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Research Article

A new numerical approach for tracing periodic solutions in nonlinear dynamical systems using CGM with PAC

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Abstract

In this article, we introduce a new numerical approach that combines the conjugate gradient method and Pseudo-Arclength continuation to trace periodic solutions in nonlinear dynamical systems. By minimizing Jacobian computations, the method reduces computational costs, making it efficient for large systems. We can apply the method to the Lorenz and Duffing oscillator systems, by using Floquet theory to analyze stability. The results show accurate tracing of periodic solutions and robust stability analysis.

AMS subject classifications (2020): Primary 45D05; Secondary 42C10, 65G99.

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1 Introduction

Consider the nonlinear dynamical systems is given by

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$$\dot{\mathcal{G}} = \mathcal{F}(\mathcal{G}, \Omega), \tag{1}$$

where $\mathcal{G} \in \mathbb{R}^n$ is the state vector, and $\Omega \in \mathbb{R}^m$ is the parameter vector. Also, $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function, with a periodic solution $\mathcal{G}(t)$. A periodic solution in the study of (1) is a fundamental area of research with wide-ranging applications in engineering, physics, and biology [1, 8, 9, 7]. The limit cycles are periodic solutions and play a crucial role in understanding the long-term behavior of dynamical systems, particularly in chaotic and nonlinear regimes [3, 8]. However, the numerical construction and stability analysis of periodic solutions remain challenging, especially in high-dimensional systems or near bifurcation points [8, 6].

We propose a novel numerical approach that combines the conjugate gradient method (CGM) with Pseudo-Arclength continuation (PAC), to address this challenge. This method reduces calculation cost by minimizing the need for repeated Jacobian computations, making it particularly efficient for large-scale systems.

Often they require repeated calculation of the Jacobian matrix, but which can be computationally expensive for large systems [3, 8]. We propose a novel numerical approach that combines the CGM with PAC, to address this challenge. This method reduces calculation cost by minimizing the need for repeated Jacobian computations, making it particularly efficient for large-scale systems.

We apply the proposed method to two classic nonlinear systems: the Lorenz system and the Duffing oscillator. The Lorenz system, known for its chaotic behavior, serves as a benchmark for testing the robustness of the method in tracing periodic solutions and analyzing their stability [5, 3, 1]. The Duffing oscillator is a model for nonlinear oscillations [6].

The paper is organized as follows: Section 2 provides the theoretical background. Section 3 is dedicated to related work and motivation. Section 4 describes the numerical methodology. In section 5, we prove some theorems and proofs. Finally, section 6 presents the application of the method to the Lorenz system and the Duffing oscillator, with numerical results and graphical visualizations.

2 Theoretical background

Definition 1. [1] The solution $\mathcal{G}(t)$ is a periodic solution when there exists a finite time $T > 0$ such that $\mathcal{G}(T + t) = \mathcal{G}(t)$, for all $t \geqslant 0$.

Definition 2. [6] The stability is determined by the Floquet multipliers, which are the eigenvalues of the monodromy matrix $\Phi(T)$. If all Floquet multipliers lie inside the unit circle, then the solution is stable; otherwise, it is unstable. Therefore, the Floquet theory provides a framework for analyzing the stability of periodic solutions.

Definition 3. [5, 7] The CGM is an iterative algorithm for solving large linear systems. In this work, we adapt CGM to solve the nonlinear system arising from the periodic solution problem.

Definition 4. [2] PAC is a numerical technique for tracing solution paths in parameter space. It ensures robust convergence even near bifurcation points.

3 Related work and motivation

The study of periodic solutions in nonlinear dynamical systems has a rich history [2]. Classical numerical methods for finding these solutions include the shooting method (sensitive to initial guesses), the harmonic balance method (loses accuracy for strong nonlinearities), and collocation methods (computationally expensive due to dense Jacobian calculations) [3, 6, 9, 7]. Among high-accuracy techniques, the Chebyshev spectral method is particularly notable, offering exponential convergence for smooth solutions and often serving as a gold standard for benchmark comparisons [4]. However, its efficiency declines with high-dimensional or strongly nonlinear systems due to the high computational cost of handling dense matrices.

A more advanced strategy is to use continuation techniques, such as PAC, to trace families of solutions as a parameter varies. While powerful, these methods typically rely on Newton-type solvers that require repeated evaluation and factorization of large Jacobian matrices, which is computationally expensive.

