0} \frac{\Delta^{\theta}\left(u(x,t)-u(x,t_0)\right)}{\left(t-t_0\right)^{\theta}},$$

where

$$\Delta^{\theta}\left(u(x,t)-u(x,t_0)\right) \simeq \Gamma(1+\theta)\left[u(x,t)-u(x,t_0)\right].$$

Moreover, the local fractional partial derivative of higher order is defined as

$$D_t^{(n\theta)}u(x,t) = \frac{\partial^{n\theta}u(x,t_{0)}}{\partial t^{n\theta}} = u^{(n\theta)}(x,t) = \underbrace{D_t^{(\theta)}D_t^{(\theta)}\dots D_t^{(\theta)}u(x,t)}_{n-times}.$$

Definition 5. The local fractional Mc-Laurin's series of the Mittag-Leffler function is given by

$$E_{\theta}(z^{\theta}) = \sum_{n=0}^{\infty} \frac{z^{n\theta}}{\Gamma(1+n\theta)}, \quad z\in\mathbb{R}, 0<\theta\leq 1.$$

3 Main results

This section presents our main results concerning the definition of local fractional Khalouta transform denoted in this paper by ${}^{LF}\mathbb{KH}_{\theta}\left[\cdot\right]$ and discusses some interesting properties of this transform.

Definition 6. Let

$$\frac{1}{\Gamma(1+\theta)}\int_0^{+\infty} |u(t)|\,(dt)^{\theta} < A < \infty.$$

The local fractional Khalouta transform of the function $u(t)$ of order θ is defined as by the following integral:

$$
\begin{aligned}
{}^{LF}\mathbb{KH}_\theta\left[u(t)\right] = \mathcal{K}_\theta(s,\gamma,\eta) &= \frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)(dt)^\theta \\
&= \lim_{\sigma\to\infty}\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\sigma E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)(dt)^\theta, \qquad (1)
\end{aligned}
$$

where s^θ, γ^θ, and η^θ with $0 < \theta \le 1$, are the Khalouta transform variables, σ is a real number, and the integral is taken along the line $t = \sigma$.

Definition 7. The inverse local fractional Khalouta transform of the function $u(t)$ of order θ is defined by

$$
{}^{LF}\mathbb{KH}_\theta^{-1}\left[\mathcal{K}_\theta(s,\eta,\gamma)\right] = u(t).
$$

Now, we give some interesting properties of the local fractional Khalouta transform.

Theorem 1. Let ${}^{LF}\mathbb{KH}_\theta\left[u(t)\right] = \mathcal{K}_\theta(s,\gamma,\eta)$ and ${}^{LF}\mathbb{KH}_\theta\left[v(t)\right] = \mathcal{H}_\theta(s,\gamma,\eta)$. Then, we have

$$
{}^{LF}\mathbb{KH}_\theta\left[\lambda u(t) \pm \theta v(t)\right] = \lambda\mathcal{K}_\theta(s,\gamma,\eta) \pm \theta\mathcal{H}_\theta(s,\gamma,\eta),
$$

where λ and θ are constants.

Proof. Using the definition of the local fractional Khalouta transform, we have

$$
\begin{aligned}
\mathbb{KH}\left[\lambda u(t) \pm \theta v(t)\right] &= \frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)\left[\lambda u(t) \pm \theta v(t)\right](dt)^\theta \\
&= \frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty \begin{bmatrix} \lambda E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t) \\ \pm\theta E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)v(t)\end{bmatrix}(dt)^\theta \\
&= \lambda\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)(dt)^\theta \\
&\quad \pm\theta\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)v(t)(dt)^\theta \\
&= \lambda\mathcal{K}_\theta(s,\gamma,\eta) \pm \theta\mathcal{H}_\theta(s,\gamma,\eta).
\end{aligned}
$$

□

Theorem 2. Let ${}^{LF}\mathbb{KH}_\theta\left[u(t)\right] = \mathcal{K}_\theta(s,\gamma,\eta)$ where $0 < \theta \le 1$. Then, the local fractional Khalouta transforms of the local fractional derivatives are given by

$$
{}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}u(t)\right] = \left(\frac{s}{\gamma\eta}\right)^\theta\mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^\theta u(0), \qquad (2)
$$

and

$$ {}^{LF}\mathbb{KH}_\theta\left[D^{(n\theta)}u(t)\right]=\left(\frac{s}{\gamma\eta}\right)^{n\theta}\mathcal{K}_\theta(s,\gamma,\eta)-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{(n-k)\theta}u^{(k\theta)}(0). \tag{3} $$

Proof. We proof the formula (2). From the definition of local fractional Khalouta transform, we have

$$ {}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}u(t)\right]=\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)D^{(\theta)}u(t)(dt)^\theta. $$

Using the fractional integration by parts formula [11], we get the following equality:

$$ \begin{aligned} {}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}u(t)\right] &= \frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\left(\lim_{\sigma\to\infty}\Gamma(1+\theta)\left[E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)\right]_0^\sigma\right.\\ &\quad\left.+\frac{s^\theta}{\gamma^\theta\eta^\theta}\lim_{\sigma\to\infty}\int_0^\sigma E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)(dt)^\theta\right)\\ &= \frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\\ &\quad\times\begin{pmatrix} -\Gamma(1+\theta)u(0)\\ +\frac{s^\theta}{\gamma^\theta\eta^\theta}\lim_{\sigma\to\infty}\int_0^\sigma E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)(dt)^\theta\end{pmatrix}\\ &= -\frac{s^\theta}{\gamma^\theta\eta^\theta}u(0)\\ &\quad+\frac{s^\theta}{\gamma^\theta\eta^\theta}\left(\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)u(t)(dt)^\theta\right)\\ &= -\frac{s^\theta}{\gamma^\theta\eta^\theta}u(0)+\frac{s^\theta}{\gamma^\theta\eta^\theta}\mathcal{K}_\theta(s,\gamma,\eta)\\ &= \left(\frac{s}{\gamma\eta}\right)^\theta\mathcal{K}_\theta(s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^\theta u(0). \end{aligned} $$

To prove the validity of the formula (3), we use mathematical induction.
For $n=1$ and from the formula (3), we have

$$ {}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}u(t)\right]=\left(\frac{s}{\gamma\eta}\right)^\theta\mathcal{K}_\theta(s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^\theta u(0). $$

So, from (2), we note that the formula is true when $n=1$.
Assume that the formula (3) is true for n. We get

$$ {}^{LF}\mathbb{KH}_\theta\left[D^{(n\theta)}u(t)\right]=\left(\frac{s}{\gamma\eta}\right)^{n\theta}\mathcal{K}_\theta(s,\gamma,\eta)-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{(n-k)\theta}u^{(k\theta)}(0). \tag{4} $$

