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Research Article

# Portfolio optimization: A mean-variance approach for non-Markovian regime-switching markets

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**Abstract**

This paper develops a novel multi-period mean-variance portfolio optimization framework for non-Markovian regime-switching markets, where state transition probabilities exhibit strong path-dependence. We propose an innovative dynamic programming solution that extends classical frameworks by incorporating path-dependent value functions through a rigorously derived modified Bellman equation. The solution involves constructing an auxiliary optimization problem using Lagrangian methods, with closed-form optimal strategies derived via matrix calculus. Analytically, we demonstrate that classical Markovian solutions emerge as special cases when path-dependence is removed. Numerical examples further demonstrate that our model could generate significantly lower-risk portfolios than Markovian alternatives by adaptively adjusting positions based on market history.

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**Keywords:** Portfolio optimization; Non-Markovian regime-switching; Path-dependence; Dynamic programming; Mean-variance analysis.

## 1 Introduction

The multi-period mean-variance portfolio optimization problem generalizes the static model introduced by Markowitz [16], where investors aim to minimize terminal wealth variance (risk) while targeting a fixed expected terminal wealth (reward). In this dynamic setting, investors sequentially rebalance their portfolios at discrete time intervals over the investment horizon according to an optimal strategy. While dynamic programming is the standard approach for solving stochastic optimization problems, the nonlinearity of the variance operator in the mean-variance framework renders direct application infeasible. However, Li and Ng [13] circumvented this challenge by embedding the original problem into an auxiliary problem solvable via dynamic programming, yielding explicit solutions. A critical limitation of their approach, however, lies in the assumption of independent asset returns across periods-a condition starkly contradicted by empirical financial market data, where return dependencies are well-documented. To address this, recent work by Cakmak and Ozekici [2] and others has extended the framework to regime-switching markets, where asset returns depend on the market state, modeled as a Markov chain. This formulation captures intertemporal return dependencies while preserving analytical tractability. Regime-switching models have become widely adopted for portfolio selection, asset allocation, and utility maximization in multi-period settings [3, 4, 6, 8, 9, 11, 21, 22, 25]. Hidden Markov chain approaches have been developed in

[1, 5, 27, 28, 29], offering alternative frameworks for modeling unobservable market regimes.

The Markovian assumption for market state processes in prior literature inherently ignores the historical path and evolution of market dynamics. This simplification is economically unrealistic, as market history and memory invariably influence current market behavior and, consequently, investor decisions. Although incorporating path-dependence significantly increases model complexity and poses analytical challenges, some recent works have started to explore non-Markovian frameworks. In particular, several studies have introduced models where asset returns depend on the historical trajectory of the market rather than solely on the current state. Specifically, these models generalize state-dependent coefficients by conditioning them on the filtration generated by a Markov chain

[7, 12, 17, 18, 19, 20]. However, it should be noted that the majority of these studies have focused exclusively on continuous-time settings, with the exception of [12], which examines a discrete-time, multi-period framework. A fundamental limitation persists that these models still assume the underlying market state follows a Markov process (i.e., transition probabilities remain path-independent). This contradicts empirical evidence where market history exerts persistent effects-exemplified by momentum (trend persistence) or regime shifts after prolonged bull/bear markets. Such phenomena demonstrate that market states are inherently non-Markovian. In addition to regime-switching models, several studies have proposed non-Markovian approaches that do not rely on market states, but instead capture temporal dependence by modeling serial correlation among asset returns directly

[10, 23, 24, 26]. While these approaches account for time dependence, they do not explicitly model market states or regime transitions.

To the best of our knowledge, the mean-variance portfolio optimization problem has not been studied in regime-switching markets where the state process is non-Markovian. In this work, we examine this scenario where asset returns depend on current market states while transition probabilities exhibit path-dependence rather than following Markovian properties. Models such as hidden Markov models and semi-Markov processes also incorporate memory effects, typically by introducing latent variables or duration-based transitions. In contrast, our model allows transition probabilities to depend explicitly on the observed history of market states, thereby providing a more direct form of path dependence. Unlike the embedding technique developed by Li and Ng [13], we employ the alternative Lagrangian multipliers method introduced by Li, Zhou, and Lim [14] to solve this problem. Our solution methodology proceeds in two key stages: First, we construct and solve an auxiliary optimization problem through the method of Lagrangian multipliers; then, we recover the solution to the original problem by applying

Lagrangian duality theorem. To incorporate path-dependence, we derive an extended Bellman equation based on path-dependent value functions. The computational complexity arising from path-dependent parameters requires matrix calculus to derive explicit forms for both the optimal strategy and the mean-variance efficient frontier. Importantly, our model generalizes classical Markovian frameworks, which emerge as special cases when path-dependence is eliminated.

This paper is organized as follows: Section 2 presents our non-Markovian regime-switching market model including all assumptions, problem formulation, and mathematical preliminaries. Section 3 introduces the auxiliary problem and solves it using dynamic programming, ultimately solving the original problem through Lagrangian duality. The special case of Markovian regime-switching markets is examined in Section 4. A numerical illustration comparing our proposed model with classical approaches appears in Section 5. Finally, Section 6 concludes with key findings and suggestions for future research.

## 2 Model framework: Assumptions and formulation

We consider a discrete-time, multi-period investment horizon with $T$ periods, where the financial market consists of $N+1$ assets. Among these assets, one is risk-free, and the remaining $N$ are risky assets. Let $\{\Theta_n\}_{n=0}^{T}$ denote the sequence of market states (or regimes) over the investment horizon, where $\Theta_n \in \{1, 2, \dots, M\}$ represents the regime of the market at time $n$. The market exhibits regime-switching dynamics, but the transitions between states are not governed by a Markov process. Instead, the transition probabilities depend on the entire past trajectory of the market states, encapsulating a path-dependent behavior. The return of the $i$th asset at time $n$, denoted by $R_n^i(\Theta_n)$, depends on the market state $\Theta_n$, where $i = 0, 1, \dots, N$, with $i = 0$ representing the risk-free asset. The transition probabilities between market states depend not only on the current state $\Theta_{n-1}$ but also on the past trajectory of the market, represented by $(\Theta_0, \Theta_1, \dots, \Theta_{n-1}) = (\theta_0, \theta_1, \dots, \theta_{n-1})$. The probability of transitioning from state $\theta_{n-1}$ at time $n-1$ to state $\theta_n$ at time $n$ under the history of market states $(\theta_0, \theta_1, \dots, \theta_{n-1})$ is defined as

$$Q_n(\theta_0, \dots, \theta_{n-1}, \theta_n) = \mathrm{P}(\Theta_n = \theta_n \mid \Theta_{n-1} = \theta_{n-1}, \dots, \Theta_1 = \theta_1, \Theta_0 = \theta_0),$$

where $\mathrm{P}(\cdot)$ is the probability measure and

$$\sum_{\theta_n=1}^{M} Q_n(\theta_0, \dots, \theta_{n-1}, \theta_n) = 1.$$

This non-Markovian model allows for a richer and more realistic representation of market dynamics, accommodating situations where the probability of state transitions is influenced by prior trends, volatility clusters, or other historical features.

At each time $n \in \{0, 1, \dots, T-1\}$, the investor allocates their wealth among the $N+1$ assets in the market. Let $\pi_n = (\pi_n^1, \pi_n^2, \dots, \pi_n^N)' \in \mathbb{R}^N$ (where $'$ denotes transpose) denote the portfolio assigned to the $N$ risky assets at time $n$, where $\pi_n^i$ represents the wealth allocated to the $i$-th risky asset. The remaining wealth

$$W_n - \sum_{i=1}^{N} \pi_n^i$$

is allocated to the risk-free asset, where $W_n$ denote the total investor's wealth at time $n$. Here, $\pi = \{\pi_0, \pi_1, \dots, \pi_{T-1}\}$ denotes the overall investment strategy over the entire investment horizon. The evolution of the investor's wealth is driven by the returns of the assets. Assuming that the returns are expressed relative to the risk-free asset, the excess return of the $i$-th risky asset at time $n$ in regime $\Theta_n$ is given by

$$R_n^{e,i}(\Theta_n) = R_n^i(\Theta_n) - R_n^0(\Theta_n).$$

The wealth dynamics over time can then be expressed as

$$W_{n+1} = (W_n - \sum_{i=1}^{N} \pi_n^i) R_n^0(\Theta_n) + \sum_{i=1}^{N} \pi_n^i R_n^i(\Theta_n) = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n),$$

where $R_n^e(\Theta_n) = \left(R_n^{e,1}(\Theta_n), \dots, R_n^{e,N}(\Theta_n)\right)'$.