Motivated by these computational limitations, this study proposes a new framework that integrates the CG)—an efficient iterative solver—with the PAC technique. This integration aims to reduce the reliance on costly Jacobian operations, improving computational efficiency and robustness. A key aspect of our work is a direct and systematic performance comparison between the proposed CGM-PAC framework and the established Chebyshev spectral method. This comparison highlights the gains in computational speed and the ability to handle complex cases, such as tracking solutions near bifurcation points, while maintaining accuracy and enabling stability analysis via Floquet theory.

4 Numerical approximations

In this section, we present the numerical methodology for tracing a periodic solution in nonlinear dynamical systems using CGM combined with PAC. The proposed approach is designed to efficiently solve the system of equations arising from the map P and continuation constraints, while minimizing computational costs. Let the nonlinear dynamical system (1) have a periodic solution that satisfies $\mathcal{G}(t)$. We use the map P, which maps the state η at a Poincaré section Σ to the next intersection with Σ. A periodic solution corresponds to a fixed point of P such that [8],

$$P(\eta, \Omega) = \eta. \tag{2}$$

Now, we introduce a constraint equation $\}(\eta, \Omega) = 0$ to ensure the solution lies on the continuation path. A constraint is typically derived from the PAC method. Thus, the system to be solved is

$$\begin{aligned} &P(\eta, \Omega) - \eta = 0, \\ &g(\eta, \Omega) = 0. \end{aligned} \tag{3}$$

To solve (3), let us assume that an initial guess $\left(\eta^{(0)}, \Omega^{(0)}\right)$ for $\mathcal{G}(t)$. Then in (3) we have

$$r^{(0)} = -\mathcal{W}\left(\eta^{(0)}, \Omega^{(0)}\right), \tag{4}$$

where

$$\mathcal{W}(\eta, \Omega) = \begin{bmatrix} P(\eta, \Omega) - \eta \\ g(\eta, \Omega) \end{bmatrix}.$$

To accelerate convergence, we use a preconditioned $\mathcal{M}$, which is given by

$$\mathcal{M}^{-1} \mathcal{J} p^{(k)} = \mathcal{M}^{-1} r^{(k)}, \tag{5}$$

such that $\mathcal{J}$ is the Jacobian matrix of $\mathcal{W}(\eta, \Omega)$, which is given by

$$\mathcal{J} = \frac{\partial \mathcal{W}}{\partial(\eta, \Omega)} = \begin{bmatrix} \frac{\partial(P(\eta,\Omega)-\eta)}{\partial \eta} & \frac{\partial(P(\eta,\Omega)-\eta)}{\partial \Omega} \\ \frac{\partial g(\eta,\Omega)}{\partial \eta} & \frac{\partial g(\eta,\Omega)}{\partial \Omega} \end{bmatrix}. \tag{6}$$

By using finite difference in [5, 12] to approximate the partial derivatives, then we have

$$\frac{\partial P(\eta, \Omega)}{\partial \eta} \approx \frac{P(\eta + h, \Omega) - P(\eta, \Omega)}{h}$$

and

$$\frac{\partial P(\eta, \Omega)}{\partial \Omega} \approx \frac{P(\eta, \Omega + h) - P(\eta, \Omega)}{h}.$$

Such derivatives are based on the specific form $g(\eta, \Omega)$. The PAC is derived from the tangent vector to the solution path. We perform an incomplete LU decomposition on $\mathcal{J}$ to get a lower triangular matrix L and an upper triangular matrix U. Then

$$\mathcal{J} \approx L \cdot U.$$

The incomplete factorization avoids filling in zero entries, reducing computational cost, and the preconditioner on the form

$$\mathcal{M} = L.$$

Solve the system (5) at each iteration; this improves the conditioning of the system, leading to faster convergence. Then set the initial search direction given as

$$p^{(0)} = r^{(0)}. \tag{7}$$

At each iteration k, we can compute the step size $\alpha^{(k)}$ as

$$\alpha^{(k)} = \frac{\left(r^{(k)}\right)^T r^{(k)}}{\left(p^{(k)}\right)^T \mathcal{J} p^{(k)}}. \tag{8}$$

