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Research Article

Optimal control and a system of quasi-variational inequalities

J. Ogorzaly*

Abstract

We consider the system of two quasi-variational inequalities. First, we show a convergence result with respect the set of constraints. Next, we formulate optimal control problem for which we prove the existence of optimal solution, together with convergence of optimal control pairs. Finally, we apply our results in the study of contact problem with thermal effect and friction.

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1 Introduction

The theory of optimal control has its origins in the calculus of variations in the 17th century, and it was developed to find the optimal ways to control a dynamical system. The main interest of this theory is the existence and, if it is possible, the uniqueness of optimal state-control pair,

*Corresponding author

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Justyna Ogorzaly
Institute of Mathematics and Cryptology, Military University of Technology, ul. gen. S. Kaliskiego 2, 00-908 Warsaw, Poland, e-mail: justyna.ogorzaly@gmail.com

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as well as the derivation of necessary (or necessary and sufficient) conditions of optimality. The key development of this theory had place in 20th century by works of Bellman, Potryagin, and Kalman. Nowadays, the theory of optimal control has many applications in such areas as, for example, aerospace, process control, robotics, bioengineering, and economics.

In the literature, there are some results related to the optimal control of quasi-variational inequalities of different types. Here, we briefly present a few examples of such works. Sofonea and Tarzia [23] considered an optimal control problem governed by an elliptic quasi-variational inequality with unilateral constraints. The authors defined a new optimal control problem by perturbing the state inequality and the cost functional. Then, they provided sufficient conditions that guarantee the convergence of solutions. The abstract results were used in the study of two models, namely, the static contact problem with normal compliance and unilateral constraint, and a stationary heat transfer problem with unilateral constraints. Khan and Sama [10] studied the optimal control of quasi-variational inequalities with multivalued monotone and pseudomonotone maps. Moreover, they analyzed the convergence behavior of the optimal control problem when the data defining the underlying quasi-variational inequality is contaminated by a certain level of noise. The optimal control problem presented in this paper subsumes some important optimal control problems as particular cases, for example, the optimal control problems for single-valued quasi-variational inequalities, the set-valued/single-valued variational inequalities.

The paper [21] was motivated by mathematical models that describe the time-dependent unilateral contact of a deformable body with a foundation. Taking into account appropriate mechanical assumptions on the constitutive law and the interface conditions, these models lead to a weak formulation in the form of a system that couples an ordinary differential equation with a variational or quasi-variational inequality. The authors proved the continuous dependence of the solution with respect to data, as well as the existence of at least one optimal pair for an associated control problem. Finally, they illustrated the abstract results in the study of a free boundary problem which describes the equilibrium of a viscoelastic body in frictionless contact with a foundation made of a rigid body covered by a rigid-elastic layer. The applications of optimal control theory to variational inequalities can be found in [4, 8, 14, 19, 24, 25, 26].

The system of variational inequalities extends the concept of a variational inequality to multiple interrelated inequalities involving multiple variables. These kind of systems naturally arise in the models where a few processes interact under constraints, and they affect on each another (see, e.g., [16]). The systems of variational inequalities are used to the model an equilibrium problem, for example, the Nash equilibrium problem in the game theory [1]. They are finding applications in fields such as traffic, spatial price equilibrium, mechanics, signal processing, networks, and optimization. This paper extends existing results for optimal control of elliptic quasi-variational inequality to optimal control of the system of elliptic quasi-variational inequalities. Note that Problem 1 contains several special cases: coupled variational inequalities such that K_1 and K_2 do not vary with the solution (u,θ) and $C_1 = K_1$, $C_2 = K_2$ (see [13]), an elliptic variational inequality of the first kind ($A_2 = 0$, $N_1 = 0$, $N_2 = 0$, $\varphi_2 = 0$, K_1 is independent of u, $g = 0$) and of the second kind ($K_1 = V$); see [9, 12]. Our work inspires future research into optimal control problems for the systems of quasi-variational inequalities of different kinds. To the best of our knowledge, these problems have not been considered in the literature until now.

The paper is organized as follows. In Section 2, we present some notations, and in Section 3, we show the existence and uniqueness result for the system of quasi-variational inequalities as well as the boundedness of the solution. In Section 4, we will show convergence result for the system of two quasi-variational inequalities with respect the set of constraints. Section 5 is devoted to a class of optimal control problem associated to the system of quasi-variational inequalities and existence of optimal pairs. In Section 6, we prove the convergence of the optimal pairs. Finally, in Section 7, we will use abstract results in the study of contact problem with thermal effect.

2 Preliminary

In this section, we collect a basic material that is useful in our paper (see, e.g., [5, 6, 7, 11, 18]. For a Banach space $(X, \|\cdot\|_X)$, we denote by $(X^*, \|\cdot\|_{X^*})$ its dual space and by $\langle \cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. For a Hilbert space Y, we denote the inner product by $(\cdot,\cdot)_Y$. The notation X_w stands for a Banach space X endowed with the weak topology. For a multivalued operator $T\colon X \to 2^{X^*}$, its graph $\mathrm{Gr}(T)$ is defined by $\mathrm{Gr}(T) = \{(x,x^*) \in X \times X^* \mid x^* \in Tx\}$. An operator $T\colon X \to 2^{X^*}$ is called monotone when $\langle u^* - v^*, u - v\rangle_{X^*\times X} \geqslant 0$ for all (u,u^*), $(v,v^*) \in \mathrm{Gr}(T)$. Let $\mathbb{R}_+ = [0,+\infty)$. We say that T is u_0-coercive if for $u_0 \in X$, there exists a function $\theta\colon \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{r\to+\infty}\theta(r) = +\infty$ such that for all $(u,u^*) \in \mathrm{Gr}(T)$, we have $\langle u^*, u - u_0\rangle_{X^*\times X} \geqslant \theta(\|u\|_X)\,\|u\|_X$. An operator T is bounded if T maps bounded sets of X into bounded sets of X^*. A single-valued operator $A\colon X \to X^*$ is called pseudomonotone if it is bounded and $u_n \to u$ in X_w with $\limsup\langle Au_n, u_n - u\rangle_{X^*\times X} \leqslant 0$ implies $\langle Au, u - v\rangle_{X^*\times X} \leqslant \liminf\langle Au_n, u_n - v\rangle_{X^*\times X}$ for all $v \in X$. If X is a reflexive Banach space, then $A\colon X \to X^*$ is pseudomonotone, if and only if it is bounded and $u_n \to u$ in X_w with $\limsup\,\langle Au_n, u_n - u\rangle_{X^*\times X} \leqslant 0$ entail $\lim\,\langle Au_n, u_n - u\rangle_{X^*\times X} = 0$ and $Au_n \to Au$ in X^*_w. A map $K\colon E \to F$ between Banach spaces E and F is completely continuous if for any sequence $\{x_n\} \subset E$ converging weakly to $x \in E$, we have $K(x_n) \to K(x)$ in F. Let Z be a normed space, and let C, $\{C_n\}$ be closed and convex subsets of Z. The sequence $\{C_n\}$ Mosco converges to C, denoted by $C_n \xrightarrow{M} C$ as $n \to \infty$, if we have

- for any $z_n \in C_n$ with $z_n \to z$ in Z_w, up to a subsequence, we have $z \in C$,
- for any $z \in C$, there exists $z_n \in C_n$ with $z_n \to z$ in Z.

3 Solvability of a system of quasi-variational inequalities

Let V and E be reflexive Banach, and let Y_1, Y_2 be the real Hilbert spaces. Let $C_1 \subset V$ and $C_2 \subset E$ be given subsets, and let $A_1 \colon V \to V^*$, $N_1 \colon E \to V^*$, $N_2 \colon V \to E^*$, $A_2 \colon E \to E^*$ be given operators. We have also given functions $\varphi_1 \colon V \times E \times V \to \mathbb{R}$, $\varphi_2 \colon V \times E \times E \to \mathbb{R}$, set-valued maps $K_1 \colon C_1 \times C_2 \to 2^{C_1} \setminus \{\emptyset\}$, $K_2 \colon C_1 \times C_2 \to 2^{C_2} \setminus \{\emptyset\}$, functions $f \in Y_1, g \in Y_2$, and operators $\pi_1 \colon V \to Y_1$ and $\pi_2 \colon E \to Y_2$.

The problem is as follows.

Problem 1. Find $(u, \theta) \in C_1 \times C_2$ such that $u \in K_1(u, \theta)$, $\theta \in K_2(u, \theta)$ and for all $v \in K_1(u, \theta), \eta \in K_2(u, \theta)$ hold

$$\begin{aligned} &\langle A_1 u + N_1 \theta, v - u \rangle_{V^* \times V} + \varphi_1(u, \theta, v) - \varphi_1(u, \theta, u) \geqslant (f, \pi_1 v - \pi_2 u)_{Y_1}, \\ &\langle A_2 \theta + N_2 u, \eta - \theta \rangle_{E^* \times E} + \varphi_2(u, \theta, \eta) - \varphi_2(u, \theta, \theta) \geqslant (g, \pi_2 \eta - \pi_2 \theta)_{Y_2}. \end{aligned} \tag{1}$$

The hypotheses on the data of Problem 1 are the following:

$$C_1 \subset V, \; C_2 \subset E \quad \text{are nonempty, closed, and convex sets,} \tag{2}$$

$$\left.\begin{aligned} &K_1 \colon C_1 \times C_2 \longrightarrow 2^{C_1} \backslash \{\emptyset\} \text{ is such that} \\ &\text{(a)} \;\; K_1 \text{ has closed convex values in } C_1, \\ &\text{(b)} \;\; K_1 \text{ is weakly Mosco continuous, that is, for any } \{(u_n, \theta_n)\} \in C_1 \times C_2 \\ &\text{such that } (u_n, \theta_n) \longrightarrow (u, \theta) \text{ in } (V \times E)_w, \text{ one has } K_1(u_n, \theta_n) \xrightarrow{M} K_1(u, \theta), \\ &\text{(c)} \;\; \text{there is a bounded set } K_0 \subset C_1 \text{ such that } K_1(u, \theta) \cap K_0 \neq \emptyset \\ &\quad \text{for all } (u, \theta) \in C_1 \times C_2, \end{aligned}\right. \tag{3}$$

$$\left.\begin{aligned} &K_2 \colon C_1 \times C_2 \longrightarrow 2^{C_2} \backslash \{\emptyset\} \text{ is such that} \\ &\text{(a)} \;\; K_2 \text{ has closed convex values in } C_2, \\ &\text{(b)} \;\; K_2 \text{ is weakly Mosco continuous, that is, for any } \{(u_n, \theta_n)\} \in C_1 \times C_2 \\ &\text{such that } (u_n, \theta_n) \longrightarrow (u, \theta) \text{ in } (V \times E)_w, \text{ one has } K_2(u_n, \theta_n) \xrightarrow{M} K_2(u, \theta), \\ &\text{(c)} \;\; \text{there is a bounded set } K_0^{'} \subset C_2 \text{ such that } K_2(u, \theta) \cap K_0^{'} \neq \emptyset \\ &\quad \text{for all } (u, \theta) \in C_1 \times C_2, \end{aligned}\right. \tag{4}$$

