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Research Article

Characterizing maximum output sets in fractional-order discrete-time linear systems utilizing fractional feedback control

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Abstract

This paper examines a linear controlled discrete-time fractional-order system, as defined by

$$\begin{cases} \Delta^{\gamma} x_{k+1} = A x_k + B \Delta^{\gamma} u_k, \\ x_0 = x_0. \end{cases}$$

The corresponding output, denoted as $y_k = Cx_k$, is intended to be stable; that is, $\lim_{k\to\infty} y_k = 0$.

More precisely, we suppose the existence of a fractional feedback control denoted as $\Delta^{\gamma} u_k = \Delta^{\gamma} L x_k$, where the matrices A, B, C, and L are suitably chosen. The characterization of the maximal output set $\Upsilon(\mathcal{R})$ is investigated by defining the fractional derivative in the Grunwald–Letnikov sense. Specifically, $\Upsilon(\mathcal{R}) = \{x_0 \in \mathbb{R}^n / y_k \in \mathcal{R}, \text{ for all } k \geq 0\}$, where $\mathcal{R} \subset \mathbb{R}^n$ represents a constraint set. It can be shown, through the use of stability and observability hypotheses, that a finite number of equations can derive $\Upsilon(\mathcal{R})$. The maximal output set is determined using an advanced algorithmic approach. In order to clarify the theoretical results, relevant algorithms and numerical simulations have been included.

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1 Introduction

Fractional calculus extends the arithmetic of integers by introducing derivatives of noninteger orders. This approach allows for a more precise representation of real systems in engineering and scientific domains. Debnath [4] highlighted the advantages of fractional-order models for capturing the nuanced behavior of systems characterized by long-lived transients and inherited properties, thereby enhancing the accuracy of their description.

Modeling physical phenomena often requires capturing intricate details of preceding histories, which extends beyond the consideration of instant time alone. In this regard, fractional calculus has proven to be invaluable. It is worth noting that recently, there has been substantial interest across a spectrum of disciplines in exploring applications involving fractional derivatives (of noninteger order). Mathematicians and researchers have dedicated their efforts to depicting authentic processes in various fields, such as mechanics, electricity, biology, chemistry, and control systems, by employing models of fractional order. These models can be considered within the context of either continuous or discrete-time representations.

In recent literature, there has been an increasing focus on discrete fractional systems. Notable works in this area include those by Dzieliński and Sierociuk [6], Kaczorek [18, 19], and Ferreira and Torres [12]. Significant advancements in the theory of fractional calculus are outlined by Kilbas, Srivastava, and Trujillo [20] and Oldham and Spanier [24]. Guermah, Djennoune, and Bettayeb [14] extended certain fundamental results on the controllability and observability of discrete-time fractional-order linear systems.

Maximum output sets (MOSs) are important in various practical applications, including stability analysis, stabilizability, and control of constrained systems (refer to [14, 28]). Despite extensive research on this concept, as demonstrated by in-depth investigations in [15, 26] and associated references, its examination within the realm of fractional discrete systems is currently limited. Therefore, further investigation and consideration of this domain is recommended within our scientific community.

The MOS is an important element in various constrained control methodologies, particularly in the implementation of diverse protocols for constraint management. It is defined as the set encompassing all initial states, ensuring that trajectories originating from this set consistently adhere to the prescribed constraints on outputs at all times. The MOS plays a vital role in navigating constrained control scenarios.

Related Work

The purpose of this paper is to investigate the maximal output admissible set within a specific class of fractional linear systems, utilizing the Grunwald–Letnikov fractional derivative. Valuable insights into the properties and characterizations of the maximal output admissible set in the context of linear systems have been provided by Gilbert and Tan [13]. Moreover, the authors present an algorithm that can be applied to both discrete and continuous-time linear time-invariant systems. This research is intended for scientists and researchers who are looking for a detailed comprehension of the intricate dynamics inherent in fractional linear systems. The maximal output set has recently been characterized for observability and asymptotically stable systems in the domain of linear discrete-time systems (see [27, 3, 2]). This conceptual advancement significantly deepens our understanding of the analysis of constrained control systems. Additionally, it has practical utility in the field of control system design techniques, as highlighted in [21]. Recent attention has been given to exploring the established maximal output set for a specific class of semilinear systems and bilinear systems, as noted in [10, 1], respectively. The focus of these studies is to characterize the MOS for such systems, using a stability hypothesis. The authors have introduced a novel sufficient condition to ensure the asymptotic stability of the system. The maximal set was characterized using a finite number of inequalities. Furthermore, an algorithmic approach was proposed to determine the admissibility index k_0^*. The effectiveness of this methodology was demonstrated through several numerical examples. In [23], the authors explored infinite-dimensional linear systems and provides a characterization of the set of gain operators that can make the system insensitive to uncertain initial states. Further insights into the exploration of the maximal output set can be gleaned from additional references in [8, 9]. The work of Hirata and Ohta [16] in the realm of nonlinear systems is worthy of recognition. Their study delineates both necessary and sufficient conditions for meeting prescribed state and control constraints within a distinct class of nonlinear systems. Additionally, the authors propose a precise methodology for determining the maximal output admissible set. The significance of fractional-order derivatives in the modeling and analysis of biological phenomena has been emphasized in several recent studies [31, 29, 30, 17, 11, 22].

Statement problem

The objective of this study is to explore fractional-order discrete-time linear systems under fractional feedback control, as represented by the system dynamics expressed in (1). The focus is on understanding the properties of the MOS associated with such systems.

The study aims to contribute novel insights into this area, motivated by the lack of comprehensive research on the MOS for fractional-order systems. We propose a methodology to construct the MOS using a finite set of inequalities informed by stability and observability considerations.

Additionally, we aim to establish new sufficient conditions that ensure the finite determination of the MOS. These conditions are crucial for clarifying the system's behavior and may inform the development of more effective control strategies. It is worth noting that this study solely focuses on characterizing the MOS and does not consider the existence of fractional feedback control.

The central objective of this study is to investigate fractional-order discrete-time linear systems employing fractional feedback control, characterized by

$$\begin{cases} \Delta^{\gamma} x_{k+1} = A x_k + B \Delta^{\gamma} u_k, \\ x_0 = x_0. \end{cases} \tag{1}$$

The associated output is $y_k = C x_k$.

More precisely, we suppose the existence of a fractional feedback control

$$\Delta^{\gamma} u_k = L \Delta^{\gamma} x_k. \tag{2}$$

The paper unfolds in a structured manner to comprehensively address the research objectives. In Section 2, we provide a thorough examination of the fundamental definition of fractional derivatives within the Grunwald–Letnikov framework, along with an in-depth analysis of the discrete-time model discussed in [5]. Section 3 is dedicated to the characterization of the maximal output set, accompanied by the establishment of sufficient conditions to ensure its finite determination. Building upon this, Section 4 introduces conditions aimed at representing the maximal output set using a finite number of inequalities. In Section 5, we propose a conceptual algorithm for determining the output admissibility index. The subsequent section presents numerical examples to illustrate and validate the theoretical findings. Finally, the concluding section synthesizes the research outcomes and provides a concise summary of the study's contributions.