We assume the following regarding the returns of the assets and market states. Let $R_n(\Theta_n) = \left(R_n^0(\Theta_n), \ldots, R_n^N(\Theta_n)\right)'$ represent the vector of asset returns under the market state $\Theta_n$. For different time points $m \neq n$, given the market states $\Theta_n = \theta_n$ and $\Theta_m = \theta_m$, the random vectors $R_n(\theta_n)$ and $R_m(\theta_m)$ are independent. Additionally, the future market state $\Theta_{n+1}$ is assumed to be independent of the given current wealth $W_n = w_n$. Also, we assume that the covariance matrix corresponding to the returns of the assets for a given market state is positive definite. Finally, we assume frictionless trading and exclude transaction costs from the model for analytical tractability.

The objective of the investor in this framework is to construct a portfolio strategy $\pi^*$ that minimizes the variance of the terminal wealth $W_T$ while achieving a predefined level of expected terminal wealth, $E_T$. The multi-period Markowitz's mean-variance optimization problem can be formulated as

$$P(MV): \begin{cases} \min\limits_{\pi} \mathbb{V}\text{ar}_0 \left[W_T\right] \\ \text{s.t. } \mathbb{E}_0 \left[W_T\right] = E_T, \\ \qquad W_{n+1} = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n), \end{cases}$$

where $\mathbb{E}_0$ and $\mathbb{V}\text{ar}_0$ represent the expectation and variance operators under the initial market condition $\Theta_0 = \theta_0$.

In this framework, the path-dependence of the market regimes $\Theta_n$ introduces additional complexity. The optimal strategy $\pi^*$ is inherently influenced by the entire history of market states $(\Theta_0, \Theta_1, \ldots, \Theta_n)$, reflecting the non-Markovian nature of the regime-switching dynamics. To analyze and solve this problem, we employ dynamic programming principles, incorporating the path-dependent transition probabilities and the regime-dependent asset returns into the optimization framework.

Before addressing the solution to the optimization problem, we introduce some certain matrix notations that will play a fundamental role in simplifying the subsequent computations.

For the given market state $\Theta_n = \theta_n$, we define

$$h_n(\theta_n) = \bar{R}_n^e(\theta_n)' V_n(\theta_n)^{-1} \bar{R}_n^e(\theta_n),$$

$$g_n(\theta_n) = R_n^0(\theta_n)\left(1 - h_n(\theta_n)\right),$$
$$f_n(\theta_n) = R_n^0(\theta_n)^2\left(1 - h_n(\theta_n)\right),$$

where

$$V_n(\theta_n) = \mathbb{E}\left[R_n^e(\theta_n)R_n^e(\theta_n)'\right],$$
$$\bar{R}_n^e(\theta_n) = \mathbb{E}\left[R_n^e(\theta_n)\right].$$

Assuming the positive definiteness of the covariance matrices, we state the following lemma. For the proof, please refer to [2, Lemmas 1 and 2]. Note that the following lemma guarantees the invertibility of the matrix $V_n(\theta_n)$ in the above notations.

**Lemma 1.** The matrix $V_n(\theta_n)$ is positive definite. Additionally, the scalars $f_n(\theta_n)$ and $g_n(\theta_n)$ are strictly positive, and $h_n(\theta_n)$ satisfies $0 < h_n(\theta_n) < 1$.

Let $C_n \in \mathbb{R}^M$ be a column vector, and let $B_n$ be a tensor of order $(n+1)$ with shape $M \times M \times \cdots \times M$, where $M$ is the number of market states. We denote $(Q_n \bullet B_n)$ as an $M \times M \times \cdots \times M$ tensor of order $(n+1)$, and $\overline{(Q_n \bullet B_n)}$ and $\overline{(Q_{C_n} \bullet B_n)}$ as $M \times M \times \cdots \times M$ tensors of order $n$ defined as follows:

$$(Q_n \bullet B_n)(\theta_0, \ldots, \theta_n) = Q_n(\theta_0, \ldots, \theta_n)B_n(\theta_0, \ldots, \theta_n),$$
$$\overline{(Q_n \bullet B_n)}(\theta_0, \ldots, \theta_{n-1}) = \sum_{\theta_n=1}^{M} Q_n(\theta_0, \ldots, \theta_{n-1}, \theta_n)B_n(\theta_0, \ldots, \theta_{n-1}, \theta_n),$$
$$\overline{(Q_{C_n} \bullet B_n)}(\theta_0, \ldots, \theta_{n-1}) = \sum_{\theta_n=1}^{M} Q_n(\theta_0, \ldots, \theta_n)C_n(\theta_n)B_n(\theta_0, \ldots, \theta_n).$$

Using these notations, for $1 \le n < k$ $(k \in \mathbb{N})$, we define

$$\overline{\boxdot_{j=k-n}^{k-1} Q_{C_j} \bullet (Q_k \bullet B_k)} = \overline{(Q_{C_{k-n}} \bullet \overline{(\ldots \bullet \overline{(Q_{C_{k-1}} \bullet \overline{(Q_k \bullet B_k)})} \ldots)})}, \quad (1)$$

as an $M \times M \times \cdots \times M$ tensor of order $(k-n)$. For convenience, we set

$$\overline{\boxdot_{\emptyset} Q_{C_j} \bullet (Q_k \bullet B_k)} = \overline{(Q_k \bullet B_k)}.$$

Furthermore, we use the conventions

$$\sum_{\emptyset}(\cdot) = \mathbf{0}, \quad \prod_{\emptyset}(\cdot) = I,$$

where $I$ denotes the identity matrix.

By applying mathematical induction, we can derive the following lemmas. See also [12, Lemmas 2 and 3].

**Lemma 2.** For $n \geq 0$,

$$\begin{aligned}&\overline{\textstyle\bigsqcap_{j=k-n}^{k-1} Q_{C_j} \bullet (Q_k \bullet B_k)}(\theta_0, \dots, \theta_{k-n-1})\\ &= \sum_{\theta_{k-n}=1}^{M} \dots \sum_{\theta_{k-1}=1}^{M} \sum_{\theta_k=1}^{M} Q_{k-n}(\theta_0, \dots, \theta_{k-n}) C_{k-n}(\theta_{k-n}) \dots Q_{k-1}(\theta_0, \dots, \theta_{k-1})\\ &\quad \times C_{k-1}(\theta_{k-1}) Q_k(\theta_0, \dots, \theta_k) B_k(\theta_0, \dots, \theta_k).\end{aligned}$$

**Lemma 3.** Let $\{C_n\}_{n=0}^{T-1}$ be a sequence of $M$-column vectors and let $\{B_n\}_{n=0}^{T}$ be a sequence of $M \times M \times \cdots \times M$ tensors of order $(n+1)$. Define the sequence $\{A_n\}_{n=0}^{T}$ of $M \times M \times \cdots \times M$ tensors of order $(n+1)$ recursively as follows:

$$\begin{aligned}A_n(\theta_0, \dots, \theta_n) &= B_n(\theta_0, \dots, \theta_n) + C_n(\theta_n)\overline{(Q_{n+1} \bullet A_{n+1})}(\theta_0, \dots, \theta_n),\\ A_T(\theta_0, \dots, \theta_T) &= B_T(\theta_0, \dots, \theta_T).\end{aligned}$$