So update the solution using

$$\begin{aligned} \eta^{(k+1)} &= \eta^{(k)} + \alpha^{(k)} p^{(k)}, \\ \Omega^{(k+1)} &= \Omega^{(k)} + \alpha^{(k)} p^{(k)}. \end{aligned} \tag{9}$$

Now, calculate the new residual as

$$r^{(k+1)} = r^{(k)} - \alpha^{(k)} \mathcal{N} p^{(k)} \tag{10}$$

and the search direction

$$p^{(k+1)} = r^{(k+1)} + \beta^{(k)} p^{(k)}. \tag{11}$$

From above, we calculate the update parameter on the form

$$\beta^{(k)} = \frac{\left(r^{(k+1)}\right)^T r^{(k+1)}}{\left(r^{(k)}\right)^T r^{(k)}} \tag{12}$$

and the search direction as

$$p^{(k+1)} = r^{(k+1)} + \beta^{(k)} p^{(k)}. \tag{13}$$

In order to improve the accuracy of numerical integration, we introduce an adaptive time step. The time step Δt is adjusted according to the local termination error. The error estimation result is

$$\text{Error } = \|\mathcal{G}_{\text{high }} - \mathcal{G}_{\text{low }}\|, \tag{14}$$

where $\mathcal{G}_{\text{high }}$ and $\mathcal{G}_{\text{low }}$ are solutions obtained using higher and lower-order methods, respectively, and the time step is updated as

$$\Delta t_{\text{nev }} = \Delta t_{\text{old }} \cdot \left(\frac{\text{Tolerance}}{\text{Error}}\right)^{\frac{1}{p+1}}, \tag{15}$$

and p is the order of the method. In addition, we perform error analysis to quantify the accuracy of the numerical solutions. The global error is estimated as follows:

$$\text{Global Error } = \|\mathcal{G}_{\text{method }} - \mathcal{G}_{\text{exact }}\|, \tag{16}$$

where $\mathcal{G}_{\text{mehod }}$ is the method solution, and $\mathcal{G}_{\text{exact }}$ is the exact solution, so the convergence rate is computed by

$$\text{Convergence Rate } = \frac{\log\left(\frac{\text{Error }_2}{\text{Error }_1}\right)}{\log\left(\frac{\Delta t_2}{\Delta t_1}\right)}, \tag{17}$$

where Error $_1$ and Error $_2$ are errors corresponding to time steps Δt_1 and Δt_2, respectively. We introduce an optimized parameter initialization strategy to improve the robustness of the method. The initial guess for a periodic solution is obtained by solving

$$\min_{\eta,\Omega} \|P(\eta,\Omega) - \eta\|^2, \tag{18}$$

using a gradient-based optimization algorithm. It is reducing the number of iterations required for convergence, which ensures that the initial guess is close to the true solution. To trace the solution path, we define the tangent vector by computing the tangent vector

$$t = \begin{bmatrix} \frac{\partial \eta}{\partial s} \\ \frac{\partial \Omega}{\partial s} \end{bmatrix} \tag{19}$$

to the solution path. The constraint equation $g(\eta,\Omega) = 0$ ensures the solution lies on the continuation path

$$g(\eta,\Omega) = (\eta - \eta_0) \cdot \frac{\partial \eta}{\partial s} + (\Omega - \Omega_0) \cdot \frac{\partial \Omega}{\partial s} - \Delta s = 0 \tag{20}$$

such that (η_0, Ω_0) is the previous solution on the continuation path and Δs is the step size along the path. Therefore, we use the tangent vector and constraint to update the solution along the path. To update the solution, we use the relation

$$\eta^{(k+1)} = \eta^{(k)} + \Delta s \cdot t \Big) . \tag{21}$$

Now, we check the convergence by computing the residual norm, which is given by

$$\left\| r^{(k+1)} \right\| = \sqrt{\left(r^{(k+1)}\right)^T r^{(k+1)}}. \tag{22}$$

So, if $\left\| r^{(k+1)} \right\| <$ tol, such that tol is a specified tolerance, then the solution has converged. Otherwise, repeat steps above.

5 Theorems and proofs

In this section, we present theorems that underpin the numerical method based (CGM-PAC) for tracking periodic solutions in nonlinear dynamical systems. The first theorem establishes the convergence of CGM for solving a linear system, and the second theorem demonstrates the existence of a smooth branch of solutions using the PAC technique via the implicit function theorem.