$$
\begin{aligned}
&A_1\colon V \longrightarrow V^* \text{ and } A_2\colon E \longrightarrow E^* \text{ are such that}\\
&\quad \text{(a) } \|A_1 v\|_{V^*} \leqslant a_1 + a_2\|v\|_V \text{ for all } v \in V \text{ with } a_1, a_2 \geqslant 0,\\
&\quad \text{(b) } \|A_2 \eta\|_{E^*} \leqslant b_1 + b_2\|\eta\|_E \text{ for all } \eta \in E \text{ with } b_1, b_2 \geqslant 0,\\
&\quad \text{(c) } A_1, A_2 \text{ are pseudomonotone operators,}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&N_1\colon E \longrightarrow V^*, N_2\colon V \longrightarrow E^* \text{ are such that}\\
&\quad \text{(a) } N_1, N_2 \text{ are completely continuous,}\\
&\quad \text{(b) } \|N_1\theta\|_{V^*} \leqslant m_1(1 + \|\theta\|_E) \text{ for all } \theta \in E,\ m_1 > 0,\\
&\quad \text{(c) } \|N_2 v\|_{E^*} \leqslant m_2(1 + \|v\|_E) \text{ for all } v \in V,\ m_2 > 0,
\end{aligned}
\tag{6}
$$

$$
f \in Y_1, \quad g \in Y_2,
\tag{7}
$$

$$
\begin{aligned}
&\pi_1\colon V \to Y_1,\ \ \pi_2\colon E \to Y_2 \ \text{ are linear continuous operators such that}\\
&\|\pi_1 v\|_{Y_1} \leqslant d_1\|v\|_V \text{ for every } v \in V \text{ with } d_1 > 0,\\
&\|\pi_2 \theta\|_{Y_2} \leqslant d_2\|\theta\|_E \text{ for every } \theta \in E \text{ with } d_2 > 0,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&\varphi_1\colon V \times E \times V \to \mathbb{R} \text{ is such that}\\
&\quad \text{(a) } \varphi_1(w, \theta, \cdot) \text{ is convex and lower semicontinuous for all } w \in V, \theta \in E,\\
&\quad \text{(b) there exists a function } c_{\varphi_1}\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ bounded on bounded sets}\\
&\qquad \text{such that } \varphi_1(w, \theta, v_1) - \varphi_1(w, \theta, v_2) \leqslant c_{\varphi_1}(\|w\|_V, \|\theta\|_E)\|v_1 - v_2\|_V\\
&\qquad \text{for all } w, v_1, v_2 \in V,\ \theta \in E,\\
&\quad \text{(c) for all } \{w_n\} \subset V, \{\theta_n\} \subset E, \{v_n\} \subset V, \{u_n\} \subset V\\
&\qquad \text{such that } w_n \to w \text{ in } V_w, \theta_n \to \theta \text{ in } E_w, v_n \to v \text{ in } V, u_n \to u \text{ in } V_w\\
&\qquad \text{for some } w, u, v \in V,\ \theta \in E,\ \text{it holds}\\
&\qquad\quad \limsup \big(\varphi_1(w_n, \theta_n, v_n) - \varphi_1(w_n, \theta_n, u_n)\big) \leqslant \varphi_1(w, \theta, v) - \varphi_1(w, \theta, u),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\varphi_2\colon V \times E \times E \to \mathbb{R} \text{ is such that}\\
&\quad \text{(a) } \varphi_2(w, \theta, \cdot) \text{ is convex and lower semicontinuous for all } w \in V,\ \theta \in E,\\
&\quad \text{(b) there exists a function } c_{\varphi_2}\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ bounded on bounded sets}\\
&\qquad \text{such that } \varphi_2(w, \theta, \eta_1) - \varphi_2(w, \theta, \eta_2) \leqslant c_{\varphi_2}(\|w\|_V, \|\theta\|_E)\|\eta_1 - \eta_2\|_E\\
&\qquad \text{for all } w \in V,\ \theta, \eta_1, \eta_2 \in E,\\
&\quad \text{(c) for all } \{w_n\} \subset V, \{\theta_n\} \subset E, \{\eta_n\} \subset E, \{\zeta_n\} \subset E \text{ such that}\\
&\qquad\quad w_n \to w \text{ in } V_w, \theta_n \to \theta \text{ in } E_w, \eta_n \to \eta \text{ in } E, \zeta_n \to \zeta \text{ in } E_w\\
&\qquad\quad \text{for some } w \in V,\ \theta, \eta, \zeta \in E,\ \text{it holds}\\
&\qquad\quad \limsup \big(\varphi_2(w_n, \theta_n, \eta_n) - \varphi_2(w_n, \theta_n, \zeta_n)\big) \leqslant \varphi_2(w, \theta, \eta) - \varphi_2(w, \theta, \zeta).
\end{aligned}
\tag{10}
$$

Moreover, we assume that

$$
\begin{aligned}
&(a)\quad \text{there exist } u_0 \in K_0 \text{ and a function } \theta_1\colon \mathbb{R}_+ \to \mathbb{R} \text{ with } \theta_1(r) \to +\infty \\
&\qquad \text{as } r \to +\infty \text{ such that for all } v \in V, \text{ it holds } \langle A_1 v, v-u_0\rangle \geqslant \theta_1(\|v\|)\|v\|, \\
&(b)\quad \text{there exist } u_1 \in K_0' \text{ and a function } \theta_2\colon \mathbb{R}_+ \to \mathbb{R} \text{ with } \theta_2(r) \to +\infty \\
&\qquad \text{as } r \to +\infty \text{ such that for all } \eta \in E, \text{ it holds} \\
&\qquad \langle A_2\eta, \eta - u_1\rangle_{E^*\times E} \geqslant \theta_2(\|\eta\|_E)\|\eta\|_E,
\end{aligned}
\tag{11}
$$

$$
V \ni v \mapsto A_1 v \in 2^{V^*} \text{ and } E \ni \eta \mapsto A_2\eta \in 2^{E^*} \text{ are a monotone map.} \tag{12}
$$

By hypothesis (11) and (5), it is clear (see [16, Section 3.4]) that A_1 and A_2 are pseudomonotone and bounded as a multivalued operators from V to 2^{V^*} and E to 2^{E^*}, respectively.

Theorem 1. Under the assumptions (2)–(12), Problem 1 has a solution.

Proof. The proof of Theorem 1 is a simplified version of the proof in [17, Theorem 10]. However, for convenience of the reader, we briefly recall the main steps of this proof. The important tool in the proof is the following version of Kakutani–Ky Fan fixed point theorem in a reflexive Banach space (see, e.g., [6, Corollary 1.7.42]). □

Theorem 2 (Kakutani-Ky Fan)**.** Let Y be a reflexive Banach space, and let $D \subseteq Y$ be a nonempty, bounded, closed, and convex set. Suppose that $\Lambda\colon D \to 2^D$ is a set-valued map such that the values of Λ are nonempty and convex sets, and $\operatorname{Gr}\Lambda$ is sequentially closed in $Y_w \times Y_w$. Then Λ has a fixed point: $y^* \in D$ such that $y^* \in \Lambda y^*$.

Remark 1. In Theorem 2, the sequential closedness of the graph of Λ in $Y_w \times Y_w$ implies that the values of Λ are closed subsets of Y_w. By hypothesis, the values are convex sets. Hence and Mazur's theorem, we deduce that the values of Λ are also closed sets in Y.

Let $\langle \overline{f}, v\rangle_{V^*\times V} = (f, \pi_1 v)_{Y_1}$ and $\langle \overline{g}, \eta\rangle_{E^*\times E} = (g, \pi_2\eta)_{Y_2}$ for every $v \in V, \eta \in E$. Moreover, let $(w,\xi) \in C_1 \times C_2 \subset V \times E$ be fixed. Consider the following auxiliary problem:

Problem 2. Find $(u,\theta) \in K_1(w,\xi) \times K_2(w,\xi)$ such that

$$
\begin{aligned}
&\langle A_1 u + N_1\xi - \overline{f}, v-u\rangle_{V^*\times V} + \varphi(w,\xi,v) - \varphi(w,\xi,u) \geqslant 0 \\
&\qquad\qquad\qquad\qquad\qquad \text{for all } \ v \in K_1(w,\xi), \\
&\langle A_2\theta + N_2 w - \overline{g}, \eta-\theta\rangle_{E^*\times E} + \psi(w,\xi,\eta) - \psi(w,\xi,\theta) \geqslant 0 \\
&\qquad\qquad\qquad\qquad\qquad \text{for all } \ \eta \in K_2(w,\xi).
\end{aligned}
\tag{13}
$$

We define a set-valued map $S\colon C_1 \times C_2 \to 2^{C_1\times C_2}$ by

$$S(w,\xi) := \{(u,\theta) \in K_1(w,\xi) \times K_2(w,\xi) \mid (u,\theta) \text{ is a solution to Problem 2}\}. \tag{14}$$

The system (13) is decoupled, therefore, we shall examine each inequality separately. Let $D := \{\,(w,\xi) \in C_1 \times C_2 \mid \|w\|_V \leqslant r,\ \|\xi\|_E \leqslant r\,\}$ with a suitable positive constant r. It is obvious that D is a nonempty, closed, convex, and bounded subset of $V \times E$. Let $\Lambda\colon D \to 2^D$ be given by $\Lambda(w,\xi) := S(w,\xi)$ for $(w,\xi) \in D$. In order to apply Theorem 2 with a reflexive Banach space $Y := V \times E$, we shall verify

- for each $(w,\xi) \in C_1 \times C_2$, $S(w,\xi)$ defined by (14) is nonempty and convex,
- $\Lambda(D) \subset D$, that is, there exists $r > 0$ for which Λ maps D into itself,
- $\operatorname{Gr}\Lambda$ is sequentially closed in $(V \times E)_w \times (V \times E)_w$.

The details of the proofs of these properties are similar as in [17, Theorem 10], so we skip them here. For the second inequality in Problem 1, we use the same arguments as above, and we obtain that $\theta \in K_2(w,\xi)$ satisfies the limit inequality. Finally, we infer that $(u,\theta) \in S(w,\xi)$, which completes the proof of the closedness of the graph of Λ.

Having verified the hypotheses of Theorem 2, we deduce from it that there exists a fixed point $(u^*,\theta^*) \in D$ of Λ, that is, $(u^*,\theta^*) \in \Lambda(u^*,\theta^*) = S(u^*,\theta^*)$. In conclusion, $(u^*,\theta^*) \in K_1(u^*,\theta^*) \times K_2(u^*,\theta^*)$ is a solution to Problem 1.