2 Analysis of dynamic models through fractional calculus

This section presents basic concepts related to fractional calculus that are utilized throughout the paper. In this section of the paper, the Grünwald–Letnikov method [24, 25] defines the discrete fractional derivative, while also defining a set of admissible outputs.

Definition 1. The Letnikov and Grunwald fractional difference operator, denoted by Δ^γ, acting on the function $x(\cdot)$ at the discrete time instance k within the interval $[0,+\infty[$, is defined as

$$\Delta^\gamma x(k) = \frac{1}{h^\gamma}\sum_{j=0}^{k}(-1)^j\binom{\gamma}{j}x\left(k-j\right). \tag{3}$$

The derivative approximation is calculated for a sampling period of unity, where the order of the derivative is between 0 and 1, and k is the number of samples.

The term $\binom{\gamma}{j}$ in Definition 1 can be obtained from the following relation:

$$\binom{\gamma}{j} = \begin{cases} 1 & \text{for } j=0, \\ \frac{\gamma(\gamma-1)\cdots(\gamma-j+1)}{j!} & \text{for } j>0. \end{cases} \tag{4}$$

The first-order difference for $x(k+1)$ defined as

$$\begin{aligned}
\Delta^1 x(k+1) &= x(k+1) - x(k) \\
&= Ax_k + BL\Delta^1 x_k - x_k \\
&= \left(A - I_n\right)x_k + BL\left(x_k - x_{k-1}\right) \\
&= \left(A - I_n\right)x_k + BL\left(x_k - x_{k-1}\right) \\
&= \left(A + BL - I_n\right)x_k - BLx_{k-1} \\
&= A_d x_k - BLx_{k-1},
\end{aligned}$$

with $A_d = A + BL - I_n$.

We have

$$\Delta^\gamma x_{k+1} = x_{k+1} + \sum_{j=1}^{k+1}(-1)^j\binom{\gamma}{j}x\left(k-j+1\right)$$

and

$$\Delta^\gamma x_k = x_k + \sum_{j=0}^{k}(-1)^j\binom{\gamma}{j}x\left(k-j\right). \tag{5}$$

Then the fractional feedback control is given by

$$\Delta^{\gamma} u_k = L \sum_{j=0}^{k} (-1)^j \binom{\gamma}{j} x(k-j). \tag{6}$$

We replace $\Delta^{\gamma} x_k$ and $\Delta^{\gamma} u_k$ by its value, system (6) can be rewritten by

$$\begin{aligned}
x_{k+1} &= A_d x_k - \sum_{j=1}^{k+1} (-1)^j \binom{\gamma}{j} x(k-j+1) + BL \sum_{j=1}^{k} (-1)^j \binom{\gamma}{j} x(k-j) \\
&= \left(A_d - \binom{\gamma}{1} \right) x_k - \sum_{j=1}^{k+1} (-1)^j \binom{\gamma}{j} x(k-j+1) \\
&\quad + BL \sum_{j=1}^{k} (-1)^j \binom{\gamma}{j} x(k-j) \\
&= \tilde{A} x_k - \sum_{j=2}^{k+1} (-1)^j \binom{\gamma}{j} x(k-j+1) + BL \sum_{j=1}^{k} (-1)^j \binom{\gamma}{j} x(k-j). \quad (7)
\end{aligned}$$

Set $a_j = (-1)^j \binom{\gamma}{j}$ and $b_j = (-1)^j \binom{\gamma}{j}$. Equation (7) can be rewritten as

$$x_{k+1} = \tilde{A} x_k - \sum_{j=2}^{k+1} a_j x(k-j+1) + BL \sum_{j=1}^{k} b_j x(k-j).$$

Let us now write

$$A_0 = A_d - a_1 I_n \qquad \text{and} \qquad B_0 = b_1 I_n,$$

and further, for all $j \geq 0$,

$$A_j = -a_{j+1} I_n \qquad \text{and} \qquad B_j = b_{j+1} I_n,$$

which leads

$$\begin{cases}
x_{k+1} = \sum\limits_{j=0}^{k} A_j x_{i-j} + BL \sum\limits_{j=0}^{k} B_j x_{i-j} \\
\qquad = \sum\limits_{j=0}^{k} (A_j + BLB_j)\, x_{i-j}, \\
y_k = C x_k,
\end{cases}$$

and u_k is the control low (feedback control) defined by

$$\Delta^{\gamma} u_k = L x_k = L \sum_{j=0}^{k} B_j x_{k-j},$$

where

$$
\begin{aligned}
&A_0 = A + \binom{\gamma}{1} I_n \qquad \text{and} \\
&A_j = -\left(-1\right)^{j+1} diag\left\{\binom{\gamma}{j+1}, \ldots, \binom{\gamma}{j+1}\right\}, \qquad \text{for all } j \geq 1. \\
&B_j = \left(-1\right)^{j} diag\left\{\binom{\gamma}{j}, \ldots, \binom{\gamma}{j}\right\}, \qquad \text{for all } j \geq 0.
\end{aligned} \tag{8}
$$

Then, putting $\tilde{A}_{B_j} = A_j + BLB_j$, we can write

$$x_{k+1} = \sum_{j=0}^{k} \tilde{A}_{B_j} x_{k-j}.$$

Thus, successively, we build the recurrence beginning with

$x(1) = \tilde{A}_{B_0} x_0 = \left(A_0 + BLB_0\ \right) x_0 \ = \left(A + \binom{\gamma}{1} I_n + BL\ \right) x_0,$

$x(2) = \tilde{A}_{B_1} x_1 = \left(A_0 + BLB_0\ \right) x_1 \ + \left(A_1 + BLB_1\ \right) x_0\ , \ldots.$

This leads to the following relation between the two column sequences:

$$
\begin{bmatrix} x\left(1\right) \\ x\left(2\right) \\ x\left(3\right) \\ \vdots \\ x\left(k+1\right) \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
A_0 + BLB_0 & 0 & 0 & 0 \cdots \\
A_1 + BLB_1 & A_0 + BLB_0 & 0 & 0 \cdots \\
A_2 + BLB_2 & A_1 + BLB_1 & A_0 + BLB_0 & 0 \cdots \\
\vdots & \vdots & \vdots & \vdots\ \vdots \\
\vdots & \vdots & \vdots & \vdots\ \vdots \\
\vdots & \vdots & \vdots & \vdots\ \vdots
\end{bmatrix}
\begin{bmatrix} x\left(0\right) \\ x\left(1\right) \\ x\left(2\right) \\ \vdots \\ x\left(k\right) \\ \vdots \end{bmatrix}.
$$

The state space model below represents the discrete-time fractional-order system:

$$
\begin{cases}
x_{k+1} = \sum_{j=0}^{k} \tilde{A}_{B_j} x_{k-j}, \\
x\left(0\right) = \quad x_0 \in \mathbb{R}^n, \\
\quad y_k \quad = \qquad Cx_k.
\end{cases} \tag{9}
$$

Remark 1. (i) It can be inferred from (9) that the fractional system exhibits a resemblance to a system characterized by an increasing number of delays.

(ii) Equations (8) – (9) suggest that the coefficients of A_j and B_j become negative when γ and γ lie in the interval $(0, 1)$, and their magnitudes decrease rapidly as j increases.

For practical applicability, it is necessary to limit the number of considered samples to a predefined quantity l, which is referred to as the memory length [7].

