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Research Article

A mathematical model and optimal control analysis of merchant resistance to e-commerce adoption

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Abstract

In this study, we develop a mathematical model that captures the dynamics of merchants' attitudes toward the adoption of e-commerce for trading activities. The merchant population is classified into four compartments: Traditional merchants, nonadopting merchants, active e-commerce users, and merchants who have temporarily suspended their online activities. The proposed model provides a quantitative framework for analyzing the persistence of e-commerce nonadoption among certain merchant groups and for identifying effective strategies to promote

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digital market participation. To this end, an optimal control problem is formulated with the objective of enhancing e-commerce adoption through suitable intervention policies. The Pontryagin maximum principle is applied to derive the necessary conditions for optimality, and numerical simulations implemented in MATLAB illustrate and validate the theoretical findings. Furthermore, a cost-effectiveness analysis is conducted to assess the efficiency of the proposed strategies, providing insight into the balance between implementation costs and adoption outcomes.

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Keywords: Mathematical modeling, Optimal control theory, E-commerce, Pontryagin maximum.

1 Introduction

E-commerce constitutes a fundamental pillar for national economic growth and integration into the global marketplace. It enhances the visibility of domestic products internationally, facilitates access to diverse markets, and provides consumers with the convenience of purchasing goods and services from virtually any location at any time. Such digital transformation not only contributes to greater efficiency in the use of resources, time, and effort but also indirectly supports public health and environmental objectives by reducing physical contact, pollution, and traffic congestion.

E-commerce broadly encompasses all activities related to the buying, selling, and exchange of goods and services conducted through the Internet using various digital devices such as computers, smartphones, and wearable technologies. In recent years, its global expansion has been remarkable, driven by technological innovation and increased connectivity. However, the effective implementation of electronic commerce remains dependent on the availability of robust communication infrastructures and the provision of adequate training for the management and operation of online commercial platforms.

E-commerce has encountered three major developments. The first was e-commerce based on EDI (Electronic Data Interchange) [33]. EDI is the exchange of files between business partners from one computer to another in a basic format. Generally, companies will have their systems to translate this format into that used by the firm itself [15]. The recipient of that file sent by another partner will be processed immediately as an invoice or a purchase order. The format, for its part, must contain the type of information that the document has. This system made it possible to improve purchasing processes as well as relationships throughout the supply chain [34].

Therefore, Internet-based e-commerce constituted the second phase. The primary drawbacks of EDI were its expensive price, its inability to adjust to B2C transactions, and its network that was underdeveloped outside of major multinational corporations. The smallest companies desired to take advantage of technology advancements as well. Transnational companies also required information to be shared at increasing rates. Then, the United States lifted the restriction on employing the Internet for business purposes in the 1990's. Consequently, e-commerce has seen brighter days and its first step into the future in 2004 [24]. Thus, more people started to use the Internet, which resulted in greater profits for merchants and industries [38]. Finally, greater development was noticed after the Internet became more popular in every household [1].

Lots of research are being made on this subject and governments are beginning to develop policies encouraging businesses to go digital. The emergence of shopping platforms such as Amazon [19], eBay [25], and Alibaba [39] made the experience of online shopping easier and worldwide.

However, the developments in e-commerce depend on its advantages and limitations [37]. e-commerce presents many advantages, unlike traditional commerce shops, due to its availability anytime anywhere [14, 20, 29]. A person can now purchase products whenever and wherever they choose, thanks to e-commerce, even from the comfort of their own home. Additionally, they have access to additional information about the features of the products and can use their computer to compare prices across multiple retailers. On the other hand, e-commerce encounters some limitations and drawbacks, especially for the government regarding the taxation problem [43]. Some academics investigate that point, where it is difficult to track the taxes on online shops [2].

Regarding the big importance that e-commerce takes in our lives and how it affects people's behavior, many academics investigated the relationship between people and e-commerce. In [6, 30], the authors investigated the correlation between e-commerce and people's trust in it, given the differences between a virtual and a conventional marketplace and how customers' trust affects their consumption and vision toward e-commerce. In that same vision, some other researchers discussed and illustrated the needs of users in e-commerce platforms regarding their age, education, and income [28]. Therefore, the important attention people and governments give to e-commerce opened many opportunities for entrepreneurs. Thus, academics gave attention to this topic and studied the relationship between e-commerce and entrepreneurship [36, 17].

Noting the important role e-commerce took in people's lives and the correlation it has with our behavior and consumption aptitude, it is primordial to better understand the process of e-commerce buying and selling, and how merchants and entrepreneurs act and invest. Consequently, a proper and effective modeling approach must be used, such as the compartmental model approach, which was first introduced by Kermack and McKendrick [23] in 1927 in the Proceedings of the Royal Society of London, which is a seminal work that laid the groundwork for the mathematical modeling of infectious diseases, principally named the SIR models. Inspired by the epidemiological modeling approach, a population can be divided into three main compartments susceptible, infected and recovered.

In recent years, fractional-order derivatives have gained considerable attention in the context of biological and epidemiological modeling, owing to their ability to capture memory and hereditary effects inherent in real-world processes. Their nonlocal properties allow for a more realistic description of biological dynamics, particularly those involving anomalous diffusion or delayed responses. For instance, fractional models have been employed to investigate complex infection and epidemic dynamics, such as multi-strain influenza with treatment effects [42], COVID-19 transmission incorporating Caputo-type fractional operators [21], and within-host viral infection models accounting for adaptive immunity [40, 35]. These studies highlight the growing importance of fractional calculus in enhancing the flexibility and realism of classical compartmental models. This modeling perspective can be extended beyond biological systems to other domains governed by human behavior and interaction, such as socio-economic and technological diffusion processes, thereby motivating the adoption of similar analytical approaches in the study of e-commerce dynamics.

Statement of problem.

The application of control theory to various problems has been explored by several authors, including the spread of rumours on social media. In [7], a study is presented where the author formulated a mathematical model and applied control theory to reduce the spread of rumours on social networks. Additionally, in [3], the author applied control theory to address nonregistration behavior in an effort to increase the number of people registered on the electoral roll. Further research on control theory can be found in [9, 22, 10, 26, 4].

Recent advances in compartmental modeling and optimal control have highlighted their utility in analyzing complex social and behavioral dynamics. For example, El Bhih et al. [8] developed a framework to study the spread of rumors and counter-rumors on social networks, formulating multiple equilibria and designing optimal interventions using Pontryagin's maximum principle to influence collective behavior. Similarly, studies on scam rumor propagation have employed deterministic models combined with cost-effectiveness analysis to evaluate strategies for mitigating false information on social platforms, emphasizing the efficiency of targeted interventions [5]. In the context of epidemiology, Laaroussi, El Bhih, and Rachik [27] demonstrated how optimal vaccination and treatment policies can be formulated under constrained resources for a multi-strain SEIR model of COVID-19, providing insight into balancing limited resources while achieving maximal impact. Furthermore, Yaagoub, El Bhih, and Allal [41] introduced a fractional-order infection model for computer virus propagation, illustrating how memory effects and fractional dynamics can improve the realism and predictive power of compartmental models.

Collectively, these studies underscore the versatility of compartmental and optimal control approaches for understanding the evolution of complex systems and designing effective intervention strategies. Inspired by these works, our study applies a similar framework to investigate merchants' adoption of e-commerce, where the dynamics of acceptance and refusal can be influenced through targeted control measures and assessed quantitatively for cost-effectiveness.

Building on these approaches, Hassouni, Belhdid, and Balatif [16] applied compartmental and optimal control modeling to study customer behavior toward e-commerce. Their model focused on the dynamics of refusal and adoption, analyzed stability and parameter sensitivity, and employed Pontryagin's maximum principle to optimize interventions. Numerical simulations confirmed the effectiveness of the proposed strategies, highlighting the relevance of control-based modeling for understanding behavioral dynamics in digital marketplaces. Inspired by this, our study extends the framework to examine merchant behavior, capturing interactions between adopters and nonadopters in commercial ecosystems and evaluating intervention strategies through cost-effectiveness analysis.