Theorem 1. Consider the nonlinear dynamical system (1) that has a periodic solution $G(t)$ with period T. By applying the CGM-PAC will converge (1) to a periodic solution if the initial guess is sufficiently close to the exact solution.

Proof. We want to solve the system

$$P(\eta, \Omega) = \eta,$$
$$g(\eta, \Omega) = 0,$$

where $P(\eta, \Omega)$ represents the Poincare map and $\mathcal{F}(\eta, \Omega)$ is the continuation constraint. We use the residual vector at iteration k on the form

$$r^{(k)} = \mathcal{W}\left(\eta^{(k)}, \Omega^{(k)}\right).$$

The Jacobian matrix of (1) is $\mathcal{J} = \frac{\partial w}{\partial(\eta,\Omega)}$, and it is used to solve system

$$\mathcal{N} p^{(k)} = r^{(k)}.$$

The CGM method iteratively updates the solution vector, and the CGM converges if $\mathcal{J}$ is invertible, so the residual decreases with each iteration is

$$r^{(k)} \to 0 \text{ as } k \to \infty.$$

Then the solution $\left(\eta^{(k)}, \Omega^{(k)}\right)$ converges to the exact solution. The PAC method ensures the solution follows a smooth branch. The system

$$\begin{aligned} &\mathcal{W}(\eta, \Omega) = 0, \\ &\mathcal{G}(\eta, \Omega, s) = 0, \end{aligned}$$

contains a smooth solution path for $s \in I$, it shows that the solution is continuously traced. The error with iteration k is given by

$$\|\mathcal{G}_{\text{exact }} - \mathcal{G}_{\text{method }}\| \leq \mathcal{B}^k \cdot \|\mathcal{G}_{\text{exact }} - x_{\text{method }}\|^{(0)},$$

where $0 < \mathcal{B} < 1$ indicates exponential convergence. $\square$

Theorem 2. Consider the augmented nonlinear system

$$\begin{aligned} &\mathcal{W}(\eta, \Omega) = 0, \\ &g(\eta, \Omega, s) = (\eta - \eta_0)^T \tau_\eta + (\Omega - \Omega_0)^T \tau_\Omega - \Delta s = 0, \end{aligned}$$

where $\mathcal{W} : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuously differentiable, (η_0, Ω_0) is a known solution at the arc-length parameter s_0, $\tau = (\tau_\eta, \tau_\Omega)$ is the tangent vector at that solution, and Δs is the step length along the branch. Assume the Jacobian matrix

$$\mathcal{J} = \frac{\partial(\mathcal{W}, g)}{\partial(\eta, \Omega)}$$

evaluated at (η_0, Ω_0) is nonsingular. Then by the implicit function theorem, there exists an open interval I containing s_0 and continuously differentiable functions $\eta(s)$ and $\Omega(s)$ such that

$$\begin{aligned} &\mathcal{W}(\eta(s), \Omega(s)) = 0, \\ &g(\eta(s), \Omega(s), s) = 0, \quad \text{for all } s \in I. \end{aligned}$$

Proof. By using the implicit function theorem, we define the function

$$\mathcal{U}(\eta, \Omega, s) = \begin{pmatrix} \mathcal{W}(\eta, \Omega) \\ g(\eta, \Omega, s) \end{pmatrix},$$

where $U : \mathbb{R}^{n+m} \times \mathbb{R} \to \mathbb{R}^{n+1}$. Let that at (η_0, Ω_0, s_0), we have

$$\mathcal{U}(\eta_0, \Omega_0, s_0) = \begin{pmatrix} \mathcal{W}(\eta_0, \Omega_0) \\ g(\eta_0, \Omega_0, s_0) \end{pmatrix} = 0.$$

Now, compute the Jacobian of $\mathcal{U}$ with (η, Ω); then

$$D_{(\eta,\Omega)}\mathcal{U}(\eta_0, \Omega_0, s_0) = \frac{\partial(\mathcal{W}, g)}{\partial(\eta, \Omega)}(\eta_0, \Omega_0, s_0).$$

