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Research Article

Nonlinear optimization of revenue per unit of time in discrete Dutch auctions with risk-aware bidders

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Abstract

This study develops a computational framework to optimize the auctioneer's revenue per unit of time in modified discrete Dutch auction by incorporating bidders' risk preferences through the constant absolute risk aversion utility function. Bidders are categorized into three distinct risk profiles—risk-loving, risk-neutral, and risk-averse—allowing for a comprehensive analysis of how risk attitudes influence auction outcomes. A nonlinear programming methodology is utilized to ascertain the optimal revenue per unit time while incorporating discrete bid levels. The findings

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demonstrate that, at the outset, an increase in the number of bidders substantially boosts the revenue per unit time; nevertheless, after reaching a specific point, the incremental benefits decrease, resulting in a plateau. Additionally, the analysis suggests that, in auctions featuring larger pools of bidders, achieving maximum revenue per unit time necessitates fewer bid levels, as surplus bid levels do not yield further revenue improvements. Bidders exhibiting risk-averse tendencies tend to generate lower returns due to their cautious bidding patterns, whereas risk-seeking participants contribute to higher revenue per unit time by engaging in more assertive bidding. Collectively, these results highlight the significant influence of bidders' risk preferences on auction design and establish a comprehensive mathematical framework that can be readily adapted to various algorithmic auction mechanisms. Behavioral interpretation via the prospect theory and alignment with published field evidence support the model's external validity.

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Keywords: Auctions; Constant absolute risk aversion; Discrete Dutch auction; Nonlinear programming; Revenue per unit of time.

1 Introduction

An auction constitutes a competitive bidding mechanism wherein an item of uncertain value is awarded to the participant prepared to offer the highest price. Auctions represent one of the three primary methods of trade, alongside fixed-price sales and negotiation-based transactions [39]. They hold significant importance in the contemporary global economy, enabling the exchange of assets ranging from real estate and agricultural commodities to mineral rights and spectrum licenses [22, 56, 29, 12]. Among the diverse auction types, the Dutch auction (DA), also referred to as the descending-price or clock auction, stands out for its swift transaction process and particular suitability for the sale of perishable goods and time-sensitive assets [20, 48].

In a DA, the auctioneer initiates the process by setting an initially high asking price, which is then systematically reduced following a specified schedule until a participant agrees to the prevailing price [48]. Unlike ascendingprice English auctions that favor unique items, such as antiques, DAs excel in markets for goods with diminishing value over time, such as fresh produce, concert tickets, and container space [1, 32]. Practical applications extend to cash management [2], stock repurchases [6], and airline overbooking [23].

Traditional auction models often assume continuous bidding and risk-neutral bidders, which may not align with real-world dynamics [43, 46, 40]. The introduction of discrete Dutch auction (DDA) has addressed practical constraints by limiting bid levels to a discrete set of values. Early studies by Li and Kuo [32, 33] explored revenue maximization of DDA through optimal bid level design, demonstrating that revenue increases with the number of bid levels and bidders. However, these models ignored the variability in bidder risk preferences and emotional attachments. Li, Yue, and Kuo [34] extended DDA models by incorporating time as a critical parameter, examining trade-offs between auction duration and revenue. Their findings revealed that optimizing revenue per unit of time could significantly enhance auctioneer profitability, particularly in high-frequency auction environments. Despite these advancements, their models also remained limited to emotional attachment of the bidders with the item to be sold and bidders' risk preferences. Addressing some of these limitations, Shamim and Ali [48] integrated bidders' emotional attachments using the log-normal valuation distribution along with the consideration of time in DDA frameworks. By accounting for the emotional attachments, their research demonstrated the significant impact of emotions on auction outcomes and bidding strategies. However, they did not discuss the impact of bidders' risk preferences on the auction outcomes.

This study builds upon the aforementioned foundational works by examining the influence of bidders' risk preferences through the constant absolute risk aversion (CARA) utility function while incorporating the critical role of time in auction profitability. Recognizing that an auctioneer seeks to maximize revenue not only per auction but also per unit of time, this research integrates risk-sensitive bidder behavior with time-optimized revenue strategies. By formulating a computational framework that captures the complexities of real-world auctions, this study aims to enhance auction theory and offer practical insights for designing more efficient DDAs.

This study also extends the standard DDA model by explicitly addressing bidder risk asymmetry, clarifying the behavioral interpretation of the CARA parameter, and situating the framework alongside the prospect theory. While this study relies on simulation-based analysis, it also demonstrates that our results are consistent with published empirical studies of fish and flower markets, thereby reinforcing the practical relevance of its findings even in the absence of new transaction-level data.