In this case the system (9) is rewritten as

$$\begin{cases} x_{k+1} = \sum\limits_{j=0}^{l} \tilde{A}_{B_j} x_{k-j}, \\ x(0) = \quad x_0 \in \mathbb{R}^n, \\ \quad y_k \quad = \qquad Cx_k. \end{cases} \tag{10}$$

Let us define the transition matrix form N_{B_k} such that

$$N_{B_k} = \begin{cases} I_{n_d} & \text{for } \ k = 0, \\ \sum\limits_{j=0}^{l} \tilde{A}_{B_j} N_{B((k-j-1)} & \text{for } \ k \geq 1. \end{cases}$$

The general solution of (10) can be written as follows:

$$x(k) = N_{B_k} x_0, \qquad \text{for all } k \geq 0.$$

Remark 2. If $\gamma = 1$, then

$$x_k = (A + I + BL)^k x_0.$$

Definition 2. Let $\mathcal{R} \subset \mathbb{R}^p$ and let $x_0 \in \mathbb{R}^n$. The initial state x_0 is said $\mathcal{R}$-output admissible if

$$y_k \in \mathcal{R}, \qquad \text{for all } k \in \mathbb{N}.$$

The set of all such initial states is described by

$$\Upsilon(\mathcal{R}) = \{x_0 \in \mathbb{R}^n / Cx_0 \in \mathcal{R}, CN_{B_k} x_0 \in \mathcal{R}, \text{ for all } k \geq 1\},$$

or

$$\Upsilon(\mathcal{R}) = \left\{ x_0 \in \mathbb{R}^n / Cx_0 \in \mathcal{R}, C \sum_{j=0}^{l} \tilde{A}_{B_j} N_{B(k-j-1)} x_0 \in \mathcal{R}, \text{ for all } k \geq 1 \right\}.$$

This manuscript presents a new approach to the identification of the set of initial states that satisfy point-in-time criteria for the resultant output function, subject to certain conditions on the matrix B_k.

3 A characterization approach to $\Upsilon(\mathcal{R})$: Output admissible set analysis

The main goal of this section is to characterize, under certain assumptions, the set $\Upsilon(\mathcal{R})$ of all initial states that lead to trajectories satisfying the constraint set. We establish the finite determinate of $\Upsilon(\mathcal{R})$ and, as a consequence, devise an algorithmic procedure for its computation. To achieve this, we introduce the sets $\Upsilon_k(\mathcal{R})$ for each $k \geq 0$ and provide a detailed description of their properties as follows:

$$\Upsilon_k(\mathcal{R}) = \left\{x_0 \in \mathbb{R}^n / Cx_0 \in \mathcal{R}, CN_{B_k}x_0 \in \mathcal{R}, \text{ for all } k \in \{1, \ldots, i\}\right\}.$$

Definition 3. Assuming the existence of an integer k^* for which $\Upsilon_{k^*}(\mathcal{R})$ equals $\Upsilon(\mathcal{R})$, the set $\Upsilon(\mathcal{R})$ is deemed to be finitely determined.

Definition 4. The dynamical system defined by (10) is acknowledged as asymptotically stable provided that for every $k \geq 1$

$$\lim_{k\to\infty} y_k = 0.$$

Consequently, the system defined by (10) achieves asymptotic stability if and only if

$$\|N_{B_k}\| \leq 1, \qquad for \quad k \geq 1.$$

By imposing specific conditions on N_{B_k}, for $k \geq 0$ and $\mathcal{R}$, we inherently introduce corresponding constraints on $\Upsilon(\mathcal{R})$. The following result presupposes that the interior of $\mathcal{R}$ is considered; a condition that is typically met in practical applications and has favorable consequences (see [13] for further insights).

Proposition 1. Should the system described by (10) achieve asymptotic stability, along with $0 \in \mathring{\mathcal{R}}$ and the norm of matrix C being bounded by 1. Then

$$0 \in \mathring{\Upsilon}(\mathcal{R}).$$

Remark 3. In the following proof, we denote the open ball centered at 0 with radius η as $\mathcal{B}_\eta(0)$. This notation represents the set of all points within a distance η from the origin.

Proof. Suppose that 0 belongs to the interior of $\mathring{\mathcal{R}}$. Consequently, there exists $\eta > 0$ such that the ball $\mathcal{B}_\eta(0)$ is contained within $\Upsilon(\mathcal{R})$.

If the system described by Equation (10) is asymptotically stable, then $\|N_{B_k}\| \leq 1, \qquad$ for all $k \geq 0$.

Let z be an element of $\mathcal{B}_\eta(0)$ and let $k \geq 0$. We observe

$$\begin{aligned}\|CN_{B_k}z\| &\leq \|C\|\,\|N_{B_k}\|\,\|z\| \\ &\leq \|C\|\,\|N_{B_k}\|\,\eta \\ &\leq \eta.\end{aligned}$$

This implies that $CN_{B_k}z$ lies within $\mathcal{B}_\eta(0)$ for every z in $\mathcal{B}_\eta(0)$ and for all $k \geq 0$. Consequently, we have

$$CN_{B_k}z \in \mathcal{R}, \qquad \text{for all } z \in \mathcal{B}_\eta(0)\,, \text{ for all } k \geq 0.$$

This inclusion yields

$$\mathcal{B}_\eta(0) \subset \Upsilon(\mathcal{R}).$$

Thus, we conclude that $0 \in \mathring{\Upsilon}(\mathcal{R})$. □

Proposition 2. If $\mathcal{R}$ is convex, symmetry, and closed, then the set $\Upsilon(\mathcal{R})$ is also

(i) Convex,

(ii) Symmetric,

(iii) Closed.

Proof. (i) Let $z_0^1, z_0^2 \in \Upsilon(\mathcal{R})$, $\theta \in \,]0,1[\,$, and $k \geq 0$. We show that $\theta z_0^1 + (1-\theta)\,z_0^2 \in \Upsilon(\mathcal{R})$ and z_0^1, $z_0^2 \in \Upsilon(\mathcal{R})$ implies that $z_0^1, z_0^2 \in \mathbb{R}^n$ such that $CN_{B_k}z_0^1 \in \mathcal{R}$ and $CN_{B_k}z_0^2 \in \mathcal{R}$.

We have

$$CN_{B_k}\left(\theta z_0^1 + (1-\theta)\,z_0^2\right) = \theta CN_{B_k}z_0^1 + (1-\theta)\,CN_{B_k}z_0^2.$$

We assume that $\mathcal{R}$ is convex, and that $CN_{B_k}\left(\theta z_0^1 + (1-\theta)\,z_0^2\right) \in \mathcal{R}$ and $\theta z_0^1 + (1-\theta)\,z_0^2 \in \Upsilon(\mathcal{R})$.

Then $\Upsilon(\mathcal{R})$ is convex.

(ii) From the definition of $\Upsilon(\mathcal{R})$, it is clear.

(iii) We define the function G_k by

$$\begin{aligned} G_k : \mathbb{R}^n &\to \quad \mathbb{R}^p \\ z \quad &\to CN_{B_k}z. \end{aligned}$$

Then $(G_k)_{k\geq 0}$ are continuous.