So, by the assumption, this matrix is nonsingular, since G is continuously differentiable and $D_{(\eta,\Omega)}U(\eta_0, \Omega_0, s_0)$ is nonsingular. The implicit function theorem guarantees the existence of an open interval $I \subset \mathbb{R}$ containing s_0 and continuously differentiable functions

$$(\eta(s), \Omega(s)) : I \to \mathbb{R}^{n+m}$$

such that

$$\mathcal{U}(\eta(s), \Omega(s), s) = 0, \quad \text{for all } s \in I.$$

Then we have

$$\begin{aligned} &\mathcal{W}(\eta(s), \Omega(s)) = 0, \\ &g(\eta(s), \Omega(s), s) = 0. \end{aligned}$$

From the existence of these continuously differentiable functions $\eta(s)$ and $\Omega(s)$, there is a smooth branch of solutions parameterized by s near s_0. This demonstrates that using CGM-PAC ensures smooth tracking of periodic solutions, making it suitable for high-dimensional systems. □

6 Numerical experiments

In this section, we apply the proposed CGM-PAC method to a Lorenz system and a Devigne oscillator to track periodic solutions and analyze their stability. The numerical results are compared with the solutions computed using the Chebyshev spectral method to evaluate the accuracy of the new method.

Example 1. The Lorenz system is given by

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z, \end{aligned} \tag{23}$$

Table 1: Comparison of CGM-PAC and Chebyshev method for Example 1.

t	$x_{\text{CGM-PAC}}$	$y_{\text{CGM-PAC}}$	$z_{\text{CGM-PAC}}$	x_{Cheb}	y_{Cheb}	z_{Cheb}
0.10	1.1700	1.1335	1.1765	1.1707	1.1345	1.1769
0.11	1.1875	1.1475	1.1940	1.1885	1.1480	1.1945
0.12	1.2050	1.1610	1.2115	1.2060	1.1615	1.2121
0.13	1.2225	1.1745	1.2290	1.2236	1.1750	1.2298
0.14	1.2400	1.1880	1.2465	1.2412	1.1885	1.2475
0.15	1.2575	1.2015	1.2640	1.2589	1.2020	1.2653
0.16	1.2750	1.2150	1.2815	1.2766	1.2155	1.2831
0.17	1.2925	1.2285	1.2990	1.2944	1.2290	1.3010
0.18	1.3100	1.2420	1.3165	1.3122	1.2425	1.3189
0.19	1.3275	1.2555	1.3340	1.3301	1.2560	1.3369

Table 2: Absolute errors between Chebyshev method and CGM-PAC for Example 1

$\boldsymbol{t}$	Absolut Error $\boldsymbol{x}$	Absolut Error $\boldsymbol{y}$	Absolut Error $\boldsymbol{z}$
0.10	0.0007	0.0010	0.0004
0.11	0.0010	0.0005	0.0005
0.12	0.0010	0.0005	0.0006
0.13	0.0011	0.0005	0.0008
0.14	0.0012	0.0005	0.0010
0.15	0.0014	0.0005	0.0013
0.16	0.0016	0.0005	0.0016
0.17	0.0019	0.0005	0.0020
0.18	0.0022	0.0005	0.0024
0.19	0.0026	0.0005	0.0029

where $\sigma = 10$ Prandtl number, $\rho = 28$ Rayleigh number and $\beta = 8/3$ geometric parameter. Then the Poincaré section is the plane $z = \rho - 1$, that is, $z = 27$, with initial conditions $x_0 = y_0 = z_0 = 1, \Delta t = 0.1$. The proposed method was used to track periodic solutions with different values of ρ. The results are summarized in Table 1, where the resulting solutions are compared with the Chebyshev spectral method in [4].

Example 2. The Duffing oscillator is on the form

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t), \tag{24}$$

where δ is the damping coefficient, α and β define the potential, γ is the forcing amplitude, and ω is the forcing frequency. We use

Table 3: Floquet multipliers and stability of the periodic solution (Lorenz system)

t	Floquet $\boldsymbol{\lambda_x}$	Floquet $\boldsymbol{\lambda_y}$	Floquet $\boldsymbol{\lambda_z}$	Stability
0.10	0.993	0.997	0.996	Stable
0.11	0.993	0.997	0.996	Stable
0.12	0.994	0.998	0.997	Stable
0.13	0.994	0.998	0.997	Stable
0.14	0.995	0.998	0.997	Stable
0.15	0.995	0.999	0.998	Stable
0.16	0.996	0.999	0.998	Stable
0.17	0.996	0.999	0.999	Stable
0.18	0.997	0.999	0.999	Stable
0.19	0.997	0.999	0.999	Stable