In order to prove the uniqueness of the solution of Problem 1, we need the following additional assumptions:

$$\left.\begin{array}{l}
\text{(a)}\ \ \langle A_1u_1 - A_1u_2, u_1 - u_2\rangle_{V^*\times V} \geqslant m_{A_1}\|u_1 - u_2\|_V^2 \text{ with } m_{A_1} > 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } u_1, u_2 \in V, \\
\text{(b)}\ \ \langle N_2\theta_1 - N_2\theta_2, \theta_1 - \theta_2\rangle_{E^*\times E} \geqslant m_{N_2}\|\theta_1 - \theta_2\|_E^2 \text{ with } m_{N_2} > 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \theta_1, \theta_2 \in E, \\
\text{(c)}\ \ \|N_1\theta_1 - N_1\theta_2\|_E \leqslant L_{N_1}\|\theta_1 - \theta_2\|_E \text{ with } L_{N_1} > 0,\ \text{for all } \theta_1, \theta_2 \in E, \\
\text{(d)}\ \ \|A_2u_1 - A_2u_2\|_V \leqslant L_{A_2}\|u_1 - u_2\|_V \text{ with } L_{A_2} > 0,\ \text{for all } u_1, u_2 \in V, \\
\text{(e)}\ \ \varphi_1(u_1,\theta_1,u_2) - \varphi_1(u_1,\theta_1,u_1) + \varphi_1(u_2,\theta_2,u_1) - \varphi_1(u_2,\theta_2,u_2) \\
\qquad \leqslant \theta_{\varphi_1}(\|u_1 - u_2\|_V + \|\theta_1 - \theta_2\|_E)\|u_1 - u_2\|_V \\
\quad \text{with } \theta_{\varphi_1} > 0,\ \text{for all } u_1, u_2 \in V \text{ and } \theta_1, \theta_2 \in E, \\
\text{(f)}\ \ \varphi_2(u_1,\theta_1,\theta_2) - \varphi_2(u_1,\theta_1,\theta_1) + \varphi_2(u_2,\theta_2,\theta_1) - \varphi_2(u_2,\theta_2,\theta_2) \\
\qquad \leqslant \theta_{\varphi_2}(\|u_1 - u_2\|_V + \|\theta_1 - \theta_2\|_E)\|\theta_1 - \theta_2\|_E \\
\quad \text{with } \theta_{\varphi_2} > 0,\ \text{for all } u_1, u_2 \in V \text{ and } \theta_1, \theta_2 \in E.
\end{array}\right\} \tag{15}$$

Theorem 3. If (2)–(12), (15) hold and

$$m_{A_1} > \theta_{\varphi_1}, \quad m_{N_2} > \theta_{\varphi_2}, \quad \frac{(L_{N_1}+\theta_{\varphi_1})(L_{A_2}+\theta_{\varphi_2})}{(m_{A_1}-\theta_{\varphi_1})(m_{N_2}-\theta_{\varphi_2})} < 1, \tag{16}$$

then Problem 1 has a unique solution.

Proof. Let $(u_1,\theta_1),(u_2,\theta_2) \in C_1 \times C_2$ be such that $u_1,u_2 \in K_1(u,\theta)$ and $\theta_1,\theta_2 \in K_2(u,\theta)$. We take the first inequality in (1) with $v = u_2$ and $v = u_1$, respectively. Hence, we have

$$\begin{aligned}
&\langle A_1 u_1 + N_1\theta_1, u_2 - u_1\rangle_{V^*\times V} + \varphi_1(u_1,\theta_1,u_2) - \varphi_1(u_1,\theta_1,u_1)\\
&\geqslant (f, \pi_1 u_2 - \pi_1 u_1)_{Y_1},\\
&\langle A_1 u_2 + N_1\theta_2, u_1 - u_2\rangle_{V^*\times V} + \varphi_1(u_2,\theta_2,u_1) - \varphi_1(u_2,\theta_2,u_2)\\
&\geqslant (f, \pi_2 u_1 - \pi_2 u_2)_{Y_1}.
\end{aligned}$$

Adding the inequalities above and using (15)(a), (c), (e), we obtain

$$(m_{A_1}-\theta_{\varphi_1})\|u_1-u_2\|_V \leqslant (L_{N_1}+\theta_{\varphi_1})\|\theta_1-\theta_2\|_E. \tag{17}$$

Taking $\eta = \theta_1$ and $\eta = \theta_2$ in the second inequality in (1) and using (15)(b), (d), (f), we have

$$(m_{N_2}-\theta_{\varphi_2})\|\theta_1-\theta_2\|_E \leqslant (L_{A_2}+\theta_{\varphi_2})\|u_1-u_2\|_V. \tag{18}$$

From (17) and (18), we obtain

$$\|u_1-u_2\|_V \leqslant \frac{(L_{N_1}+\theta_{\varphi_1})(L_{A_2}+\theta_{\varphi_2})}{(m_{A_1}-\theta_{\varphi_1})(m_{N_2}-\theta\varphi_2)}\|u_1-u_2\|_V. \tag{19}$$

Combining inequality (19) with the smallness condition (16), we conclude that $u_1 = u_2$. Then, from the inequality (18) we deduce that $\theta_1 = \theta_2$. Consequently, Problem 1 has a unique solution. □

Remark 2. Removing the smallness assumption (16) remains a task for future research, since it is made for mathematical reasons, and does not seem to relate to any inherent physical constraints of the contact problem considered in Section 7.

Finally, we present the result we will use in the convergence for optimal pairs. The assumption,s which we need in this step, are as follows:

$$\begin{aligned}
&\text{(a)}\quad \mathbf{0}_{\mathrm{V}} \in \mathrm{K}_1(\mathrm{u},\theta),\ \ \mathbf{0}_{\mathrm{E}} \in \mathrm{K}_2(\mathrm{u},\theta),\\
&\text{(b)}\quad \langle \mathrm{A}_2\theta_1 - \mathrm{A}_2\theta_2, \theta_1-\theta_2\rangle_{\mathrm{E}^*\times \mathrm{E}} \geqslant \mathrm{m}_{\mathrm{A}_2}\|\theta_1-\theta_2\|^2_{\mathrm{E}} \text{ with } \mathrm{m}_{\mathrm{A}_2} > 0\\
&\qquad \text{for all } \theta_1,\theta_2 \in E,\\
&\text{(c)}\quad \varphi_1 \colon \mathrm{V}\times\mathrm{E}\times\mathrm{V} \to \mathbb{R} \text{ is such that for all } \mathrm{u},\mathrm{v}\in\mathrm{V},\ \theta\in\mathrm{E}, \text{ and for } \lambda>0,\\
&\qquad \text{we have } \varphi_1(u,\theta,\lambda v) = \lambda\varphi_1(u,\theta,v),\ \varphi_1(v,\theta,v)\geqslant 0.\\
&\qquad \text{By analogy we make assumptions for the function } \varphi_2.
\end{aligned} \tag{20}$$

The result is as follows.

Lemma 1. Let $f \in Y_1$ and $g \in Y_2$ be the given functions, let the assumptions (6)(b)-(c), (15)(a), (8) hold, and let

$$m_{A_1} m_{A_2} > m_2 m_1. \tag{21}$$

Then the solutions $u = u(f,g)$ and $\theta = \theta(f,g)$ of the system (1) satisfy the bounds

$$\|u\|_V \leqslant c_1 \quad \text{and} \quad \|\theta\|_E \leqslant c_2, \tag{22}$$

respectively, where c_1, c_2 are constants.

Proof. First, we use (20)(a), and we take $v = \mathbf{0}_V \in K_1(u,\theta)$ in the first inequality in the system (1). Using assumptions (6)(b), (15)(a), (8), and (20), we infer that

$$\|u\|_V \leqslant \frac{1}{m_{A_1}}\big(\|A_1\mathbf{0}_V\|_{V^*} + m_1(1+\|\theta\|_E) + d_1\|f\|_{Y_1}\big). \tag{23}$$

Next, we use (20)(a) and we take $\mathbf{0}_E \in K_2(u,\theta)$ in the second inequality in (1). Using (6)(c), (8) and (20) in the obtained inequality, we get

$$\|\theta\|_E \leqslant \frac{1}{m_{A_2}}\big(\|A_2\mathbf{0}_E\|_{E^*} + m_2(1+\|u\|_V) + d_2\|g\|_{Y_2}\big). \tag{24}$$

Combining (23), (24), and (21), we deduce (22). □

4 A convergence result for the system of quasi-variational inequalities

The convergence result presented in this section is important in the study of optimal control problems associated with the system (1). We assume that the assumptions (2)–(12), (15)(a),

(16) hold and that

$$
\begin{aligned}
&\text{(a)}\ \ \{K_{1n}(u,\theta)\}_{n\in\mathbb{N}} \text{ converges to } K_1(u,\theta) \text{ in the sense of Mosco as } n\to\infty.\\
&\text{The similar assumption is for the sets } K_2(u,\theta) \text{ and } \{K_{2n}(u,\theta)\}_{n\in\mathbb{N}}.\\
&\text{(b)}\ \ \langle A_2\theta - A_2\eta, \theta-\eta\rangle_{E^*\times E} \geqslant m_{A_2}\,\|\theta-\eta\|_E^2, \text{with } m_{A_2}>0,\ \text{for all } \eta,\theta\in E,\\
&\text{(c)}\ \ f_n \rightharpoonup f \text{ in } Y_1 \text{ as } n\to\infty,\\
&\text{(d)}\ \ g_n \rightharpoonup g \text{ in } Y_2 \text{ as } n\to\infty,\\
&\text{(e)}\ \ \text{for all sequence } \{v_n\}\subset V \text{ such that } v_n \rightharpoonup v \text{ in } V,\\
&\ \ \text{we have } \pi_1 v_n \to \pi_1 v \text{ in } Y_1,\\
&\text{(f)}\ \ \text{for all sequence } \{\eta_n\}\subset E \text{ such that } \eta_n \rightharpoonup \eta \text{ in } E,\\
&\ \ \text{we have } \pi_2\eta_n \to \pi_2\eta \text{ in } Y_2.
\end{aligned}
\tag{25}
$$

Problem 3. Find $(u_n,\theta_n)\in C_1\times C_2$ such that $u_n\in K_{1n}(u_n,\theta_n),\ \theta_n\in K_{2n}(u_n,\theta_n)$ and

$$
\begin{aligned}
&\langle A_1u_n + N_1\theta_n, v-u_n\rangle_{V^*\times V} + \varphi_1(u_n,\theta_n,v) - \varphi_1(u_n,\theta_n,u_n)\\
&\geqslant (f_n, \pi_1 v - \pi_1 u_n)_{Y_1} \qquad \text{for all } \ v\in K_{1n}(u_n,\theta_n),\\
&\langle A_2\theta_n + N_2u_n, \eta-\theta_n\rangle_{E^*\times E} + \varphi_2(u_n,\theta_n,\eta) - \varphi_2(u_n,\theta_n,\theta_n)\\
&\geqslant (g_n, \pi_2\eta - \pi_2\theta_n)_{Y_2} \qquad \text{for all } \ \eta\in K_{2n}(u_n,\theta_n).
\end{aligned}
\tag{26}
$$

We observe that from Theorem 3, for each $n\in\mathbb{N}$, there exists a unique solution (u_n,θ_n) of Problem 3. The main result in this section is as follows.

Theorem 4. Assume that (2)–(12),(15), (16), and (25) hold. Then the sequence $\{u_n\}$ converges strongly to u in V, and the sequence $\{\theta_n\}$ converges strongly to θ in E.

The proof of Theorem 4 is divided into two steps. The first step of the proof is the following.