Let $G_k^{-1}(\mathcal{R}) = \{z \in \mathbb{R}^n \,/ CN_{B_k}z \in \mathcal{R}\}$. Then

$$\Upsilon(\mathcal{R}) = \bigcap_{k\geq 0} G_k^{-1}(\mathcal{R}).$$

Since $(G_k)_{k\geq 0}$ are continuous and $\mathcal{R}$ is closed, $G_k^{-1}(\mathcal{R})$, $k \geq 0$ are closed.

Hence, $\Upsilon(\mathcal{R})$ is proven to be closed. Moreover, in the case where $\mathcal{R}$ exhibits symmetry, it follows that $\Upsilon(\mathcal{R})$ also inherits this property of symmetry.

Thus, the demonstration is concluded.

□

The imposition of specific requirements on both N_{B_k} (where $k > 0$) and $\mathcal{R}$ naturally extends to analogous conditions for $\Upsilon(\mathcal{R})$. The ensuing outcome presupposes the inclusion of the origin, 0, within the interior of $\mathcal{R}$, denoted as $\mathring{\mathcal{R}}$. This assumption is not only consistent with most practical applications but also yields favorable outcomes, as elaborated in reference [13].

Proposition 3. Under the condition that $\Upsilon(\mathcal{R})$ is determined to be finite, there exists a natural number k such that $\Upsilon_k(\mathcal{R}) = \Upsilon_{k+1}(\mathcal{R})$.

Proof. Let us assume that $\Upsilon(\mathcal{R})$ is determined to be finite. Then, there exists $(k \in \mathbb{N})$, such that $\Upsilon(\mathcal{R}) = \Upsilon_k(\mathcal{R})$. On the other hand, $\Upsilon_k(\mathcal{R}) = \Upsilon(\mathcal{R}) \subset \Upsilon_k(\mathcal{R})$. Since $\{\Upsilon_k(\mathcal{R})\}_{k\geq 0}$ is decreasing sequence then $\Upsilon_{k+1}(\mathcal{R}) \subset \Upsilon_k(\mathcal{R})$.

As a result, we have $\Upsilon_k(\mathcal{R}) = \Upsilon_{k+1}(\mathcal{R})$, for some $k \geq 0$, thereby concluding the proof. □

We now introduce an intriguing result that enables us to determine the set $\Upsilon(\mathcal{R})$ through a finite number of inequalities. This, in turn, paves the way for the development of an algorithmic approach to calculate the admissibility index k^*.

In the following result, the set is taken as $\mathcal{R} = \{y \in \mathbb{R}^p \,/ \|y\| \leq \xi\}$.

In our pursuit of a solution, we explore two distinct scenarios:

a) In the first scenario, we assume that $\dim \mathcal{R} = n$, signifying that both the observation space and the state space exhibit identical dimensions.

b) Under the second scenario, we assume that $\dim \mathcal{R} < n$, indicating a discrepancy in dimensions between the observation space and the state space.

Considering the first scenario where $\dim \mathcal{R} = n$, the matrix C takes the form of an $n \times n$ matrix.

Proposition 4. Assume that the following conditions are met:

Condition (i) $\sum_{j=0}^{l} \left\|\tilde{A}_{B_j}\right\| \leq 1$ and $\Upsilon_k(\mathcal{R}) = \Upsilon_{k+1}(\mathcal{R})$ for some k,

Condition (ii) C commutes with $\tilde{A}_{B_j}$ for all $0 \leq j \leq L$.

Then, it follows that $\Upsilon(\mathcal{R})$ is finitely determined.

Proof. It is evident that $\Upsilon(\mathcal{R}) \subset \Upsilon_k(\mathcal{R})$.

Assume $x_0 \in \Upsilon_k(\mathcal{R})$. Then $z_k = CN_{B_k}x_0 \in \mathcal{R}$ for all $k \in \{0,1,2,\dots,i+1\}$, implying $\|z_k\| \leq \xi$ for all $k \leq i+1$.

For $k = i+2$, we have

$$\begin{aligned}
\|z_k\| &= \|CN_{B_k}x_0\| \\
&= \left\| C\sum_{j=0}^{l} \tilde{A}_{B_j} N_{B((k-j-1)}x_0 \right\| \\
&= \left\| C\sum_{j=0}^{l} \tilde{A}_{B_j} N_{B((k-j-1)}x_0 \right\| \\
&= \left\| \sum_{j=0}^{l} \tilde{A}_{B_j} CN_{B((k-j-1)}x_0 \right\| \\
&\leq \sum_{j=0}^{l} \left\| \tilde{A}_{B_j} \right\| \left\| CN_{\mathcal{B}((0k-j-1)}x_0 \right\| \\
&\leq \xi \sum_{j=0}^{l} \left\| \tilde{A}_{B_j} \right\|.
\end{aligned}$$

Given that $\left\|CN_{B((k-j-1)}x_0\right\| \leq \xi$ holds for all $j \in \{0,\dots,L\}$, we employ the inequality $\sum_{j=0}^{l} \left\|\tilde{A}_{B_j}\right\| < 1$ to deduce that $\left\|CN_{B((k+2)}x_0\right\| \leq \xi$.

Through iterative reasoning, we establish that $\left\|CN_{\mathcal{B}((0k+j)}x_0\right\| \leq \xi$ for all $j \geq 2$.

This implies $\|CN_{B_i}x_0\| \leq \xi$ for all $i \geq k+2$, consequently leading to $\|CN_{B_i}x_0\| \leq \xi$ for all $i \geq 0$.

As a result, $x_0 \in \Upsilon(\mathcal{R})$, thereby confirming the finiteness of $\Upsilon(\mathcal{R})$.

□

In the second scenario where $\dim \mathcal{R} = p < n$, considering that the matrix $C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, let us define $\tilde{C}$ and $\tilde{\mathcal{R}}$ as follows:

$\tilde{C} = \begin{pmatrix} C \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\tilde{\mathcal{R}} = \mathcal{R} \times \{0_{\mathbb{R}^{n-p}}\} \subset \mathbb{R}^n$.

Now, let us explore the implications of the new observation $\tilde{y}_i = \tilde{C}x_i$. Upon careful examination, it is evident that for every integer i,

$$y_i \in \mathcal{R} \Longleftrightarrow \tilde{y}_i \in \tilde{\mathcal{R}}.$$

Remark 4. It is interesting to note that the output admissibility of an initial state, denoted by x_0, regarding C and $\mathcal{R}$, is equivalent to its output admissibility regarding $\tilde{C}$ and $\tilde{\mathcal{R}}$.

The findings from the initial case inform the derivation of the subsequent statement, given that $\dim \tilde{\mathcal{R}} = n$.

Proposition 5. Assume that the following conditions are met:
Condition (i) $\sum_{j=0}^{l} \left\| \tilde{A}_{B_j} \right\| \leq 1$ and $\Upsilon_k(\mathcal{R}) = \Upsilon_{k+1}(\mathcal{R})$ for some k,
Condition (ii) $\tilde{C}$ commutes with $\tilde{A}_{B_j}$ for all $0 \leq j \leq L$.
Then, it follows that $\Upsilon(\mathcal{R})$ is finitely determined.

In section 4, we will present an algorithmic method designed to determine the minimum integer value, denoted as i^*, where $\Upsilon(\mathcal{R}) = \Upsilon_{k^*}(\mathcal{R})$.

