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Research Article

Spline-interpolation solution of a plane Stefan problem

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Abstract

We construct a spline-interpolation solution of the two-dimensional Stefan problem. We reduce it to the set of boundary value problems for analytic functions and integral equations. Our construction of an approximate solution is based on the introduction of analytic functions and the application of approximate solutions of boundary-value problems. We modify the initial problem in order to reduce it to a set of boundary value problems. The first of these problems is the Cauchy problem for the Laplace equation. The second problem is the Riemann problem for analytic function, We also give estimates of the approximate solution error and some examples of the solution.

AMS subject classifications (2020): Primary 35J25; Secondary 30E10.

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1 Introduction

The task of studying the conditions for the formation and dynamics of the liquid-solid phase transition has wide practical applications, as it is directly related to issues such as storing thermal

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energy, controlling temperature and humidity, and ensuring the durability and strength of the systems under study. In addition, the energy stored during melting can be released during the solidification process, which enables a wide range of energy applications.

Possible applications span various technologies, such as in the steel industry, soil freezing, and thermal control systems for spacecraft, as well as ground-based industrial and civil facilities.

The solution to this problem has practical applications in the construction industry, where the pressing issue of determining the current position of the freezing front in external enclosing structures of buildings and construction elements arises. The position of the freezing front is directly linked to the durability of the enclosing structures. In fact, in the zone where the freezing front is moving, extremely unfavorable operating conditions for the material of the enclosing structures are created due to the potential alternation of freezing and thawing. This can gradually lead to a decrease in strength, and ultimately, to the destruction of the structure.

The Stefan problem for the temperature field $T(t,x,y)$, is formulated as follows: Consider a medium occupying a region Ω consisting of two phases in the corresponding subdomains of Ω: Phase 1 in the subdomain $D_1(t)$ and phase 2 in $D_2(t)$. These subdomains have the common dynamic boundary $\Gamma'(t)$. Let the two phases have thermal diffusivities θ_1 and θ_2. Here $\frac{\partial T}{\partial t} = \nabla \cdot (\theta_1 \nabla T)$ in the region D_1 and $\frac{\partial T}{\partial t} = \nabla \cdot (\theta_2 \nabla T)$ in the region D_2.

Assume that we know the boundary $\Gamma'(0)$ between the zones $D_1(0)$ and $D_2(0)$. So given the initial value of $T(0,x,y)$ at Ω and the boundary data $T(t,x,y)|_{(x,y)\in\partial\Omega=\Gamma\cup\Gamma''}$ it should be possible to reconstruct both the function $T(t,x,y)$, $t>0$, $(x,y)\in\Omega$ and the boundary $\Gamma'(t)$ between two subdomains D_1 and D_2 at any time $t>0$. The Stefan condition determines the evolution of the curve $\Gamma'(t)$ by giving an equation governing the velocity V of the free boundary in the normal direction ν, specifically $V = \theta_1 \partial_\nu T_1 - \theta_2 \partial_\nu T_2$. By boundary values of T_1 we mean the limit of the gradient as (x,y) approaches $\Gamma'(t)$ from the region $D_1(t)$, and by boundary values of T_2 we mean the limit of the gradient as (x,y) approaches $\Gamma'(t)$ from the region $D_2(t)$.

There exists a number of numerical methods of solution of this problem [1, 3, 5, 14, 16]. Most of these approaches are variations of finite-difference and finite-element methods [17, 18]. The authors usually consider different types of meshes. Here, we try to exclude the mesh at least from the spatial coordinates.

Here, we apply the Cauchy integral formula to the solution of the Stefan problem. We first modify the initial problem in order to reduce it to a set of certain boundary value problems. The first of these problems is the Cauchy problem for Laplace equation. So we again see the connection of the Stefan problem with the subject of incorrectly over-determined problems [4]. This is the crucial part of our construction. We show the solvability of this problem and present certain examples of the solution. We apply methods similar to our constructions of [8]. The

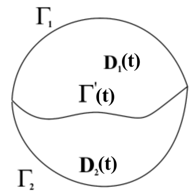

Figure 1: Domain D bounded by the curves Γ and Γ'

second problem is the usual Schwartz problem of reconstruction of the boundary imaginary part of the function analytic in a simply-connected domain by the real part of this function.

Our construction of an approximate solution is based on the introduction of analytic functions and the application of approximate solutions of boundary-value problems. We do not apply here the common methods of solution of the integral equations corresponding to so-called weak solutions [9, 10, 11, 12] of finite-element methods [5, 14, 16]. Our solution is thus an approximate-analytic one.