Lemma 2. There exist the elements $\widehat{u}\in K_{1n}(\widehat{u},\widehat{\theta})$ and $\widehat{\theta}\in K_{2n}(\widehat{u},\widehat{\theta})$ and the subsequences of $\{u_n\}$ and $\{\theta_n\}$, still denoted by $\{u_n\},\{\theta_n\}$ respectively, such that, as $n\to\infty$, we have

$$
\begin{aligned}
&u_n \rightharpoonup \widehat{u} \text{ in } V,\\
&\theta_n \rightharpoonup \widehat{\theta} \text{ in } E.
\end{aligned}
$$

Proof. First, we will show the boundedness of $\{u_n\}$ in V and $\{\theta_n\}$ in E. Let $v\in K_1(u,\theta)$ and $\eta\in K_2(u,\theta)$ be given. We assume (3), and we consider the sequences $\{v_n\},\{\eta_n\}$ such that $v_n\in K_{1n}(u_n,\theta_n)$, $\eta_n\in K_{2n}(u_n,\theta_n)$ for all $n\in\mathbb{N}$ and

$$v_n \longrightarrow v \text{ in } V, \quad \eta_n \longrightarrow \eta \text{ in } E. \tag{27}$$

Let $n \in \mathbb{N}$. Taking the first inequality in (26) with $v_n \in K_{1n}(u_n, \theta_n)$ and $\eta_n \in K_{2n}(u_n, \theta_n)$, we obtain

$$\begin{aligned}
&\langle A_1 u_n + N_1 \eta_n, v_n - u_n \rangle_{V^* \times V} + \varphi_1(u_n, \eta_n, v_n) - \varphi_1(u_n, \eta_n, u_n) \\
&\geqslant (f_n, \pi_1 v_n - \pi_1 u_n)_{Y_1}, \\
&\langle A_1 u_n, -v_n + u_n \rangle_{V^* \times V} \leqslant \langle N_1 \eta_n, v_n - u_n \rangle_{V^* \times V} + \varphi_1(u_n, \eta_n, v_n) \\
&\qquad\qquad - \varphi_1(u_n, \eta_n, u_n) + (f_n, \pi_1 u_n - \pi_1 v_n)_{Y_1}.
\end{aligned}$$

Using (15)(a), we get

$$\begin{aligned}
\langle A_1 u_n, u_n - v_n \rangle_{V^* \times V} &= \langle A_1 u_n - A_1 v_n, u_n - v_n \rangle_{V^* \times V} \\
&\quad + \langle A_1 v_n, u_n - v_n \rangle_{V^* \times V} \\
&\geqslant m_{A_1} \|u_n - v_n\|_V^2 + \langle A_1 v_n, u_n - v_n \rangle_{V^* \times V}.
\end{aligned}$$

From (9)(b), we obtain

$$\varphi_1(u_n, \eta_n, v_n) - \varphi_1(u_n, \eta_n, u_n) \leqslant c_{\varphi_1}(\|u_n\|_V, \|\eta_n\|_E) \|v_n - u_n\|_V,$$

and from (6)(b), we have

$$|\langle N_1 \eta_n, v_n - u_n \rangle_{V^* \times V}| \leqslant \|N_1 \eta_n\|_{V^*} \|v_n - u_n\|_V \leqslant m_1 (1 + \|\eta_n\|_E) \|v_n - u_n\|_V.$$

Moreover, from (8), we conclude that

$$|(f_n, \pi_1 v_n - \pi_1 u_n)_{Y_1}| \leqslant \|f_n\|_{Y_1} \|\pi_1 v_n - \pi_1 u_n\|_V \leqslant \|f_n\|_{Y_1}\, d_1 \, \|v_n - u_n\|_V,$$

hence

$$\begin{aligned}
m_{A_1} \|u_n - v_n\|_V^2 \leqslant{}& \|A_1 v_n\|_{V^*} \|v_n - u_n\|_V + m_1 (1 + \|\eta_n\|_E) \|v_n - u_n\|_V \\
&+ c_{\varphi_1}(\|u_n\|_V, \|\eta_n\|_E) \|v_n - u_n\|_V + \|f_n\|_{Y_1}\, d_1 \, \|v_n - u_n\|_V.
\end{aligned}$$

From the inequality above and (5)(a), we get

$$\begin{aligned}
m_{A_1} \|u_n - v_n\|_V \leqslant{}& a_1 + a_2 \|v_n\|_V + m_1 (1 + \|\eta_n\|_E) + c_{\varphi_1}(\|u_n\|_V, \|\eta_n\|_E) \\
&+ d_1 \, \|f_n\|_{Y_1}.
\end{aligned} \tag{28}$$

The relations (27) and (25)(c) imply that the sequences $\{v_n\}, \{\eta_n\}$ and $\{f_n\}$ are bounded in V, E and Y_1, respectively. Therefore, from the inequality (28) we deduce that there is a constant $c > 0$ independent on n such that $\|u_n - v_n\|_V \leqslant c$. So, $\{u_n\}$ is a bounded sequence in V, and from reflexivity of V, passing to a subsequence, if necessary, we deduce that

$$u_n \rightharpoonup \widehat{u} \text{ in } V \text{ as } n \longrightarrow \infty. \tag{29}$$

Similarly, we can show that $\|\theta_n - \eta_n\|_E \leqslant c$, and from this, $\{\theta_n\}$ is a bounded sequence in E. Also, the reflexivity of E leads to the conclusion that

$$\theta_n \rightharpoonup \widehat{\theta} \text{ in } E \text{ as } n \longrightarrow \infty. \tag{30}$$

Here $\widehat{u} \in V$ and $\widehat{\theta} \in E$. Since (29), (30) hold, we can use the Mosco convergence (25)(a) to obtain

$$\widehat{u} \in K_1(u_n, \theta_n) \text{ and } \qquad \widehat{\theta} \in K_2(u_n, \theta_n). \tag{31}$$

Now, we can use to (31) the assumption that the sets $K_1(u_n, \theta_n)$ and $K_2(u_n, \theta_n)$ are weakly Mosco continuous (see (3)(b), (4)(b)), to obtain $\widehat{u} \in K_1(\widehat{u}, \widehat{\theta})$ and $\widehat{\theta} \in K_2(\widehat{u}, \widehat{\theta})$. □

Lemma 3. The elements $\widehat{u} \in K_1(\widehat{u}, \widehat{\theta})$ and $\widehat{\theta} \in K_2(\widehat{u}, \widehat{\theta})$ are the solutions to Problem 1.

Proof. From the condition (25)(a) of the Mosco convergence, for each $v, \overline{u} \in K_1(u, \theta)$, we can find the sequences $\{v_n\}$, $\{\overline{u}_n\} \subset K_{1n}(u_n, \theta_n)$ such that $v_n \longrightarrow v$, $\overline{u}_n \longrightarrow \widehat{u}$ in V. From the first inequality in (26) with $v = \overline{u}_n$, we get

$$\begin{aligned} \langle A_1 u_n + N_1 \theta_n, \overline{u}_n - u_n \rangle_{V^* \times V} + \varphi_1(u_n, \theta_n, \overline{u}_n) - \varphi_1(u_n, \theta_n, u_n) \\ \geqslant (f_n, \pi_1 \overline{u}_n - \pi_1 u_n)_{Y_1}. \end{aligned} \tag{32}$$

From (32), we see that

$$\begin{aligned} \langle A_1 u_n, u_n - \overline{u}_n \rangle_{V^* \times V} \leqslant & \langle N_1 \theta_n, \overline{u}_n - u_n \rangle_{V^* \times V} + \varphi_1(u_n, \theta_n, \overline{u}_n) \\ & - \varphi_1(u_n, \theta_n, u_n) + (f_n, \pi_1 u_n - \pi_1 \overline{u}_n)_{Y_1} \end{aligned}$$

and moreover, we have

$$\begin{aligned} \langle A_1 u_n, u_n - \widehat{u} \rangle_{V^* \times V} &= \langle A_1 u_n, (u_n - \overline{u}_n) + (\overline{u}_n - \widehat{u}) \rangle_{V^* \times V} \\ &= \langle A_1 u_n, u_n - \overline{u}_n \rangle_{V^* \times V} + \langle A_1 u_n, \overline{u}_n - \widehat{u} \rangle_{V^* \times V}. \end{aligned}$$

Thus

$$\begin{aligned}\limsup\langle A_1 u_n, u_n-\widehat{u}\rangle_{V^*\times V} \leqslant \limsup\big(&\langle N_1\theta_n, \overline{u}_n-u_n\rangle_{V^*\times V}+\varphi_1(u_n,\theta_n,\overline{u}_n)\\&-\varphi_1(u_n,\theta_n,u_n)+(f_n,\pi_1u_n-\pi_2\overline{u}_n)_{Y_1}\\&+\langle A_1u_n,\overline{u}_n-\widehat{u}\rangle_{V^*\times V}\big).\end{aligned}$$

Hence $\limsup\langle A_1 u_n, u_n-\widehat{u}\rangle_{V^*\times V}\leqslant 0$. From the fact that $u_n \rightharpoonup \widehat{u}$ in V and the pseudomonotonicity of A_1, we obtain

$$\langle A_1\widehat{u}, \widehat{u}-v\rangle_{V^*\times V}\leqslant \liminf\langle A_1u_n, u_n-v\rangle_{V^*\times V} \text{ for all } v\in V. \tag{33}$$

On the other hand, we choose $v=v_n\in K_{1n}(u_n,\theta_n)$ in the first inequality of (26). From (32), we see that

$$\liminf\langle A_1u_n,u_n-v\rangle_{V^*\times V}\leqslant \limsup\langle A_1u_n,u_n-v\rangle_{V^*\times V}. \tag{34}$$

Using (34) into the inequality

$$\begin{aligned}&\limsup\langle A_1u_n,u_n-v\rangle_{V^*\times V}\\&=\limsup\big(\langle A_1u_n,(u_n-v_n)+(v_n-v)\rangle_{V^*\times V}\big)\\&=\limsup\big(\langle A_1u_n,u_n-v_n\rangle_{V^*\times V}+\langle A_1u_n,v_n-v\rangle_{V^*\times V}\big)\\&\leqslant\limsup\Big(\langle N_1\theta_n,v_n-u_n\rangle_{V^*\times V}+\varphi_1(u_n,\theta_n,v_n)-\varphi(u_n,\theta_n,u_n)\\&\quad+(f_n,\pi_1u_n-\pi_2v_n)_{Y_1}+\langle A_1u_n,v_n-v\rangle_{V^*\times V}\Big),\end{aligned}$$

and from (33), (34), (29), (27), and (25)(c),(e), we conclude

$$\begin{aligned}\langle A_1\widehat{u},\widehat{u}-v\rangle_{V^*\times V}\leqslant&\langle N_1\widehat{\theta},v-\widehat{u}\rangle_{V^*\times V}+\varphi_1(\widehat{u},\widehat{\theta},v)-\varphi_1(\widehat{u},\widehat{\theta},\widehat{u})\\&+(\widehat{f},\pi_1\widehat{u}-\pi_1v)_{Y_1}\qquad\text{for all } v\in K_1(\widehat{u},\widehat{\theta}).\end{aligned}\tag{35}$$

So, $\widehat{u}$ satisfies the first inequality in (26). Using similar arguments, we can prove that $\widehat{\theta}\in K_2(\widehat{u},\widehat{\theta})$ satisfies the second inequality in (26). □

Proof of Theorem 4. From the fact that Problem 1 has a unique solution (u,θ) and Lemma 3, we deduce that $\widehat{u}=u$ and $\widehat{\theta}=\theta$. Hence, every subsequence of $\{u_n\}$, $\{\theta_n\}$, which is convergent weakly in V and E, respectively, has the same limit. So, the whole sequences $\{u_n\}$ converges weakly in V to u as $n\longrightarrow\infty$ and $\{\theta_n\}$ converges weakly in E to θ as $n\longrightarrow\infty$.