4 Algorithmic determination

Drawing directly from the implications of Propositions 3 and 4, this section outlines a systematic approach to ascertain k^*, the admissibility index, and consequently, the set $\Upsilon(\mathcal{R})$.

Let $\mathbb{R}^p$ be endowed with the following norm:

$$\|x\|_\infty = \max_{1 \leq i \leq p} |x_i| , \qquad \text{for all } x \in \mathbb{R}^p.$$

Define $\mathcal{R}$ as $\mathcal{R} = \{y \in \mathbb{R}^p | h_j(y) \leq 0, j = 1, 2, \ldots, 2p\}$, where $h_j : \mathbb{R}^p \to \mathbb{R}$ are given functions. Such sets hold particular relevance in practical scenarios.

In this instance, $\Upsilon_k(\mathcal{R})$ is described for every integer i as

$$\Upsilon_k(\mathcal{R}) = \{x_0 \in \mathbb{R}^n | h_j(CN_{B_i}x) \leq 0, j = 0, \ldots, 2p, i = 0, \ldots, k\} .$$

On the other hand,

$$\begin{aligned}\Upsilon_{k+1}(\mathcal{R}) &= \left\{x_0 \in \Upsilon_k(\mathcal{R}) \,/ CN_{B_{k+1}}x_0 \in \mathcal{R} \right\} \\ &= \left\{x_0 \in \Upsilon_k(\mathcal{R}) \,/ h_j(CN_{B_{k+1}}x_0) \leq 0 \quad for\ j = 1, \ldots, 2p\right\} .\end{aligned}$$

Now, since $\Upsilon_{k+1}(\mathcal{R}) \subset \Upsilon_k(\mathcal{R})$ for every k, then

$$\begin{aligned}\Upsilon_{k+1}(\mathcal{R}) = \Upsilon_k(\mathcal{R}) &\iff \Upsilon_k(\mathcal{R}) \subset \Upsilon_{k+1}(\mathcal{R}) \\ &\iff \text{ for all } x_0 \in \Upsilon_k(\mathcal{R}), \quad h_j(CN_{B_{k+1}}x_0) \leq 0 \ ,\end{aligned}$$

$$\text{for all} \quad j = \{1, \dots, 2p\}$$

$$\Longleftrightarrow \sup_{\substack{x_0 \in \mathbb{R}^n, h_k(CN_{B_l} x_0) \leq 0 \\ \text{for all } k=\{1,\dots,2p\} \text{ for all } l \in \{0,\dots,i\}}} h_j(CN_{B_{k+1}} x_0) \leq 0 \,,$$

$$\text{for all} \quad j = \{1, \dots, 2p\} \,.$$

As a result, the equivalence $\Upsilon_{k+1}(\mathcal{R}) = \Upsilon_k(\mathcal{R})$ generates a collection of mathematical programming problems.

Algorithm 1 Determination of k^*

Require: $n, p, l \in \mathbb{N}^*, C, N_{B_i}, \xi > 0$
$k \longleftarrow 0$
For $j = 1$ **to** p Maximize $J_j(x) = \left|\left(CN_{B_{k+1}} x\right)_j\right| - \xi$
Subject to the constraints:
$$\begin{cases} \left|\left(CN_{B_{k+1}} x\right)_j\right| - \xi \\ \text{for all} \quad k = \{1, \dots, 2p\} \quad \text{for all} \quad l \in \{0, \dots, i\} \end{cases} \quad J_j^* \longleftarrow \max\{J_j(x)\}$$
If $J_j^* \leq 0$ for all $j = 1, \dots, p$, **then** $k^* \longleftarrow 0$

To showcase our findings, we will provide concrete numerical examples in section 6.

5 Criteria ensuring finite determination of $\Upsilon_k(\mathcal{R})$

The theorem presented below represents the pivotal result in our endeavor, focusing on the notion that having straightforward conditions to attain the finite determination of the set $\Upsilon(\mathcal{R})$ is highly preferable.

Theorem 1. If $\|N_{B_k}\| \leq \gamma_k$, for all $k \geq 0$, with $\gamma_k \longrightarrow 0$ when $k \longrightarrow \infty$, then $\Upsilon(\mathcal{R})$ is finitely determined.

Proof. We have $\|N_{B_k}\| \leq \gamma_k \Longrightarrow \|CN_{B_k} x\| \leq \gamma_k \|C\| \|x\|$, $\gamma_k \longrightarrow 0$, when $k \longrightarrow \infty$. Then there exists an integer i_0 such that $\|CN_{B_k} x\| \leq \varepsilon$, for all $k \geq i_0$, for all $x \in \mathbb{R}^n$.

Hence $\left\|CN_{B((i_0+1)} x\right\| \leq \varepsilon$, for all $x \in \Upsilon_{k_0}(\mathcal{R})$, and we have $\Upsilon(\mathcal{R}) \subset \Upsilon_{k_0+1}(\mathcal{R}) \subset \Upsilon_{k_0}(\mathcal{R})$.

Consequently, $\Upsilon(\mathcal{R}) = \Upsilon_{k_0+1}(\mathcal{R}) = \Upsilon_{k_0}(\mathcal{R})$.

Then $\Upsilon(\mathcal{R})$ is finitely determined. $\square$

6 Numerical examples

In order to clarify our outcomes, we showcase numerical instances, utilizing a carefully crafted algorithm. This approach allows us to precisely define the set $\Upsilon(\mathcal{R})$ through a finite set of inequalities, providing substantial support for our underlying hypothesis $\sum_{j=0}^{l}\left\|\tilde{A}_{B_j}\right\| \leq 1$ holds in all instances presented. We strategically choose the matrix $A_0 = A + BL + \gamma I_n$ and $\tilde{A}_{B_0} = A_0 + BLB_0$.

We have

$$\sum_{j=0}^{l}(-1)^j\binom{\gamma}{j} = \frac{\Gamma(l+1-\gamma)}{\Gamma(1-\gamma)\ \Gamma(l+1)},$$

$$\sum_{j=0}^{l}(-1)^j\binom{\gamma}{j} = \frac{\Gamma(l+1-\gamma)}{\Gamma(1-\gamma)\ \Gamma(l+1)},$$

and the fact

$$\sum_{j=0}^{l}\left\|\tilde{A}_{B_j}\right\| = \sum_{j=0}^{l}\|A_j + BLB_j\| = \|A_0 + BLB_0\| + \sum_{j=1}^{l}\|A_j + BLB_j\|.$$

Example 1. Consider the following system:

$$\begin{cases} x(k+1) = \sum\limits_{j=0}^{l}\tilde{A}_{B_j}x_{k-j}, \\ \quad x_0 \quad = \quad x(0) \in \mathbb{R}^n. \end{cases}$$

The matrices A_j and B_j are defined as $A_j = -a_{j+1}I_n$ and $B_j = b_{j+1}I_n$, where $a_j = (-1)^j\binom{\gamma}{j}$ and $b_j = (-1)^j\binom{\gamma}{j}$.