2 Approximate solution construction

Here we construct an approximate solution of a two-phase Stefan problem for the constant thermal diffusivities $\theta_j,\ j=1,2$. Consider the domain $D_j(t),\ j=1,2$, bounded by the simple curves Γ_j and $\Gamma'(t)$, so $\partial D_j(t)=\Gamma_j\bigcup\Gamma'(t)$ (see Figure 1). Assume that the curve Γ_j is fixed, and that the curve $\Gamma'(t)$ is the free part of the boundary $\partial D_j(t)$. Position of $\Gamma'(t)$ is known for $t=0$.

The first step in our construction is to fix the subdomain $D_1=D_1(0)$ and to construct an approximate solution $T^1(t,x,y)$ in $[0,1]\times D_1$. Recall the main relation

$$\frac{\partial T}{\partial t}=\nabla\cdot(\theta_1\nabla T) \tag{1}$$

in this domain.

Consider the modified version of the problem: Let the function T take values at $T(0,x(s),y(s))$ and $T(1,x(s),y(s))$, with $(x(s),y(s))$ belonging to Γ_1, denoted by $U_0(s)$ and $U_1(s)$, respectively. Similarly, let the values of the normal derivative of T, that is, $\frac{\partial T(t,x,y)}{\partial n}$, at $t=0$ and $t=1$, for points $(x(s),y(s))\in\Gamma_1$, be denoted by $V_0(s)$ and $V_1(s)$, respectively.

These conditions allow us to formulate a linearized solution, specifically a linear approximation in terms of t, given by $T^1(t,z,\overline{z}) = T_0(z,\overline{z}) + tT_1(z,\overline{z})$ for $t \in [0,1]$. According to equation (1), the function $T_1(z,\overline{z})$ is harmonic with respect to z and $\overline{z}$ in the domain D_1. To determine $T_1(z,\overline{z})$, we are faced with an ill-posed problem, where the function $T_1(z,\overline{z})$ is specified only on a part of the boundary.

The first boundary value problem that we need to solve is as follows: Given the value of the holomorphic function $T_1(z,\overline{z})$ on the boundary curve Γ_1 of the domain D_1, it should be possible to extend $T_1(z,\overline{z})$ to the boundary curve $\Gamma'(0)$.

Then we apply the initial boundary conditions on ∂D_1 and the reconstructed values of $T_1(z,\overline{z})$ and find the 2-harmonic function $T_0(z,\overline{z})$ by the formula $T_0(z,\overline{z}) = \mathrm{Re}[\int\int T_1(z,\overline{z}) d\overline{z} dz] + h(z) + \overline{h(z)}$, where $h(z)$ is the analytic in D function reconstructed from the initial boundary conditions. So we construct the linear on t spline $T^1(t,x,y)$ for the function $T(t,x,y)$ in the subdomain D_1.

The second step in our construction is to repeat the procedure of the first step in order to reconstruct the spline $T^2(t,x,y)$, $(t,x,y) \in [0,1] \times D_2(0)$, for the function $T(t,x,y)$ in the subdomain $D_2(t)$. Both of these steps provide us with the values of $T(t,x,y)$ on $[0,1] \times D$.

Finally, we find $\Gamma'(1)$ through the relation on the velocity $v = \theta_1 \partial_\nu T^1(1,x(s),y(s)) - \theta_2 \partial_\nu T^2(1,x(s),y(s))$, $(x(s),y(s)) \in \Gamma'(0)$, of the points of this curve. Since the approximate solution is linear on t for any $(x(s),y(s)) \in \Gamma'(0)$ the shift simply equals $\theta_1 \partial_\nu T^1(1,x(s),y(s)) - \theta_2 \partial_\nu T^2(1,x(s),y(s))$. Note that in general $T^1(1,x(s),y(s)) \neq T^2(1,x(s),y(s))$, $s \in [S_1,S]$. Indeed the temperature $T(1,x(s),y(s))|_{(x(s),y(s)) \in \Gamma'(1)} \equiv \text{const}$. So we again know the boundary data necessary for the construction of the spline for $t \in [1,2]$. We then reconstruct the spline for $t \in [1,2]$, and so on.