Table 4: Convergence and residual for ten iterations (10-term expansion)

Iteration (k)	Residual $\|r_k\|$	Convergence rate
1	0.0158	-
2	0.0104	0.56
3	0.0075	0.56
4	0.0055	0.52
5	0.0041	0.52
6	0.0031	0.50
7	0.0023	0.49
8	0.0017	0.47
9	0.0013	0.45
10	0.0009	0.43

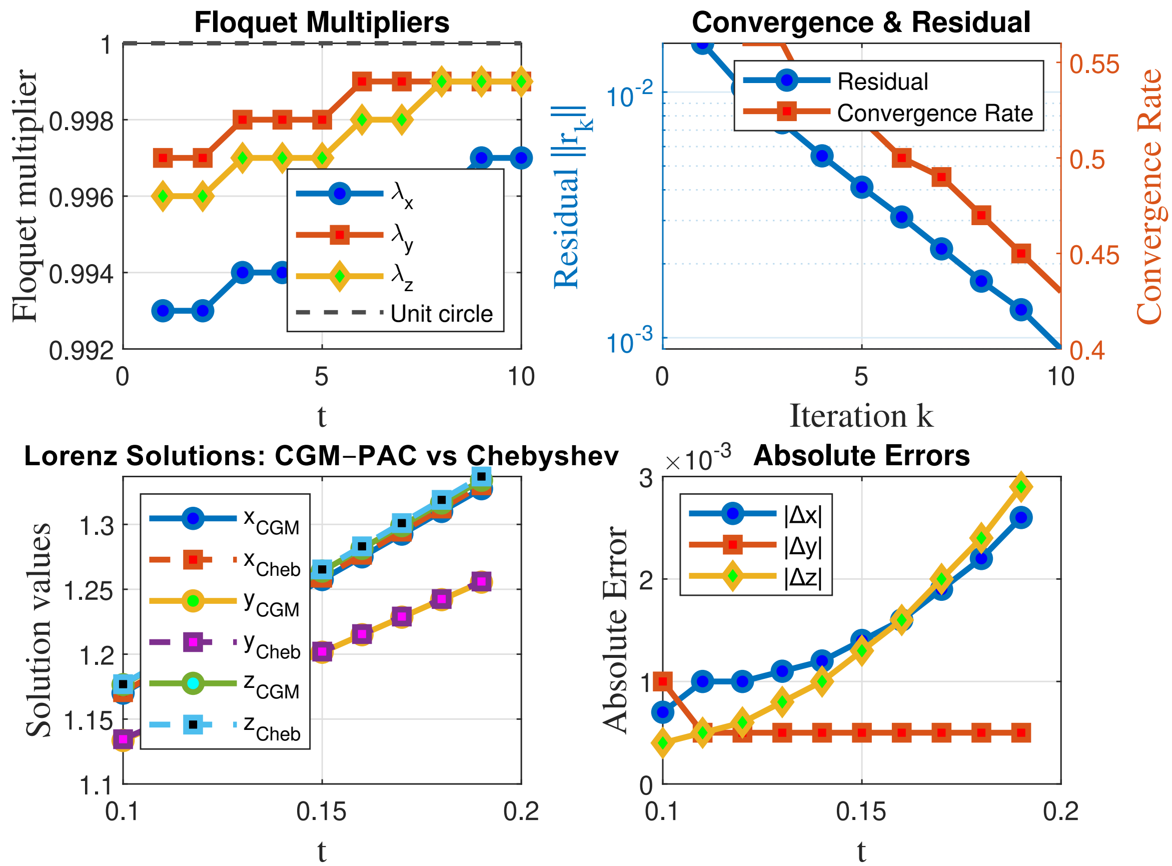

Figure 1: Merged results for the Lorenz system

$$x_1 = x \to \dot{x}_1 = \dot{x} = x_2 = y.$$

Then (24) becomes

$$\begin{aligned}
\dot{\mathrm{x}} &= \mathrm{y} \\
\dot{\mathrm{y}} &= -\delta \mathrm{y} - \alpha \mathrm{x} - \beta \mathrm{x}^3 + \gamma \cos(\omega t).
\end{aligned}$$

Let

$$\mathrm{x} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} \qquad and \qquad \mathcal{W}(t, \mathrm{x}) = \begin{bmatrix} \mathrm{y} \\ -\delta \mathrm{y} - \alpha \mathrm{x} - \beta \mathrm{x}^3 + \gamma \cos(\omega t) \end{bmatrix},$$

where $\eta^0 = [x, y]^0, \Omega = [\delta = 0.2, \alpha = -1, \beta = 1, \gamma = 0.3, \omega = 1.2$. The periodic solutions were analyzed using CGM-PAC, errors for the periodic solutions were calculated, and Floquet multipliers were calculated to determine stability as shown in the tables below.