The parameters γ, γ, ε, n, and the matrices A, B, L, and C are specified as follows: $\gamma = 0.25$, $\varepsilon = 0.5$, $\gamma = 0.25$, $n = 3$, $C = \begin{pmatrix} 1 & -1 \end{pmatrix}$, $A = \begin{pmatrix} \frac{1}{54} & \frac{1}{16} \\ -\frac{1}{32} & \frac{1}{12} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix}$, and $L = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Then

$$\begin{aligned}
N_{B_0} &= I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
N_{B_1} &= A_{B_0} = A_0 + BLB_0 = A + BL + I_2 + BLB_0 = A + I_2 + BL\left(I_2 + B_0\right) \\
&= \begin{pmatrix} \frac{1}{54} & \frac{1}{16} \\ -\frac{1}{32} & \frac{1}{12} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.25\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1.0498 & 0.09375 \\ -1.0417 \times 10^{-2} & 1.1042 \end{pmatrix},
\end{aligned}$$

$$
\begin{aligned}
N_{B_2} &= A_{B_0}^2 + A_{B_1} = (A_0 + BLB_0)^2 + A_1 + BLB_1 \\
&= \begin{pmatrix} 1.0498 & 0.09375 \\ -1.0417\times 10^{-2} & 1.1042 \end{pmatrix}^2 - 0.09375 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad -0.25 \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.99694 & 0.19152 \\ -2.9383\times 10^{-2} & 1.1176 \end{pmatrix}, \\
N_{B_3} &= A_{B_0}^3 + A_{B_1}^2 + A_{B_2} = (A_0 + BLB_0)^3 + (A_1 + BLB_1)^2 + (A_2 + BLB_2) \\
&= \begin{pmatrix} 1.0498 & 0.09375 \\ -1.0417\times 10^{-2} & 1.1042 \end{pmatrix}^3 \\
&\quad + \left(-0.09375 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.25 \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^2 \\
&\quad + \left(-5.4688\times 10^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.09375 \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1.1314 & 0.3497 \\ -2.0582\times 10^{-2} & 1.3129 \end{pmatrix}, \\
N_{B_4} &= A_{B_0}^4 + A_{B_1}^3 + A_{B_2}^2 + A_{B_3} = (A_0 + BLB_0)^4 + (A_1 + BLB_1)^3 \\
&\quad + (A_2 + BLB_2)^2 + (A_3 + BLB_3) \\
&= \begin{pmatrix} 1.0498 & 0.09375 \\ -1.0417\times 10^{-2} & 1.1042 \end{pmatrix}^4 \\
&\quad + \left(-0.09375 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.25 \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^3 \\
&\quad + \left(-5.4688\times 10^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.09375 \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^2 \\
&\quad + \left(3.7598\times 10^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - -3.7598\times 10^{-2} \begin{pmatrix} \frac{1}{12} \\ \frac{1}{18} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1.2141 & 0.47237 \\ -4.9375\times 10^{-2} & 1.4845 \end{pmatrix},
\end{aligned}
$$

$$
CN_{B_0} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x - y,
$$

$$\begin{aligned}
CN_{B_1}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1.0498 & 0.09375 \\ -1.0417\times 10^{-2} & 1.1042 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= 1.0602x - 1.0105y, \\
CN_{B_2}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 0.99694 & 0.19152 \\ -2.9383\times 10^{-2} & 1.1176 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= 1.0263x - 0.92608y, \\
CN_{B_3}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1.1314 & 0.3497 \\ -2.0582\times 10^{-2} & 1.3129 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= 1.1520x - 0.9632y, \\
CN_{B_4}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1.2141 & 0.47237 \\ -4.9375\times 10^{-2} & 1.4845 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= 1.2635x - 1.0121y.
\end{aligned}$$

Algorithm:

$k = 0$,

$i = 1$,

Maximize $J_1(x) = 1.0602x - 1.0105y - 0.1$,

S.C: $x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

$k = 0$,

$i = 2$,

Maximize $J_2(x) = -1.0602x + 1.0105y - 0.1$,

S.C: $x - y - 0.1 \leq 0.1$ and $-x + y - 0.1 \leq 0$,

We have $J_1^*, J_2^* \geq 0$. Then go to the next step.

$k = 1$,

$i = 1$,

Maximize $J_1(x) = 0.16595x - 6.8985\times 10^{-2}y - 0.1$,

S.C: $1.0602x - 1.0105y - 0.1 \leq 0$ and $-1.0602x + 1.0105y - 0.1 \leq 0$,

$x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

$k = 1$,

$i = 2$,

Maximize $J_2(x) = -0.16595x + 6.8985\times 10^{-2}y - 0.1$,

S.C: $1.0602x - 1.0105y - 0.1 \leq 0$ and $-1.0602x + 1.0105y - 0.1 \leq 0$,

$x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

We have $J_1^*, J_2^* \geq 0$. Then go to the next step.

$k = 2$,

$i = 1$,

Maximize $J_1(x) = 1.152\,0x - 0.963\,2y - 0.1$,

S.C: $0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$ and $-0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$,

$1.060\,2x - 1.010\,5y - 0.1 \leq 0$ and $-1.060\,2x + 1.010\,5y - 0.1 \leq 0$,

$x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

$k = 2$,

$i = 2$,

Maximize $J_2(x) = -1.152\,0x + 0.963\,2y - 0.1$,

S.C : $0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$ and $-0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$,

$1.060\,2x - 1.010\,5y - 0.1 \leq 0$ and $-1.060\,2x + 1.010\,5y - 0.1 \leq 0$,

$x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

We have $J_1^*, J_2^* \geq 0$. Then go to the next step.

$k = 3$,

$i = 1$,

Maximize $J_1(x) = 1.263\,5x - 1.012\,1y - 0.1$,

S.C : $1.152\,0x - 0.963\,2y - 0.1 \leq 0$ and $-1.152\,0x + 0.963\,2y - 0.1 \leq 0$,

$0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$ and $-0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$,

$1.060\,2x - 1.010\,5y - 0.1 \leq 0$ and $-1.060\,2x + 1.010\,5y - 0.1 \leq 0$,

$x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

$k = 3$,

$i = 1$,

Maximize $J_2(x) = -1.263\,5x + 1.012\,1y - 0.1$,

S.C: $1.152\,0x - 0.963\,2y - 0.1 \leq 0$ and $-1.152\,0x + 0.963\,2y - 0.1 \leq 0$,

$0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$ and $-0.165\,95x + 6.898\,5 \times 10^{-2}y - 0.1 \leq 0$,

$1.060\,2x - 1.010\,5y - 0.1 \leq 0$ and $-1.060\,2x + 1.010\,5y - 0.1 \leq 0$,

$x - y - 0.1 \leq 0$ and $-x + y - 0.1 \leq 0$,

Since $J_1^*, J_2^* \leq 0$, we stop.