Naturally, the linear solution is not smooth on the variable t. In order to overcome this we construct a solution as a polynomial on t spline in each slice $[N-1,N] \times D_i$, $i = 1,2$. Solution then takes the form $T^1(t,z,\overline{z}) = \sum_{j=0}^{2K-1} t^j T_j(z,\overline{z})$, here K is the order of differentiability of the spline. Again, the function $T_{2K-1}(z,\overline{z})$ is harmonic on $z,\overline{z}$. The function $T_{2K-2}(z,\overline{z}) = \mathrm{Re}[\int\int T_{2K-1}(z,\overline{z}) d\overline{z} dz] + h_1(z) + \overline{h_1(z)}$, for an analytic function $h(z)$ is biharmonic on $z,\overline{z}$. Finally, the function $T_0(z,\overline{z})$ is a $2K$-harmonic function. We start our reconstruction of the solution from the function $T_{2K-1}(z,\overline{z})$ and move down to $T_0(z,\overline{z})$ in a way similar to that for the linear spline.

3 Analytic foundation of solution

In order to solve the boundary value problem for $T_1(z,\overline{z})$, we apply the techniques developed in [7, 8].

Let the union of curves $\Gamma_1 \cup \Gamma'(0)$ be parameterized by the arc length $s \in [0,S]$. Assume that $s \in [0,S_1]$ corresponds to the curve Γ_1. The function $f(z(s)) = \psi(s) + \mathrm{i}\psi(s)$ represents the boundary values of the function $f(z)$ analytic in the domain $D_1(0)$ if and only if the following holds true [6, 13]:

$$f(z(s)) = \frac{1}{\pi \mathrm{i}} \int\limits_0^S \frac{f(z(t))z'(t)}{z(t)-z(s)}dt, \quad s \in \partial D_1(0) = z(s),\, s \in [0,S]. \tag{2}$$

Assume that $f(z(s))|_{\Gamma_1} = \psi(s) + \mathrm{i}\psi(s)$, $s \in [0,S_1]$, and $f(z(s))|_{\Gamma'(0)} = \tilde{\psi}(s) + \mathrm{i}\tilde{\chi}(s)$, $s \in [S_1,S]$. Here $\psi(s) = U_1(s) - U_0(s)$, $\psi(s) = \int\limits_0^s V_1(t) - V_0(t)dt$, $s \in [0,S_1]$. In this case we arrive to the integral equation similar to that of [8]:

$$\begin{aligned}\psi(s) + \mathrm{i}\psi(s) = \frac{1}{\pi \mathrm{i}} &\int\limits_{\Gamma_1} \frac{(\psi(t)+\mathrm{i}\psi(t))z'(t)}{z(t)-z(s)}dt \\ &+\frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma'(0)} \frac{(\tilde{\psi}(t)+\mathrm{i}\tilde{\chi}(t))z'(t)}{z(t)-z(s)}dt, \qquad s \in [0,S_1],\end{aligned} \tag{3}$$

$$\begin{aligned}\tilde{\psi}(s) + \mathrm{i}\tilde{\chi}(s) = \frac{1}{\pi \mathrm{i}} &\int\limits_{\Gamma_1} \frac{(\psi(t)+\mathrm{i}\psi(t))z'(t)}{z(t)-z(s)}dt \\ &+\frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma'(0)} \frac{(\tilde{\psi}(t)+\mathrm{i}\tilde{\chi}(t))z'(t)}{z(t)-z(s)}dt, \qquad s \in [S_1,S].\end{aligned} \tag{4}$$

Lemma 1. Solutions $\tilde{\psi}(s)$, $\tilde{\chi}(s)$ of integral equation (3), (4) minimize the functions $\|f(z) - \frac{1}{\pi \mathrm{i}} \int \frac{f(z(s))ds}{z(t)-z(s)}\|_{2,\Gamma_1}$, $\|f(z) - \frac{1}{pii} \int \frac{f(z(s))ds}{z(t)-z(s)}\|_{2,\Gamma'(0)}$, respectively, on the set of all analytic in $D_1(0)$ functions that equal $\psi(s) + \mathrm{i}\psi(s)$ on $\partial D_1(0)$.