We show that (3) is true for $n+1$. Let $D^{(n\theta)}u(t)=U(t)$, where ${}^{LF}\mathbb{KH}_\theta\left[U(t)\right]=\mathcal{U}_\theta(s,\gamma,\eta)$ and from (2) and (4), we have

$$
\begin{aligned}
{}^{LF}\mathbb{KH}_\theta\left[D^{((n+1)\theta)}u(t)\right] &= {}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}U(t)\right] \\
&= \left(\frac{s}{\gamma\eta}\right)^{\theta}\mathcal{U}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{\theta}U(0) \\
&= \left(\frac{s}{\gamma\eta}\right)^{(n+1)\theta}\mathcal{K}_\theta(s,\gamma,\eta) \\
&\quad - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{(n+1-k)\theta}u^{(k\theta)}(0) \\
&\quad - \left(\frac{s}{\gamma\eta}\right)^{\theta}D^{(n\theta)}u(0), \\
&= \left(\frac{s}{\gamma\eta}\right)^{(n+1)\theta}\mathcal{K}_\theta(s,\gamma,\eta) \\
&\quad - \sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{(n+1-k)\theta}u^{(k\theta)}(0).
\end{aligned}
$$

So, the formula (3) is true for $n+1$.

Thus, by the principle of mathematical induction for all $n > 1$, the formula (3) is true. □

Theorem 3. Let ${}^{LF}\mathbb{KH}_\theta\left[u(t)\right] = \mathcal{K}_\theta(s,\gamma,\eta)$. Then, the local fractional Khalouta transform of the local fractional integral is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[{}_0I_t^{(\theta)}u(t)\right] = \left(\frac{\gamma\eta}{s}\right)^{\theta}\mathcal{K}_\theta(s,\gamma,\eta), \quad 0 < \theta \leq 1.
$$

Proof. Let

$$
K(t) = {}_0 I_t^{(\theta)}u(t). \tag{5}
$$

Operating the local fractional derivative on (5), we attain

$$
D^{(\theta)}K(t) = D^{(\theta)}{}_0I_t^{(\theta)}u(t) = u(t), \tag{6}
$$

and

$$
K(0) = 0.
$$

Operating the local fractional Khalouta transform on (6), we attain

$$
{}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}K(t)\right] = {}^{LF}\mathbb{KH}_\theta\left[u(t)\right],
$$

which implies

$$
\left(\frac{s}{\gamma\eta}\right)^{\theta}\left({}^{LF}\mathbb{KH}_\theta\left[K(t)\right]\right) = \mathcal{K}_\theta(s,\gamma,\eta).
$$

Thus, we get

$$ {}^{LF}\mathbb{KH}_{\theta}\left[{}_{0}I_{t}^{(\theta)}u(t)\right]=\left(\frac{\gamma\eta}{s}\right)^{\theta}\mathcal{K}_{\theta}(s,\gamma,\eta). $$

□

Theorem 4. Let ${}^{LF}\mathbb{KH}_{\theta}\left[u(t)\right]=\mathcal{K}_{\theta}(s,\gamma,\eta)$ and ${}^{LF}\mathbb{KH}_{\theta}\left[v(t)\right]=\mathcal{H}_{\theta}(s,\gamma,\eta)$. Then, the local fractional Khalouta transform of their convolution is given by

$$ {}^{LF}\mathbb{KH}_{\theta}\left[(u(t)*v(t))_{\theta}\right]=\left(\frac{\gamma\eta}{s}\right)^{\theta}\mathcal{K}_{\theta}(s,\gamma,\eta)\mathcal{H}_{\theta}(s,\gamma,\eta), $$

where

$$ (u(t)*v(t))_{\theta}=\frac{1}{\Gamma(1+\theta)}\int_{0}^{\infty}u(\tau)v(t-\tau)\left(d\tau\right)^{\theta}. $$

Proof. Using the definition of the local fractional Khalouta transform in (1), we arrive at

$$ \begin{aligned}
&{}^{LF}\mathbb{KH}_{\theta}\left[(u(t)*v(t))_{\theta}\right]\\
&=\frac{1}{\Gamma(1+\theta)}\frac{s^{\theta}}{\gamma^{\theta}\eta^{\theta}}\int_{0}^{\infty}E_{\theta}\left(-\frac{s^{\theta}t^{\theta}}{\gamma^{\theta}\eta^{\theta}}\right)(u(t)*v(t))_{\theta}\,(dt)^{\theta}\\
&=\frac{1}{\Gamma(1+\theta)}\frac{s^{\theta}}{\gamma^{\theta}\eta^{\theta}}\\
&\quad\times\int_{0}^{\infty}\left(\begin{array}{c}E_{\theta}\left(-\frac{s^{\theta}t^{\theta}}{\gamma^{\theta}\eta^{\theta}}\right)\\ \times\left(\frac{1}{\Gamma(1+\theta)}\int_{0}^{\infty}u(\tau)v(t-\tau)\left(d\tau\right)^{\theta}\right)\end{array}\right)(dt)^{\theta}\\
&=\frac{1}{\Gamma(1+\theta)}\frac{1}{\Gamma(1+\theta)}\frac{s^{\theta}}{\gamma^{\theta}\eta^{\theta}}\\
&\quad\times\int_{0}^{\infty}\int_{0}^{\infty}\left(E_{\theta}\left(-\frac{s^{\theta}t^{\theta}}{\gamma^{\theta}\eta^{\theta}}\right)u(\tau)v(t-\tau)\left(d\tau\right)^{\theta}\right)(dt)^{\theta}.
\end{aligned} \tag{7} $$

Substituting $x=t-\tau$ and $dx=dt$ in (7), we attain

$$ \begin{aligned}
&{}^{LF}\mathbb{KH}_{\theta}\left[(u(t)*v(t))_{\theta}\right]\\
&=\frac{1}{\Gamma(1+\theta)}\frac{1}{\Gamma(1+\theta)}\frac{s^{\theta}}{\gamma^{\theta}\eta^{\theta}}\\
&\quad\times\int_{0}^{\infty}\int_{0}^{\infty}E_{\theta}\left(-\frac{s^{\theta}\left(x+\tau\right)^{\theta}}{\gamma^{\theta}\eta^{\theta}}\right)u(\tau)v(x)\left(d\tau\right)^{\theta}(dx)^{\theta}\\
&=\frac{1}{\Gamma(1+\theta)}\frac{1}{\Gamma(1+\theta)}\frac{s^{\theta}}{\gamma^{\theta}\eta^{\theta}}\\
&\quad\times\int_{0}^{\infty}\int_{0}^{\infty}\left(\begin{array}{c}E_{\theta}\left(-\frac{s^{\theta}x^{\theta}}{\gamma^{\theta}\eta^{\theta}}\right)E_{\theta}\left(-\frac{s^{\theta}\tau^{\theta}}{\gamma^{\theta}\eta^{\theta}}\right)\\ \times u(\tau)v(x)\end{array}\right)\left(d\tau\right)^{\theta}(dx)^{\theta}
\end{aligned} $$

$$
\begin{aligned}
&= \frac{\gamma^\theta \eta^\theta}{s^\theta}\left(\frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \int_0^\infty E_\theta\left(-\frac{s^\theta \tau^\theta}{\gamma^\theta \eta^\theta}\right) u(\tau)\,(d\tau)^\theta\right) \\
&\quad \times\left(\frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \int_0^\infty E_\theta\left(-\frac{s^\theta x^\theta}{\gamma^\theta \eta^\theta}\right) v(x)(dx)^\theta\right) \\
&= \left(\frac{\gamma\eta}{s}\right)^\theta \mathcal{K}_\theta(s,\gamma,\eta)\mathcal{H}_\theta(s,\gamma,\eta).
\end{aligned}
$$

□

Now, we present the results related to the local fractional Khalouta transform of some special functions.