Then,

$$\begin{aligned}A_n(\theta_0, \dots, \theta_n) =& B_n(\theta_0, \dots, \theta_n)\\ &+ C_n(\theta_n) \sum_{k=n+1}^{T} \overline{\textstyle\bigsqcap_{j=n+1}^{k-1} Q_{C_j} \bullet (Q_k \bullet B_k)}(\theta_0, \dots, \theta_n).\end{aligned}$$

## 3 Dynamic programming formulation

The dynamic programming method serves as a powerful tool for solving stochastic optimization problems, particularly in multi-stage decision-making scenarios. This approach facilitates breaking down complex problems into smaller, more manageable subproblems. However, when applied to the mean-variance portfolio selection problem, dynamic programming faces challenges due to the nonseparability of variance in the dynamic programming framework. To address this issue, Li and Ng [13] introduced an embedding technique that redefines the problem, allowing the variance to be addressed indirectly. By constructing an auxiliary problem, they derived solutions for the classical mean-variance problem. While effective, their approach can be computationally intricate and is not always intuitive. In subsequent work, Li, Zhou, and Lim [14] proposed a more practical method by employing the Lagrange duality technique to derive optimal solutions. This approach reduces computational complexity and simplifies the solution process. In this study, we adopt a similar method to solve the problem $P(MV)$.

First, we reformulate $P(MV)$ using the variance definition, resulting in the following formulation:

$$P(MV):\begin{cases}\min\limits_{\pi}\ \mathbb{E}_0\left[(W_T-E_T)^2\right]\\ \text{s.t. } \mathbb{E}_0\left[W_T-E_T\right]=0,\\ \qquad W_{n+1}=W_nR_n^0(\Theta_n)+\pi_n'R_n^e(\Theta_n).\end{cases}$$

By introducing the Lagrange multiplier $2\lambda\in\mathbb{R}$, the constrained problem can be transformed into the following unconstrained formulation:

$$\tilde{P}(MV):\begin{cases}\min\limits_{\pi}\ \mathbb{E}_0\left[(W_T-E_T)^2\right]+2\lambda\mathbb{E}_0\left[W_T-E_T\right]\\ \text{s.t. } W_{n+1}=W_nR_n^0(\Theta_n)+\pi_n'R_n^e(\Theta_n).\end{cases}$$

Introducing the substitutions $d_1=2(\lambda-E_T)$ and $d_0=E_T^2-2\lambda E_T$, we obtain

$$\tilde{P}(MV):\begin{cases}\min\limits_{\pi}\ \mathbb{E}_0\left[W_T^2+d_1W_T+d_0\right]\\ \text{s.t. } W_{n+1}=W_nR_n^0(\Theta_n)+\pi_n'R_n^e(\Theta_n).\end{cases}$$

## 3.1 Solution to problem $\tilde{P}(MV)$

To determine the optimal solution for problem $\tilde{P}(MV)$, the approach involves minimizing the expected cost function of the terminal wealth, that is,

$$\min_{\pi} \ \mathbb{E}_0 \left[ g(W_T) \right],$$

using the dynamic programming approach, where the cost function is defined as

$$g(W_T) = W_T^2 + d_1 W_T + d_0.$$

Define $J_n(\theta_0, \ldots, \theta_n; w_n; \pi_n)$ as the expected cost incurred when employing the investment policy $\pi_n$ at time $n$, followed by optimal strategies from time $n+1$ to $T$. This is conditional on the market path $(\theta_0, \ldots, \theta_n)$ and the wealth $w_n$ available at time $n$. Accordingly,

$$v_n(\theta_0, \ldots, \theta_n; w_n) = \min_{\pi_n} J_n(\theta_0, \ldots, \theta_n; w_n; \pi_n),$$

represents the optimal expected cost under the given market path $(\theta_0, \ldots, \theta_n)$ and the available wealth $w_n$ at stage $n$. Using the dynamic programming principle, the relationship between $J_n$ and $v_{n+1}$ is given by

$$\begin{aligned} & J_n(\theta_0, \ldots, \theta_n; w_n; \pi_n) \\ & = \mathbb{E} \left[ v_{n+1} \left( \theta_0, \ldots, \theta_n, \Theta_{n+1}; W_{n+1}(\pi_n) \right) \mid \Theta_0 = \theta_0, \ldots, \Theta_n = \theta_n, W_n = w_n \right], \end{aligned}$$

where $W_{n+1}(\pi_n)$ is the wealth at time $n+1$ resulting from applying policy $\pi_n$. The dynamic programming equation (DPE) for this problem can thus be expressed as

$$\begin{aligned} & v_n(\theta_0, \ldots, \theta_n; w_n) \\ & = \min_{\pi_n} \mathbb{E} \left[ v_{n+1} \left( \theta_0, \ldots, \theta_n, \Theta_{n+1}; W_{n+1}(\pi_n) \right) \mid \Theta_0 = \theta_0, \ldots, \Theta_n = \theta_n, W_n = w_n \right]. \end{aligned}$$

Rewriting this, the DPE becomes

$$\begin{aligned} v_n(\theta_0, \ldots, \theta_n; w_n) = \min_{\pi_n} \Bigg\{ & \sum_{\theta_{n+1}=1}^{M} Q_{n+1}(\theta_0, \ldots, \theta_n, \theta_{n+1}) \\ & \times \mathbb{E} \left[ v_{n+1} \left( \theta_0, \ldots, \theta_{n+1}; w_n R_n^0(\theta_n) + \pi_n' R_n^e(\theta_n) \right) \right] \Bigg\}, \end{aligned} \tag{2}$$

with the boundary condition

$$v_T(\theta_0, \ldots, \theta_T; w_T) = w_T^2 + d_1 w_T + d_0. \tag{3}$$

The DPE is solved recursively, starting from the terminal condition at $T$ and proceeding backward to $n = 0$, to determine the optimal strategy.

The following theorem provides the main result of our analysis, offering an explicit solution to problem $\tilde{P}(MV)$.

**Theorem 1.** For $n = 0, 1, \dots, T-1$, $v_n(\theta_0, \dots, \theta_n; w_n)$ is given by

$$v_n(\theta_0, \dots, \theta_n; w_n) = a_n(\theta_0, \dots, \theta_n) w_n^2 + b_n(\theta_0, \dots, \theta_n) w_n + c_n(\theta_0, \dots, \theta_n), \tag{4}$$

under the optimal policy

$$\begin{aligned} &\pi_n^*(\theta_0, \dots, \theta_n; w_n) \\ &= -\left[ w_n R_n^0(\theta_n) + \frac{\overline{\bigodot_{j=n+1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{d}_T)}(\theta_0, \dots, \theta_n)}{2\overline{\bigodot_{j=n+1}^{T-1} Q_{f_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n)} \right] V_n(\theta_n)^{-1} \bar{R}_n^e(\theta_n), \end{aligned} \tag{5}$$

where

$$\begin{aligned} a_n(\theta_0, \dots, \theta_n) &= f_n(\theta_n) \overline{\bigodot_{j=n+1}^{T-1} Q_{f_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n), \\ b_n(\theta_0, \dots, \theta_n) &= g_n(\theta_n) \overline{\bigodot_{j=n+1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{d}_T)}(\theta_0, \dots, \theta_n), \\ c_n(\theta_0, \dots, \theta_n) &= e_n(\theta_0, \dots, \theta_n) + \sum_{k=n+1}^{T} \overline{\bigodot_{j=n+1}^{k-1} Q_{\mathbf{1}_j} \bullet (Q_k \bullet e_k)}(\theta_0, \dots, \theta_n), \\ e_n(\theta_0, \dots, \theta_n) &= -\frac{\left[ \overline{\bigodot_{j=n+1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{d}_T)}(\theta_0, \dots, \theta_n) \right]^2}{4\overline{\bigodot_{j=n+1}^{T-1} Q_{f_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n)} h_n(\theta_n), \end{aligned}$$

$e_T(\theta_0, \dots, \theta_T) = d_0$, $\mathbf{d}_T(\theta_0, \dots, \theta_T) = d_1$, $\mathbf{1}_T(\theta_0, \dots, \theta_T) = 1$ and $\mathbf{1}_j(\theta_j) = 1$ $(j = 1, 2, \dots, T-1)$.