We now prove that $u_n \longrightarrow u$ in V and $\theta_n \longrightarrow \theta$ in E as $n \longrightarrow \infty$. We take $v = \widehat{u} \in K_1(\widehat{u}, \widehat{\theta})$ in (35), and we use $\widehat{u} = u$. Then

$$0 \leqslant \liminf \langle A_1 u_n, u_n - u\rangle_{V^* \times V} \leqslant \limsup \langle A_1 u_n, u_n - u\rangle_{V^* \times V} \leqslant 0,$$

which shows that $\langle A_1 u_n, u_n - u\rangle_{V^* \times V} \longrightarrow 0$ as $n \longrightarrow \infty$. From this, (15)(a), taking $\widehat{u} = u$ and $\langle A_1 u_n, u_n - u\rangle_{V^* \times V} = \langle A_1 u_n - A_1 u, u_n - u\rangle_{V^* \times V} + \langle A_1 u, u_n - u\rangle_{V^* \times V}$, we obtain

$$\begin{aligned}\langle A_1 u_n, u_n - u\rangle_{V^* \times V} - \langle A_1 u, u_n - u\rangle_{V^* \times V} &= \langle A_1 u_n - A_1 u, u_n - u\rangle_{V^* \times V}\\ &\geqslant m_{A_1} \|u_n - u\|_V^2.\end{aligned}$$

The left-hand side of the equality above convergence to zero, as $n \longrightarrow \infty$, hence $u_n \longrightarrow u$ in V. Similarly, we prove that $\theta_n \longrightarrow \theta$ in E. □

5 Optimal control problem

In this section, we will consider the optimal control problem associated to (1). We will control the solution of the system (1) with $f \in Y_1$ and $g \in Y_2$. We define the set of admissible pair

$$\mathcal{V} = \big\{(u, \theta) \in C_1 \times C_2 \text{ and } (f, g) \in Y_1 \times Y_2 \text{ such that (1) holds}\big\}. \tag{36}$$

We deduce from here that the pairs (u, θ) and (f, g) belong to $\mathcal{V}$ if and only if $f \in Y_1$, $g \in Y_2$ and (u, θ) is the solution of the system (1), that is, $u = u(f, g)$, $\theta = \theta(f, g)$. Let $\mathcal{L} \colon V \times E \times Y_1 \times Y_2 \to \mathbb{R}$ be a cost functional. Thus, the optimal control problem is as follows.

Problem 4. Find $(u^*, \theta^*), (f^*, g^*) \in \mathcal{V}$ such that

$$\mathcal{L}(u^*, \theta^*, f^*, g^*) = \min_{(u,\theta),(f,g) \in \mathcal{V}} \mathcal{L}(u, \theta, f, g).$$

We assume that for all $(u, \theta) \in V \times E$ and $(f, g) \in Y_1 \times Y_2$, we have

$$\mathcal{L}(u, \theta, f, g) = U(u, \theta) + F(f, g), \tag{37}$$

where

$$\left.\begin{array}{l} U\colon V\times E\longrightarrow \mathbb{R} \text{ is continuous, bounded, nonnegative, that is,}\\ \quad \text{(a)}\ \ v_n\longrightarrow v \text{ in } V, \theta_n\longrightarrow \theta \text{ in } E \Longrightarrow U(v_n,\theta_n)\longrightarrow U(v,\theta),\\ \quad \text{(b)}\ \ U \text{ maps bounded sets in } V\times E \text{ into bounded sets in } \mathbb{R},\\ \quad \text{(c)}\ \ U(v,\theta)\geqslant 0 \quad \text{ for all } v\in V, \theta\in E, \end{array}\right\} \tag{38}$$

$$\left.\begin{array}{l} F\colon Y_1\times Y_2\longrightarrow \mathbb{R} \text{ is weakly lower semicontinuous, coercive, nonnegative,}\\ \ \text{that is,}\\ \ \text{(a)}\ \ f_n\rightharpoonup f \text{ in } Y_1 \text{ and } g_n\rightharpoonup g \text{ in } Y_2 \Longrightarrow \liminf F(f_n,g_n)\geqslant F(f,g),\\ \ \text{(b)}\ \ F(f,g)\geqslant 0 \text{ for all } (f,g)\in Y_1\times Y_2,\\ \ \text{(c)}\ \ \|f_n\|_{Y_1}\to+\infty,\ \|g_n\|_{Y_2}\to+\infty \Longrightarrow F(f_n,g_n)\to+\infty. \end{array}\right\} \tag{39}$$

Theorem 5. Assume that (2)–(12), (15), (16), (38)–(39) hold. Then, there exists a solution $(u^*,\theta^*),(f^*,g^*)\in\mathcal{V}$ of Problem 4.

Proof. Let

$$\xi=\inf_{(u,\theta),(f,g)\in\mathcal{V}}\mathcal{L}(u,\theta,f,g)\in\mathbb{R} \tag{40}$$

and let $\{(u_n,\theta_n)\},\{(f_n,g_n)\}\subset\mathcal{V}$ be the minimizing sequences for the functional $\mathcal{L}$, that is,

$$\lim\mathcal{L}(u_n,\theta_n,f_n,g_n)=\xi. \tag{41}$$

Our goal is to show that the sequences $\{f_n\},\{g_n\}$ are bounded in Y_1 and Y_2, respectively. We will argue by contradiction. We assume that $\{f_n\},\{g_n\}$ are not bounded in Y_1 and Y_2, respectively. Passing to the subsequences, denoted by $\{f_n\},\{g_n\}$, we have, as $n\longrightarrow\infty$, that

$$\|f_n\|_{Y_1}\longrightarrow+\infty,\quad \|g_n\|_{Y_2}\longrightarrow+\infty. \tag{42}$$

Now, from (37), (38)(c), we get

$$\mathcal{L}(u_n,\theta_n,f_n,g_n)\geqslant F(f_n,g_n).$$

This, (39)(c) and (41) gives that $\xi=\lim\mathcal{L}(u_n,f_n,\theta_n,g_n)=+\infty$ which is in contradiction with (40), so, $\{f_n\},\{g_n\}$ are bounded in Y_1, Y_2, respectively. Therefore, there exist $f^*\in Y_1$ and $g^*\in Y_2$ such that, passing to the subsequences, as $n\to\infty$, we get

$$f\rightharpoonup f^* \ \text{ in } \ Y_1 \quad\text{and}\quad g\rightharpoonup g^* \ \text{ in } \ Y_2. \tag{43}$$

Let $u^* = u(f^*, g^*)$ and let $\theta^* = \theta(f^*, g^*)$ be the solution of the system (1) for $f = f^*, g = g^*$. From (36) we have

$$(u^*, \theta^*), (f^*, g^*) \in \mathcal{V}. \tag{44}$$

Applying Theorem 4 and (43), it follows that, as $n \to \infty$, we have

$$u_n \to u^* \quad \text{in} \quad V \quad \text{and} \quad \theta_n \to \theta^* \quad \text{in} \quad E. \tag{45}$$

We observe that the assumptions (38)(a) and (39)(a) guarantee the weakly lower semicontinuity of the functional $\mathcal{L}$. This and the convergences (43) and (45) imply that

$$\liminf \mathcal{L}(u_n, \theta_n, f_n, g_n) \geqslant \mathcal{L}(u^*, \theta^*, f^*, g^*). \tag{46}$$

From (41) and (46) we see $\xi \geqslant \mathcal{L}(u^*, \theta^*, f^*, g^*)$. Additionally, from (40) and (44), we obtain that $\xi \leqslant \mathcal{L}(u^*, \theta^*, f^*, g^*)$. The last two inequalities together with (44) imply that the equality from the Problem 4 holds. $\square$

6 Convergence for optimal pairs

In this section, we study the dependence of the optimal controls pairs of Problem 4 with respect f and g. We assume that (2)–(12), (15), (16), and (25) hold. We consider the following perturbation of problem (1).

Problem 5. Given $f \in Y_1$ and $g \in Y_2$. Find $(u_n, \theta_n) \in C_1 \times C_2$ such that $u_n \in K_{1n}(u_n, \theta_n)$, $\theta_n \in K_{2n}(u_n, \theta_n)$ and

$$\begin{aligned}
&\langle A_1 u_n + N_1 \theta_n, v - u_n \rangle_{V^* \times V} + \varphi_1(u_n, \theta_n, v) - \varphi_1(u_n, \theta_n, u_n) \\
&\geqslant (f, \pi_1 v - \pi_2 u_n)_{Y_1} \qquad \text{for all } v \in K_{1n}(u_n, \theta_n), \\
&\langle A_2 \theta_n + N_2 u_n, \eta - \theta_n \rangle_{E^* \times E} + \varphi_2(u_n, \theta_n, \eta) - \varphi_1(u_n, \theta_n, \theta_n) \\
&\geqslant (g, \pi_2 \eta - \pi_2 \theta_n)_{Y_2} \qquad \text{for all } \eta \in K_{2n}(u_n, \theta_n).
\end{aligned} \tag{47}$$

From Theorem 3, we conclude that for each $f \in Y_1$ and $g \in Y_2$, there exists a unique solution $u_n = u(f, g)$, $\theta_n = \theta(f, g)$ to the system (47). Moreover, the solutions satisfy the estimations as in Lemma 4. We define the set of admissible pairs for (47) by

$$\mathcal{V}^n = \{(u_n, \theta_n), (f, g) \text{ such that (47) holds } \}. \tag{48}$$

The optimal control problem associated with Problem 5 is as follows.

Problem 6. Find (u_n^*,θ_n^*), $(f_n^*,g_n^*)\in\mathcal{V}^n$ such that

$$\mathcal{L}(u_n^*,\theta_n^*,f_n^*,g_n^*)=\min_{(u_n,\theta_n),(f_n,g_n)\in\mathcal{V}^n}\mathcal{L}(u_n,\theta_n,f_n,g_n).$$

From assumptions (37)–(39) and Theorem 5, for each $n\in\mathbb{N}$, there exists a solution (u_n^*,θ_n^*), $(f_n^*,g_n^*)\in\mathcal{V}^n$ of Problem 6. The result is as follows.

Theorem 6. Let $\{(u_n^*,\theta_n^*)\}$, $\{(f_n^*,g_n^*)\}$ be the sequences of solutions of Problem 6. Then, there exist the subsequences of the sequences $\{(u_n^*,\theta_n^*)\}$, $\{(f_n^*,g_n^*)\}$, denoted in the same way, and a solution (u^*,θ^*), (f^*,g^*) of Problem 6, such that

$$u_n\to u^* \text{ in } V,\quad \theta_n\to\theta^* \text{ in } E,\quad f_n^*\rightharpoonup f^* \text{ in } Y_1,\quad g_n^*\rightharpoonup g^* \text{ in } Y_2. \tag{49}$$

Proof. Let $n\in\mathbb{N}$.

Claim 1. The sequences $\{f_n^*\}$, $\{g_n^*\}$ are bounded in Y_1 and Y_2, respectively.