Theorem 5. Let $u(t)=1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[1\right] = 1.
$$

Proof. By using (1), we attain

$$
\begin{aligned}
{}^{LF}\mathbb{KH}_\theta\left[1\right] &= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta \eta^\theta}\right)(dt)^\theta \\
&= \frac{s^\theta}{\gamma^\theta \eta^\theta} \lim_{\sigma\to\infty}\left[-\frac{\gamma^\theta\eta^\theta}{s^\theta} E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta \eta^\theta}\right)\right]_0^\sigma \\
&= 1.
\end{aligned}
$$

□

Theorem 6. Let $u(t)=\dfrac{t^\theta}{\Gamma(1+\theta)}, 0<\theta\leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[\frac{t^\theta}{\Gamma(1+\theta)}\right] = \frac{\gamma^\theta\eta^\theta}{s^\theta}.
$$

Proof. By using (1) and the fractional integration by parts formula [11], we attain

$$
\begin{aligned}
&{}^{LF}\mathbb{KH}_\theta\left[\frac{t^\theta}{\Gamma(1+\theta)}\right] \\
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta \eta^\theta}\right) \frac{t^\theta}{\Gamma(1+\theta)}(dt)^\theta \\
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \int_0^\infty \left(\begin{array}{c} \left(-\dfrac{\gamma^\theta\eta^\theta}{s^\theta} E_\theta\left(-\dfrac{s^\theta t^\theta}{\gamma^\theta \eta^\theta}\right)\right)^{(\theta)} \\ \times\dfrac{t^\theta}{\Gamma(1+\theta)} \end{array}\right)(dt)^\theta \\
&= \frac{s^\theta}{\gamma^\theta \eta^\theta}\left(\lim_{\sigma\to\infty}\left[-\frac{\gamma^\theta\eta^\theta}{s^\theta} E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta \eta^\theta}\right)\frac{t^\theta}{\Gamma(1+\theta)}\right]_0^\sigma\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(1+\theta)}\frac{\gamma^\theta\eta^\theta}{s^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)(dt)^\theta\Bigg)\\
&=\frac{s^\theta}{\gamma^\theta\eta^\theta}\left(\frac{\gamma^{2\theta}\eta^{2\theta}}{s^{2\theta}}\lim_{\sigma\to\infty}\left[-E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)\right]_0^\sigma\right)\\
&=\frac{\gamma^\theta\eta^\theta}{s^\theta}.
\end{aligned}
$$

□

Theorem 7. Let $u(t)=E_\theta\left(at^\theta\right), 0<\theta\leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[E_\theta\left(at^\theta\right)\right]=\frac{s^\theta}{s^\theta-a\gamma^\theta\eta^\theta}.
$$

Proof. By using (1), we attain

$$
\begin{aligned}
&{}^{LF}\mathbb{KH}_\theta\left[E_\theta\left(at^\theta\right)\right]\\
&=\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)E_\theta\left(at^\theta\right)(dt)^\theta\\
&=\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\left(\frac{s^\theta-a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta}\right)t^\theta\right)(dt)^\theta\\
&=\frac{s^\theta}{\gamma^\theta\eta^\theta}\left(\lim_{\sigma\to\infty}\left[-\frac{\gamma^\theta\eta^\theta}{s^\theta-a\gamma^\theta\eta^\theta}E_\theta\left(-\left(\frac{s^\theta-a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta}\right)t^\theta\right)\right]_0^\sigma\right)\\
&=\frac{s^\theta}{s^\theta-a\gamma^\theta\eta^\theta}.
\end{aligned}
$$

□

Theorem 8. Let $u(t)=\dfrac{t^\theta}{\Gamma(1+\theta)}E_\theta\left(at^\theta\right), 0<\theta\leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[\frac{t^\theta}{\Gamma(1+\theta)}E_\theta\left(at^\theta\right)\right]=\frac{s^\theta\gamma^\theta\eta^\theta}{\left(s^\theta-a\gamma^\theta\eta^\theta\right)^2}.
$$

Proof. By using (1) and the fractional integration by parts formula [11], we attain

$$
\begin{aligned}
&{}^{LF}\mathbb{KH}_\theta\left[\frac{t^\theta}{\Gamma(1+\theta)}E_\theta\left(at^\theta\right)\right]\\
&=\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\\
&\quad\times\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)\frac{t^\theta}{\Gamma(1+\theta)}E_\theta\left(at^\theta\right)(dt)^\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \\
&\quad \times \int_0^\infty E_\theta \left(-\left(\frac{s^\theta - a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta} \right) t^\theta \right) \frac{t^\theta}{\Gamma(1+\theta)} (dt)^\theta \\
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \\
&\quad \times \int_0^\infty \left(\begin{array}{c} \left(\begin{array}{c} -\dfrac{\gamma^\theta\eta^\theta}{s^\theta - a\gamma^\theta\eta^\theta} \\ \times E_\theta \left(-\left(\dfrac{s^\theta - a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta} \right) t^\theta \right) \end{array} \right)^{(\theta)} \\ \times \dfrac{t^\theta}{\Gamma(1+\theta)} \end{array} \right) (dt)^\theta \\
&= \frac{s^\theta}{\gamma^\theta\eta^\theta} \left(\begin{array}{c} \lim\limits_{\sigma\to\infty} \left[\begin{array}{c} -\dfrac{\gamma^\theta\eta^\theta}{s^\theta - a\gamma^\theta\eta^\theta} \\ \times E_\theta \left(-\left(\dfrac{s^\theta - a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta} \right) t^\theta \right) \\ \dfrac{t^\theta}{\Gamma(1+\theta)} \end{array} \right]_0^\sigma \\ + \dfrac{1}{\Gamma(1+\theta)} \dfrac{\gamma^\theta\eta^\theta}{s^\theta - a\gamma^\theta\eta^\theta} \\ \times \int_0^\infty E_\theta \left(-\left(\dfrac{s^\theta - a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta} \right) t^\theta \right) (dt)^\theta \end{array} \right) \\
&= \frac{s^\theta}{\gamma^\theta\eta^\theta} \left(\begin{array}{c} \dfrac{\gamma^{2\theta}\eta^{2\theta}}{\left(s^\theta - a\gamma^\theta\eta^\theta\right)^2} \\ \times \lim\limits_{\sigma\to\infty} \left[-E_\theta \left(-\left(\dfrac{s^\theta - a\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta} \right) t^\theta \right) \right]_0^\sigma \end{array} \right) \\
&= \frac{s^\theta\gamma^\theta\eta^\theta}{\left(s^\theta - a\gamma^\theta\eta^\theta\right)^2}.
\end{aligned}
$$