Proof. Consider the variation equation $\int\limits_{\partial D_1} |f(z(s))|^2 ds$. Here it turns into the relation

$$\int\limits_{\partial D_1(0)} \left\{ \left(\mathrm{Re}[f(z(t))] - \int\limits_{\Gamma_1} \mathrm{Re}[\frac{(\psi(s)+\mathrm{i}\psi(s))z'(s)}{z(t)-z(s)}]ds + \int\limits_{\Gamma'(0)} \mathrm{Re}[\frac{f(s)z'(s)}{z(t)-z(s)}]ds \right)^2 \right.$$

$$+\left(\mathrm{Im}[f(z(t))]-\int\limits_{\Gamma_1}\mathrm{Im}[\frac{(\psi(s)+\mathrm{i}\psi(s))z'(s)}{z(t)-z(s)}]ds+\int\limits_{\Gamma'(0)}\mathrm{Im}[\frac{f(z(s))z'(s)}{z(t)-z(s)}]ds\right)^2\Bigg\}\,dt.$$

This relation splits into the sum of equations relative to (3) and (4).

We then simply variate this relation with respect to $f(z(s))$ and achieve relations (3) and (4).

□

Hence, we solve the integral equation (4) nd find the function closest to an analytic one on $D_1(0)$ that equals $\psi(s)+\mathrm{i}\psi(s)$ on Γ_1. So the real problem is to find the conditions under which equation (4) is resolvable.

Theorem 1. Assume that the given curve Γ_1 is a simple curve that does not intersect the simple curve $\Gamma'(0)$. Assume also that the functions $U_j(z(s),\overline{z(s)})|_{z(s)\in\Gamma_1}$, $V_j(z(s),\overline{z(s)})|_{z(s)\in\Gamma_1}$, $j=0,1$, meet the Hölder condition. Then we can reconstruct $T_1(z(s),\overline{z(s)})|_{z(s)\in\Gamma'(0)}$ as solution of equation (4).

Proof. Let us solve the second integral equation.

Any curve connecting the endpoints of the curve Γ_1 is longer than the straight line connecting these points. Consider then the case of the line segment.

Assume that we approximate the function $\tilde{\psi}(s)+\mathrm{i}\tilde{\chi}(s)$ by a piece-wise linear function, that is, a linear spline. The approximate calculation of the singular principal value Cauchy integral over the part of the curve relative to Γ_1 is as follows:

$$\frac{1}{\pi\mathrm{i}}\int\limits_{\Gamma_1}\frac{f(e^{\mathrm{i}\theta})\mathrm{i}e^{\mathrm{i}\theta}d\theta}{e^{\mathrm{i}\theta}-e^{\mathrm{i}t_k}},$$

at the intermediate node $t_k=\pi k/(N)$, $k=1,\dots,N$, $p=0,2,\dots,2N$, can be performed by application of the uniform fragmentation of the intervals $[\varphi_{j-1},\varphi_j]$, $j=1,\dots N$. So the matrix of the system corresponding to the straight segment equals

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n}\\ -\frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{n-1}\\ -\frac{1}{3} & -\frac{1}{2} & 0 & \cdots & \frac{1}{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{1}{n} & -\frac{1}{n-1} & \cdots & -\frac{1}{2} & 0 \end{pmatrix}.$$

The norm of the Hilbert matrix then majorizes the norm of this one. We estimated the norm of the matrix in [8].

For any curve different from the straight segment the matrix of the system equals

$$\begin{pmatrix} 0 & \frac{1}{2l_{1,2}} & \frac{1}{3l_{1,3}} & \cdots & \frac{1}{nl_{1,n}} \\ -\frac{1}{2l_{1,2}} & 0 & \frac{1}{2l_{2,3}} & \cdots & \frac{1}{(n-1)l_{2,n-1}} \\ -\frac{1}{3l_{1,3}} & -\frac{1}{2l_{2,3}} & 0 & \cdots & \frac{1}{(n-1)l_{3,n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{nl_{1,n}} & -\frac{1}{n-1,l_{2,n}} & \cdots & -\frac{1}{2l_{n-1,n}} & 0 \end{pmatrix}.$$

Here $l_{i,j}$ is the length of the segment connecting the ith and the jth node divided by the length of the line segment connecting these nodes. Since $l_{i,j} \geq 1$ for all $i \neq j$ the norm of the matrix for any curve is less than or equal to π.

So we can replace either of the singular integral equations (3) or (4) with respect to the values of the unknown functions at the intermediate nodes with the linear system with the block matrix of order N. We investigate the solvability of this finite system for sufficiently large N. Let N tend to infinity. The finite system tends to the infinite one with the matrix $I - \frac{1}{\pi}M$, where $M = [a_{kj}]_{k,j}^{\infty}$, and

$$a_{kj} = \begin{cases} -\ln\left(1 + \frac{2}{2k-2j-1}\right) & \text{if } j < k, \\ \ln\left(1 + \frac{2}{2j-2k-1}\right) & \text{if } j > k, \\ 0 & \text{if } j = k. \end{cases}$$

Here, I is the infinite identity matrix. The L_2-norm of this matrix can be compared to that of the Hilbert matrix.