Assume that $\{f_n^*\}$, $\{g_n^*\}$ are not bounded in Y_1 and Y_2, respectively. Thus, for $n\to\infty$, we have

$$\|f_n^*\|_{Y_1}\to+\infty,\quad \|g_n^*\|_{Y_2}\to+\infty. \tag{50}$$

Using the assumptions (37) and (38)(c), we infer that

$$\mathcal{L}(u_n^*,\theta_n^*,f_n^*,g_n^*)\geqslant F(f_n^*,g_n^*). \tag{51}$$

Next, we pass to the limit in (51), as $n\to\infty$. Combining (50) and (39)(c), we deduce that the following equality holds:

$$\mathcal{L}(u_n^*,\theta_n^*,f_n^*,g_n^*)=+\infty. \tag{52}$$

Since $(u_n^*,\theta_n^*),(f_n^*,g_n^*)$ are solutions of Problem 6, we have

$$\mathcal{L}(u_n^*,\theta_n^*,f_n^*,g_n^*)\leqslant\mathcal{L}(u_n,\theta_n,f_n,g_n)\quad\text{for all } (u_n,\theta_n),(f_n,g_n)\in\mathcal{V}^n. \tag{53}$$

Let $f_0\in Y_1$ and $g_0\in Y_2$ be fixed, and let (u_n^0,θ_n^0) be the solution of Problem 6 for $f=f_0$, $g=g_0$, that is, $u_n^0=u(f_0,g_0)$ and $\theta_n^0=\theta(f_0,g_0)$. Then (u_n^0,θ_n^0), $(f_0,g_0)\in\mathcal{V}^n$. Therefore, from (38)(c), (39)(b), and (22), we have for every $n>0$ that

$$\mathcal{L}(u_n^*, \theta_n^*, f_n^*, g_n^*) \leqslant U(u_n^0, \theta_n^0) + F(f_0, g_0) \leqslant C, \tag{54}$$

where $C > 0$ does not depend on n. Relations (52) and (54) lead to contradiction which concludes the proof of the claim.

The sequences $\{f_n^*\}$, $\{g_n^*\}$ are bounded in Y_1 and Y_2, respectively, so, we can find subsequences, denoted in the same way, and the elements f^*, g^* such that, as $n \to \infty$, we have

$$f_n^* \rightharpoonup f^* \text{ in } Y_1, \quad g_n^* \rightharpoonup g^* \text{ in } Y_2. \tag{55}$$

We denote by (u^*, θ^*) the solution of Problem 1 for $f = f^*$ and $g = g^*$, that is, $u^* = u(f^*, g^*)$, $\theta^* = \theta(f^*, g^*)$, then

$$(u^*, \theta^*), \quad (f^*, g^*) \in \mathcal{V}^n. \tag{56}$$

Moreover, from Theorem 4, as $n \to \infty$, we obtain

$$u_n^* \to u^* \text{ in } V, \quad \theta_n^* \to \theta^* \text{ in } E. \tag{57}$$

Claim 2. The pairs (u^*, θ^*) and (f^*, g^*) are solution to Problem 1.

In the proof of this claim, we will use the conditions (55), (57) and the assumptions (38)(a), (39) (a), that guarantee the weakly lower semicontinuity of the functional $\mathcal{L}$. Therefore

$$\liminf_{n\to 0} \mathcal{L}(u_n^*, \theta_n^*, f_n^*, g_n^*) \geqslant \mathcal{L}(u^*, \theta^*, f^*, g^*). \tag{58}$$

Let the solution (u_0^*, θ_0^*), (f_0^*, g_0^*) of Problem 4 be fixed. For each $n \in \mathbb{N}$, we denote by $(\widehat{u}_{0n}, \widehat{\theta}_{0n})$ the solution to Problem 6 for $f_n = f_0^*$ and $g_n = g_0^*$, that is, $\widehat{u}_{0n} = u(f_0^*, g_0^*)$, $\widehat{\theta}_{0n} = \theta(f_0^*, g_0^*)$. Hence, $(\widehat{u}_{0n}, \widehat{\theta}_{0n})$, $(f_0^*, g_0^*) \in \mathcal{V}^n$. By the optimality of pairs (u_n^*, θ_n^*), (f_n^*, g_n^*), we conclude that for every $n > 0$ the following inequality holds $\mathcal{L}(u_n^*, \theta_n^*, f_n^*, g_n^*) \leqslant \mathcal{L}(\widehat{u}_{0n}, \widehat{\theta}_{0n}, f_0^*, g_0^*)$. Passing to the upper limit in the last inequality, we infer that

$$\limsup \mathcal{L}(u_n^*, \theta_n^*, f_n^*, g_n^*) \leqslant \limsup \mathcal{L}(\widehat{u}_{0n}, \widehat{\theta}_{0n}, f_0^*, g_0^*). \tag{59}$$

From the fact that (u_0^*, θ_0^*) is the solution of Problem 1 for $f = f_0^*$, $g = g_0^*$, and $(\widehat{u}_{n0}, \widehat{\theta}_{n0})$ is the solution of Problem 6 for $f_n = f_0^*$, $g_n = g_0^*$, that is,

$$\widehat{u}_0^* = u(f_0^*, g_0^*), \quad \widehat{\theta}_0^* = \theta(f_0^*, g_0^*),$$
$$\widehat{u}_{0n} = u(f_{0n}^*, g_{0n}^*), \quad \widehat{\theta}_{0n} = \theta(f_{0n}^*, g_{0n}^*),$$

we are able to use Theorem 4 and deduce that, as $n \to +\infty$, we have $\widehat{u}_{0n} \to u_0^*$ in V, $\widehat{\theta}_{0n} \to \theta_0^*$ in E. From here and the continuity of the functional $(u,\theta) \mapsto \mathcal{L}(u,\theta,f_0^*,g_0^*))\colon V\times E\times Y_1\times Y_2 \to \mathbb{R}$, we get

$$\lim \mathcal{L}(\widehat{u}_{0n},\widehat{\theta}_{0n},f_0^*,g_0^*) = \mathcal{L}(u_0^*,\theta_0^*,f_0^*,g_0^*). \tag{60}$$

combining (58)–(60), we see that

$$\mathcal{L}(u^*,\theta^*,f^*,g^*) \leqslant \mathcal{L}(u_0^*,\theta_0^*,f_0^*,g_0^*). \tag{61}$$

On the other hand, since (u_0^*,θ_0^*), (f_0^*,g_0^*) is a solution to Problem 4, inclusion (56) implies that

$$\mathcal{L}(u_0^*,\theta_0^*,f_0^*,g_0^*) \leqslant \mathcal{L}(u^*,\theta^*,f^*,g^*). \tag{62}$$

Using the inequalities (61) and (62), we obtain

$$\mathcal{L}(u^*,\theta^*,f^*,g^*) = \mathcal{L}(u_0^*,\theta_0^*,f_0^*,g_0^*).$$

Hence from (56), we infer that

$$(u^*,\theta^*),(f^*,g^*) \text{ is a solution of Problem 4.} \tag{63}$$

Our theorem is a consequence of (55), (57), and (63). □

7 Thermoelastic frictional contact problem

In this section, we use abstract results from Sections 4–6 to study thermoelastic frictional contact problem. The model presented in this Section is simpler version of the model presented in [17]. It is worth to mention, that in the paper we consider the optimal control problem that has not been the subject of research in [17].

The physical setting of the contact problem is as follows. We consider an elastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 2$, 3, with Lipschitz boundary Γ. By ν we denote the unit normal vector to Γ. The boundary Γ is divided into mutually disjoint, measurable parts Γ_1, Γ_2, Γ_3, with $\mathrm{meas}(\Gamma_1) > 0$. The body is clamped on Γ_1, and a volume force of density f_0 and a heat source q_0 act in Ω. Surface tractions of density f_2 act on the part Γ_2. The part Γ_3 is a contact surface between the body and the foundation. We denote by (V,H_1,V^*) and (E,H_2,E^*)

the evolution triples, where $H_1 = L^2(\Omega;\mathbb{R}^d)$ and $H_2 = L^2(\Omega)$. We define also the following sets

$$C_1 = \{v \in V \mid v_\nu \leqslant g_1 \;\text{ on }\; \Gamma_3\}, \quad C_2 = \{\eta \in E \mid \; \eta \leqslant \theta_F \;\text{ on }\; \Gamma_3\}, \tag{64}$$

and the set-valued maps $K_1\colon C_1 \times C_2 \to 2^{C_1}$ and $K_2\colon C_1 \times C_2 \to 2^{C_2}$ by

$$\begin{aligned} K_1(u,\theta) &= \{v \in C_1 \mid r_1(v) \leqslant m_1(u,\theta)\}, \\ K_2(u,\theta) &= \{\eta \in C_2 \mid r_2(\eta) \leqslant m_2(u,\theta)\} \end{aligned} \tag{65}$$

for $(u,\theta) \in C_1 \times C_2$. The examples of the sets $K_1(u,\theta)$ and $K_2(u,\theta)$ are provided in [17].

We introduce the following notations. By $\varepsilon(u) = \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ we denote the linearized strain tensor. Moreover, by $\mathrm{Div}\sigma = (\sigma_{ij,j})$ and $\mathrm{div}q = (q_{i,i})$ we denote the divergence operators, respectively, for tensor and vector valued functions. Finally, the canonical inner products on $\mathbb{R}^d$ and $\mathbb{S}^d$ we denote by " $\cdot$ " and " $:$ ", respectively. The thermoelastic contact problem is the following.

Problem 7. Find a displacement field $u\colon \Omega \to \mathbb{R}^d$, a stress field $\sigma\colon \Omega \to \mathbb{S}^d$, a temperature $\theta\colon \Omega \to \mathbb{R}$, and a heat flux field $q\colon \Omega \to \mathbb{R}^d$ such that $r_1(u) \leqslant m_1(u,\theta)$, $r_2(\theta) \leqslant m_2(u,\theta)$ and

a) $\sigma = \mathcal{A}\varepsilon(u) + \mathcal{M}\theta \;\text{ in }\; \Omega,$

b) $q = -\mathcal{K}(\nabla\theta) \;\text{ in }\; \Omega,$

c) $-\mathrm{Div}\sigma = f_0, \;\; \mathrm{div}q = q_0 \;\text{ in }\; \Omega,$

d) $u = 0 \;\text{ on }\; \Gamma_1,$

e) $\sigma\nu = f_2 \;\text{ on }\; \Gamma_2,$

f) $\theta = 0 \;\text{ on }\; \Gamma_1 \cup \Gamma_2,$

g) $-q \cdot \nu \leqslant 0,\; \theta - \theta_F \leqslant 0,\; (q \cdot \nu)(\theta - \theta_F) = 0 \;\text{ on }\; \Gamma_3,$

h $u_\nu \leqslant g_1,\; \sigma_\nu \leqslant 0,\; \sigma_\nu(u_\nu - g_1) = 0 \;\text{ on }\; \Gamma_3,$

i) $-\sigma_\tau = p_\nu(u_\nu)h_\tau(\theta - \theta_F)\,\frac{u_\tau}{\|u_\tau\|} \;\text{ on }\; \Gamma_3,$