□

Theorem 9. Let $u(t) = \sin_\theta(at^\theta), 0 < \theta \leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta \left[\sin_\theta(at^\theta) \right] = \frac{as^\theta\gamma^\theta\eta^\theta}{s^{2\theta} + a^2\gamma^{2\theta}\eta^{2\theta}}.
$$

Proof. By using (1) and the fractional integration by parts formula [11], we attain

$$
\begin{aligned}
&{}^{LF}\mathbb{KH}_\theta \left[\sin_\theta(at^\theta) \right] \\
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta\eta^\theta} \int_0^\infty E_\theta \left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta} \right) \sin_\theta(at^\theta) (dt)^\theta \\
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta\eta^\theta}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)\left(\frac{E_\theta\left(ai^\theta t^\theta\right)-E_\theta\left(-ai^\theta t^\theta\right)}{2i^\theta}\right)(dt)^\theta\\
&= \frac{1}{2i^\theta}\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\\
&\times\left(\int_0^\infty E_\theta\left(-\left(\frac{s^\theta-ai^\theta\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta}\right)t^\theta\right)(dt)^\theta\right.\\
&\left.-\int_0^\infty E_\theta\left(-\left(\frac{s^\theta+ai^\theta\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta}\right)t^\theta\right)(dt)^\theta\right)\\
&= \frac{1}{2i^\theta}\frac{s^\theta}{\gamma^\theta\eta^\theta}\\
&\left(-\frac{\gamma^\theta\eta^\theta}{s^\theta-ai^\theta\gamma^\theta\eta^\theta}\lim_{\sigma\to\infty}\left[E_\theta\left(-\left(\frac{s^\theta-ai^\theta\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta}\right)t^\theta\right)\right]_0^\sigma\right.\\
&\left.+\frac{\gamma^\theta\eta^\theta}{s^\theta+ai^\theta\gamma^\theta\eta^\theta}\lim_{\sigma\to\infty}\left[E_\theta\left(-\left(\frac{s^\theta+ai^\theta\gamma^\theta\eta^\theta}{\gamma^\theta\eta^\theta}\right)t^\theta\right)\right]_0^\sigma\right)\\
&= \frac{1}{2i^\theta}\frac{s^\theta}{\gamma^\theta\eta^\theta}\left(\frac{\gamma^\theta\eta^\theta}{s^\theta-ai^\theta\gamma^\theta\eta^\theta}-\frac{\gamma^\theta\eta^\theta}{s^\theta+ai^\theta\gamma^\theta\eta^\theta}\right)\\
&= \frac{as^\theta\gamma^\theta\eta^\theta}{s^{2\theta}+a^2\gamma^{2\theta}\eta^{2\theta}}.
\end{aligned}
$$

□

Theorem 10. Let $u(t)=\sinh_\theta(at^\theta), 0<\theta\leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[\sinh_\theta(at^\theta)\right]=\frac{as^\theta\gamma^\theta\eta^\theta}{s^{2\theta}-a^2\gamma^{2\theta}\eta^{2\theta}}.
$$

Proof. To prove this theorem, we follow the same previous steps, knowing that

$$
\sinh_\theta(at^\theta)=\frac{E_\theta\left(at^\theta\right)-E_\theta\left(-at^\theta\right)}{2}.
$$

□

Theorem 11. Let $u(t)=\cos_\theta(at^\theta), 0<\theta\leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[\cos_\theta(at^\theta)\right]=\frac{s^{2\theta}}{s^{2\theta}+a^2\gamma^{2\theta}\eta^{2\theta}}.
$$

Proof. By using (1) and the fractional integration by parts formula [11], we attain

$$
\begin{aligned}
&{}^{LF}\mathbb{KH}_\theta\left[\cos_\theta(at^\theta)\right]\\
&=\frac{1}{\Gamma(1+\theta)}\frac{s^\theta}{\gamma^\theta\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta\eta^\theta}\right)\cos_\theta(at^\theta)(dt)^\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \\
&\quad \times \int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\gamma^\theta \eta^\theta}\right)\left(\frac{E_\theta\left(ai^\theta t^\theta\right)+E_\theta\left(-ai^\theta t^\theta\right)}{2}\right)(dt)^\theta \\
&= \frac{1}{2} \frac{1}{\Gamma(1+\theta)} \frac{s^\theta}{\gamma^\theta \eta^\theta} \\
&\quad \times \left(\int_0^\infty E_\theta\left(-\left(\frac{s^\theta - ai^\theta \gamma^\theta \eta^\theta}{\gamma^\theta \eta^\theta}\right) t^\theta\right)(dt)^\theta\right. \\
&\quad \left. + \int_0^\infty E_\theta\left(-\left(\frac{s^\theta + ai^\theta \gamma^\theta \eta^\theta}{\gamma^\theta \eta^\theta}\right) t^\theta\right)(dt)^\theta\right) \\
&= \frac{1}{2} \frac{s^\theta}{\gamma^\theta \eta^\theta} \\
&\quad \times \left(-\frac{\gamma^\theta \eta^\theta}{s^\theta - ai^\theta \gamma^\theta \eta^\theta} \lim_{\sigma\to\infty}\left[E_\theta\left(-\left(\frac{s^\theta - ai^\theta \gamma^\theta \eta^\theta}{\gamma^\theta \eta^\theta}\right) t^\theta\right)\right]_0^\sigma\right. \\
&\quad \left. -\frac{\gamma^\theta \eta^\theta}{s^\theta + ai^\theta \gamma^\theta \eta^\theta} \lim_{\sigma\to\infty}\left[E_\theta\left(-\left(\frac{s^\theta + ai^\theta \gamma^\theta \eta^\theta}{\gamma^\theta \eta^\theta}\right) t^\theta\right)\right]_0^\sigma\right) \\
&= \frac{1}{2} \frac{s^\theta}{\gamma^\theta \eta^\theta}\left(\frac{\gamma^\theta \eta^\theta}{s^\theta - ai^\theta \gamma^\theta \eta^\theta} + \frac{\gamma^\theta \eta^\theta}{s^\theta + ai^\theta \gamma^\theta \eta^\theta}\right) \\
&= \frac{s^{2\theta}}{s^{2\theta} + a\gamma^{2\theta}\eta^{2\theta}}.
\end{aligned}
$$

□

Theorem 12. Let $u(t) = \cosh_\theta(at^\theta), 0 < \theta \leq 1$. Then, its local fractional Khalouta transform is given by

$$
{}^{LF}\mathbb{KH}_\theta\left[\cosh_\theta(at^\theta)\right] = \frac{s^{2\theta}}{s^{2\theta} - a^2\gamma^{2\theta}\eta^{2\theta}}.
$$

Proof. To prove this theorem, we follow the same previous steps, knowing that

$$
\cosh_\theta(at^\theta) = \frac{E_\theta\left(at^\theta\right) + E_\theta\left(-at^\theta\right)}{2}.
$$

□

4 Relation with other known local fractional integral transforms

This section illustrates the relation between the local fractional Khalouta transform and other known local fractional integral transforms.