**Remark 1.** In the quadratic value function, the coefficient $a_n$ captures the impact of risk (variance), $b_n$ relates to expected return, and $c_n$ reflects the accumulated path-dependent effect independent of current wealth.

*Proof.* By mathematical induction, we first establish (4) under the following recursive relationships:

$$a_n(\theta_0,\dots,\theta_n) = f_n(\theta_n)\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n) > 0, \quad a_T(\theta_0,\dots,\theta_T) = 1,$$
$$b_n(\theta_0,\dots,\theta_n) = g_n(\theta_n)\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\dots,\theta_n), \quad b_T(\theta_0,\dots,\theta_T) = d_1,$$
$$c_n(\theta_0,\dots,\theta_n) = e_n(\theta_0,\dots,\theta_n) + \overline{(Q_{n+1}\bullet c_{n+1})}(\theta_0,\dots,\theta_n), c_T(\theta_0,\dots,\theta_T) = d_0,$$

where

$$e_n(\theta_0,\dots,\theta_n) = -\frac{\left[\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\dots,\theta_n)\right]^2}{4\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n)} h_n(\theta_n).$$

For $n = T$, the boundary condition (3) provides the terminal values for $a_T$, $b_T$, and $c_T$. Now, let $n = T-1$. Given an arbitrary market path $(\theta_0,\dots,\theta_{T-1})$ and the available wealth $w_{T-1}$, (2) leads to

$$\begin{aligned}
&v_{T-1}(\theta_0,\dots,\theta_{T-1};w_{T-1}) \\
&= \min_{\pi_{T-1}} \mathbb{E}\left\{\sum_{\theta_T=1}^{M} Q_T(\theta_0,\dots,\theta_T)\right. \\
&\quad \left.\times v_T\left(\theta_0,\dots,\theta_T; w_{T-1}R^0_{T-1}(\theta_{T-1}) + \pi'_{T-1}R^e_{T-1}(\theta_{T-1})\right)\right\} \\
&= \min_{\pi_{T-1}} \mathbb{E}\left\{\sum_{\theta_T=1}^{M} Q_T(\theta_0,\dots,\theta_T)\right. \\
&\quad \times a_T(\theta_0,\dots,\theta_T)\left[w_{T-1}R^0_{T-1}(\theta_{T-1}) + \pi'_{T-1}R^e_{T-1}(\theta_{T-1})\right]^2 \\
&\quad + \sum_{\theta_T=1}^{M} Q_T(\theta_0,\dots,\theta_T) \\
&\quad \times b_T(\theta_0,\dots,\theta_T)\left[w_{T-1}R^0_{T-1}(\theta_{T-1}) + \pi'_{T-1}R^e_{T-1}(\theta_{T-1})\right] \\
&\quad \left.+ \sum_{\theta_T=1}^{M} Q_T(\theta_0,\dots,\theta_T)c_T(\theta_0,\dots,\theta_T)\right\} \\
&= \min_{\pi_{T-1}} \mathbb{E}\left\{\overline{(Q_T\bullet a_T)}(\theta_0,\dots,\theta_{T-1})\left[w_{T-1}R^0_{T-1}(\theta_{T-1}) + \pi'_{T-1}R^e_{T-1}(\theta_{T-1})\right]^2\right. \\
&\quad + \overline{(Q_T\bullet b_T)}(\theta_0,\dots,\theta_{T-1})\left[w_{T-1}R^0_{T-1}(\theta_{T-1}) + \pi'_{T-1}R^e_{T-1}(\theta_{T-1})\right] \\
&\quad \left.+\overline{(Q_T\bullet c_T)}(\theta_0,\dots,\theta_{T-1})\right\} \\
&=\overline{(Q_T\bullet a_T)}(\theta_0,\dots,\theta_{T-1})w^2_{T-1}R^0_{T-1}(\theta_{T-1})^2 \\
&\quad + \overline{(Q_T\bullet b_T)}(\theta_0,\dots,\theta_{T-1})w_{T-1}R^0_{T-1}(\theta_{T-1}) + \overline{(Q_T\bullet c_T)}(\theta_0,\dots,\theta_{T-1})
\end{aligned}$$

$$+\min_{\pi_{T-1}}\Big\{\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})\pi'_{T-1}V_{T-1}(\theta_{T-1})\pi_{T-1}$$
$$+\Big[2\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})w_{T-1}R^0_{T-1}(\theta_{T-1})+\overline{(Q_T \bullet b_T)}(\theta_0,\dots,\theta_{T-1})\Big]$$
$$\times\pi'_{T-1}\bar{R}^e_{T-1}(\theta_{T-1})\Big\}. \tag{6}$$

Since Lemma 1 establishes that $V_{T-1}(\theta_{T-1})$ is positive definite and the positivity of $a_T$ ensures that $\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})$ remains positive, it follows that the Hessian matrix of the objective function in (6) is positive definite. Thus, setting the gradient to zero provides the necessary and sufficient optimality condition

$$\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})V_{T-1}(\theta_{T-1})\pi_{T-1}$$
$$+\Big[\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})w_{T-1}R^0_{T-1}(\theta_{T-1})+\frac{1}{2}\overline{(Q_T \bullet b_T)}(\theta_0,\dots,\theta_{T-1})\Big]$$
$$\times\bar{R}^e_{T-1}(\theta_{T-1})=\mathbf{0},$$

which leads to the optimal policy

$$\pi^*_{T-1}(\theta_0,\dots,\theta_{T-1};w_{T-1})=$$
$$-\left[w_{T-1}R^0_{T-1}(\theta_{T-1})+\frac{\overline{(Q_T \bullet b_T)}(\theta_0,\dots,\theta_{T-1})}{2\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})}\right]V_{T-1}(\theta_{T-1})^{-1}\bar{R}^e_{T-1}(\theta_{T-1}).$$

By substituting this optimal policy in (6), we derive

$$v_{T-1}(\theta_0,\dots,\theta_{T-1};w_{T-1})$$
$$=a_{T-1}(\theta_0,\dots,\theta_{T-1})w^2_{T-1}+b_{T-1}(\theta_0,\dots,\theta_{T-1})w_{T-1}+c_{T-1}(\theta_0,\dots,\theta_{T-1}),$$

where

$$\begin{aligned}
a_{T-1}(\theta_0,\dots,\theta_{T-1}) &= f_{T-1}(\theta_{T-1})\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})>0,\\
b_{T-1}(\theta_0,\dots,\theta_{T-1}) &= g_{T-1}(\theta_{T-1})\overline{(Q_T \bullet b_T)}(\theta_0,\dots,\theta_{T-1}),\\
c_{T-1}(\theta_0,\dots,\theta_{T-1}) &= -\frac{\Big[\overline{(Q_T \bullet b_T)}(\theta_0,\dots,\theta_{T-1})\Big]^2}{4\overline{(Q_T \bullet a_T)}(\theta_0,\dots,\theta_{T-1})}h_{T-1}(\theta_{T-1})\\
&\quad+\overline{(Q_T \bullet c_T)}(\theta_0,\dots,\theta_{T-1}).
\end{aligned}$$

Observe that the positivity of $a_{T-1}$ is a direct consequence of Lemma 1 and the positivity of $a_T$.