In our research, we formulate a new mathematical model in order to study the influence of merchants' behavior on e-commerce and their refusal to use e-commerce. We study the basic characteristics of the proposed $CAEP$ model to discover the various aspects of the proposed model and to better understand the process of store owner behavior towards e-commerce. We also apply control theory to this model in order to increase the number of merchants who use electronic commerce, which will reflect positively on them, whether in terms of income or increasing the number of customers. The spread of the electronic neighbor will inevitably lead to an increase in the number of commercial markets available to citizens. This will reflect positively on the state's economy.

The originality of the proposed $CAEP$ model lies in its focus on the dynamics of merchants who refuse to adopt e-commerce, in addition to those who already use it, thereby capturing interactions between adopters and nonadopters within the commercial ecosystem. Unlike previous studies that primarily examine general adoption patterns or consumer behavior, our model integrates an optimal control framework to identify effective strategies for promoting e-commerce adoption among reluctant merchants. Furthermore, the incorporation of a cost-effectiveness analysis provides quantitative guidance for policymakers and decision-makers, allowing them to evaluate trade-offs between intervention costs and adoption outcomes. Collectively, these features distinguish our work from existing literature and offer new insights into targeted interventions to enhance e-commerce penetration.

The paper is organized as follows. Section 2 introduces a mathematical model that analyzes merchants' behavior with respect to e-commerce. In Section 3, optimal control theory is applied to the model, and the positivity and boundedness of the solutions are established. Section 4 presents numerical simulations to assess the effectiveness of the proposed strategies. Finally, concluding remarks are provided in Section 5.

2 Formulation of the mathematical model

The study of epidemic spread has enabled researchers to adopt epidemiological models as a foundation for modeling the dynamics of e-commerce. Our research offers a qualitative extension to previous studies by introducing a novel mathematical model that describes merchant behavior in relation to electronic commerce.

In this section, we focus on the issue of merchants who refuse to use e-commerce for buying or selling their products. To address this, we propose the mathematical model $CAEP$, which aims to analyze the evolution of e-commerce usage among merchants and investigate the reasons behind its limited adoption by certain segments.

Furthermore, the model examines how merchants-whether or not they currently use e-commerce-may be influenced through direct interaction with those who resist adopting it. The proposed model consists of four compartments:

Compartment C: These are merchants who sell goods to customers or buy goods to sell. It also includes traders who produce goods and then sell them. Individuals in this compartment are considered to be nonusers of e-commerce, as they do not have the knowledge or skills required to set up an online shop. The number of merchants in this compartment increases with the number Λ_1, which represents the number of new merchants, and the number of merchants in this compartment decreases with the number θ, which represents the rate of natural deaths or cessation of trade, and $\gamma_1 CA$, which is the proportion of the number of traders affected by the refusal to use e-commerce as a result of contacts with traders who refuse to use e-commerce, and $\gamma_2 CE$, which is the proportion of the number of traders affected by the refusal to use e-commerce as a result of contacts with traders who use e-commerce, and $\theta_1 C$, which is the proportion of the number of traders influenced to refuse to use e-commerce as a result of negative thoughts about e-commerce, and $\theta_2 C$, which is the proportion of the number of traders influenced to use e-commerce as a result of positive thoughts about e-commerce. Hence

$$\frac{dC(t)}{dt} = \Lambda_1 - \gamma_1 C(t)A(t) - \theta_1 C(t) - \theta_2 C(t) - \gamma_2 C(t)E(t) - \theta C(t). \tag{1}$$

Compartment A: These are merchants who refuse to use e-commerce to sell or buy their products because of negative experiences with e-commerce such as fraud, cybercrime, or because of negative thoughts about e-commerce coming from the merchant who refuses to use e-commerce or a person who has negative thoughts about e-commerce. The number of this compartment is increased by Λ_2, which represents the number of new merchants who know about e-commerce but do not want to use it, and $\gamma_1 CA$, which represents the proportion of the number of merchants who have been affected by the refusal to use e-commerce as a result of contacting merchants who refuse to use e-commerce, and $\gamma_4 AE$, which represents the proportion of the number of merchants who were influenced by the refusal to use e-commerce as a result of direct contact with merchants who refuse to use e-commerce, and $\theta_1 C$, which represents the proportion of the number of merchants who were influenced by the refusal to use e-commerce as a result of negative thoughts about e-commerce, and $\theta_2 E$ is the number of merchants who used e-commerce and were affected by the refusal to use e-commerce as a result of negative thoughts about e-commerce. The number for this compartment is reduced by θ, representing natural deaths or stopping trade, and $\gamma_3 AE$, which represents the proportion of the number of merchants who refused to use e-commerce and were influenced by e-commerce as a result of contact with merchants who use e-commerce, and $\theta_1 A$, which represents the proportion of the number of merchants who refused to use e-commerce and were influenced by e-commerce as a result of positive thoughts about e-commerce received from a customer or community member. Therefore

$$\begin{aligned}\frac{dA(t)}{dt} =&\Lambda_2 + \gamma_1 C(t)A(t) + \theta_1 C(t) + \gamma_4 A(t)E(t) + \theta_2 E(t) - \gamma_3 A(t)E(t)\\ &- \theta_1 A(t) - \theta A(t).\end{aligned} \tag{2}$$

Compartment E: These are merchants who use e-commerce to sell or buy their products because they know the pros and cons of e-commerce.

The number of this compartment is increased by Λ_3, which represents the proportion of new merchants with e-commerce stores, and $\theta_2 C$, which represents the number of merchants who have been influenced by e-commerce as a result of positive thoughts about e-commerce coming from customers or people, $\gamma_2 CE$, which represents the number of merchants who have been influenced by e-commerce as a result of direct contact with merchants who use e-commerce, and $\gamma_3 AE$, which represents the number of merchants who declined to use e-commerce and were influenced by e-commerce as a result of contact with merchants who use e-commerce, and $\theta_1 A$, which represents the number of merchants who declined to use e-commerce and were influenced by e-commerce as a result of positive ideas about e-commerce received from the customer or a member of the community, and $\gamma_2 P$, which represents the number of merchants who return to using e-commerce after solving the problems they faced, such as maintenance. The number of members of this compartment decreases with the number θ, which represents natural deaths or stopping trade, and $\gamma_4 AE$, which represents the proportion of the number of merchants who used e-commerce and were influenced by the refusal to use e-commerce as a result of contacting merchants who refuse to use e-commerce, and $\theta_2 E$, which represents the proportion of the number of merchants who have been affected by the refusal to use e-commerce as a result of negative thoughts about e-commerce, and $\gamma_1 E$, which represents the proportion of the number of merchants who have stopped using e-commerce due to problems they face, such as the difficulty of maintaining the site.Thus

$$\begin{aligned}\frac{dE(t)}{dt} =& \Lambda_3 + \theta_2 C(t) + \gamma_2 C(t)E(t) + \gamma_3 A(t)E(t) + \theta_1 A(t) + \gamma_2 P(t) \\ &- \gamma_4 A(t)E(t) - \theta_2 E(t) - \gamma_1 E(t) - \theta E(t). \end{aligned} \tag{3}$$

Compartment P: These are merchants who have temporarily stopped using e-commerce to sell or buy their products due to problems they have encountered on their websites, such as damage to the site or its maintenance, lack of human capacity to work on the site or, in some cases, the absence of the Internet. The number of merchants in this category increases with the number $\gamma_1 E$, which represents the percentage of merchants who stopped using e-commerce because of problems they encountered, such as difficulties in maintaining the site, and decreases with the number θ, which represents the percentage of merchants who died of natural causes or stopped trading, and with the number $\gamma_2 P$, which represents the percentage of merchants who returned to using e-commerce after solving the problems they encountered, such as maintenance. Then

$$\frac{dP(t)}{dt} = \gamma_1 E(t) - \gamma_2 P(t) - \theta P(t). \tag{4}$$

The chart below will show the trends of merchant flow between compartments. These directions will be represented by directed arrows to illustrate these movements; see the following Figure 1.