- **Local fractional Khalouta–Laplace relation:**
Let $\mathcal{K}_\theta(s,\gamma,\eta)$ be the local fractional Khalouta transform of the function $u(t)$. If $\gamma=\eta=1$, then

$$\begin{aligned}\mathcal{K}_\theta(s,1,1) &= s^\theta\left(\frac{1}{\Gamma(1+\theta)}\int_0^\infty E_\theta\left(-s^\theta t^\theta\right)u(t)(dt)^\theta\right)\\ &= s^\theta\mathcal{L}_\theta(s),\end{aligned}$$

where $\mathcal{L}_\theta(s)$ denotes the local fractional Laplace transform of the function $u(t)$ [28].

- **Local fractional Khalouta–Sumudu relation:**
Let $\mathcal{K}_\theta(s,\gamma,\eta)$ be the local fractional Khalouta transform of the function $u(t)$. If $s=\gamma=1$, then

$$\begin{aligned}\mathcal{K}_\theta(1,1,\eta) &= \frac{1}{\Gamma(1+\theta)}\frac{1}{\eta^\theta}\int_0^\infty E_\theta\left(-\frac{t^\theta}{\eta^\theta}\right)u(t)(dt)^\theta\\ &= \mathcal{S}_\theta(\eta),\end{aligned}$$

where $\mathcal{S}_\theta(\eta)$ denotes the local fractional Sumudu transform of the function $u(t)$ [24].

- **Local fractional Khalouta–Elzaki relation:**
Let $\mathcal{K}_\theta(s,\gamma,\eta)$ be the local fractional Khalouta transform of the function $u(t)$. If $s=\eta=1$, then

$$\begin{aligned}\mathcal{K}_\theta(1,\gamma,1) &= \frac{1}{\gamma^{2\theta}}\left(\frac{1}{\Gamma(1+\theta)}\gamma^\theta\int_0^\infty E_\theta\left(-\frac{t^\theta}{\gamma^\theta}\right)u(t)(dt)^\theta\right)\\ &= \frac{1}{\gamma^{2\theta}}\mathcal{T}_\theta(\eta),\end{aligned}$$

where $\mathcal{T}_\theta(\eta)$ denotes the local fractional Elzaki transform of the function $u(t)$ [30].

- **Local fractional Khalouta–Aboodh relation:**
Let $\mathcal{K}_\theta(s,\gamma,\eta)$ be the local fractional Khalouta transform of the function $u(t)$. If $\gamma=\eta=1$, then

$$\begin{aligned}\mathcal{K}_\theta(s,1,1) &= s^{2\theta}\left(\frac{1}{\Gamma(1+\theta)}\frac{1}{s^\theta}\int_0^\infty E_\theta\left(-s^\theta t^\theta\right)u(t)(dt)^\theta\right)\\ &= s^{2\theta}\mathcal{A}_\theta(s),\end{aligned}$$

where $\mathcal{A}_\theta(\eta)$ denotes the local fractional Aboodh transform of the function $u(t)$ [29].

- **Local fractional Khalouta–Natural relation:**
  Let $\mathcal{K}_\theta(s,\gamma,\eta)$ be the local fractional Khalouta transform of the function $u(t)$. If $\gamma=1$, then

$$\begin{aligned}\mathcal{K}_\theta(s,1,\eta) &= s^\theta\left(\frac{1}{\Gamma(1+\theta)}\frac{1}{\eta^\theta}\int_0^\infty E_\theta\left(-\frac{s^\theta t^\theta}{\eta^\theta}\right)u(t)(dt)^\theta\right)\\ &= s^\theta\mathcal{N}_\theta(s,\eta),\end{aligned}$$

  where $\mathcal{N}_\theta(s,\eta)$ denotes the local fractional natural transform of the function $u(t)$ [19].

5 Applications

This section demonstrates the applicability, efficiency, and high accuracy of the local fractional Khalouta transform method to some linear local fractional differential equations.

Example 1. Let us examine the local fractional differential equation

$$D^{(\theta)}u(t)+u(t)=\frac{t^\theta}{\Gamma(1+\theta)},\quad 0<\theta\le 1, \tag{8}$$

within the initial condition

$$u(0)=-1. \tag{9}$$

Operating the local fractional Khalouta transform on (8) and utilizing Theorem 1, we attain

$${}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}u(t)\right]+{}^{LF}\mathbb{KH}_\theta\left[u(t)\right]={}^{LF}\mathbb{KH}_\theta\left[\frac{t^\theta}{\Gamma(1+\theta)}\right]. \tag{10}$$

Implementing Theorems 2 and 6, (10) becomes

$$\left(\frac{s}{\gamma\eta}\right)^\theta\mathcal{K}_\theta(s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^\theta u(0)+\mathcal{K}_\theta(s,\gamma,\eta)=\frac{\gamma^\theta\eta^\theta}{s^\theta}. \tag{11}$$

Making use of the initial condition (9), (11) can be written as

$$\left(\frac{s}{\gamma\eta}\right)^\theta\mathcal{K}_\theta(s,\gamma,\eta)+\left(\frac{s}{\gamma\eta}\right)^\theta+\mathcal{K}_\theta(s,\gamma,\eta)=\frac{\gamma^\theta\eta^\theta}{s^\theta}.$$

Then, we have

$$\mathcal{K}_\theta(s,\gamma,\eta)=\frac{\gamma^{2\theta}\eta^{2\theta}-s^{2\theta}}{s^\theta(s^\theta+\gamma^\theta\eta^\theta)}$$

$$= \frac{\gamma^{\theta}\eta^{\theta}}{s^{\theta}} - 1. \tag{12}$$

Performing the inverse local fractional Khalouta transform on (12), we achieve

$$u(t) = \frac{t^{\theta}}{\Gamma(1+\theta)} - 1. \tag{13}$$

The result (13) indicates the exact solution to (8).