Now, suppose that (4) holds for $n+1$. We will establish its validity for $n$, considering the market path $(\theta_0,\dots,\theta_n)$ and the corresponding wealth level $w_n$. Utilizing the induction hypothesis and equation (2), we derive

$$\begin{aligned}
&v_n(\theta_0,\dots,\theta_n;w_n)\\
&=\min_{\pi_n}\mathbb{E}\left\{\sum_{\theta_{n+1}=1}^{M}Q_{n+1}(\theta_0,\dots,\theta_{n+1})v_{n+1}\left(\theta_0,\dots,\theta_{n+1};w_nR_n^0(\theta_n)+\pi_n'R_n^e(\theta_n)\right)\right\}\\
&=\min_{\pi_n}\mathbb{E}\left\{\sum_{\theta_{n+1}=1}^{M}Q_{n+1}(\theta_0,\dots,\theta_{n+1})a_{n+1}(\theta_0,\dots,\theta_{n+1})\left[w_nR_n^0(\theta_n)+\pi_n'R_n^e(\theta_n)\right]^2\right.\\
&\quad+\sum_{\theta_{n+1}=1}^{M}Q_{n+1}(\theta_0,\dots,\theta_{n+1})b_{n+1}(\theta_0,\dots,\theta_{n+1})\left[w_nR_n^0(\theta_n)+\pi_n'R_n^e(\theta_n)\right]\\
&\quad\left.+\sum_{\theta_{n+1}=1}^{M}Q_{n+1}(\theta_0,\dots,\theta_{n+1})c_{n+1}(\theta_0,\dots,\theta_{n+1})\right\}\\
&=\min_{\pi_n}\mathbb{E}\left\{\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n)\left[w_nR_n^0(\theta_n)+\pi_n'R_n^e(\theta_n)\right]^2\right.\\
&\quad+\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\dots,\theta_n)\left[w_nR_n^0(\theta_n)+\pi_n'R_n^e(\theta_n)\right]\\
&\quad\left.+\overline{(Q_{n+1}\bullet c_{n+1})}(\theta_0,\dots,\theta_n)\right\}\\
&=\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n)w_n^2R_n^0(\theta_n)^2+\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\dots,\theta_n)w_nR_n^0(\theta_n)\\
&\quad+\overline{(Q_{n+1}\bullet c_{n+1})}(\theta_0,\dots,\theta_n)\\
&\quad+\min_{\pi_n}\left\{\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n)\pi_n'V_n(\theta_n)\pi_n\right.\\
&\quad+\left[2\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n)w_nR_n^0(\theta_n)+\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\dots,\theta_n)\right]\\
&\quad\left.\times\pi_n'\bar{R}_n^e(\theta_n)\right\}.
\end{aligned}\tag{7}$$

The minimization problem in (7) shares the same structural form as that in (6). Following a similar reasoning, the optimal policy is derived as

$$\begin{aligned}
&\pi_n^*(\theta_0,\dots,\theta_n;w_n)\\
&\quad=-\left[w_nR_n^0(\theta_n)+\frac{\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\dots,\theta_n)}{2\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\dots,\theta_n)}\right]V_n(\theta_n)^{-1}\bar{R}_n^e(\theta_n).
\end{aligned}$$

By substituting this optimal policy, (7) simplifies to

$$v_n(\theta_0,\ldots,\theta_n;w_n)=a_n(\theta_0,\ldots,\theta_n)w_n^2+b_n(\theta_0,\ldots,\theta_n)w_n+c_n(\theta_0,\ldots,\theta_n),$$

where

$$\begin{aligned}
a_n(\theta_0,\ldots,\theta_n) =&f_n(\theta_n)\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\ldots,\theta_n),\\
b_n(\theta_0,\ldots,\theta_n) =&g_n(\theta_n)\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\ldots,\theta_n),\\
c_n(\theta_0,\ldots,\theta_n) =&-\frac{(\overline{(Q_{n+1}\bullet b_{n+1})}(\theta_0,\ldots,\theta_n))^2}{4\overline{(Q_{n+1}\bullet a_{n+1})}(\theta_0,\ldots,\theta_n)}h_n(\theta_n)\\
&+\overline{(Q_{n+1}\bullet c_{n+1})}(\theta_0,\ldots,\theta_n).
\end{aligned}$$

Once again, we confirm that $a_n(\theta_0,\ldots,\theta_n)>0$.

The claims of the theorem now follow from Lemma 3, applied to the recursive definitions of $a_n$, $b_n$, and $c_n$. To derive $a_n$, we define $B_n(\theta_0,\ldots,\theta_n)=0$ and set the terminal condition as $B_T(\theta_0,\ldots,\theta_T)=a_T(\theta_0,\ldots,\theta_T)=1$. Similarly, for $b_n$, we impose $B_n(\theta_0,\ldots,\theta_n)=0$ with the final condition $B_T(\theta_0,\ldots,\theta_T)=b_T(\theta_0,\ldots,\theta_T)=d_1$. Using these results, we reformulate $e_n$ and the optimal strategy $\pi_n^*$. For $c_n$, we assume $C_n(\theta_n)=1$. □

### 3.2 Solution to problem $P(MV)$

To derive the solution for problem $P(MV)$ under the initial conditions $\Theta_0=\theta_0$ and $W_0=w_0$, we reformulate the optimal value function of $\tilde{P}(MV)$, given by

$$v_0(\theta_0;w_0)=a_0(\theta_0)w_0^2+b_0(\theta_0)w_0+c_0(\theta_0)$$

in terms of the Lagrange multiplier $\lambda$. To achieve this, we apply Lemma 2 to re-express the coefficients derived in Theorem 1, incorporating their dependence on $\lambda$. This yields

$$\begin{aligned}
c_0(\theta_0)&=e_0(\theta_0)+\sum_{k=1}^{T-1}\overline{\boxdot_{j=1}^{k-1}Q_{\mathbf{1}_j}\bullet(Q_k\bullet e_k)}(\theta_0)+\overline{\boxdot_{j=1}^{T-1}Q_{\mathbf{1}_j}\bullet(Q_T\bullet e_T)}(\theta_0)\\
&=-(d_1^2/4)c_0^*(\theta_0)+d_0,
\end{aligned}$$

where

$$c_0^*(\theta_0) = e_0^*(\theta_0) + \sum_{k=1}^{T-1} \overline{\bigodot_{j=1}^{k-1} Q_{\mathbf{1}_j} \bullet (Q_k \bullet e_k^*)}(\theta_0),$$

$$e_n^*(\theta_0, \dots, \theta_n) = \frac{\left[\overline{\bigodot_{j=n+1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n)\right]^2}{\overline{\bigodot_{j=n+1}^{T-1} Q_{f_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n)} h_n(\theta_n).$$

Moreover, we obtain $b_0(\theta_0) = d_1 b_0^*(\theta_0)$, where

$$b_0^*(\theta_0) = g_0(\theta_0) \overline{\bigodot_{j=1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0).$$

Thus, the function simplifies to

$$\begin{aligned} v_0(\theta_0; w_0) =& a_0(\theta_0) w_0^2 + b_0(\theta_0) w_0 + c_0(\theta_0) \\ =& a_0(\theta_0) w_0^2 + d_1 b_0^*(\theta_0) w_0 - \frac{d_1^2}{4} c_0^*(\theta_0) + d_0 \\ =& a_0(\theta_0) w_0^2 + 2(\lambda - E_T) b_0^*(\theta_0) w_0 - (\lambda - E_T)^2 c_0^*(\theta_0) + E_T^2 - 2\lambda E_T. \end{aligned} \tag{8}$$

Since $v_0(\theta_0; w_0)$ depends on $\lambda$, we define

$$L(\lambda) := v_0(\theta_0; w_0).$$

By the Lagrange duality theorem (see [15]), maximizing (8) over $\lambda \in \mathbb{R}$ provides the optimal value for problem $P(MV)$, denoted by $\mathbb{V}\mathrm{ar}_0^*(E_T)$, that is,

$$\mathbb{V}\mathrm{ar}_0^*(E_T) = \max_{\lambda \in \mathbb{R}} L(\lambda).$$

Lemma 1 guarantees that $c_0^*(\theta_0)$ remains strictly positive. Consequently, the function $L(\lambda)$ attains its maximum at

$$\lambda^* = \frac{b_0^*(\theta_0) w_0 - E_T}{c_0^*(\theta_0)} + E_T.$$

To derive the optimal portfolio strategy for $P(MV)$, we substitute

$$d_1 = 2(\lambda^* - E_T) = \frac{2\,(b_0^*(\theta_0)w_0 - E_T)}{c_0^*(\theta_0)}$$

into (5). Additionally, replacing $\lambda^*$ in (8) yields the minimum variance associated with problem $P(MV)$. These results lead to the following theorem, which provides the solution to the main problem $P(MV)$.