Then we use Algorithm II to prove that $k^* = 3$, and we have

$$\mathcal{R} = \{x \in \mathbb{R} \,||x| \leq 1\} = [-1, 1],$$

$$\Upsilon_3(\mathcal{R}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \left| \begin{array}{c} |x - y| \leq 0.1, |1.060\,2x - 1.010\,5y| \leq 0.1, \\ |1.026\,3x - 0.926\,08y| \leq 0.1, \\ |1.152\,0x - 0.963\,2y| \leq 0.1 \end{array} \right. \right\}.$$

Example 2. Consider the following system:

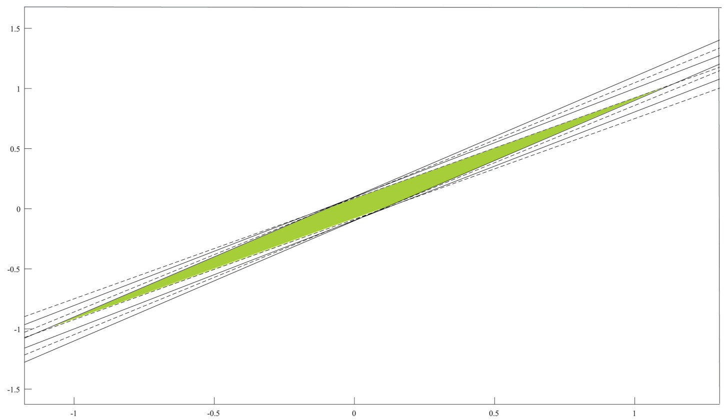

Figure 1: The shaded area represents the set $\Upsilon_3(\mathcal{R})$ as illustrated in Example 1 with $\gamma = 0.25$

$$\begin{cases} x(k+1) = \sum_{j=0}^{l} \tilde{A}_{B_j} x_{k-j}, \\ x_0 \quad = \quad x(0) \in \mathbb{R}^n. \end{cases}$$

We define the parameters γ, γ, ε, n, and the matrices A, B, L, and C as follows: $\gamma = 0.25$, $\gamma = 0.25$, $\varepsilon = 0.4$, $n = 3$, $C = \begin{pmatrix} 1 & -1 \end{pmatrix}$, $A = \begin{pmatrix} \frac{35}{12} & -\frac{13}{7} \\ \frac{3}{32} & -\frac{11}{7} \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$, and $L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then

$$\begin{aligned}
N_{B_0} &= I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
N_{B_1} &= A_{B_0} = A_0 + BLB_0 \\
&= A + BL + I_2 + BLB_0 = A + I_2 + BL\left(I_2 + B_0\right) \\
&= \begin{pmatrix} \frac{35}{12} & -\frac{13}{7} \\ \frac{3}{32} & -\frac{11}{7} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.25 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1.6667 & -0.35714 \\ \frac{3}{32} & 0.17857 \end{pmatrix}, \\
A_{B_1} &= A_1 + BLB_1 = 0.09375 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.09375 \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.375 & -0.1875 \\ 0 & 0 \end{pmatrix}, \\
N_{B_2} &= A_{B_0}^2 + A_{B_1} = \left(A_0 + BLB_0\right)^2 + A_1 + BLB_1
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1.6667 & -0.35714 \\ \frac{3}{32} & 0.17857 \end{pmatrix}^2 + \begin{pmatrix} 0.375 & -0.1875 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 3.1194 & -0.84652 \\ 0.17299 & -1.5946 \times 10^{-3} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
N_{B_3} &= A_{B_0}^3 + A_{B_1}^2 + A_{B_2} = (A_0 + BLB_0)^3 + (A_1 + BLB_1)^2 + (A_2 + BLB_2) \\
&= \begin{pmatrix} 1.6667 & -0.35714 \\ \frac{3}{32} & 0.17857 \end{pmatrix}^3 + \begin{pmatrix} 0.375 & -0.1875 \\ 0 & 0 \end{pmatrix}^2 \\
&\quad + \left(\begin{pmatrix} 5.4688 \times 10^{-2} & 0 \\ 0 & 5.4688 \times 10^{-2} \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -5.4688 \times 10^{-2} & 0 \\ 0 & -5.4688 \times 10^{-2} \end{pmatrix} \right) \\
&= \begin{pmatrix} 4.8717 & -1.2775 \\ 0.28818 & -6.2068 \times 10^{-2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
CN_{B_0} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x - y \\
CN_{B_1} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1.6667 & -0.35714 \\ \frac{3}{32} & 0.17857 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1.5730x - 0.53571y \\
CN_{B_2} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 3.1194 & -0.84652 \\ 0.17299 & -1.5946 \times 10^{-3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 2.9464x - 0.84493y \\
CN_{B_3} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 4.8717 & -1.2775 \\ 0.28818 & -6.2068 \times 10^{-2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 4.5835x - 1.2154y
\end{aligned}
$$

Then we use Algorithm II to prove that $k^* = 1$ and we have

$$
\Upsilon_1(\mathcal{R}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x - y| \leq 0.4, \quad |1.5730x - 0.53571y| \leq 0.4 \right\}.
$$

Example 3. Consider the following system:

$$
\begin{cases} x(k+1) = \sum_{j=0}^{L} \tilde{A}_{B_j} x_{k-j}, \\ \quad x_0 \quad = \quad x(0) \in \mathbb{R}^n. \end{cases}
$$

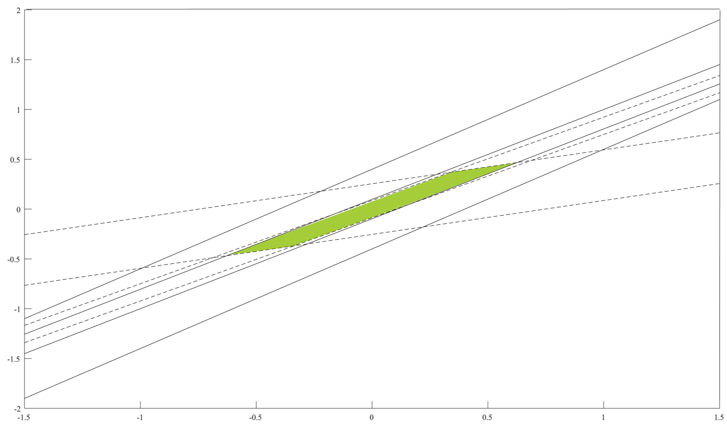

Figure 2: The shaded area represents the set $\Upsilon_1(\mathcal{R})$ as illustrated in Example 2 with $\gamma = 0.25$

Let the parameters $\gamma, \gamma, \varepsilon, n$ and the matrix A, B, L and C be defined by

$\gamma = 0.1,\ \ \varepsilon = 0.3, n = 2,\ \gamma = 0.1,\ C = \begin{pmatrix} 2 & -1 \end{pmatrix}, A = \begin{pmatrix} \frac{1}{48} & \frac{1}{16} \\ 0 & \frac{1}{40} \end{pmatrix}, B = \begin{pmatrix} \frac{1}{16} \\ \frac{1}{10} \end{pmatrix}, L = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$