Example 2. Let us examine the local fractional differential equation

$$D^{(2\theta)}u(t) - u(t) = -\sin_{\theta}(t^{\theta}), \quad 0 < \theta \leq 1, \tag{14}$$

within the initial conditions

$$u(0) = 0, \qquad D^{(\theta)}u(0) = -1. \tag{15}$$

Operating the local fractional Khalouta transform on (14) and utilizing Theorem 1, we attain

$${}^{LF}\mathbb{KH}_{\theta}\left[D^{(2\theta)}u(t)\right] - {}^{LF}\mathbb{KH}_{\theta}\left[u(t)\right] = {}^{LF}\mathbb{KH}_{\theta}\left[-\sin_{\theta}(t^{\theta})\right]. \tag{16}$$

Implementing Theorems 2 and 9, (16) becomes

$$\begin{aligned} &\left(\frac{s}{\gamma\eta}\right)^{2\theta}\mathcal{K}_{\theta}(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{2\theta}u(0) - \left(\frac{s}{\gamma\eta}\right)^{\theta}u^{(\theta)}(0) - \mathcal{K}_{\theta}(s,\gamma,\eta) \\ &= -\frac{s^{\theta}\gamma^{\theta}\eta^{\theta}}{s^{2\theta}+\gamma^{2\theta}\eta^{2\theta}}. \end{aligned} \tag{17}$$

Making use of the initial conditions (15), (17) can be written as

$$\left(\frac{s}{\gamma\eta}\right)^{2\theta}\mathcal{K}_{\theta}(s,\gamma,\eta) + \left(\frac{s}{\gamma\eta}\right)^{\theta} - \mathcal{K}_{\theta}(s,\gamma,\eta) = -\frac{s^{\theta}\gamma^{\theta}\eta^{\theta}}{s^{2\theta}+\gamma^{2\theta}\eta^{2\theta}}.$$

Then, we have

$$\begin{aligned} \mathcal{K}_{\theta}(s,\gamma,\eta) &= \frac{-s^{3\theta}\gamma^{\theta}\eta^{\theta} - 2s^{\theta}\gamma^{3\theta}\eta^{3\theta}}{\left(s^{2\theta}+\gamma^{2\theta}\eta^{2\theta}\right)\left(s^{2\theta}-\gamma^{2\theta}\eta^{2\theta}\right)} \\ &= \frac{3}{4}\frac{s^{\theta}}{s^{\theta}+\gamma^{\theta}\eta^{\theta}} - \frac{3}{4}\frac{s^{\theta}}{s^{\theta}-\gamma^{\theta}\eta^{\theta}} + \frac{1}{2}\frac{s^{\theta}\gamma^{\theta}\eta^{\theta}}{s^{2\theta}+\gamma^{2\theta}\eta^{2\theta}}. \end{aligned} \tag{18}$$

Performing the inverse local fractional Khalouta transform on (18), we achieve

$$u(t) = \frac{3}{4}E_{\theta}(-t^{\theta}) - \frac{3}{4}E_{\theta}(t^{\theta}) + \frac{1}{2}\sin_{\theta}(t^{\theta}). \tag{19}$$

The result (19) indicates the exact solution to (14).

Example 3. Let us examine the local fractional differential equation

$$D^{(2\theta)}u(t) + D^{(\theta)}u(t) - u(t) = E_\theta(-2t^\theta), \quad 0 < \theta \leq 1, \tag{20}$$

within the initial conditions

$$u(0) = 1, \qquad D^{(\theta)}u(0) = -2. \tag{21}$$

Operating the local fractional Khalouta transform on (20) and utilizing Theorem 1, we attain

$${}^{LF}\mathbb{KH}_\theta\left[D^{(2\theta)}u(t)\right] + {}^{LF}\mathbb{KH}_\theta\left[D^{(\theta)}u(t)\right] - {}^{LF}\mathbb{KH}_\theta\left[u(t)\right] = {}^{LF}\mathbb{KH}_\theta\left[E_\theta(-2t^\theta)\right]. \tag{22}$$

Implementing Theorems 2 and 7, (22) becomes

$$\begin{aligned}\frac{s^\theta}{s^\theta + 2\gamma^\theta\eta^\theta} &= \left(\frac{s}{\gamma\eta}\right)^{2\theta}\mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{2\theta}u(0) - \left(\frac{s}{\gamma\eta}\right)^{\theta}u^{(\theta)}(0)\\ &\quad + \left(\frac{s}{\gamma\eta}\right)^{\theta}\mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{\theta}u(0) - \mathcal{K}_\theta(s,\gamma,\eta).\end{aligned} \tag{23}$$

Making use of the initial conditions (21), (23) can be written as

$$\begin{aligned}&\left(\frac{s}{\gamma\eta}\right)^{2\theta}\mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{2\theta} + \left(\frac{s}{\gamma\eta}\right)^{\theta} + \left(\frac{s}{\gamma\eta}\right)^{\theta}\mathcal{K}_\theta(s,\gamma,\eta) - \mathcal{K}_\theta(s,\gamma,\eta)\\ &\quad = \frac{s^\theta}{s^\theta + 2\gamma^\theta\eta^\theta}.\end{aligned}$$

Then, we have

$$\begin{aligned}\mathcal{K}_\theta(s,\gamma,\eta) &= \frac{\gamma^{2\theta}\eta^{2\theta}}{s^{2\theta} + s^\theta\gamma^\theta\eta^\theta - \gamma^{2\theta}\eta^{2\theta}}\left(\frac{s^\theta}{s^\theta + 2\gamma^\theta\eta^\theta} + \frac{s^{2\theta} - s^\theta\gamma^\theta\eta^\theta}{\gamma^{2\theta}\eta^{2\theta}}\right)\\ &= \frac{\gamma^{2\theta}\eta^{2\theta}}{s^{2\theta} + s^\theta\gamma^\theta\eta^\theta - \gamma^{2\theta}\eta^{2\theta}}\left(\frac{s^\theta\left(s^{2\theta} + s^\theta\gamma^\theta\eta^\theta - \gamma^{2\theta}\eta^{2\theta}\right)}{\left(s^\theta + 2\gamma^\theta\eta^\theta\right)\gamma^{2\theta}\eta^{2\theta}}\right)\\ &= \frac{s^\theta}{s^\theta + 2\gamma^\theta\eta^\theta}.\end{aligned} \tag{24}$$

Performing the inverse local fractional Khalouta transform on (24), we achieve

$$u(t) = E_\theta(-2t^\theta). \tag{25}$$

The result (25) indicates the exact solution to (20).