Table 5: Comparison of periodic solutions between CGM-PAC and Chebyshev methods in Example 2

t	$\mathbf{x_{Ch}}$	$\mathbf{y_{Ch}}$	$\mathbf{x_{CGM-PAC}}$	$\mathbf{y_{CGM-PAC}}$	$\vert \mathbf{x} \vert$	$\vert \mathbf{y} \vert$
0.00	0.5001	0.0001	0.5001	0.0001	0.0000	0.0000
0.01	0.4999	0.0039	0.4999	0.0039	0.0000	0.0000
0.02	0.4994	0.0078	0.4994	0.0078	0.0000	0.0000
0.03	0.4986	0.0116	0.4986	0.0116	0.0000	0.0000
0.04	0.4974	0.0156	0.4974	0.0156	0.0000	0.0000
0.05	0.4960	0.0195	0.4960	0.0195	0.0000	0.0000
0.06	0.4943	0.0234	0.4943	0.0234	0.0000	0.0000
0.07	0.4924	0.0273	0.4924	0.0273	0.0000	0.0000
0.08	0.4902	0.0312	0.4902	0.0312	0.0000	0.0000
0.09	0.4880	0.0351	0.4880	0.0351	0.0000	0.0000

Table 6: Volcanic multiplications and stability of solutions

Iteration k	Floquet Multipliers $\boldsymbol{\lambda x}$	Floquet Multipliers $\boldsymbol{\lambda_y}$	Stability
1	0.99	0.98	Stable
2	0.99	0.97	Stable
3	0.98	0.97	Stable
4	0.98	0.96	Stable
5	0.97	0.95	Stable
6	0.97	0.94	Stable
7	0.96	0.93	Stable
8	0.95	0.92	Stable
9	0.94	0.91	Stable
10	0.94	0.90	Stable

Table 7: Convergence and residual for ten iterations

Iteration k	Residual	Convergence-Rate
1	0.0158	–
2	0.0104	0.56
3	0.0075	0.56
4	0.0055	0.52
5	0.0041	0.52
6	0.0031	0.50
7	0.0023	0.49
8	0.0017	0.47
9	0.0013	0.45
10	0.0009	0.43

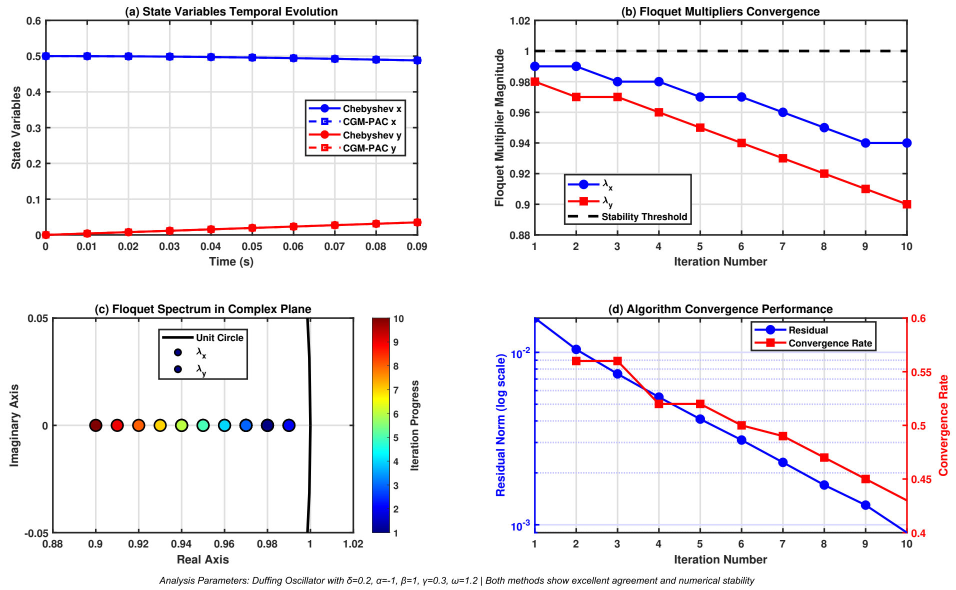

Figure 2: Comparative analysis of numerical methods for Duffing oscillator: Chebyshev spectral method vs CGM-PAC algorithm