**Theorem 2.** The optimal variance (or risk) corresponding to problem $P(MV)$ is given by

$$\mathbb{V}\text{ar}_0^*(E_T) = \frac{1 - c_0^*(\theta_0)}{c_0^*(\theta_0)}\left(E_T - \frac{b_0^*(\theta_0)w_0}{1 - c_0^*(\theta_0)}\right)^2 + \left(a_0(\theta_0) - \frac{b_0^*(\theta_0)^2}{1 - c_0^*(\theta_0)}\right)w_0^2. \tag{9}$$

This follows from the optimal portfolio strategy given by

$$\begin{aligned}&\pi_n^*(\theta_0, \dots, \theta_n; w_n)\\ &= -\left[w_n R_n^0(\theta_n) + d^* \frac{\overline{\bigodot_{j=n+1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n)}{\overline{\bigodot_{j=n+1}^{T-1} Q_{f_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0, \dots, \theta_n)}\right] V_n(\theta_n)^{-1} \bar{R}_n^e(\theta_n),\end{aligned} \tag{10}$$

where

$$d^* = \frac{b_0^*(\theta_0)w_0 - E_T}{c_0^*(\theta_0)}. \tag{11}$$

**Remark 2.** The optimal policy presented in (10) clearly reflects the influence of regime dynamics on asset allocation. Specifically, the allocation decision is shaped by three distinct components: Return parameters such as $R_n^0(\theta_n)$, $V_n(\theta_n)$, and $\bar{R}_n^e(\theta_n)$, which depend only on the current market state; the scalar $d^*$, which encapsulates initial market conditions and investor targets; and a path-dependent term involving transition tensors, which captures the effect of historical regime evolution. This structure highlights how the policy simultaneously accounts for both the present market regime and the trajectory of past regimes in determining optimal investment actions.

The primary objective in portfolio selection is to determine *efficient* portfolio strategies. A portfolio strategy $\pi^*$ is deemed efficient if no alternative strategy $\pi$ exists that yields the same expected terminal wealth with lower risk, or the same risk with a higher expected terminal wealth. An efficient point refers to the ordered pair in the Mean-Variance plane associated with an efficient portfolio strategy. The efficient frontier is the collection of all such efficient points. It is important to note that while an efficient point corresponds to the solution of a Mean-Variance problem, not the solution of a Mean-Variance problem necessarily represents an efficient point. Specifically, if an optimal portfolio strategy falls on the lower branch of the parabola (9), meaning

$$E_T < \frac{b_0^*(\theta_0)w_0}{1 - c_0^*(\theta_0)},$$

then there exists another optimal strategy with the same variance but a greater expected terminal wealth. Therefore, in the Mean-Variance plane, the efficient frontier corresponds to the upper branch of the parabola (9), where

$$E_T \geq \frac{b_0^*(\theta_0)w_0}{1 - c_0^*(\theta_0)}.$$

This discussion leads to the following theorem, which concludes this section.

**Corollary 1.** The Mean-Variance efficient frontier is given by (9) for

$$E_T \geq \frac{b_0^*(\theta_0)w_0}{1 - c_0^*(\theta_0)}.$$

The global minimum risk portfolio strategy corresponds to the ordered pair

$$\left( \frac{b_0^*(\theta_0)w_0}{1 - c_0^*(\theta_0)}, \left( a_0(\theta_0) - \frac{b_0^*(\theta_0)^2}{1 - c_0^*(\theta_0)} \right) w_0^2 \right)$$

in the Mean-Variance plane and can be determined using (10) with

$$d^* = \frac{b_0^*(\theta_0)w_0}{c_0^*(\theta_0) - 1}.$$

**Remark 3.** While our model allows for path-dependent transition probabilities, its computational complexity remains manageable in practice. In real-world applications, these transition probabilities typically depend only on a short and finite memory of recent market states–such as the last two or three time periods–rather than the entire historical path. This finite-memory assumption not only aligns with empirical observations in financial markets, but also significantly reduces the number of relevant paths that need to be considered. Furthermore, the use of tensor-based representations and recursive backward computation contributes to numerical tractability and enables efficient implementation in multi-period settings. If higher-dimensional path dependence is required in specific applications, then the model can also accommodate various approximation methods, such as Monte Carlo simulation, dimension reduction techniques, or path truncation strategies.

## 4 Markovian regime-switching markets

In this section, we examine the Mean-Variance portfolio selection problem within a standard Markovian regime-switching market, where the transition probability satisfies the Markov property:

$$\mathrm{P}(\Theta_n = \theta_n \mid \Theta_{n-1} = \theta_{n-1}, \ldots, \Theta_1 = \theta_1, \Theta_0 = \theta_0) = \mathrm{P}(\Theta_n = \theta_n \mid \Theta_{n-1} = \theta_{n-1}).$$

This assumption leads to the simplification

$$Q_n(\theta_0, \ldots, \theta_{n-1}, \theta_n) = Q_n(\theta_{n-1}, \theta_n),$$

which implies that $Q_n$ can be considered as a standard $M \times M$ matrix.

This classical framework has been previously analyzed by Cakmak and Ozekici [2]. We demonstrate that the findings in [2] emerge as special cases of our generalized results. A key distinction from their work is that we do not impose the assumption of time-homogeneity on the Markov process, and we allow asset returns to be influenced by both the market state and the specific time period.

To simplify the calculations, we adopt a notation consistent with the conventions introduced in Section 2. For an $M \times M$ matrix $A$ and the $M$-column vector $\mathbf{1} = (1, \ldots, 1)'$ we define $\overline{A} = A\mathbf{1}$. Then, for any $M \times M$ matrix $B$ and an $M$-column vector $C$ the following identities hold:

$$\overline{AB} = AB\mathbf{1} = A\overline{B}, \quad \overline{A_C} = AC.$$

Below, we present our results under the assumption of Markovian transition probabilities. To achieve this, we express key parameters and coefficients using Lemma 2 alongside the notations introduced earlier. For instance, we obtain

$$\begin{aligned}
&\overline{\bigsqcap_{j=n+1}^{T-1} Q_{g_j} \bullet (Q_T \bullet \mathbf{1}_T)(\theta_0,\dots,\theta_n)}\\
&= \sum_{\theta_{n+1}=1}^{M} \dots \sum_{\theta_{T-1}=1}^{M} \sum_{\theta_T=1}^{M} Q_{n+1}(\theta_0,\dots,\theta_{n+1}) g_{n+1}(\theta_{n+1}) \dots Q_{T-1}(\theta_0,\dots,\theta_{T-1})\\
&\quad \times g_{T-1}(\theta_{T-1}) Q_T(\theta_0,\dots,\theta_T) \mathbf{1}_T(\theta_0,\dots,\theta_T)\\
&= \sum_{\theta_{n+1}=1}^{M} \dots \sum_{\theta_{T-1}=1}^{M} \sum_{\theta_T=1}^{M} Q_{n+1}(\theta_n,\theta_{n+1}) g_{n+1}(\theta_{n+1}) \dots Q_{T-1}(\theta_{T-2},\theta_{T-1})\\
&\quad \times g_{T-1}(\theta_{T-1}) Q_T(\theta_{T-1},\theta_T)\\
&= \sum_{\theta_{n+1}=1}^{M} \dots \sum_{\theta_{T-1}=1}^{M} Q_{g_{n+1}}(\theta_n,\theta_{n+1}) \dots Q_{g_{T-1}}(\theta_{T-2},\theta_{T-1}) \sum_{\theta_T=1}^{M} Q_T(\theta_{T-1},\theta_T)\\
&= \sum_{\theta_{n+1}=1}^{M} \dots \sum_{\theta_{T-1}=1}^{M} Q_{g_{n+1}}(\theta_n,\theta_{n+1}) \dots Q_{g_{T-1}}(\theta_{T-2},\theta_{T-1}) \mathbf{1}(\theta_{T-1})\\
&= \left( (\prod_{j=n+1}^{T-1} Q_{g_j}) \mathbf{1} \right)(\theta_n)\\
&= \overline{(\prod_{j=n+1}^{T-1} Q_{g_j})}(\theta_n).
\end{aligned} \tag{12}$$