Example 4. Let us examine the local fractional differential equation

$$D^{(4\theta)}u(t) - u(t) = 0, \quad 0 < \theta \leq 1, \tag{26}$$

within the initial conditions

$$u(0) = 0, \quad D^{(\theta)}u(0) = -1, \quad D^{(2\theta)}u(0) = 0, \quad D^{(3\theta)}u(0) = 0. \tag{27}$$

Operating the local fractional Khalouta transform on (26) and utilizing Theorem 1, we attain

$$^{LF}\mathbb{KH}_\theta \left[D^{(4\theta)}u(t)\right] -^{LF} \mathbb{KH}_\theta \left[u(t)\right] = 0. \tag{28}$$

Implementing Theorem 2, (28) becomes

$$\begin{aligned} 0 = &\left(\frac{s}{\gamma\eta}\right)^{4\theta} \mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{4\theta} u(0) - \left(\frac{s}{\gamma\eta}\right)^{3\theta} u^{\theta}(0) \\ &- \left(\frac{s}{\gamma\eta}\right)^{2\theta} u^{(2\theta)}(0) - \left(\frac{s}{\gamma\eta}\right)^{\theta} u^{(3\theta)}(0) - \mathcal{K}_\theta(s,\gamma,\eta). \end{aligned} \tag{29}$$

Making use of the initial conditions (27), (29) can be written as

$$\left(\frac{s}{\gamma\eta}\right)^{4\theta} \mathcal{K}_\theta(s,\gamma,\eta) + \left(\frac{s}{\gamma\eta}\right)^{3\theta} - \mathcal{K}_\theta(s,\gamma,\eta) = 0.$$

Then, we have

$$\begin{aligned} \mathcal{K}_\theta(s,\gamma,\eta) &= -\frac{s^{3\theta}\gamma^{\theta}\eta^{\theta}}{s^{4\theta} - \gamma^{4\theta}\eta^{4\theta}} \\ &= \frac{1}{4}\frac{s^{\theta}}{s^{\theta} + \gamma^{\theta}\eta^{\theta}} - \frac{1}{4}\frac{s^{\theta}}{s^{\theta} - \gamma^{\theta}\eta^{\theta}} - \frac{1}{2}\frac{s^{\theta}\gamma^{\theta}\eta^{\theta}}{s^{2\theta} + \gamma^{2\theta}\eta^{2\theta}}. \end{aligned} \tag{30}$$

Performing the inverse local fractional Khalouta transform on (30), we achieve

$$u(t) = \frac{1}{4}E_\theta(-t^{\theta}) - \frac{1}{4}E_\theta(t^{\theta}) - \frac{1}{2}\sin_\theta(t^{\theta}). \tag{31}$$

The result (31) indicates the exact solution to (26).

Example 5. Let us examine the local fractional Volterra integro-differential equation

$$D^{(\theta)}u(t) = 1 + \frac{1}{\Gamma(1+\theta)} \int_0^t u(\tau)\,(d\tau)^{\theta}, \quad 0 < \theta \leq 1, \tag{32}$$

within the initial condition

$$u(0) = 0. \tag{33}$$

Operating the local fractional Khalouta transform on (32) and utilizing Theorem 1, we attain

$$^{LF}\mathbb{KH}_{\theta}\left[D^{(\theta)}u(t)\right] =^{LF} \mathbb{KH}_{\theta}\left[1\right] +^{LF} \mathbb{KH}_{\theta}\left[\frac{1}{\Gamma(1+\theta)}\int_0^t u(\tau)\,(d\tau)^{\theta}\right]. \tag{34}$$

Then

$$^{LF}\mathbb{KH}_{\theta}\left[D^{(\theta)}u(t)\right] =^{LF} \mathbb{KH}_{\theta}\left[1\right] +^{LF} \mathbb{KH}_{\theta}\left[(u(t)*1)_{\theta}\right]. \tag{35}$$

Implementing Theorems 2, 4, and 5, we arrive at

$$\left(\frac{s}{\gamma\eta}\right)^{\theta}\mathcal{K}_{\theta}(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{\theta}u(0) = 1 + \left(\frac{\gamma\eta}{s}\right)^{\theta}\mathcal{K}_{\theta}(s,\gamma,\eta). \tag{36}$$

Making use of the initial condition (33), (36) can be written as

$$\left(\frac{s}{\gamma\eta}\right)^{\theta}\mathcal{K}_{\theta}(s,\gamma,\eta) = 1 + \left(\frac{\gamma\eta}{s}\right)^{\theta}\mathcal{K}_{\theta}(s,\gamma,\eta).$$

Then, we have

$$\mathcal{K}_{\theta}(s,\gamma,\eta) = \frac{s^{\theta}\gamma^{\theta}\eta^{\theta}}{s^{2\theta}-\gamma^{2\theta}\eta^{2\theta}}. \tag{37}$$

Performing the inverse local fractional Khalouta transform on (37), we achieve

$$u(t) = \sinh_{\theta}(t^{\theta}). \tag{38}$$

The result (38) indicates the exact solution to (32).

Example 6. Let us examine the local fractional Volterra integro-differential equation

$$D^{(2\theta)}u(t) = 1 + \frac{1}{\Gamma(1+\theta)}\int_0^t \frac{(t-\tau)^{\theta}}{\Gamma(1+\theta)}u(\tau)\,(d\tau)^{\theta}, \quad 0<\theta\leq 1, \tag{39}$$

within the initial conditions

$$u(0) = 1, \qquad D^{(\theta)}u(0) = 0. \tag{40}$$

Operating the local fractional Khalouta transform on (39) and utilizing Theorem 1, we attain

$$^{LF}\mathbb{KH}_{\theta}\left[D^{(2\theta)}u(t)\right] =^{LF} \mathbb{KH}_{\theta}\left[1\right] +^{LF} \mathbb{KH}_{\theta}\left[\frac{1}{\Gamma(1+\theta)}\int_0^t \frac{(t-\tau)^{\theta}}{\Gamma(1+\theta)}u(\tau)\,(d\tau)^{\theta}\right].$$

Then

$$ {}^{LF}\mathbb{KH}_\theta\left[D^{(2\theta)}u(t)\right] = {}^{LF}\,\mathbb{KH}_\theta\,[1] + {}^{LF}\,\mathbb{KH}_\theta\left[\left(\frac{t^\theta}{\Gamma(1+\theta)} * u(t)\right)_\theta\right]. \tag{41} $$

Using Theorems 2, 4, 5 and 6, (41) becomes

$$ \begin{aligned} &\left(\frac{s}{\gamma\eta}\right)^{2\theta}\mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{2\theta}u(0) - \left(\frac{s}{\gamma\eta}\right)^{\theta}u^{(\theta)}(0) \\ &= 1 + \left(\frac{\gamma\eta}{s}\right)^{\theta}\left(\frac{\gamma\eta}{s}\right)^{\theta}\mathcal{K}_\theta(s,\gamma,\eta). \end{aligned} \tag{42} $$

Making use of the initial conditions (40), (42) can be written as

$$ \left(\frac{s}{\gamma\eta}\right)^{2\theta}\mathcal{K}_\theta(s,\gamma,\eta) - \left(\frac{s}{\gamma\eta}\right)^{2\theta} = 1 + \left(\frac{\gamma\eta}{s}\right)^{2\theta}\mathcal{K}_\theta(s,\gamma,\eta). $$

Then, we have

$$ \begin{aligned} \mathcal{K}_\theta(s,\gamma,\eta) &= \left(\frac{s^{2\theta}\gamma^{2\theta}\eta^{2\theta}}{s^{4\theta}-\gamma^{4\theta}\eta^{4\theta}}\right)\left(\frac{\gamma^{2\theta}\eta^{2\theta}+s^{2\theta}}{\gamma^{2\theta}\eta^{2\theta}}\right) \\ &= \frac{s^{2\theta}}{s^{2\theta}-\gamma^{2\theta}\eta^{2\theta}}. \end{aligned} \tag{43} $$

Performing the inverse local fractional Khalouta transform on (43), we achieve

$$ u(t) = \cosh_\theta(t^\theta). \tag{44} $$

The result (44) indicates the exact solution to (39).