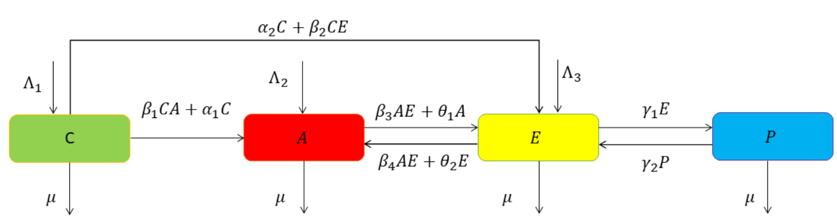

Figure 1: A diagram of the evolution of the transfer of e-commerce between merchants.

Based on the previously described compartments and the interactions between them, we now present the mathematical model that captures the dynamics of e-commerce usage rejection. The model is governed by the following system of differential equations:

$$\begin{cases} \dfrac{dC(t)}{dt} = \Lambda_1 - \gamma_1 C(t)A(t) - \theta_1 C(t) - \theta_2 C(t) - \gamma_2 C(t)E(t) - \theta C(t), \\ \dfrac{dA(t)}{dt} = \Lambda_2 + \gamma_1 C(t)A(t) + \theta_1 C(t) + \gamma_4 A(t)E(t) + \theta_2 E(t) - \gamma_3 A(t)E(t) \\ \theta_1 A(t) - \theta A(t), \\ \dfrac{dE(t)}{dt} = \Lambda_3 + \theta_2 C(t) + \gamma_2 C(t)E(t) + \gamma_3 A(t)E(t) + \theta_1 A(t) + \gamma_2 P(t) \\ \gamma_4 A(t)E(t) - \theta_2 E(t) - \gamma_1 E(t) - \theta E(t), \\ \dfrac{dP(t)}{dt} = \gamma_1 E(t) - \gamma_2 P(t) - \theta P(t). \end{cases} \tag{5}$$

With the initial conditions

$$C(0) \geq 0, \quad A(0) \geq 0, \quad E(0) \geq 0, \quad P(0) \geq 0,$$

the total number of merchants at time t is denoted as $N(t)$. Therefore, we have

$$N(t) = C(t) + A(t) + P(t) + E(t).$$

The parameters used in this model are summarized in Table 1.

2.1 Positivity of solutions

Since the model (5) tracks the behavior of merchants, it is important that all state variables and associated parameters remain positive. This will be verified by the following theorem.

Theorem 1. If $C(0) \geq 0$, $A(0) \geq 0$, $E(0) \geq 0$, and $P(0) \geq 0$, then the solutions $C(t)$, $A(t)$, $E(t)$, and $P(t)$ of system (5) will be positive for all $t \geq 0$.

Proof. From the first equation of system (5), we obtain the inequality

$$\frac{dC(t)}{dt} + (\theta + \theta_1 + \theta_2 + \gamma_1 A(t) + \gamma_2 E(t))\, C(t) \geq \Lambda_1.$$

Rearranging this inequality, we have

$$\frac{dC(t)}{dt} + (\theta + \theta_1 + \theta_2 + \gamma_1 A(t) + \gamma_2 E(t))\, C(t) \geq 0. \tag{6}$$

Table 1: Parameters definitions of the $CAEP$ model.

Parameter	Description
Λ_1	The number of new merchants introduced to the market. (individual)
Λ_2	The number of new merchants not using e-commerce. (individual)
Λ_3	The number of new merchants using e-commerce. (individual)
γ_1	The rate at which a potential merchant adopts the idea of refusing e-commerce by contacting individuals from the against group. (individual/day)
γ_2	The rate at which a potential merchant adopts the idea of using e-commerce by contacting individuals from the e-commerce users' group. (individual/day)
γ_3	The rate at which a refusing merchant turns to a user of e-commerce by contacting e-commerce users. (individual/day)
γ_4	The rate at which an e-commerce user switches to not using it by contacting individuals from the against group. (individual/day)
θ_1	The rate of merchants who were affected by negative ideas about e-commerce. (individual/day)
θ_2	The rate of merchants who were influenced by positive ideas about e-commerce. (individual/day)
θ_1	The rate of merchants who refused to use e-commerce but were influenced by positive ideas about it. (individual/day)
θ_2	The rate of merchants who use e-commerce but are affected by negative thoughts about it. (individual/day)
γ_1	The rate of merchants who used e-commerce but temporarily stopped using it. (individual/day)
γ_2	The rate of merchants who temporarily stopped using e-commerce and returned to it. (individual/day)
θ	The rate of final departure from the compartment. (individual/day)

Multiplying both sides of inequality (6) by the factor

$$\exp\left(\int_0^t \left(\theta+\theta_1+\theta_2+\gamma_1 A(h)+\gamma_2 E(h)\right)\, dh\right),$$

we get

$$\begin{aligned}\frac{dC(t)}{dt}\ .\ &\exp\left(\int_0^t \left(\theta+\theta_1+\theta_2+\gamma_1 A(h)+\gamma_2 E(h)\right) dh\right)\\ &+\left(\theta+\theta_1+\theta_2+\gamma_1 A(t)+\gamma_2 E(t)\right)\\ &\exp\left(\int_0^t \left(\theta+\theta_1+\theta_2+\gamma_1 A(h)+\gamma_2 E(h)\right) dh\right).C(t)\geq 0.\end{aligned}$$

This simplifies to

$$\frac{d}{dt}\left(C(t)\cdot\exp\left(\int_0^t \left(\theta+\theta_1+\theta_2+\gamma_1 A(h)+\gamma_2 E(h)\right)\, dh\right)\right)\geq 0,\quad \forall t\geq 0.$$

Integrating the inequality, we obtain

$$C(t) \geq C(0) \cdot \exp\left(\int_0^t -\left(\theta+\theta_1+\theta_2+\gamma_1 A(h)+\gamma_2 E(h)\right)\, dh\right).$$

Thus, $C(t)$ remains positive, as it is greater than or equal to the initial condition $C(0)$. Similarly, we can prove that $A(t) \geq 0$, $E(t) \geq 0$, and $P(t) \geq 0$ for all $t \geq 0$.

□

2.2 Boundedness of solutions

In this subsection, we present a result which proves that all possible solutions of the system description model (5) are bounded.

Lemma 1. Let the region Ω be defined by

$$\Omega = \left\{ (C(t), A(t), E(t), P(t)) \in \mathbb{R}_+^4 \,\middle|\, 0 \leq C(t)+A(t)+E(t)+P(t) \leq \frac{\Lambda_1+\Lambda_2+\Lambda_3}{\theta} \right\},$$

where the initial conditions are given as $C(0) \geq 0$, $A(0) \geq 0$, $E(0) \geq 0$, and $P(0) \geq 0$. Then the region Ω is positively invariant for the model (5).

Proof. We have the following differential inequality for the total number of merchants:

$$\frac{dN(t)}{dt} \leq \Lambda_1+\Lambda_2+\Lambda_3-\theta N(t),$$

hence, solving this inequality gives

$$\frac{dN(t)}{dt} \leq N(0)\exp(-\theta t)+\left(\frac{\Lambda_1+\Lambda_2+\Lambda_3}{-\theta}\right)\left(\exp(-\theta t)-1\right).$$

Thus, we obtain the asymptotic bound

$$\lim_{t\to+\infty} \sup N(t) = \frac{\Lambda_1+\Lambda_2+\Lambda_3}{\theta},$$

and consequently,

$$N(t) \leq \frac{\Lambda_1+\Lambda_2+\Lambda_3}{\theta}.$$

From this, we conclude that the set Ω is positively invariant for the system (5) [9]. □

2.3 Existence of solutions

In the following theorem, we prove the existence and uniqueness of solutions.

Theorem 2. The system (5) that fulfills the initial conditions $(C(0),\ A(0),\ E(0),\ P(0))$ has a unique solution.