Then

$$
\begin{aligned}
N_{B_0} &= I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
N_{B_1} &= N_{B_1} = A_{B_0} = A_0 + BLB_0 \\
&= A + BL + I_2 + BLB_0 = A + I_2 + BL\left(I_2 + B_0\right) \\
&= \begin{pmatrix} \frac{1}{48} & \frac{1}{16} \\ 0 & \frac{1}{40} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{16} \\ \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.25 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1.0443 & 8.5938 \times 10^{-2} \\ 0.0375 & 1.0625 \end{pmatrix}, \\
N_{B_2} &= A_{B_0}^2 + A_{B_1} = \left(A_0 + BLB_0\right)^2 + A_1 + BLB_1 \\
&= \begin{pmatrix} 1.0443 & 8.5938 \times 10^{-2} \\ 0.0375 & 1.0625 \end{pmatrix}^2 + 0.08 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad -0.2 \begin{pmatrix} \frac{1}{16} \\ \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1.1675 & 0.1748 \\ 6.9005 \times 10^{-2} & 1.2021 \end{pmatrix}, \\
N_{B_3} &= A_{B_0}^3 + A_{B_1}^2 + A_{B_2}
\end{aligned}
$$

$$
\begin{aligned}
&= (A_0 + BLB_0)^3 + (A_1 + BLB_1)^2 + (A_2 + BLB_2) \\
&= \begin{pmatrix} 1.0443 & 8.5938 \times 10^{-2} \\ 0.0375 & 1.0625 \end{pmatrix}^3 \\
&\quad + \left(0.08 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.2 \begin{pmatrix} \frac{1}{16} \\ \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^2 \\
&\quad + \left(-0.08 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{16} \\ \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} 0.08 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1.077 & 0.28797 \\ 0.12752 & 1.1386 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
CN_{B_0} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x - y, \\
CN_{B_1} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1.0443 & 8.5938 \times 10^{-2} \\ 0.0375 & 1.0625 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 2.0511x - 0.89062y, \\
CN_{B_3} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1.1675 & 0.1748 \\ 6.9005 \times 10^{-2} & 1.2021 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 2.2660x - 0.8525y, \\
CN_{B_4} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1.077 & 0.28797 \\ 0.12752 & 1.1386 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 2.0265x - 0.56266y.
\end{aligned}
$$

Then we use Algorithm II to prove that $k^* = 2$, and we have

$$
\begin{aligned}
\Upsilon_2(\mathcal{R}) = \Big\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \quad & |2x - y| \leq 0.3, \;\; |2.0511x - 0.89062y| \leq 0.3, \\
& |2.2660x - 0.8525y| \leq 0.3 \Big\}.
\end{aligned}
$$

Example 4. Let the parameters $\gamma, \gamma, \varepsilon, n$ and the matrix $\tilde{A}, B, L$ and C be defined by

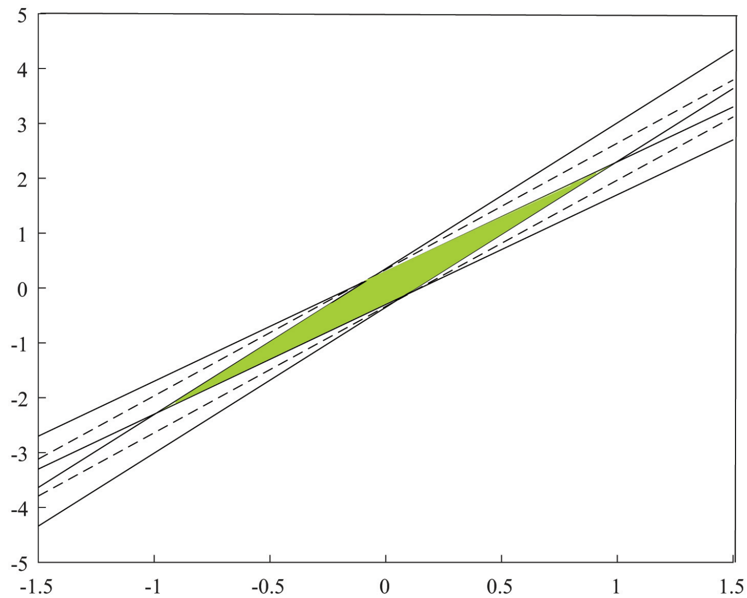

Figure 3: The shaded area represents the set $\Upsilon_2(\mathcal{R})$ as illustrated in Example 3 with $\gamma = 0.1$

$$\gamma = 0.7,\ \varepsilon = 0.5, n = 3,\quad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},\quad \tilde{A} = \begin{pmatrix} \frac{1}{12} & \frac{1}{9} & \frac{1}{7} \\ \frac{1}{15} & \frac{1}{13} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{15} & \frac{1}{14} \end{pmatrix},\quad B = \begin{pmatrix} \frac{1}{10} \\ \frac{1}{5} \\ \frac{1}{8} \end{pmatrix},\quad L = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

Also, $CN_{B_0}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CN_{B_1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CN_{B_2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CN_{B_3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are given by

$$\begin{aligned} CN_{B_0}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + y + z, \\ CN_{B_1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.85 & \frac{19}{90} & \frac{27}{140} \\ \frac{1}{5} & 0.976\,92 & \frac{21}{110} \\ \frac{19}{84} & \frac{23}{120} & 0.833\,93 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= 1.\,276\,2x + 1.\,379\,7y + 1.\,217\,7z, \\ CN_{B_2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.706\,84 & 0.427\,90 & 0.367\,69 \\ 0.415\,57 & 0.938\,69 & 0.389\,53 \\ 0.423\,60 & 0.401\,39 & 0.673\,93 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= 1.546x + 1.7680y + 1.4312z,\\
CN_{B_3}\begin{pmatrix} x\\ y\\ z\end{pmatrix} &= \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}0.90853 & 0.65023 & 0.53937\\ 0.63646 & 1.2192 & 0.59321\\ 0.60801 & 0.61878 & 0.85754\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}\\
&= 2.153x + 2.4882y + 1.9901z.
\end{aligned}
$$

Algorithm II is employed to demonstrate that $k^* = 2$ and to identify the MOS $\Upsilon_2(\mathcal{R})$ as follows:

$$
\Upsilon_2(\mathcal{R}) = \left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} \;\middle|\; \begin{array}{c} |x+y+z| \leq 0.5,\\ |1.2762x + 1.3797y + 1.2177z| \leq 0.5,\\ |1.546x + 1.7680y + 1.4312z| \leq 0.5, \end{array} \right\}.
$$

Comment

In Examples 1 and 2, the admissibility index k_0 and the MOS $\Upsilon(\mathcal{R})$ for vectors satisfying certain constraints were determined using the simplex method. This choice was made in order to solve the maximization problems inherent in Algorithm 2. The constraints characterizing the sets $\Upsilon(\mathcal{R})$ in Figures 1–3 have been visualized to provide a graphical insight into $\Upsilon(\mathcal{R})$, as illustrated in Examples 1–3.

7 Conclusion

This paper addressed a novel problem concerning the maximum output admissible set in discrete-time fractional control systems utilizing the Grünwald–Letnikov fractional derivative. It introduced a new sufficient condition to ensure system stability and characterizes the maximal output admissible set, denoted as $\Upsilon(\mathcal{R})$. The study demonstrated that, under specific conditions, finite determinations of $\Upsilon(\mathcal{R})$ can be achieved. Additionally, it proposes an effective algorithmic approach to determine the admissibility index k^* and establish the set using a finite number of inequalities. The application of the algorithm to controlled scenarios is exemplified through numerical examples, effectively showcasing the theoretical results in both two and three dimensions. As a natural continuation of this work, future investigations will address the extension of the proposed fractional-order control methodology to nonlinear dynamics as well as to infinite-dimensional systems, where additional theoretical challenges and practical implications arise.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to

Availability of data and materials

Data is available upon request.

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Conflicts of Interest

All authors jointly worked on the results and they read and approved the final manuscript, and also declare that they have no competing interests.

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