Here, the notation $\prod$ represents standard matrix multiplication. Following a similar procedure, we obtain

$$\overline{\bigsqcap_{j=n+1}^{T-1} Q_{f_j} \bullet (Q_T \bullet \mathbf{1}_T)}(\theta_0,\dots,\theta_n) = \overline{(\prod_{j=n+1}^{T-1} Q_{f_j})}(\theta_n).$$

It follows that these parameters depend solely on $\theta_n$. Consequently, $e_n^*$ simplifies to

$$e_n^*(\theta_n) = \frac{\left[\overline{(\prod_{j=n+1}^{T-1} Q_{g_j})}(\theta_n)\right]^2}{\overline{(\prod_{j=n+1}^{T-1} Q_{f_j})}(\theta_n)} h_n(\theta_n).$$

Applying the same approach as in (12), we obtain

$$\overline{\bigodot_{j=n+1}^{k-1} Q_{\mathbf{1}_j} \bullet (Q_k \bullet e_k^*)}(\theta_0, \ldots, \theta_n) = \left( (\prod_{j=n+1}^{k} Q_j) e_k^* \right) (\theta_n).$$

Finally, the key parameters are expressed as

$$\begin{aligned}
a_0(\theta_0) &= f_0(\theta_0) \left[ \overline{(\prod_{j=1}^{T-1} Q_{f_j})(\theta_0)} \right], \\
b_0^*(\theta_0) &= g_0(\theta_0) \left[ \overline{(\prod_{j=1}^{T-1} Q_{g_j})(\theta_0)} \right], \\
c_0^*(\theta_0) &= e_0^*(\theta_0) + \sum_{k=1}^{T-1} \left( (\prod_{j=1}^{k} Q_j) e_k^* \right) (\theta_0) = \sum_{k=0}^{T-1} \left( (\prod_{j=1}^{k} Q_j) e_k^* \right) (\theta_0).
\end{aligned}$$

By replacing these expressions in (10) and (11), the optimal portfolio strategy simplifies to

$$\begin{aligned}
&\pi_n^*(\theta_0, \ldots, \theta_n; w_n) = \pi_n^*(\theta_n; w_n) = \\
&\quad - \left[ w_n R_n^0(\theta_n) + d^* \frac{\overline{(\prod_{j=n+1}^{T-1} Q_{g_j})}(\theta_n)}{\overline{(\prod_{j=n+1}^{T-1} Q_{f_j})}(\theta_n)} \right] V_n(\theta_n)^{-1} \bar{R}_n^e(\theta_n).
\end{aligned}$$

This confirms that the optimal portfolios depend only on $\theta_n$.

In a more constrained setting, Cakmak and Ozekici [2] studied a market with risky assets and a riskless asset, assuming that asset returns depend only on the market state, not on the time period, within a time-homogeneous Markov chain. In other words, their parameters are time-independent. A straightforward manipulation shows that, under time-independent parameters, the results presented in [2, Corollary 5] match our results in Theorem 2 and Corollary 1.

## 5 Numerical illustration

Consider a regime-switching market model with two states: A bull state (State 1) and a bear state (State 2). The model includes two assets: A risky asset, whose returns follow a log-normal distribution, and a risk-free asset, whose returns are deterministic and vary with the market state. Specifically, if $R^1$ denotes the return of the risky asset, then $\ln R^1$ follows a normal distribution with mean $\mu$ and variance $\sigma^2$, where $\mu$ and $\sigma^2$ depend solely on the prevailing market state. In the bull state, the risky asset exhibits positive rate of returns, reflecting favorable market conditions. In contrast, during the bear state, the risky asset experiences negative rate of returns, reflecting adverse market conditions and heightened volatility. The risk-free asset provides stable but slightly reduced returns (denoted by $R^0$) in the bear state compared to the bull state. The parameters for both market states are summarized in Table 1.

Table 1: Market parameters in bull and bear states

| $\theta_n$ | $\mu(\theta_n)$ | $\sigma^2(\theta_n)$ | $\mathbb{E}[R_n^1(\theta_n)]$ | $\mathbb{V}\text{ar}[R_n^1(\theta_n)]$ | $R_n^0(\theta_n)$ |
|---|---|---|---|---|---|
| 1 | 0.020 | 0.015 | 1.0279 | 0.016 | 1.005 |
| 2 | $-0.030$ | 0.045 | 0.9925 | 0.0453 | 1.003 |
| $\theta_n$ | $\mathbb{E}[\bar{R}_n^e(\theta_n)]$ | $V_n(\theta_n)$ | $h_n(\theta_n)$ | $g_n(\theta_n)$ | $f_n(\theta_n)$ |
| 1 | 0.0229 | 0.0165 | 0.0317 | 0.9731 | 0.978 |
| 2 | -0.0105 | 0.0454 | 0.0024 | 1.0006 | 1.0036 |

Let the investment horizon be $T = 3$. The transition matrices, as shown in Table 2, are constructed for each time period, conditional on the historical path, ensuring that the model accurately reflects the dynamic nature of market regimes, where

$$
\begin{aligned}
&P_0(\theta_0, \theta_1) = Q_1(\theta_0, \theta_1),\\
&P_1^{(\theta_0)}(\theta_1, \theta_2) = Q_2(\theta_0, \theta_1, \theta_2),\\
&P_2^{(\theta_0,\theta_1)}(\theta_2, \theta_3) = Q_3(\theta_0, \theta_1, \theta_2, \theta_3).
\end{aligned}
$$

For example, if at time $n = 1$, the market is currently in the bull state ($\Theta_1 = 1$) and has been in the bull state in the previous period ($\Theta_0 = 1$), then the probability of remaining in the bull state is 80%, while the probability of transitioning to the bear state is 20% (see $P_1^{(1)}(1,1)$ and $P_1^{(1)}(1,2)$ in Table 2). However, if at time $n = 2$ the market is currently in the bull state ($\Theta_2 = 1$) and has been in the bull state for the previous two periods ($\Theta_0 = 1$, $\Theta_1 = 1$), then the probability of remaining in the bull state is 90%, while the probability of transitioning to the bear state is 10% (see

$P_2^{(1,1)}(1,1)$ and $P_2^{(1,1)}(1,2)$ in Table 2). Here, $P_2^{(1,1)}(1,1) > P_1^{(1)}(1,1)$ but $P_2^{(1,1)}(1,2) < P_1^{(1)}(1,2)$.