If M^* is the matrix conjugate of M, then the L_2-norm of the product $M \cdot M^*$ is strictly less than that of the product $H \cdot H^*$, where $H = \left[\frac{1}{j-k}\right]_{j,k=1,j\neq k}^{\infty}$. To show this, it suffices to consider the series representation of the natural logarithm, keeping in mind that $\ln(1+x) \leq x$, and establish the estimates of the matrix M elements by those of H.

Therefore, due to the well-known results [15, Theorem 1], since the L_2-norm of the matrix H equals π, the norm of the matrix M does not exceed π. Thus, the infinite matrix $I - \frac{1}{\pi}M$ is invertible, and the corresponding finite system with respect to $f(e^{\mathrm{i}t_k})$, $k = 1, \ldots, N$, is also resolvable for sufficiently large N.

Finally, we conclude that singular integral equations (3) and (4) can be resolved approximately. The convergence to the exact solution depends first on the accuracy of the approximation of the initial boundary data. Since the boundary data is approximated by the linear spline, the error equals $O\left(\max\left(\omega(\psi, \Delta x)\Delta x, \omega(\psi, \Delta x)\Delta x\right)\right)$ [2].

The second obstacle in our approximation is the inversion of the matrix M of our system. Assume that the maximal subinterval equals Δx. Since the elements of the matrix M differ from the exact values by $O(\Delta x)$, we have $O\left(\max\left\{\Delta x, \Delta\psi\|M^{-1}\|, \Delta\psi\|M^{-1}\|\right\}\right)$ as the final estimate. $\square$

Assume that the time interval step is equal to Δt. Then we have the following result.

Corollary 1. Spline-interpolation solution error estimate is $O(\max\{\Delta x, \omega(\psi, \Delta t)\Delta t, \omega(\psi, \Delta t)\Delta t, \|M^{-1}\|\omega(\psi, \Delta x)\Delta x, \|M^{-1}\|\omega(\psi, \Delta x)\Delta x\})$.

For the variable t we apply the standard convergence estimate of $O(\Delta t^{2K})$ [2].

Note. These constructions hold even in the case of approximation of the function $\tilde{\psi}(s)+\mathrm{i}\tilde{\chi}(s)$ with the linear spline. The approximate calculation of the singular principal value Cauchy integral over the part of the curve relative to $\Gamma'(0)$

$$\frac{1}{\pi\mathrm{i}} \int\limits_{\Gamma'(0)} \frac{f(e^{\mathrm{i}\theta})\mathrm{i}e^{\mathrm{i}\theta}d\theta}{e^{\mathrm{i}\theta} - e^{\mathrm{i}t_k}},$$

at the intermediate node $t_k = \pi k/(N) - \pi/(2N)$, $k = 1, \dots, N$, can be performed by application of the uniform fragmentation of the interval $[\pi, 2\pi]$. So we assume that the subintervals of both Γ_1 and $\Gamma'(0)$ are equally spaced and uniformly divided. We apply the following approximation formula:

$$\begin{aligned}
&\frac{1}{\pi\mathrm{i}} \int\limits_0^\pi \frac{f(e^{\mathrm{i}\theta})\mathrm{i}e^{\mathrm{i}\theta}d\theta}{e^{\mathrm{i}\theta} - e^{\mathrm{i}t_k}} \\
&\approx \frac{1}{\pi\mathrm{i}} \sum_{j=1, j\neq k}^{N} f(\frac{\pi j}{N} - \frac{\pi}{2N}) \ln \frac{e^{\pi\mathrm{i}(2(j-k)+1)/(2N)} - 1}{e^{\pi\mathrm{i}(2(j-k-1)+1)/(2N)} - 1} \\
&\quad + \frac{1}{\pi\mathrm{i}} f(\frac{\pi k}{N} - \frac{\pi}{2N}) \ln \frac{e^{\pi\mathrm{i}/(2N)} - 1}{e^{-\pi\mathrm{i}/(2N)} - 1} \\
&= \frac{1}{\pi\mathrm{i}} \sum_{j=1}^{k-1} f(\frac{j}{N} - \frac{\pi}{2N})[\frac{\pi\mathrm{i}}{2N} + \ln \frac{\sin((2(k-j)-1)\pi/(4N))}{\sin((2(k-j+1)-1)\pi/(4N))}] \\
&\quad + f(\frac{k\pi}{N} - \frac{\pi}{2N})\frac{1}{2N} + \frac{1}{\pi\mathrm{i}} \sum_{j=k+1}^{N} f(\frac{j\pi}{N} - \frac{\pi}{2N})[\frac{\pi\mathrm{i}}{2N} \\
&\quad + \ln \frac{\sin((2(j-k)+1)\pi/(4N))}{\sin((2(j-k-1)+1)\pi/(4N))}].
\end{aligned}$$