6 Conclusions

In this paper, we proposed the local fractional Khalouta transform method, based on the coupling of two powerful concepts: The Khalouta transform and the local fractional derivative. Several illustrative examples of local linear fractional differential equations have been provided in order to demonstrate the advantages of the proposed method. The results obtained in the examples presented have shown that this method is capable of solving any linear local integral equation.

These results open up avenues for further research and applications in related fields of science and mathematics. Future research directions could include:

- Proposing novel numerical methods that combine the local fractional Khalouta transform method with semi-analytical methods, such as the Adomian decomposition method, the homotopy perturbation method, and the variational iteration method, to solve nonlinear local fractional differential equations.

- Investigating the stability and convergence analysis of proposed methods.

- Extending the applicability of these methods to more complex problems, such as nonlinear local fractional partial differential equations and nonlinear local fractional integro-differential equations.

Acknowledgements

The author thanks the anonymous reviewers and the editor for their constructive comments.

Funding

- This research received no external funding.

Conflicts of Interest

- The author declares no conflicts of interest.

References

[1] Abdeljawad, T. *On conformable fractional calculus*, J. Comput. Appl. Math. 279 (2015), 57–66.

[2] Atangana, A. and Koca, I. *Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order*, Chaos, Soliton. Fract. 89 (2016) 447–454.

[3] Baleanu, D. and Jassim, H.K. *A modification fractional homotopy perturbation method for solving Helmholtz and coupled Helmholtz equations on Cantor sets*, Fractal Fract. 3(30) (2019), 1–8.

[4] Baleanu, D. and Jassim, H.K. *Approximate solutions of the damped wave equation and dissipative wave equation in fractal strings*, Fractal Fract. 3(26) (2019), 1–12.

[5] Baleanu, D., Jassim, H.K. and Al Qurashi, M. *Solving Helmholtz equation with local fractional derivative operators*, Fractal Fract. 3(43) (2019), 1–13.

[6] Baleanu, D., Jassim, H.K. and Al Qurashic,M. *Approximate analytical solutions of Goursat problem within local fractional operators*, J. Nonlinear Sci. Appl. 9 (2016), 4829–4837.

[7] Hilfer, R. *Applications of fractional calculus in physics*, World Scientific, River Edge, New Jerzey, 2000.

[8] Jafari, H. and Jassim, H.K. *Application of the Local Fractional Variational Iteration Method to Solve System of Coupled Partial Differential Equations Involving Local Fractional Operator*, Appl. Math. Inf. Sci. Lett. 5(2) (2017), 45–50.

[9] Jafari, H., Jassim, H.K. and Baleanu, D. *On the existence and uniqueness of solutions for local differential equations*, Entropy. 18(420) (2016), 1–9.

[10] Jafari, H., Jassim, H.K., Tchier, F. and Baleanu,D. *On the approximate solutions of local fractional differential equations with local fractional operator*, Entropy. 18 (2016), 1–12.

[11] Jumarie, G. *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions*, Appl. Math. Lett. 22(3) (2009), 378–385.

[12] Khalouta, A. *A new exponential type kernel integral transform: Khalouta transform and its applications*, Math. Montisnigri. 57 (2023), 5–23.

[13] Khalouta, A. *New results of Khalouta transform method for non-constant coefficients ordinary differential equations*, J. Interdiscip. Math. 28(7) (2025), 2555–2566.

[14] Kilbas, A.A. *Hadamard-type fractional calculus*, J. Korean Math. Soc. 3(6) (2001), 1191–1204.

[15] Kilbas, A.A, Srivastava, H.M. and Trujillo, J.J. *Theory and application of fractional differential equations*, Elsevier, Amsterdam, 2006.

[16] Leibniz, G.W. *Letter from Hanover, Germany to G.F.A. L'Hospital, September 30, 1695*, Leibniz Mathematische Schriften. Olms-Verlag, Hildesheim, Germany, 1962. 301–302, First published in 1849.

[17] Liu, J-G., Yang, X-J. Feng, Y-Y. and Cui, P. *On the* $(N+1)-$*dimensional local fractional reduced differential transform method and its applications*, Math. Methods Appl. Sci. 43(5) (2020), 8856-8866.

[18] Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelaticity: An Introduction to Mathematical Models*, Imperial College Press, London, 2010.

[19] Maitama, S. *Local fractional natural homotopy perturbation method for solving partial differential equations with local fractional derivative*, Progr. Fract. Differ. Appl. 4(3) (2018), 219–228.

[20] Maitama, S. and Zhao,W. *Local fractional homotopy analysis method for solving non-differentiable problems on Cantor sets*, Adv. Differ. Equ. 2019(127) (2019), 1–22.

[21] Miller, K.S. and Ross, B. *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

[22] Odzijewicz, T. *Generalized fractional isoperimetric problem of several variables*, DDiscrete Contin. Dyn. Syst. Ser. B. 19(8) (2014), 2617–2629.

[23] Podlubny, I. Fractional Differential Equations, Academic Press, New York, 1999.

[24] Srivastava, H.M., Golmankhaneh, A.K. Baleanu, D. and Yang, X-J. *Local fractional Sumudu transform with application to IVPs on Cantor sets*, Abstr. Appl. Anal. 2014 (2014), 1–7.

[25] Yang, X.J. *Local Fractional Functional Analysis and Its Applications*, Asian Academic: Hong Kong, China, 2011.

[26] Yang, X.J., *Advanced local fractional calculus and its applications*, World Science: New York, NY, USA, 2012.

[27] Yang, Y-J. and Wang, S-Q. *A local fractional homotopy perturbation method for solving the local fractional Korteweg-de Vries equations with non-homogeneous term*, Therm. Sci. 23(00) (2019), 216–216.

[28] Zhao, C.G., Yang, A.M., Jafari, H. and Haghbin, A. *The Yang-Laplace transform for solving the IVPs with local fractional derivative*, Abstr. Appl. Anal. 2014 (2014), 1–5.

[29] Ziane, Dj., Belgacema, R. and Bokhari, A. *Local fractional Aboodh transform and its applications to solve linear local fractional differential equations*, Adv. Theory Nonlinear Anal. Appl. 6(2) (2022), 217–228.

[30] Ziane, Dj. and Hamdi Cherif, M. *Local fractional Elzaki transform and its application to local fractional differential equations*, J. New Results Sci. 10(3) (2021), 19–33.