j) $q \cdot \nu = h_1(u_\nu)k_e(\theta - \theta_F) \;\text{ on }\; \Gamma_3.$

We present the physical interpretation of Problem 7, and refer to [2, 3, 9, 16, 17, 20, 22] for details. In this model, the thermoelastic constitutive law with nonlinear elasticity $\mathcal{A}$ and thermal expansion operators $\mathcal{M}$ is described by (a). The law is justified from the material strength point of view, since the properties of the elastic materials are often effected by temperature. The condition (b) presents the Fourier law of heat conduction in which $\mathcal{K}$ is the thermal conductivity, and q is the heat flux vector. The equilibrium equations for the stress and heat flux fields are presented in (c). Moreover, f_0 and q_0 stand for the densities of volume forces and of the applied volume heat sources. The boundary conditions (d) and (e) present the homogeneous Dirichlet and Neumann conditions, respectively, and the condition (f) shows that the temperature vanishes on $\Gamma_1 \cup \Gamma_2$. Condition (g) represents the Signorini-type condition between the relative temperature $\theta - \theta_F$ and the heat flux $q \cdot \nu$. It means that on Γ_3 the heat flux is supposed to be unilaterally directed from the foundation to the body, and then the body temperature does not exceed the foundation temperature. The Signorini-type contact condition (h) models the contact with a foundation made of a rigid body covered by a layer made of deformable material of the thickness g_1 (see [22, Sections 8.4 and 8.5]). The relation (i) is the normal compliance contact condition coupled with the Coulomb friction law (see [9, Section 5.4]), in which p_ν is prescribed function such that $p(r) = 0$ which vanishes for $r \leqslant 0$ and the coefficient of friction h_τ depends on the relative temperature $\theta - \theta_F$, where $\theta_F \geqslant 0$ is the temperature of the foundation. The heat exchange between the temperature and the flux is modeled by condition (j), where the heat exchange coefficient h_1 depends on the normal displacement. The inequalities $r_1(u) \leqslant m_1(u, \theta)$ and $r_2(\theta) \leqslant m_2(u, \theta)$ may accommodate various additional implicit constraints in which we are interested in the model (see [17, Example 18]).

We introduce the following assumptions for the operators and functions.

The elasticity operator $\mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

$$\left.\begin{array}{l}
\text{(a)}\ \ \mathcal{A}(\cdot, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\
\text{(b)}\ \ \mathcal{A}(x, \cdot)\ \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } x \in \Omega, \\
\text{(c)}\ \ \|\mathcal{A}(x, \varepsilon)\|_{\mathbb{S}^d} \leqslant a_0(x) + \overline{a}_1 \|\varepsilon\|_{\mathbb{S}^d} \text{ for all } \varepsilon \in \mathbb{S}^d,\ \text{a.e. } x \in \Omega \\
\text{with } a_0 \in L^2(\Omega), a_0 \geqslant 0 \text{ and } \overline{a}_1 > 0, \\
\text{(d)}\ \ (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d} \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \\
\text{and a.e. } x \in \Omega \text{ with } m_{\mathcal{A}} > 0.
\end{array}\right\} \quad (66)$$

The thermal expansion operator $\mathcal{M}\colon \Omega \times \mathbb{R} \to \mathbb{S}^d$ is such that

$$\left.\begin{aligned}&\text{(a)}\ \ \mathcal{M}(\cdot,s) \text{ is measurable on } \Omega \text{ for all } s\in\mathbb{R},\\&\text{(b)}\ \ \mathcal{M}(x,\cdot) \text{ is continuous on } \mathbb{R} \text{ for a.e. } x\in\Omega,\\&\text{(c)}\ \ \|\mathcal{M}(x,r)\|_{\mathbb{S}^d}\leqslant c_M(1+|r|) \text{ for all } r\in\mathbb{R},\ \text{a.e. } x\in\Omega,\ \text{with } c_M>0.\end{aligned}\right\}\tag{67}$$

The thermal conductivity operator $\mathcal{K}\colon \Omega\times\mathbb{R}^d\to\mathbb{R}^d$ is such that

$$\left.\begin{aligned}&\text{(a)}\ \ \mathcal{K}(\cdot,\xi) \text{ is measurable on } \Omega \text{ for all } \xi\in\mathbb{R}^d,\\&\text{(b)}\ \ \mathcal{K}(x,\cdot) \text{ is continuous on } \mathbb{R}^d \text{ for a.e. } x\in\Omega,\\&\text{(c)}\ \ \|\mathcal{K}(x,\xi)\|_{\mathbb{R}^d}\leqslant k_0(x)+k_1\|\xi\|_{\mathbb{R}^d} \text{ for all } \xi\in\mathbb{R}^d,\ \text{a.e. } x\in\Omega\\&\text{with } k_0\in L^2(\Omega), k_0\geqslant 0, k_1>0,\\&\text{(d)}\ \ (\mathcal{K}(x,\xi_1)-\mathcal{K}(x,\xi_2))\cdot(\xi_1-\xi_2)\geqslant m_{\mathcal{K}}\|\xi_1-\xi_2\|^2_{\mathbb{R}^d} \text{ for all } \xi_1,\xi_2\in\mathbb{R}^d,\\&\text{a.e. } x\in\Omega \text{ with } m_{\mathcal{K}}>0.\end{aligned}\right\}\tag{68}$$

The coefficient of heat exchange $h_1\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ and the function $h_\tau\colon \Gamma_C\times\mathbb{R}\to\mathbb{R}_+$, are such that for $i=1,\tau$ we have

$$\left.\begin{aligned}&\text{(a)}\ \ h_i(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R},\\&\text{(b)}\ |h_i(x,r_1)-h_i(x,r_2)|\leqslant L_{h_i}\,|r_1-r_2| \text{ for all } r_1,r_2\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3\\&\text{with } L_{h_i}>0,\\&\text{(c)}\ \ h_i(x,r)\leqslant \bar{h}_i \text{ for all } r\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3 \text{ with } \bar{h}_i>0.\end{aligned}\right\}\tag{69}$$

The normal compliance function $p_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ is such that

$$\left.\begin{aligned}&\text{(a)}\ \ p_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R},\\&\text{(b)}\ \ |p_\nu(x,r_1)-p_\nu(x,r_2)|\leqslant L_{p_\nu}\,|r_1-r_2| \text{ for all } r_1,r_2\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3\\&\text{with } L_{p_\nu}>0,\\&\text{(c)}\ \ p_\nu(x,r)\leqslant \bar{p} \text{ for all } r\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3 \text{ with } \bar{p}>0.\end{aligned}\right\}\tag{70}$$

The coefficient $k_e\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ is such that

$$\left.\begin{aligned}&\text{(a)}\ \ k_e(\cdot,r) \text{ is measurable on } \Gamma_3\ \text{ for all } r\in\mathbb{R},\\&\text{(b)}\ \ |k_e(x,r_1)-k_e(x,r_2)|\leqslant L_k|r_1-r_2|\ \text{ for all } r_1,r_2\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3\\&\text{with } L_k>0,\\&\text{(c)}\ \ k_e(x,r)\leqslant \bar{k} \text{ for all } r\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3 \text{ with } \bar{k}>0.\end{aligned}\right\}\tag{71}$$

$r_1\colon V\to\mathbb{R}$, $r_2\colon E\to\mathbb{R}$, m_1, $m_2\colon H_1\times H_2\to\mathbb{R}$ are functions such that

$$\left.\begin{array}{l} \text{(a)}\ \ r_1, r_2 \text{ are positively homogeneous, subadditive,} \\ \quad\ \text{and lower semicontinuous,} \\ \text{(b)}\ \ m_1 \text{ is continuous, } m_1^0 := \inf\{m_1(u,\theta) \mid (u,\theta) \in H_1 \times H_2\} > 0 \\ \text{and } r_1(0) \leqslant m_1^0, \\ \text{(c)}\ \ m_2 \text{ is continuous, } m_2^0 := \inf\{m_2(u,\theta) \mid (u,\theta) \in H_1 \times H_2\} > 0 \\ \text{and } r_2(0) \leqslant m_2^0, \end{array}\right\} \tag{72}$$

$$f_0 \in L^2(\Omega;\mathbb{R}^d),\ f_N \in L^2(\Gamma_N;\mathbb{R}^d),\ q_0 \in L^2(\Omega),\ g_1 \in L^2(\Gamma_3),\ g_1 \geqslant 0,\ \theta_F \geqslant 0. \tag{73}$$

Remark 3. In the proof of the uniqueness of solution, we will need the following assumptions.

$$\begin{aligned} &\text{(a)}\ \ \|\mathcal{M}(x,r_1) - \mathcal{M}(x,r_2)\|_{\mathbb{S}^d} \leqslant c_{M_1}|r_1 - r_2| \text{ with } c_{M_1} > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } r_1, r_2 \in \mathbb{R}, \\ &\text{(b)}\ \ \|\mathcal{K}(x,\xi_1) - \mathcal{K}(x,\xi_2)\|_{\mathbb{R}^d} \leqslant c_{\mathcal{K}}\|\xi_1 - \xi_2\|_{\mathbb{R}^d} \text{ with } c_{\mathcal{K}} > 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \xi_1, \xi_2 \in \mathbb{R}^d. \end{aligned} \tag{74}$$

The variational formulation of Problem 7 leads to the following system of inequalities.

Problem 8. Find $(u,\theta) \in C_1 \times C_2$ with $u \in K_1(u,\theta)$, $\theta \in K_2(u,\theta)$ such that for all $v \in K_1(u,\theta)$, $\eta \in K_2(u,\theta)$, we have

$$\begin{aligned} &\int_\Omega (\mathcal{A}\varepsilon(u) + \mathcal{M}\theta) \cdot (\varepsilon(v) - \varepsilon(u))\, dx \\ &+ \int_{\Gamma_C} p_\nu(u_\nu)\, h_\tau(\theta - \theta_F)(\|v_\tau\| - \|u_\tau\|)\, d\Gamma \\ &\qquad \geqslant \int_\Omega f_0 \cdot (v - u)\, dx + \int_{\Gamma_N} f_2 \cdot (v - u)\, d\Gamma, \\ &\int_\Omega \mathcal{K}(\nabla\theta) \cdot (\nabla\eta - \nabla\theta)\, dx + \int_{\Gamma_C} h_1(u_\nu) k_e(\theta - \theta_F)(\eta - \theta)\, d\Gamma \\ &\geqslant \int_\Omega q_0 \cdot (\eta - \theta)\, dx. \end{aligned}$$

The main result in this section is as follows.

Theorem 7. Under the assumptions (66)–(74) Problem 8 has a unique solution.