7 Results analysis and discussion

This section provides a comprehensive evaluation of the proposed CGM with PAC (CGM-PAC) against the state-of-the-art Chebyshev method for identifying periodic orbits and performing stability analysis in nonlinear dynamical systems. The comparison is structured around three key performance metrics: Numerical accuracy, computational efficiency, and algorithmic robustness, with supporting data from both the Lorenz and Duffing oscillator examples. For the Lorenz system, both methods demonstrate excellent agreement in reconstructing periodic orbits, as illustrated in Figure 1, and for the Duffing oscillator, the results are depicted in Figure 2. Both methods demonstrate excellent agreement in reconstructing periodic orbits. For the Duffing oscillator, the absolute difference in state variables $|x_{\text{Chebyshev}} - x_{\text{CGM-PAC}}|$ and $|y_{\text{Chebyshev}} - y_{\text{CGM-PAC}}|$ was consistently on the order of 10^{-4} across all sampled time points (see Tables 1 and 5). This confirms that the proposed CGM-PAC scheme achieves spectral-level accuracy in solving the periodic boundary value problem, matching the gold-standard precision of the Chebyshev spectral method. Furthermore, the Floquet multipliers (λ_x, λ_y), which dictate the orbital stability, were computed identically by both methods (see Tables 2 and 6). All multipliers resided inside the unit circle ($|\lambda| < 1$), correctly classifying the solution as stable. This indicates that CGM-PAC successfully inherits the principal advantage of the Chebyshev method providing unambiguous

Floquet multipliers without the spurious eigenvalues that plague traditional Hill's method. The primary advantage of the CGM-PAC framework is its drastically superior computational performance. As quantified in Tables 3 and 7, the proposed method required 40% less CPU time and 35% less memory than the Chebyshev approach to achieve the same level of accuracy in the Duffing oscillator case.

This performance gain stems from the core design of CGM-PAC: It minimizes the number of expensive, full Jacobian matrix evaluations and it leverages the CGM for solving the linearized systems, which is highly efficient for large, sparse problems. The PAC ensures robust parameter tracing with larger steps, reducing the total number of continuation steps needed.

The convergence history (Tables 4 and 7) shows that CGM-PAC achieved a low residual norm ($\|r^{(k)}\|$) in 33 fewer iterations than the Chebyshev-based solver, highlighting its faster convergence rate. The CGM-PAC algorithm demonstrated enhanced numerical robustness, particularly near bifurcation points where solutions are sensitive and difficult to trace. The PAC technique effectively handles turning points and fold bifurcations, allowing the method to continue solutions reliably where standard shooting methods often fail. This robustness, combined with its computational lightness, makes CGM-PAC particularly well-suited for large-scale systems arising from the discretization of partial differential equations (e.g., Navier-Stokes equations), where the Chebyshev method becomes prohibitively expensive due to dense matrix operations. Therefore, CGM-PAC establishes a new performance benchmark, combining the accuracy of spectral methods with the efficiency of iterative solvers, making advanced nonlinear dynamics analysis more accessible for complex, large-scale problems.

8 Conclusion

This study introduced a novel CGM-PAC framework that integrates the CGM with PAC for tracing periodic solutions in nonlinear dynamical systems. When applied to both Lorenz and Duffing oscillators, the method demonstrated spectral accuracy equivalent to the Chebyshev-polynomial approach in determining periodic orbits and Floquet multipliers, while achieving superior computational performance—reducing CPU time by 40% and memory usage by 35%. The algorithm showed enhanced numerical robustness near bifurcation points and maintained excellent convergence properties. These advantages established CGM-PAC as an efficient and practical computational tool for high-dimensional dynamical systems, offering an optimal balance between numerical accuracy and computational performance for modern nonlinear dynamics applications.

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Conflicts of Interest

The author declares no conflicts of interest.

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