There exist singularities in the neighborhood of the meeting points of the curves Γ_1 and $\Gamma'(0)$. The matrix of the relative finite algebraic linear system should be modified so that the approximate solution at these points is continuous.

The second boundary value problem in our construction is the reconstruction of the imaginary part of the analytic function $f(z)$ such that

$$T_0(z,\overline{z}) = \iint T_1(z,\overline{z})\,dz\,d\overline{z} + f(z) + \overline{f(z)},$$

by the value of its real part on the curve $\partial D_1(0) = \Gamma_1 \cup \Gamma'(0)$.

We again consider the necessary and sufficient condition on the function $f(z)$ analyticity:

$$f(z(s)) = \frac{1}{\pi \mathrm{i}} \int_0^S \frac{f(z(t))z'(t)}{z(t)-z(s)}\,dt, \quad s \in [0,S].$$

Assume that $f(z(s)) = U_0(z(s)) + \mathrm{i}v(z(s))$, where $z(s) \in \partial D_1$. Then, by the values of $U_0(z(s))$, $s \in [0,S]$, it should be possible to find $v(s) = v(z(s))$. We reduce this problem to the solution of the linear equation system for the set of coefficients with $\sin(ks)$, $\cos(ks)$, for $k = 1,\dots,N$, as follows:

$$\begin{pmatrix} AA & AB \\ BA & BB \end{pmatrix} \begin{pmatrix} \theta \\ \gamma \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}, \tag{5}$$

here

$$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_M \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{pmatrix}.$$

The vectors

$$F = \begin{pmatrix} f_1 \\ \vdots \\ f_M \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ \vdots \\ g_M \end{pmatrix}.$$

in the right-hand side of this system consist of

$$f_l = \left(-\sum_j \mathrm{Im} z_j^l + \sum_k \mathrm{Im} \hat{z}_k^{-l} \right) \frac{1}{l} + \frac{1}{\pi^2} \int\limits_0^{2\pi} u(\tau)d\tau \int\limits_0^{2\pi} L(\tau,t)\cos(lt)dt,$$

$$g_l = \left(\sum_j \mathrm{Re} z_j^l + \sum_k \mathrm{Re} \hat{z}_k^{-l} \right) \frac{1}{l} + \frac{1}{\pi^2} \int\limits_0^{2\pi} u(\tau)d\tau \int\limits_0^{2\pi} L(\tau,t)\sin(lt)dt,$$

for $l = 1, \ldots, M$. Here

$$K(\tau,t) = \mathrm{Im}\left[\ln \frac{z(\tau)-z(t)}{\mathrm{e}^{\mathrm{i}\tau}-\mathrm{e}^{\mathrm{i}t}}\right]'_{\tau}$$
$$= \mathrm{Im}\left[\ln\left(\sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}kt} \sum_{l=0}^{k-1} \mathrm{e}^{\mathrm{i}l(\tau-t)} - \sum_{j=1}^{m} c_{-j} \mathrm{e}^{-\mathrm{i}j\tau} \sum_{l=0}^{j-1} \mathrm{e}^{\mathrm{i}l(\tau-t)}\right)\right]'_{\tau}, \tag{6}$$

and

$$L(\tau,t) = \mathrm{Re}\left[\ln \frac{z(\tau)-z(t)}{\mathrm{e}^{\mathrm{i}\tau}-\mathrm{e}^{\mathrm{i}t}}\right]'_{\tau}$$
$$= \mathrm{Re}\left[\ln\left(\sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}kt} \sum_{l=0}^{k-1} \mathrm{e}^{\mathrm{i}l(\tau-t)} - \sum_{j=1}^{m} c_{-j} \mathrm{e}^{-\mathrm{i}j\tau} \sum_{l=0}^{j-1} \mathrm{e}^{\mathrm{i}l(\tau-t)}\right)\right]'_{\tau}. \tag{7}$$