Proof. Let

$$X(t)=\begin{pmatrix} C(t)\\ A(t)\\ E(t)\\ P(t)\end{pmatrix} \quad \text{and} \quad \Gamma(X)=\begin{pmatrix} \frac{dC(t)}{dt}\\ \frac{dA(t)}{dt}\\ \frac{dE(t)}{dt}\\ \frac{dP(t)}{dt}\end{pmatrix}.$$

Also,

$$\Gamma(X)=F_1X+F_2(X), \tag{7}$$

in which

$$F_1=\begin{bmatrix} -(\theta+\theta_1+\theta_2) & 0 & 0 & 0\\ \theta_1 & \theta_2 & -(\theta+\theta_1) & 0\\ \theta_2 & \theta_1 & -(\theta+\theta_2+\gamma_1) & \gamma_2\\ 0 & 0 & \gamma_1 & -(\theta+\gamma_2)\end{bmatrix},$$

and

$$F_2=\begin{bmatrix} \Lambda_1-\gamma_1CA-\gamma_2CE\\ \Lambda_2+\gamma_1CA+\gamma_4AE-\gamma_3AE\\ \Lambda_3+\gamma_2CE+\gamma_3AE-\gamma_4AE\\ 0\end{bmatrix}.$$

Then

$$\begin{aligned}
&\|F_2(Y)-F_2(Z)\|\\
&\le |\gamma_1C_YA_Y+\gamma_2C_YE_Y-\gamma_1C_ZA_Z-\gamma_2C_ZA_Z|\\
&\quad+|\gamma_1CyA_Y+\gamma_4AyE_Y-\gamma_3AyE_Y-\gamma_1C_ZA_Z-\gamma_4A_ZE_Z+\gamma_3A_ZE_Z|\\
&\quad+|\gamma_2C_YE_Y+\gamma_3A_YE_Y-\gamma_4A_YE_Y-\gamma_2C_ZE_Z-\gamma_3A_ZE_Z+\gamma_4A_ZE_Z|,\\
&\le \tfrac{\Lambda}{\theta}\left(2\left(\gamma_1+\gamma_2\right)|(A_Y-A_Z|+6\gamma_1\,|(A_Y-A_Z|+\left(6\gamma_2+3\gamma_3+4\gamma_4\right)|E_Y-E_Z|\right),\\
&\le C\,\|Y-Z\|.
\end{aligned}$$

Thus, we have

$$M=\frac{\Lambda_1+\Lambda_2+\Lambda_3}{\theta}\cdot\max\left(2\left(\gamma_1+\gamma_2\right),6\gamma_1,\left(6\gamma_2+3\gamma_3+4\gamma_4\right)\right),$$

and also

$$\|\Gamma(Y) - \Gamma(Z)\| \leq N \|Y - Z\|,$$

where $N = \max(M, \|F_1\|) < \infty$. Thus, it follows that the function Γ is Lipschitz continuous, so the system admits a unique solution [12].

$\square$

3 The optimal control problem

The failure of merchants to use e-commerce leads to a set of negative effects that hinder economic growth in terms of energy bills. Abstaining from e-commerce also contributes to increasing customers' daily expenses, as the cost of traveling to purchase a product is very high compared to purchasing via e-commerce. The weak use of e-commerce also leads to the spread of infectious diseases.

E-commerce offers many benefits, both for merchants who use it, and for customers who purchase products via e-commerce, as it allows e-store owners to display their goods and services to customers at any time of the day or night. They can also open an e-store without having a physical store and work from home.

• $v_1(t)$: represents the use of advertising across all visual and audio communication means and social media to introduce e-commerce, in addition to the steps that must be taken to create an e-commerce site in a way that is safe from cybercrimes. The entities responsible for building websites can also be displayed safely and at reasonable prices. It is also necessary to provide training for merchants on how to operate websites and how to display products for sale in a good way that meets the customer's needs.

• $v_2(t)$: indicates the provision of insurance for all merchants against all cybercrimes that they may be exposed to, such as fraud and theft, in addition to theft of personal information.

By adding the two controls $v_1(t)$ and $v_2(t)$ to the first system (5), we obtain the following control model:

$$
\begin{cases}
\dfrac{dC(t)}{dt} = & \Lambda_1 - \gamma_1 C\left(t\right) A\left(t\right) - \theta_1 C\left(t\right) - \theta_2 C\left(t\right) - \gamma_2 C\left(t\right) E\left(t\right) - \theta C\left(t\right) \\
& -v_1(t) C(t), \\
\dfrac{dA(t)}{dt} = & \Lambda_2 + \gamma_1 C\left(t\right) A\left(t\right) + \theta_1 C\left(t\right) + \gamma_4 A\left(t\right) E\left(t\right) + \theta_2 E\left(t\right) \\
& -\gamma_3 A\left(t\right) E\left(t\right) - \theta_1 A\left(t\right) - \theta A\left(t\right) - v_2\left(t\right) A\left(t\right) - v_2\left(t\right) \theta_2 E\left(t\right) \\
& -\gamma_4 v_2\left(t\right) A\left(t\right) E\left(t\right), \\
\dfrac{dE(t)}{dt} = & \Lambda_3 + \theta_2 C\left(t\right) + \gamma_2 C\left(t\right) E\left(t\right) + \gamma_3 A\left(t\right) E\left(t\right) + \theta_1 A\left(t\right) + \gamma_2 P(t) \\
& -\gamma_4 A\left(t\right) E\left(t\right) - \theta_2 E\left(t\right) - \gamma_1 E\left(t\right) - \theta E\left(t\right) + v_1(t) C(t) \\
& +v_2\left(t\right) A\left(t\right) + \theta_2 v_2\left(t\right) E\left(t\right) + \gamma_4 v_2\left(t\right) A\left(t\right) E\left(t\right), \\
\dfrac{dP(t)}{dt} = & \gamma_1 E\left(t\right) - \gamma_2 P\left(t\right) - \theta P\left(t\right),
\end{cases}
\tag{8}
$$

with the initial conditions $C(0) \geq 0$, $A(0) \geq 0$, $E(0) \geq 0$, and $P(0) \geq 0$.

3.1 The optimal control: Existence and characterization

In this subsection, our objective is to minimize the total number of merchants who refuse to use e-commerce and maximize the number of those who use it. The control variables $v_1(t)$ and $v_2(t)$ are introduced to minimize the optimal control model (5) subject to the objective functional defined as follows:

$$
J(v_1, v_2) = A(t_f) - E(t_f) + \int_{t_0}^{t_f} \left[A(t) - E(t) + \frac{C_1}{2} \left(v_1(t)\right)^2 + \frac{C_2}{2} \left(v_2(t)\right)^2 \right] dt,
$$

where $C_1 > 0$ and $C_2 > 0$ are the cost coefficients. These coefficients are chosen in order to weigh the relative importance of $v_1(t)$ and $v_2(t)$ at each time t, where t_f is the final time. In other words, we aim to find the optimal controls v_1^* and v_2^* such that

$$
J(v_1^*, v_2^*) = \min_{(v_1, v_2) \in A_d} J(v_1, v_2).
$$

Here, A_d is the set of admissible controls defined by

$$
A_d = \left\{ (v_1(t), v_2(t)) : 0 \leq v_1(t) \leq 1,\ 0 \leq v_2(t) \leq 1 \text{ for } t \in [t_0, t_f] \right\}. \tag{9}
$$

3.2 Existence of optimal controls

In this subsection, we present the theorem which proves the existence of an optimal control (v_1^*, v_2^*) minimizing the cost function J.

Theorem 3. There exists an optimal control $(v_1^*, v_2^*) \in A_d$ such that

$$J(v_1^*, v_2^*) = \min_{(v_1,v_2)\in A_d} J(v_1, v_2).$$

Proof. To use the existence result in [11], we must check the following properties:
(K1): The set of controls and the corresponding state variables is nonempty.
(K2): The control set A_d is convex and closed.
(K3): The right-hand side of the state system is bounded by a linear function in the state and control variables.
(K4): The integral $L(A, E, v_1, v_2)$ of the objective functional is convex on A_d, and there exist constants $\kappa_1 > 0$, $\kappa_2 > 0$, and $\varepsilon > 1$ such that

$$L(A, E, v_1, v_2) \geq -\kappa_1 + \kappa_2 \left(|v_1|^2 + |v_2|^2\right)^{\frac{\varepsilon}{2}}.$$

The first condition (K_1) is verified using the result in [11]. The set A_d is convex and closed by definition, thus the condition (K_2). Our state system is bilinear in v_1 and v_2, moreover the solutions of the system are bounded as proved in model (5), hence the condition (K_3). Also, we have the last needed condition (K_4):

$$L(A, E, v_1, v_2) \geq -\kappa_1 + \kappa_2 \left(|v_1|^2 + |v_2|^2\right)^{\frac{\varepsilon}{2}}.$$

where

$$\kappa_1 = 2 \sup_{t\in[t_0,t_f]} (A(t), E(t)), \quad \kappa_2 = \inf\left(\frac{C_1}{2}, \frac{C_2}{2}\right), \varepsilon = 2,$$

since

$$C_1 > 0 \quad \text{and} \quad C_2 > 0.$$

We conclude that there exists an optimal control $(v_1^*, v_2^*) \in A_d$ such that

$$J(v_1^*, v_2^*) = \min_{(v_1,v_2)\in A_d} J(v_1, v_2).$$

□

3.3 Characterization of the optimal controls

In this subsection, we apply Pontryagin's maximum principle to address the problem of merchants and businesses that are capable of using e-commerce but choose not to. Our objective is to solve for the optimal control in system (8) such that the control variables $v_1(t)$ and $v_2(t)$ represent the optimal strategies, ultimately reducing the number of merchants and businesses that refrain from using e-commerce to sell their products.