Table 2: Path-dependent transition matrices

| Time | Path | Transition Matrix |
|---|---|---|
| $n=0$ | − | $P_0 = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$ |
| $n=1$ | $\Theta_0 = 1$ | $P_1^{(1)} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$ |
| $n=1$ | $\Theta_0 = 2$ | $P_1^{(2)} = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$ |
| $n=2$ | $\Theta_0 = 1, \Theta_1 = 1$ | $P_2^{(1,1)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}$ |
| $n=2$ | $\Theta_0 = 1, \Theta_1 = 2$ | $P_2^{(1,2)} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$ |
| $n=2$ | $\Theta_0 = 2, \Theta_1 = 1$ | $P_2^{(2,1)} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$ |
| $n=2$ | $\Theta_0 = 2, \Theta_1 = 2$ | $P_2^{(2,2)} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ |

Let the initial wealth be 100, and assume that the market starts in state 1 (bull state). Values of

$$\hat{g}_n(\theta_0, \ldots, \theta_n) := \overline{\boxdot_{j=n+1}^{2} Q_{g_j} \bullet (Q_3 \bullet \mathbf{1}_3)}(\theta_0, \ldots, \theta_n),$$
$$\hat{f}_n(\theta_0, \ldots, \theta_n) := \overline{\boxdot_{j=n+1}^{2} Q_{f_j} \bullet (Q_3 \bullet \mathbf{1}_3)}(\theta_0, \ldots, \theta_n)$$

are given in Table 3. Moreover, we obtain the following initial parameters:

$$a_0(1) = 0.95, \ b_0^*(1) = 0.937, \ c_0^*(1) = 0.076.$$

In the following, we compare the optimal investment strategies under a path-dependent model and a Markovian model to achieve a fixed expected terminal wealth of $E_3 = 105$ for some different market paths. M-V efficient frontiers are also compared under two different models. For the Markovian model, we set $P_0$ as the transition matrix.

Table 3: Values of $\hat{g}_n$ and $\hat{f}_n$.

| Time | Path | $\hat{g}_n(\theta_0,\ldots,\theta_n)$ | $\hat{f}_n(\theta_0,\ldots,\theta_n)$ |
|---|---|---|---|
| $n=0$ | $\Theta_0=1$ | 0.9628 | 0.9713 |
| $n=1$ | $\Theta_0=1,\Theta_1=1$ | 0.9786 | 0.9831 |
| $n=1$ | $\Theta_0=1,\Theta_1=2$ | 0.9868 | 0.9908 |
| $n=2$ | $\Theta_0=1,\Theta_1=1,\Theta_2=1$ | 1 | 1 |
| $n=2$ | $\Theta_0=1,\Theta_1=1,\Theta_2=2$ | 1 | 1 |
| $n=2$ | $\Theta_0=1,\Theta_1=2,\Theta_2=1$ | 1 | 1 |
| $n=2$ | $\Theta_0=1,\Theta_1=2,\Theta_2=2$ | 1 | 1 |

For the first market path $(1,1,1)$ (see Figure 1) with corresponding wealth levels $(100,101,102)$, the optimal investment in the risky asset under the path-dependent model is $(65.5,64.97,64.52)$, while under the Markovian model, it is $(68.39,67.87,67.39)$. The higher investment in the risky asset under the Markovian model stems from its lower expected return compared to the path-dependent model. Specifically, the path-dependent model incorporates momentum effects, increasing the probability of remaining in the bull state and thus raising the expected return. This allows for a more conservative investment strategy to achieve the fixed expected terminal wealth. In contrast, the Markovian model, which ignores historical paths, yields a lower expected return, necessitating a more aggressive (higher) investment strategy to meet the same target.

For the second market path $(1,2,2)$ (see Figure 1) with corresponding wealth levels $(100,101,99)$, the optimal investment in the risky asset under the path-dependent model is $(65.5,-10.85,-11.45)$, while under the Markovian model, it is $(68.39,-11.34,-11.93)$. The negative investments in the risky asset under unfavorable market conditions (e.g., state 2, bear state) reflect risk-averse behavior, where investors reduce exposure or take short positions to mitigate potential losses. However, the magnitude of these short positions differs between the two models due to their distinct assumptions. In the path-dependent model, the expected decline in the value of the risky asset is greater, as the model accounts for the persistence of unfavorable market conditions, leading to a higher likelihood of continued losses. Consequently, a smaller short position is required to achieve the fixed expected terminal wealth, as the model anticipates and adjusts for the larger expected decline. In contrast, the Markovian model predicts a smaller expected decline in the value of the risky asset, as it ignores historical paths. Thus, a larger short position is necessary to achieve the same target, as the model underestimates the persistence of unfavorable conditions and associated risks. This highlights how the path-dependent model's incorporation of historical market behavior enables more precise adjustments in investment strategies.

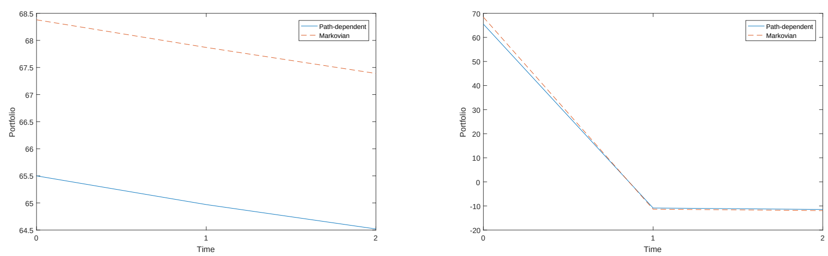


Figure 1: Optimal portfolio strategies for the market paths $(1,1,1)$ (left) and $(1,2,2)$ (right).

We also compared and plotted the mean-variance efficient frontiers for the path-dependent and Markovian models. The efficient frontier for the path-dependent model lies below that of the Markovian model, indicating that for a fixed level of expected terminal wealth, the optimal investment risk (i.e., the variance of terminal wealth) is lower in the path-dependent model compared to the Markovian model. This can be attributed to the observed behavior in the above two market paths: The absolute value of the investment in the risky asset is generally smaller in the path-dependent model than in the Markovian model. This reduction in exposure to the risky asset likely contributes to lower overall risk. As demonstrated in the optimal strategies examples, the path-dependent model's incorporation of historical market behavior leads to more conservative investment strategies, which in turn reduce risk. This comparison of the efficient frontiers is illustrated in Figure 2. The end point $(101.4, 0.0216)$ represents the global minimum risk portfolio strategy.

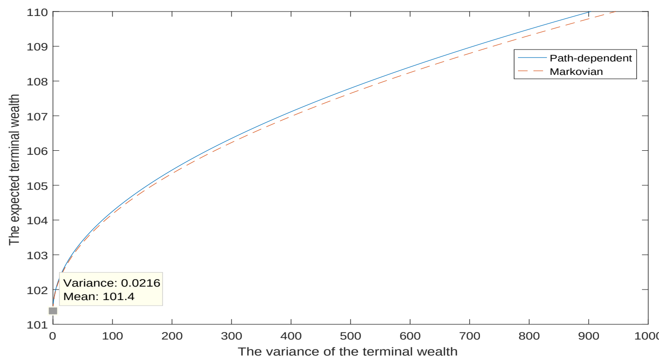


Figure 2: The M-V efficient frontier.

## 6 Conclusion

In this paper, we have investigated a multi-period mean-variance portfolio optimization problem under a non-Markovian regime-switching model. The asset returns in this market depend on market states that evolve stochastically over time among a finite set of possible states, with transition probabilities that are path-dependent rather than Markovian. To solve this optimization problem, we employed dynamic programming combined with an auxiliary problem approach, necessitated by the non-separability introduced by the variance operator in the dynamic programming framework. The solution methodology combines dynamic programming with Lagrangian multiplier method, utilizing an extended Bellman equation based on path-dependent value functions. The optimal policy parameters are obtained implicitly through a system of path-dependent backward recursive relations. Explicit closed-form solutions are derived using matrix computations, and the optimal strategy for the original problem is recovered via Lagrangian duality theorem.

Our results demonstrated that the optimal investment strategy exhibits path-dependence at each time point. Notably, we showeded that the traditional Markovian model emerges as a special case of our framework, where the optimal strategy depends only on the current state. Furthermore, we establish that the efficient frontier—characterizing the relationship between expected final wealth and optimal risk—is significantly influenced by our modeling assumptions. In particular, the path-dependent model leads to risk reduction compared to the Markovian benchmark, achieved through more conservative allocations to risky assets. This reduction stems from the model's ability to incorporate historical market behavior, enabling finer adjustments in investment strategies.

For future research, we suggest two promising directions: (1) Extending the model to incorporate path-dependent asset returns, and (2) investigating time-consistent formulations under these path-dependent assumptions. These extensions could provide even more realistic tools for portfolio management in regime-switching environments.

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