Block-matrices of M

$$AA, AB, BA, BB$$

consist of

$$AA = \left(\delta_{ln} - \frac{1}{\pi^2}\int_0^{2\pi} \cos(n\tau)d\tau \int_0^{2\pi} K(\tau,t)\cos(lt)dt\right)_{l,n=1}^{M},$$

$$AB = \left(-\frac{1}{\pi^2}\int_0^{2\pi} \sin(n\tau)d\tau \int_0^{2\pi} K(\tau,t)\cos(lt)dt\right)_{l,n=1}^{M},$$

$$BA = \left(-\frac{1}{\pi^2}\int_0^{2\pi} \cos(n\tau)d\tau \int_0^{2\pi} K(\tau,t)\sin(lt)dt\right)_{l,n=1}^{M},$$

$$BB = \left(\delta_{ln} - \frac{1}{\pi^2}\int_0^{2\pi} \sin(n\tau)d\tau \int_0^{2\pi} K(\tau,t)\sin(lt)dt\right)_{l,n=1}^{M},$$

here δ_{ln} is the Kronecker delta-function. The convergence rate of this part of the solution is $O(\frac{1}{M^2})$, and M equals the size of the block in the matrix of the system. Note that for contours with singularities or for boundaries of thin domains this estimate is not really effective and requires certain additional work [7].

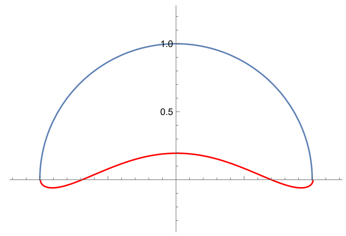

Figure 2: Domain of Example 1

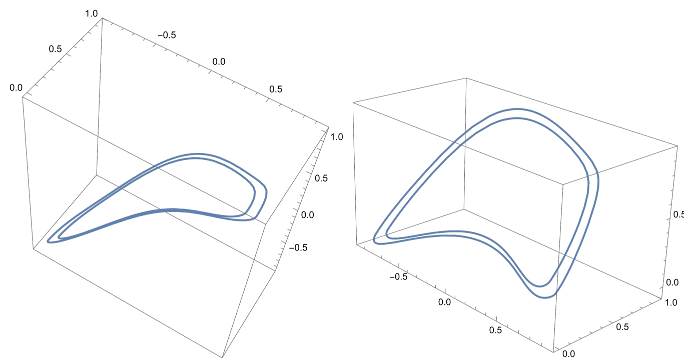

Figure 3: Real and imaginary parts of the function $T_1(z, \overline{z})$

4 Examples

Here we present the reconstruction of the functions $T_1(z, \overline{z})$, $T_0(z, \overline{z})$, that is, the principle part of the approximate solution construction.

Example 1. Consider the upper half of the unit circle as Γ_1 and the curve given by the equation $(\cos(t), 0.1\sin(3t) - 0.1\sin(t))$, $t \in [\pi, 2\pi]$ as $\Gamma'(0)$ (see Figure 2). Note that the domain does not contain the coordinate origin. So the reconstructed function may have a pole at 0. The boundary values of T_1 are given as $\cos(t) + \mathrm{i}(\sin(t) + 0.15\sin(3t) + 0.2\sin(4t))$.

The real and imaginary parts of the reconstructed function $T_1(x, y)$ are given in Figure 3. The surface of the reconstructed function T_0 is at Figure 4.

Let us now consider the example with more acute angle between the initial free and the fixed boundaries.

Example 2. Consider the upper half of the unit circle as Γ_1 and the curve given by the equation $(\cos(t), 0.1\sin(t) + 0.1\sin(2t))$, $t \in [\pi, 2\pi]$ as $\Gamma'(0)$ (see Figure 5). The boundary values of T_1 are given as $\cos(t) + \mathrm{i}(\sin(t) + 0.15\sin(3t) + 0.2\sin(4t))$