To derive the optimal control conditions, we utilize Pontryagin's maximum principle. The Hamiltonian $\mathcal{H}(t)$ at time t is given by

$$\mathcal{H}(t) = A(t) - E(t) + \frac{C_1}{2} v_1(t)^2 + \frac{C_2}{2} v_2(t)^2 + \sum_{i=1}^{4} \lambda_i h_i,$$

where h_i represents the right-hand side of the system of differential equations (8) for the i-th state variable.

Theorem 4. Given the optimal controls (v_1^*, v_2^*) and the corresponding solutions C^*, A^*, E^*, and P^* to the state system (8), there exist adjoint functions λ_1, λ_2, λ_3, and λ_4 that satisfy the following system:

$$\begin{cases} \frac{d\lambda_1}{dt} & = -\frac{\partial \mathcal{H}}{\partial C} \\ & = -\lambda_1 \left[-\gamma_1 A(t) - \theta_1 - \theta_2 - \gamma_2 E(t) - \theta - v_1(t)\right] \\ & \quad -\lambda_2 \left[\gamma_1 A(t) + \theta_1\right] - \lambda_3 \left[\theta_2 + \gamma_2 E(t) + v_1(t)\right], \\ \frac{d\lambda_2}{dt} & = -\frac{\partial \mathcal{H}}{\partial A} \\ & = -1 - \lambda_1 \left(-\gamma_1 C(t)\right) \\ & \quad -\lambda_2 \left[\gamma_1 C(t) + \gamma_4 E(t) - \gamma_3 E(t) - \theta_1 - \theta - v_2(t)\right] \\ & \quad -\lambda_2 \left[-\gamma_4 v_2(t) E(t)\right] \\ & \quad -\lambda_3 \left[\gamma_3 E(t) + \theta_1 - \gamma_4 E(t) + v_2(t) + \gamma_4 v_2(t) E(t)\right], \\ \frac{d\lambda_3}{dt} & = -\frac{\partial \mathcal{H}}{\partial E} \\ & = 1 - \lambda_1 \left(-\gamma_2 C(t)\right) \\ & \quad -\lambda_2 \left[\gamma_4 A(t) + \theta_2 - \gamma_3 A(t) - v_2(t)\theta_2 - \gamma_4 v_2(t) A(t)\right] \\ & \quad -\lambda_3 \left[\gamma_2 C(t) + \gamma_3 A(t) - \gamma_4 A(t) - \theta_2 - \gamma_1 - \theta \right. \\ & \qquad \left. +\theta_2 v_2(t) + \gamma_4 v_2(t) A(t)\right] \\ & \quad -\lambda_4 \left(\gamma_1\right), \\ \frac{d\lambda_4}{dt} & = -\frac{\partial \mathcal{H}}{\partial P} \\ & = -\lambda_3 \left(\gamma_2\right) - \lambda_4 \left(-\gamma_2 - \theta\right). \end{cases} \tag{10}$$

The transversality conditions at time t_f are given by

$$\begin{cases} \lambda_1(t_f) = \ 0, \\ \lambda_2(t_f) = \ 1, \\ \lambda_3(t_f) = -1, \\ \lambda_4(t_f) = \ 0. \end{cases} \tag{11}$$

Moreover, for $t \in [t_0, t_f]$, the optimal controls $v_1^*(t)$ and $v_2^*(t)$ are given by

$$\begin{cases} v_1^*(t) = \min\left(1, \max\left(0, \frac{1}{C_1}\left(\lambda_1 - \lambda_3\right) C(t)\right)\right), \\ v_2^*(t) = \min\left(1, \max\left(0, \frac{\lambda_2 - \lambda_3}{C_2}\left(A(t) + \theta_2 E(t) + \gamma_4 A(t)E(t)\right)\right)\right). \end{cases} \tag{12}$$

Proof. The Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H}(t) = A(t) - E(t) + \frac{C_1}{2} v_1(t)^2 + \frac{C_2}{2} v_2(t)^2 + \sum_{i=1}^{4} \lambda_i h_i,$$

where the h_i functions are given by

$$\begin{cases} h_1 = \Lambda_1 - \gamma_1 C(t)A(t) - \theta_1 C(t) - \theta_2 C(t) - \gamma_2 C(t)E(t) - \theta C(t) \\ \qquad -v_1(t)C(t), \\ \\ h_2 = \Lambda_2 + \gamma_1 C(t)A(t) + \theta_1 C(t) + \gamma_4 A(t)E(t) + \theta_2 E(t) - \gamma_3 A(t)E(t) \\ \qquad -\theta_1 A(t) - \theta A(t) - v_2(t)A(t) - v_2(t)\theta_2 E(t) - \gamma_4 v_2(t)A(t)E(t), \\ \\ h_3 = \Lambda_3 + \theta_2 C(t) + \gamma_2 C(t)E(t) + \gamma_3 A(t)E(t) + \theta_1 A(t) + \gamma_2 P(t) \\ \qquad -\gamma_4 A(t)E(t) - \theta_2 E(t) - \gamma_1 E(t) - \theta E(t) + v_1(t)C(t) + v_2(t)A(t) \\ \qquad +\theta_2 v_2(t)E(t) + \gamma_4 v_2(t)A(t)E(t), \\ \\ h_4 = \gamma_1 E(t) - \gamma_2 P(t) - \theta P(t). \end{cases}$$

By using Pontryagin's maximum principle [31], the adjoint equations and transversality conditions can be derived, leading to the system in equation (10) and the optimal controls expressed in equation (12).

The optimal controls $v_1^*(t)$ and $v_2^*(t)$ are determined by solving the optimality conditions. For v_1 and v_2, we have

$$\frac{\partial \mathcal{H}}{\partial v_1} = C_1 v_1(t) - \lambda_1 C(t) + \lambda_3 C(t) = 0,$$

which gives

$$v_1 = \frac{C(t)}{C_1} \left(\lambda_1 - \lambda_3 \right).$$

Similarly, for v_2,

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial v_2} &= C_2 v_2(t) + \lambda_2 \left(-A(t) - \theta_2 E(t) - \gamma_4 A(t) E(t) \right) \\ &\quad + \lambda_3 \left(A(t) + \theta_2 E(t) + \gamma_4 A(t) E(t) \right) \\ &= 0,\end{aligned}$$

which gives

$$v_2 = \frac{\lambda_2 - \lambda_3}{C_2} \left(A(t) + \theta_2 E(t) + \gamma_4 A(t) E(t) \right).$$

These expressions for v_1^* and v_2^* are bounded by the control limits, and hence we obtain the final form of the optimal controls in equation (12). □

4 Numerical simulations

In this section, we perform numerical simulations to illustrate the strategies we have implemented. The goal is to reduce the number of merchants and companies that do not use e-commerce to sell their products, while increasing the number of merchants that adopt e-commerce. To solve the proposed $CAEP$ model and the corresponding optimal control problem, numerical simulations were performed using MATLAB. The state equations were solved using a standard fourth-order Runge–Kutta method, which is widely used in the literature for compartmental and epidemiological models due to its stability and accuracy. The optimal control problem was addressed via the forward-backward sweep method, which iteratively solves the state system forward in time and the adjoint system backward in time until convergence is achieved. Parameter values and initial conditions were selected based on realistic assumptions and prior studies to reflect typical merchant behavior and e-commerce adoption dynamics. Figures presented in this section illustrate the temporal evolution of the merchant compartments under different control scenarios, and the results are interpreted to assess the effectiveness of the proposed strategies in promoting e-commerce adoption. This approach is consistent with methodologies employed in recent studies on optimal control in socio-economic and epidemiological systems [32].