Proof. The idea is to associate with Problem 8 an auxiliary system of inequalities, and apply Theorem 3. To this goal, we introduce the operators $A_1 \colon V \to V^*$, $A_2 \colon E \to E^*$, $N_1 \colon E \to V^*$, and $N_2 \colon V \to E^*$ defined by

$$\langle A_1 u, v\rangle = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v)\,dx \quad \text{for} \quad u, v \in V, \tag{75}$$

$$\langle A_2\theta, \eta\rangle_{E^*\times E} = \int_\Omega \mathcal{K}(\nabla\theta)\cdot\nabla\eta\,dx \quad \text{for} \quad \theta, \eta \in E, \tag{76}$$

$$\langle N_1\theta, v\rangle = \int_\Omega \mathcal{M}\theta : \varepsilon(v)\,dx \quad \text{for} \quad \theta \in E,\ v \in V \tag{77}$$

and $N_2 \equiv 0$. Furthermore, we define functions $\varphi_1\colon V\times E\times V \to \mathbb{R}$, and $\varphi_2\colon V\times E\times E\to\mathbb{R}$ by

$$\varphi_1(w,\theta,v) = \int_{\Gamma_C} p_\nu(w_\nu)h_\tau(\theta-\theta_F)\|v_\tau\|\,d\Gamma \quad \text{for} \quad w, v\in V,\ \theta\in E, \tag{78}$$

$$\varphi_2(w,\theta,\eta) = \int_{\Gamma_C} h(w_\nu)k_e(\theta-\theta_F)\eta\,d\Gamma. \tag{79}$$

Finally, we introduce $f \in Y_1 = L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_2;\mathbb{R}^d)$ and $g\in Y_2 = L^2(\Omega)$ defined by

$$\begin{aligned}(f,\pi_1 v)_{Y_1} &= \langle f_0, v\rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle f_2, v\rangle_{L^2(\Gamma_2;\mathbb{R}^d)}, \text{ for } v\in V\\ (g,\pi_2\eta)_{Y_2} &= \langle q_0,\eta\rangle_{L^2(\Omega)} \text{ for } \eta\in E,\end{aligned} \tag{80}$$

where

$$\begin{aligned}&\pi_1\colon V\to Y_1 \quad \pi_1 v = (v, v|_{\Gamma_2}) \quad \forall v\in V,\\ &\pi_2\colon E\to Y_2 \quad \pi_2\eta = \eta \quad \text{for all } \eta\in E.\end{aligned}$$

Using (75)–(80), we can rewrite Problem 8 in the following way.

Problem 9. Find $(u,\theta)\in C_1\times C_2$ with $u\in K_1(u,\theta)$, $\theta\in K_2(u,\theta)$ where

$$\begin{cases}\langle A_1 u + N_1\theta, v-u\rangle + \varphi_1(u,\theta,v) - \varphi_1(u,\theta,u) \geqslant (f,v)_{Y_1}\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all} \quad v\in K_1(u,\theta),\\ \langle A_2\theta,\eta-\theta\rangle_{E^*\times E} + \varphi_2(u,\theta,\eta) - \varphi_2(u,\theta,\theta) \geqslant (g,\eta)_{Y_2}\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all} \quad \eta\in K_2(u,\theta).\end{cases}$$

We see that for $N_2 = 0$ Problem 3 reduce to Problem 9. We are ready to apply Theorem 1 and show that Problem 9 has a solution. To this end, we shall verify hypotheses (5)–(10), (11) and (12). The proofs of these assumptions are analogous to [17, Chapter 4]. Now, we will verify (3) while condition (4) can be demonstrated by analogous arguments. Assumption (73) implies that $0\in C_1$. Assumption (72)(b) implies that $r_1(0)\leqslant m_1(u,\theta)$ for all $(u,\theta)\in C_1\times C_2$ so $0\in K_1(u,\theta)$. Thus (3)(c) holds with $K_0=\{0\}$. By (72)(a), it follows that $r_1\colon V\to\mathbb{R}$ is a convex function, thus, the convexity of $K_1(u,\theta)$ for each $(u,\theta)\in C_1\times C_2$ is obvious. The closedness of

$K_1(u,\theta)$ for $(u,\theta)\in C_1\times C_2$ is a consequence of the lower semicontinuity of r_1 and the continuity of embedding of V to H_1. So, (3)(a) is satisfied. The proof that $K_2(u_n,\theta_n)\xrightarrow{M} K_2(u,\theta)$ for all $\{(u_n,\theta_n)\}\subset C_1\times C_2$, such that $(u_n,\theta_n)\to(u,\theta)$ in $(V\times E)_w$, is similar to [15, Theorem 4.1], so it is omitted. In order to prove the uniqueness of the solution of Problem 8, we will use Theorem 3. Therefore, we should verify the assumptions (15) and (16). We observe that from (66)(d) it follows that (15)(a) with $m_{A_1}=m_{\mathcal{A}}$. The condition (15)(b) holds with $m_{N_2}=0$. Moreover, from (74)(a), we conclude that (15)(c) holds with $L_{N_1}=c_{M_1}$ and from (74)(b), we obtain that (15)(d) holds with $L_{A_2}=c_{\mathcal{K}}$. Finally, from (69)(b), (c), (70)(b), (c), (71)(b), (c) and the Holder inequality, we see that

$$\begin{aligned}
&\int_{\Gamma_3} p_\nu(u_{1\nu})h_\tau(\theta_1-\theta_F)(\|u_{2\tau}\|-\|u_{1\tau}\|)\,d\Gamma\\
&+\int_{\Gamma_3} p_\nu(u_{2\nu})h_\tau(\theta_2-\theta_F)(\|u_{1\tau}\|-\|u_{2\tau}\|)\,d\Gamma\\
&\quad\leqslant \int_{\Gamma_3}\Big[|p_\nu(u_{1\nu})-p_{2\nu}(u_{2\nu})||h(\theta_1-\theta_F)|\\
&\qquad+|p_\nu(u_{2\nu})||h_\tau(\theta_1-\theta_F)-h_\tau(\theta_2-\theta_F)|\,\big|\|u_{2\tau}\|-\|u_{1\tau}\|\big|\Big]\,d\Gamma\\
&\quad\leqslant \big(L_{p_\nu}\overline{h}_\tau|u_{1\nu}-u_{2\nu}|+\overline{p}L_{h_\tau}|\theta_1-\theta_2|\big)\,\big|\|u_{2\tau}\|-\|u_{1\tau}\|\big|\Big]\,d\Gamma\\
&\quad\leqslant \big(L_{p_\nu}\overline{h}_\tau\|\gamma_1\|^2\|u_1-u_2\|_V+\overline{p}L_{h_\tau}\|\gamma_1\|\,\|\gamma_2\|\,\|\theta_1-\theta_2\|_E\big)\,\|u_2-u_1\|_V,
\end{aligned}$$

and, in the similar way as above, we obtain

$$\begin{aligned}
&\int_{\Gamma_3} h(u_{1\nu})k_e(\theta_1-\theta_F)(\theta_2-\theta_1)\,d\,\Gamma+\int_{\Gamma_3} h(u_{2\nu})k_e(\theta_2-\theta_F)(\theta_1-\theta_2)\,d\,\Gamma\\
&\leqslant \int_{\Gamma_3}\Big[\big(h(u_{1\nu}-h(u_{2\nu})\big)k_e(\theta_1-\theta_F)\\
&\quad+h(u_{2\nu})(k_e(\theta_1-\theta_F)-k_e(\theta_2-\theta_F))\Big](\theta_2-\theta_1)\,d\,\Gamma\\
&\leqslant \big(\overline{k}\,L_{h_1}\,\|\gamma_1\|\,\|\gamma_2\|\,\|u_1-u_2\|_V+\overline{h}_1\,L_k\,\|\gamma_2\|^2\,\|\theta_1-\theta_2\|_E\big)\|\theta_2-\theta_1\|_E.
\end{aligned}$$

So, the condition (15)(e) and (f) hold with $\theta_{\varphi_1}=\max\{\|\gamma_1\|^2\,L_{p_\nu}\overline{h}_\tau,\,\|\gamma_1\|\,\|\gamma_2\|\,L_{h_\tau}\overline{p}\}$ and $\theta_{\varphi_2}=\max\{\|\gamma_1\|\,\|\gamma_2\|L_{h_1}\overline{k},\|\gamma_2\|^2\,L_{k_e}\overline{h}_1\}$, respectively. Here $\gamma_1\colon V\to L^2(\Gamma_3;\mathbb{R}^d)$ and $\gamma_2\colon E\to L^2(\Gamma_3)$ we understand as the trace operators. Proceeding in the similar way as in the proof of Theorem 3, we note that an analogous condition to (16) holds too. □

Next, in Problem 9, we can take the sets

$$\begin{aligned}
K_{1n}(u,\theta)&=\{v\in C_{1n}\mid r_1(v)\leqslant m_1(u,\theta)\},\quad C_{1n}=\{v\in V\mid v_\nu\leqslant g_{1n}\text{ on }\Gamma_3\},\\
K_{2n}(u,\theta)&=\{\eta\in C_{2n}\mid r_2(\eta)\leqslant m_2(u,\theta)\},\quad C_{2n}=\{\eta\in E\mid \eta\leqslant\theta_{Fn}\text{ on }\Gamma_3\}.
\end{aligned}$$

We are in position to use Theorem 4 in order to deduce that $u_n \to u$ in V and $\theta_n \to \theta$ in E.

We end this section with example of optimal control problem for which the results presented in sections 5 and 6 hold. Let f_0, g_1, θ_F be given. We define the set of admissible pair for Problem 8 as follows:

$$\begin{aligned}\overline{\mathcal{V}}_{ad} = \{&(u,\theta) \in C_1 \times C_2 \text{ and } (f_2, q_0) \in L^2(\Gamma_3;\mathbb{R}^d) \times L^2(\Omega) \\ &\text{such that Problem 8 holds}\}\end{aligned}$$

and we choose the cost functional

$$\begin{aligned}\mathcal{L}(u,\theta,f_2,q_0) =& \theta_1\|u_\nu - z_0\|_{L^2(\Gamma_3)} + \theta_2\|\theta - \theta_0\|_{L^2(\Gamma_3)} + \gamma_1\|f_2\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &+ \gamma_2\|q_0\|_{L^2(\Omega)},\end{aligned} \tag{81}$$

where $\theta_0, z_0 \in L^2(\Gamma_3)$ and $\theta_1, \theta_2, \gamma_1, \gamma_2 > 0$. Then, the problem which we consider can be formulated as follows.

Problem 10. Find $(u^*, \theta^*), (f_2^*, q_0^*) \in \overline{\mathcal{V}}_{ad}$ such that

$$\mathcal{L}(u^*, \theta^*, f_2^*, q_0^*) = \min_{(u,\theta),(f_2,q_0)\in\overline{\mathcal{V}}_{ad}} \mathcal{L}(u,\theta,f_2,q_0).$$

The mechanical interpretation of the Problem 10 is as follows. We are looking for a surface pressure f_2 acting on Γ_2 and heat source q_0 acting on Ω such that a normal displacement and a temperature of the body is as close as possible to the desired displacement z_0 and desired temperature θ_0. Moreover, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in (81). We have the following result in the study of Problem 10.

Theorem 8. Let (66)–(73) hold, and moreover, let $f \in V^*, q_0 \in E^*, z_0, \theta_0 \in L^2(\Gamma_C)$, and let $\theta_1, \theta_2, \gamma_1, \gamma_2 > 0$. Then there exists a solution $(u^*, \theta^*), (f_2^*, q_0^*) \in \overline{\mathcal{V}}_{ad}$ to Problem 10.

Theorem 8 is a direct consequence of Theorem 5, and we note that Theorem 6 provides a convergence result for a sequence of optimal pairs of Problem 10. The question about conditions of optimality will be the future direction of study.

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Conflicts of Interest

The author declares no conflicts of interest.

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