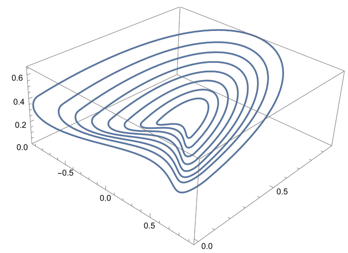

Figure 4: Real part of the function $T_0(z,\overline{z})$

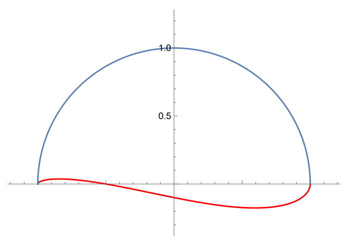

Figure 5: Domain of Example 2

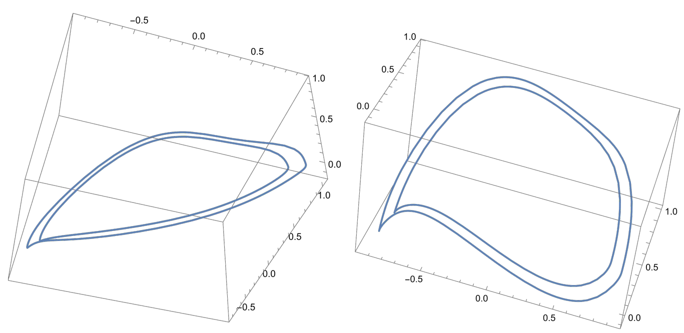

Figure 6: Real and imaginary parts of the function $T_1(z,\overline{z})$

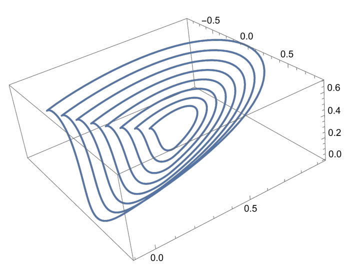

Figure 7: Real part of the function $T_0(z,\overline{z})$

The real and imaginary parts of the reconstructed function $T_1(x, y)$ are given on Figure 6. The surface of the reconstructed function T_0 is at Figure 7.

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Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Apushkinskaya, D.E. and Lazareva, G.G. *Algorithm for the numerical solution of the Stefan problem and its application to calculations of the temperature of Tungsten under impulse action*, Contemp. Math. Fundam. Dir., 67(3) (2021), 442–454.

[2] de Boor, C. *A practical guide to splines.* Springer, 2001.

[3] Gurkov, Y.V. and Petrova, A.G. *Numerical solution of the stefan problem with a variable phase-transition temperature*, J. Appl. Mech. Tech. Phys., 37 (1996), 547–553.

[4] Kinderlehrer, D. and Stampacchia, G. *An introduction to variational inequalities and their applications.* SIAM, 2000.

[5] Meirmanov, A.M. *The Stefan problem*, Berlin, New York, De Gruyter, 1992.

[6] Gakhov, F.D. *Boundary value problems*, Pergamon Press, Oxford, 1966.

[7] Ivanshin, P.N. and Shirokova, E.A. *The approximate conformal mapping of a disk onto domain with an acute angle*, Int. J. Appl. Comput. Math. 9(5), (2023), 54

[8] Shirokova, E.A. and Ivanshin, P.N. *On Cauchy problem solution for a harmonic function in a simply connected domain*, Issues Anal., 12(30) (2023), 87–96.

[9] Khasanov, A.B. and Tursunov, F.R. *Of Cauchy problem for Laplace equation*, Ufa Math. J. 11(4) (2019), 91–107.

[10] Khasanov, A.B. and Tursunov, F.R. *On Cauchy problem for the three-dimensional Laplace equation*, Rus. Math., 65(2) (2021), 49–64.

[11] Lavrent'ev, M.M., Romanov, V.G. and Shishatskiy, S.P. *Ill-posed problems of mathematical physics and analysis*, Transl. of Math. Monographs, Amer. Math. Soc., 1986.

[12] Markushevich, A.I. *Theory of analytic functions*, GTTL, Moscow, 1950.

[13] Muskhelishvili, N. I. *Singular integral equations*, Springer, 1958.

[14] Saini, S., Das, P. and Kumar, S. *Computational cost reduction for coupled system of multiple scale reaction diffusion problems with mixed type boundary conditions having boundary layers*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 117(2) (2023) 66.

[15] Schur, I. *Bemerkungen zur Theorie der Beschraenkten Bilinearformen mit unendlich vielen Veraenderlichen*, J. Reine Angew. Math. 140 (1911) 1–28.

[16] Shiromani, R., Shanthi, V. and Das, P. *A higher order hybrid-numerical approximation for a class of singularly perturbed two-dimensional convection-diffusion elliptic problem with non-smooth convection and source terms*, Comput. Math. Appl., 142 (2023), 9–30.

[17] Shlapunov, A.A. *Iterations of self-adjoint operators and their applications to elliptic systems*, Math. Nachrichten, 218 (2000), 165–174.

[18] Tarkhanov, N. *The Cauchy problem for solutions of elliptic equations*, Akademie-Verlag, Berlin, 1995.