The code is written and compiled in MATLAB using the following data to represent $N = 1,071,900$ companies in France in 2022 [18], noting that this number increased by $+17\%$

compared to the previous year. During the same year, the number of online sales sites in France is expected to continue increasing. It is predicted that over 10000 new active sites will emerge, marking a 5% increase compared to the previous year. By the end of 2022, there were 207000 electronic stores marketing commercial activities [13].

To apply the numerical simulation to the application, we use the initial data for the number of people in each compartment as follows: $C(0) = 700000$, $A(0) = 159900$, $E(0) = 207000$, $P(0) = 50000$, $\Lambda_1 = 73000$, $\Lambda_2 = 4797$, $\Lambda_3 = 10000$, $\gamma_1 = 10^{-8}$, $\gamma_2 = 2 \times 10^{-9}$, $\gamma_3 = 3 \times 10^{-8}$, $\gamma_4 = 10^{-8}$, $\theta_1 = 7 \times 10^{-11}$, $\theta_2 = 6 \times 10^{-7}$, $\gamma_1 = 6 \times 10^{-8}$, $\gamma_2 = 5 \times 10^{-8}$, and $\theta = 0.078$, $\theta_1 = 0.0001$, $\theta_2 = 0.0002$.

In this study, we have implemented two different control strategies, denoted $v_1(t)$ and $v_2(t)$, aimed at informing merchants and e-commerce companies about the protective measures available to them in case of exposure to cybercrime or fraud. To assess the specific impact of each control strategy, we have chosen to evaluate three scenarios:

4.1 First scenario: Motivating merchants to use e-commerce through advertising

In this subsection, the control strategy v_1 is applied to assist merchants and companies who may have limited experience in e-commerce and online store creation and management. The objective is to enable them to successfully create and operate an online store.

■ Figure 2.1 presents the dynamics of merchant behavior under the implementation of the first control strategy. Following the initiation of the strategy, a clear decline is observed in the number of merchants who initially refused to adopt e-commerce, represented in compartment A, beginning around day 20. By day 30, this number decreases by approximately 4.68%, indicating an early response of the nonadopting merchants to the applied intervention. This reduction appears to be correlated with a corresponding decrease in the population of nonadopting merchants in compartment C, suggesting that a portion of these individuals have been successfully influenced by the strategy to reconsider their adoption decisions.

Over the full 60-day simulation period, the cumulative effect of the first strategy results in a substantial 24% reduction in the number of merchants resistant to e-commerce adoption. This trend demonstrates the effectiveness of the control measures in shifting merchant behavior from nonadoption toward adoption. The results also underscore the importance of timely interventions: earlier implementation of the strategy leads to faster reductions in nonadopters, which can subsequently accelerate overall market penetration. Moreover, the gradual decline indicates

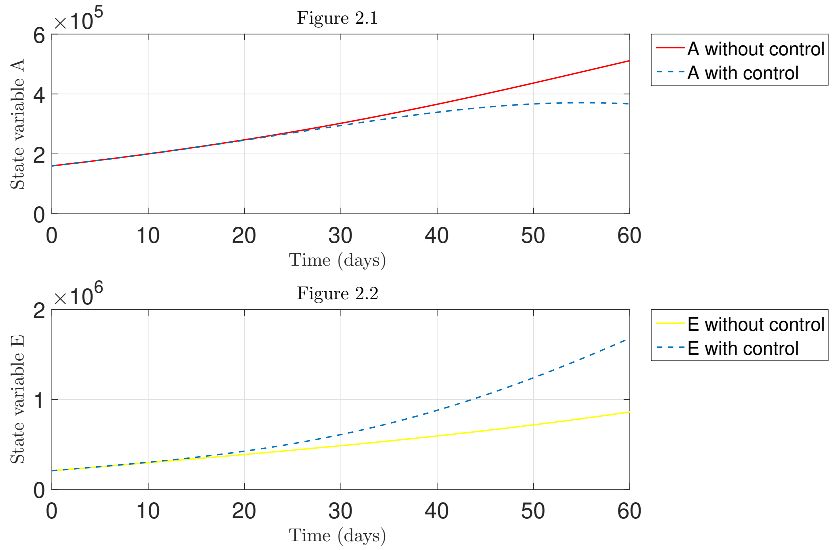

Figure 2: Dynamics with control v_1.

that while the strategy is effective, behavioral change among merchants is a progressive process, influenced by factors such as peer adoption, perceived benefits, and exposure to incentives. Overall, the figure highlights the potential of targeted policy interventions to promote e-commerce adoption and improve the overall efficiency and competitiveness of the commercial ecosystem.

■ Figure 2.2 illustrates the impact of the first control strategy on the population of merchants actively using e-commerce, represented in compartment E. Following the implementation of the strategy, a noticeable increase in the number of adopting merchants is observed beginning around day 14. By day 30, the population of e-commerce users has risen by approximately 59.76%, indicating a rapid initial response to the intervention. This early increase suggests that the strategy effectively motivates merchants who were previously hesitant to transition toward adoption, likely due to perceived benefits or incentives embedded within the control measures.

Over the full 60-day simulation period, the cumulative effect of the strategy leads to a substantial overall increase in the total number of merchants using e-commerce. This trend highlights the effectiveness of the applied measures in promoting sustained adoption over time. Moreover, the gradual growth trajectory emphasizes that while the strategy yields significant short-term gains, continued monitoring and adaptive interventions may be necessary to maintain momentum and encourage long-term adherence. Overall, the figure demonstrates that targeted control strategies can successfully enhance e-commerce adoption, thereby improving market coverage, competitiveness, and economic efficiency.

4.2 Second scenario: Providing insurance against electronic crimes for merchants to encourage them to use electronic commerce.

In this strategy, we apply the control v_2, which represents the display of the insurance element on compartment A.

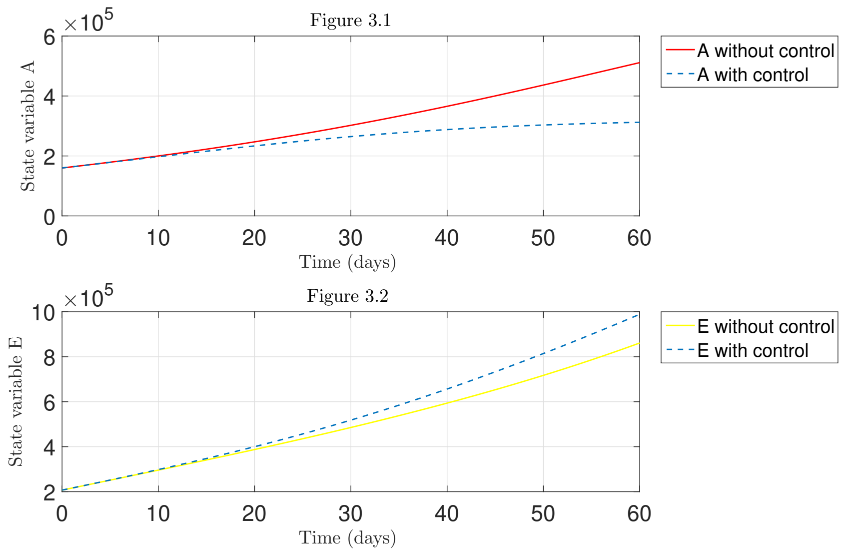

Figure 3: Dynamics with control v_2.

■ Figure 3.1 illustrates the effect of the second control strategy on merchants resistant to e-commerce adoption in compartment A. The results show that the number of nonadopting merchants begins to decline from day 10, with a 23.66% reduction observed by day 30. By the end of the 60-day period, the total number of resistant merchants decreases by 48%, indicating that the second strategy is particularly effective in accelerating adoption. This rapid and substantial reduction suggests that targeted interventions addressing specific barriers can significantly influence merchant behavior, promoting faster integration into e-commerce. These findings provide valuable insights for policymakers and business stakeholders, highlighting the potential of well-designed strategies to reduce resistance, enhance market participation, and strengthen overall competitiveness in the commercial system.

■ Figure 3.2 shows the response of merchants adopting e-commerce in compartment E following the implementation of the second control strategy. The simulation indicates that adoption begins to rise noticeably from day 15, reaching a 15.85% increase by day 30. By the end of the 60-day period, the total number of merchants using e-commerce has grown by 61.63%, demonstrating the strong effectiveness of this strategy. The progressive increase highlights how well-targeted interventions can gradually shift merchant behavior, encouraging hesitant individuals to embrace digital commerce. These results also emphasize the importance of timing and sustained application of control measures, as consistent efforts over the simulation period produce substantial adoption gains, thereby contributing to a more competitive and digitally integration.

4.3 Third scenario: Using advertising and insurance to encourage merchants to use electronic commerce.

In this strategy, we implement the controls v_1 and v_2 so that we can show merchants and companies who do not know how to use e-commerce to sell their products. How can they create online stores or how can they sell their products on some websites dedicated to e-commerce. We also provide insurance to merchants against electronic crimes and fraud.

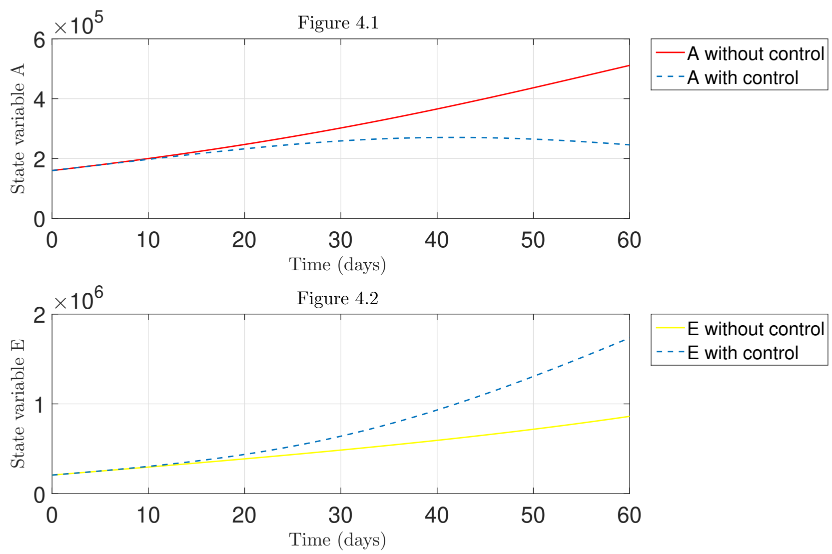

Figure 4: Dynamics with controls v_1 and v_2.

Figures 4.1 and 4.2 illustrate the impact of the third control strategy on merchant behavior regarding e-commerce adoption. Following the implementation of this strategy, the number of merchants initially refusing to adopt e-commerce in compartment A begins to decline as early as day 7, with a reduction of approximately 27.20% observed by day 30. Concurrently, the population of merchants actively using e-commerce in compartment E starts to increase from day 15, reaching a 74.63% rise by day 30. Over the full 60-day simulation period, the strategy produces a substantial overall decrease in nonadopting merchants alongside a significant increase in e-commerce adopters. This pattern demonstrates the effectiveness of the third strategy in both reducing resistance and promoting adoption, highlighting its capacity to accelerate behavioral shifts among merchants. The results also suggest that early interventions, coupled with sustained application, can produce rapid and lasting changes in merchant behavior. Overall, the third strategy appears to be the most effective among the three considered, significantly enhancing e-commerce uptake and contributing to greater market integration and competitiveness within the commercial ecosystem.

Finally, the results indicate that the combined application of the two optimal controls, v_1 and v_2, substantially enhances the effectiveness of the proposed strategy. By simultaneously promoting e-commerce adoption and addressing resistance among hesitant merchants, the integration of these controls provides a more balanced and comprehensive intervention. This dual-control approach not only accelerates the transition of nonadopting merchants into active e-commerce users but also ensures a more efficient allocation of resources to influence behavioral change. The holistic nature of this strategy increases the likelihood of achieving sustained adoption, thereby generating broader economic benefits, improving market competitiveness, and strengthening the resilience of the commercial ecosystem. These findings underscore the potential of multi-faceted control policies in designing effective interventions for complex socio-economic systems.

4.4 Cost-effectiveness analysis (CEA)

In the context of e-commerce, implementing strategies such as promotion via advertising $v_1(t)$ and protection through cybercrime insurance $v_2(t)$ requires financial resources. Cost-effectiveness analysis (CEA) provides a quantitative framework to evaluate which strategies offer the best impact in terms of e-commerce adoption for the lowest cost.

A classic indicator in CEA is the average cost-effectiveness ratio (ACER), defined as

$$\mathrm{ACER} = \frac{T_c}{T_a},$$

where T_c is the total cost of interventions and T_a is the total number of merchants converted to e-commerce thanks to the control strategies.

More precisely, the commercial impact measure T_a can be expressed as:

$$T_a = \int_0^T E_{\text{without control}}(t)\,dt - \int_0^T E_{\text{with control}}(t)\,dt,$$

which measures the reduction in the number of merchants refusing e-commerce due to the application of controls $v_1(t)$ and $v_2(t)$.

The total cost T_c associated with these strategies is given by

$$T_c = \int_0^T (c_1 v_1(t) C(t) + c_2 v_2(t) A(t))\,dt,$$

where c_1 and c_2 are unit costs associated respectively with advertising/training and cybercrime insurance.

For example, $c_1 = 8$ could represent the average promotion cost per merchant in category C, while $c_2 = 10$ would represent the average insurance cost per merchant in category A.

Table 2 shows the numerical simulation results, sorted by increasing ACER.

Table 2: ACER sorted by increasing efficiency

Scenario	T_a (merchants converted)	T_c (total cost in \$)
Scenario 2	8.2979×10^7	5.8188×10^7
Scenario 1	8.7509×10^7	6.3199×10^7
Scenario 3	8.7580×10^7	6.3271×10^7

The corresponding ACER values are calculated as follows:

- **Scenario 2**:
$$\text{ACER} = \frac{5.8188 \times 10^7}{8.2979 \times 10^7} \approx 0.701237.$$

- **Scenario 1**:
$$\text{ACER} = \frac{6.3199 \times 10^7}{8.7509 \times 10^7} \approx 0.722205.$$

- **Scenario 3**:
$$\text{ACER} = \frac{6.3271 \times 10^7}{8.7580 \times 10^7} \approx 0.722441.$$

Although Scenario 2 presents the lowest ACER, it also offers the most moderate impact in terms of e-commerce adoption. Scenario 3, which combines promotion and insurance, achieves the highest total conversion despite a slightly higher ACER, making it the most robust overall strategy.

Therefore, it is recommended that economic decision-makers and platform managers adopt an integrated approach combining advertising and insurance to maximize the transition to e-commerce while controlling costs.

5 Conclusion

In this study, we developed a nonlinear mathematical model, $CAEP$, to describe the dynamics of merchants and companies that refrain from adopting e-commerce and to analyze its impact on both adopters and nonadopters. The findings highlight the significant role of e-commerce in enhancing economic efficiency, increasing merchants' revenues, and improving product visibility for consumers. The $CAEP$ model was extended to an optimal control framework with two control variables aimed at promoting adoption and reducing resistance among merchants. The Pontryagin maximum principle was used to derive optimality conditions, and numerical simulations in MATLAB confirmed the effectiveness of the proposed strategies. An ACER provided insights into the trade-offs between implementation costs and adoption rates, identifying the most efficient strategies for decision-makers. Overall, the integration of mathematical modeling, optimal control, and economic evaluation offers a comprehensive understanding of e-commerce adoption dynamics. The results suggest that e-commerce has strong potential for future economic growth and job creation, particularly for merchants with limited access to physical retail spaces. Future research could extend this work by incorporating network effects and social influence on merchants' adoption decisions, introducing stochastic dynamics to account for market uncertainty and random fluctuations, and integrating merchant adoption models with consumer behavior models to capture feedback between supply and demand. These extensions would provide a more realistic and robust framework for designing effective strategies to promote e-commerce adoption.

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Abbate, J. *Popularizing the internet*, Commun. Hist., Routledge (2018), 297–302.

[2] Agrawal, D.R. and Fox, W.F. *Taxes in an e-commerce generation*, Int. Tax Public Finance 24 (2017), 903–926.

[3] Balatif, O., Khajji, B. and Rachik, M. *Mathematical modeling, analysis, and optimal control of abstinence behavior of registration on the electoral lists*, Discrete Dyn. Nat. Soc. 2020 (2020), Art. ID 9738934.

[4] Baroudi, M., Gourram, H., Labzai, A. and Belam, M. *Mathematical modeling and monkeypox's optimal control strategy*, Commun. Math. Biol. Neurosci. 2023 (2023), Art. ID 110.

[5] Belhdid, S., Balatif, O. and Khajji, B. *Optimal control and cost-effectiveness analysis of scam rumor propagation over social networks*, Results Control Optim. 16 (2024), 100441.

[6] Corbitt, B.J., Thanasankit, T. and Yi, H. *Trust and e-commerce: a study of consumer perceptions*, Electron. Commer. Res. Appl. 2 (2003), 203–215.

[7] El Bhih, A., Ghazzali, R., Ben Rhila, S., Rachik, M. and El Alami Laarroussi, A. *A discrete mathematical modeling and optimal control of the rumor propagation in online social network*, Discrete Dyn. Nat. Soc. 2020 (2020), 1–12.

[8] El Bhih, A., Yaagoub, Z., Rachik, M., Allali, K. and Abdeljawad, T. *Controlling the dissemination of rumors and antirumors in social networks*, Eur. Phys. J. Plus 139 (2024), 1–23.

[9] El Youssoufi, L., Moutamanni, H., Labzai, A., Balatif, O. and Rachik, M. *Optimal control for a discrete model of hepatitis C*, Commun. Math. Biol. Neurosci. 2020 (2020), Art. ID 82.

[10] Elhia, M., Balatif, O., Boujallal, L. and Rachik, M. *Optimal control problem for a tuberculosis model*, Int. J. Optim. Control Theory Appl. 11 (2021), 75–91.

[11] Fleming, W.H. and Rishel, R.W. *Deterministic and stochastic optimal control*, Springer, New York, 2012.

[12] Gourram, H., Baroudi, M., Labzai, A. and Belam, M. *Mathematical modeling and optimal control strategy for the influenza (H5N1)*, Commun. Math. Biol. Neurosci. 2023 (2023), Art. ID 113.

[13] Gouvernement, https://www.gouvernement.fr

[14] Gupta, D.C., Bindal, M., Agarwal, N. and Khandelwal, K. *Traditional commerce v/s e-commerce and impact of demonetization on e-commerce*, Int. J. Eng. Manag. Res. 8 (2018), 136–142.

[15] Hannum, K.M. and Downs, H.A. *Evaluation research in the EDI field*, Booysen, LAE (2018), 60–84.

[16] Hassouni, H., Belhdid, S. and Balatif, O. *Mathematical modeling and optimal control of customer's behavior toward e-commerce*, Iran. J. Numer. Anal. Optim. 14(4) (2024) 1168–1202.

[17] Huang, L., Huang, Y., Huang, R., Xie, G. and Cai, W. *Factors influencing returning migrants' entrepreneurship intentions*, Sustainability 14 (2022), 3682.

[18] Institut Natl. Statist. Études Écon. (INSEE), https://www.insee.fr/fr/accueil

[19] Isckia, T. *Amazon's evolving ecosystem*, Canad. J. Admin. Sci. 26 (2009), 332–343.

[20] Jang, S.Y. and Lee, C.S. *A Comparison of Customers' Perceived Value on Electronic Commerce and Traditional Commerce*, Asia-Pac. J. Inf. Syst. 10 (2000), 159–180.

[21] Jose, S.A., Panigoro, H.S., Jirawattanapanit, A., Omede, B.I. and Yaagoub, Z. *Understanding COVID-19 propagation: A comprehensive mathematical model with Caputo fractional derivatives for Thailand*, Front. Appl. Math. Stat. 10 (2024), 1374721.

[22] Kada, D., Khajji, B., Balatif, O., Rachik, M. and Labriji, E.H. *Optimal control approach of gaming disorder spread*, Discrete Dyn. Nat. Soc. 2021 (2021), 5584315.

[23] Kermack, W.O. and McKendrick, A.G. *A contribution to the mathematical theory of epidemics*, Proc. Roy. Soc. Lond. Ser. A 115 (1927), 700–721.

[24] Kim, K.K. and Prabhakar, B. *Initial trust and the adoption of B2C e-commerce: The case of internet banking*, ACM SIGMIS Database, 35 (2004), 50–64.

[25] Krishnamurthy, S. *A comparative analysis of eBay and Amazon*, IGI Global, 2004, 29–44.

[26] Laarabi, H., Rachik, M., El Kahlaoui, O. and Labriji, E. *Optimal vaccination strategies of an SIR model*, Univ. J. Appl. Math. 1 (2013), 185–191.

[27] Laaroussi, A.E.A., El Bhih, A. and Rachik, M. *Optimal vaccination and treatment policies with constrained inequalities to study limited vaccination resources for a multistrain reaction-diffusion SEIR model of COVID-19*, Partial Differ. Equ. Appl. Math. 10 (2024), 100684.

[28] Lightner, N.J. *What users want in e-commerce design: Effects of age, education and income*, Ergonomics 46 (2003), 153–168.

[29] Maity, M. and Dass, M. *Consumer decision-making across modern and traditional channels: E-commerce, m-commerce, in-store*, Decision Support Systems 61 (2014), 34–46.

[30] Olson, J.S. and Olson, G.M. *i2i trust in e-commerce*, Commun. ACM 43 (2000), 41–44.

[31] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V. and Mishchenko, E. F. *The mathematical theory of optimal processes*, Wiley, New York, 1962.

[32] Purwani, S., Inayaturohmat, F. and Tresna, S.T. *COVID-19 epidemic model: study of numerical methods and solving optimal control problem through forward-backward sweep method*, Commun. Math. Biol. Neurosci. 2022 (2022), Art. ID 123.

[33] Qin, Z., Chang, Y., Li, S. and Li, F. *E-commerce strategy*, Springer, 2014.

[34] Rahman, S. and Qing, N. *Graduate students' perceptions of supply chain skills for supply chain managers*, Benchmarking 21 (2014), 276–299.

[35] Sadki, M., Yaagoub, Z. and Allali, K. *Qualitative Analysis of a Fractional-Order for a Within-Host Infection Dynamics with Adaptive Immunity Using Caputo Derivative*, Iran. J. Sci. 49(3) (2025), 847–869.

[36] Shemi, A.P. and Procter, C. *E-commerce and entrepreneurship in SMEs: case of myBot*, J. Small Bus. Enterprise Dev. 25 (2018), 501–520.

[37] Taher, G. *E-commerce: advantages and limitations*, Int. J. Acad. Res. Account. Finance Manag. Sci. 11 (2021), 153–165.

[38] Tan, Z. and Ouyang, W. *Diffusion and impacts of e-commerce in China*, Electron. Mark. 14 (2004), 25–35.

[39] Wulf, J. *Alibaba Group*, Harv. Bus. Sch. Case 710 (2010), 436.

[40] Yaagoub, Z. and Allali, K. *Fractional HCV infection model with adaptive immunity and treatment*, Math. Model. Comput. 10 (2023), 995–1013.

[41] Yaagoub, Z., El Bhih, A. and Allali, K. *Global analysis of a fractional-order infection model for the propagation of computer viruses*, Model. Earth Syst. Environ. 11 (2025), 68.

[42] Yaagoub, Z., Farah, E.M. and Ahmad, S. *Three-strain epidemic model for influenza virus involving fractional derivative and treatment*, J. Appl. Math. Comput. 71 (2025), 1247–1266.

[43] Yapar, B.K., Bayrakdar, S. and Yapar, M. *The role of taxation problems on the development of e-commerce*, Procedia Soc. Behav. Sci. 195 (2015), 642–648.