The rest of this paper is structured as follows. Section 2 provides a comprehensive review of existing literature, highlighting the research gap addressed in this study. In Section 3, a mathematical revenue model for DDAs is developed, incorporating bidders' risk preferences and time considerations. Section 4 presents and analyzes the key results obtained by solving the proposed model using the R software. Finally, Section 5 concludes the study by summarizing the findings, discussing its limitations, and suggesting potential directions for future research.

2 Literature review

Auction theory has been a central theme in economic research, with considerable emphasis placed on analyzing the dynamics of different auction formats, such as English auctions, sealed-bid auctions, and DAs. Notably, DAs, distinguished by their descending-price structure, have become particularly valued for their effectiveness in facilitating the sale of perishable and time-sensitive goods [20, 48]. Nevertheless, much of the existing scholarship presumes continuous bidding and risk-neutral behavior among participants, assumptions that often do not align with the practical realities of auction environments [43, 46, 41]. This section provides an overview of the current literature on DAs, focusing specifically on discrete bidding and strategies for maximizing auctioneer revenue per unit of time, while also drawing attention to the insufficient consideration of bidders' risk preferences within the field.

Traditional auction models often assume that bid prices are continuous variables, allowing bidders to outbid each other by infinitesimally small increments [43, 46, 40, 14, 47]. While this assumption is suitable for unique items such as antiques, it is less applicable to fast DAs, where perishable goods or services are sold rapidly. For instance, Royal Flora Holland auctions last approximately four seconds per transaction [28], and fish markets in Italy complete 15 transactions per minute using simultaneous clocks [17, 19]. The inefficiency of continuous bidding in such contexts has led researchers to explore discrete bidding mechanisms, where bid levels are restricted to a finite set of values.

Discrete bidding is not uncommon in English auctions [14, 47], sealed-bid auctions [10, 37], and hybrid auctions [25]. However, its application in DAs has received limited attention. Early work by Yu [54] demonstrated the existence of a symmetric pure-strategy equilibrium in DAs with fixed bid decrements but did not explore the optimization of bid levels or their impact on closing prices. Yuen, Sung, and Wong [55] extended this research by analyzing DAs conducted via wireless networks, introducing a communication cost factor. While their iterative numerical approach provided insights into optimal bid decrements, their model was constrained by its focus on communication costs and did not address revenue maximization directly.

The optimization of auctioneer revenue has been a central theme in auction theory. Cramton et al. [13] and Sujarittanonta [49] examined DAs with discrete bid levels, focusing on efficiency maximization rather than revenue optimization. In contrast, Li and Kuo [32, 33] explored revenue-maximizing DAs with unequal bid decrements, demonstrating that revenue increases with the number of bid levels and bidders. The assumption of a deterministic number of bidders was challenged by McAfee and McMillan [38], who introduced probabilistic models to account for uncertain bidder participation. This line of research gained traction with the rise of e-commerce, as online auctions necessitated models that could accommodate fluctuating bidder arrivals. Studies by Bajari and Hortaçsu [5], Etzion, Pinker, and Seidmann [15], and Caldentey and Vulcano [9] approximated bidder arrivals using Poisson processes, a modeling approach validated by empirical studies [50, 24]. Despite these advancements, the focus remained on English and sealed-bid auctions, leaving only a limited number of studies in DAs [48, 33, 34].

A significant limitation of traditional auction models is their neglect of time as a critical factor in auction profitability. While increasing the number of bid levels can enhance revenue per auction, it also prolongs auction duration, potentially reducing the total number of transactions conducted within a given timeframe. Li, Yue, and Kuo [34] and Shamim and Ali [48] tackled this challenge by integrating time considerations into DDA models, illustrating that maximizing revenue per unit of time can substantially improve the auctioneer's profitability. Their research highlighted the critical need to balance the number of bid levels with the overall auction duration, especially within high-frequency auction settings.

Although previous studies have advanced the optimization of auction design, they have largely neglected the influence of bidders' risk preferences and emotional attachments. Conventional models typically assume bidders to be risk-neutral for the sake of analytical tractability; however, this assumption does not necessarily reflect the diversity of risk attitudes observed in actual auction settings. Shamim and Ali [48] contributed to this area by incorporating bidders' emotional attachments into DDA frameworks through the use of lognormal valuation distributions. Their work demonstrated the considerable effect of emotions on both auction outcomes and bidding behavior. Nevertheless, their approach did not consider the risk preferences of bidders, thereby leaving a significant gap in the existing body of literature.

This research extends the foundational contributions of Li and Kuo [32, 33], Li, Yue, and Kuo [34], and Shamim and Ali [48] by addressing two significant gaps identified in the current literature. First, it incorporates bidders' risk preferences-encompassing risk-neutral, risk-seeking, and risk-averse behaviors-within the DDA framework through the application of the CARA utility function. Second, it treats time as a pivotal parameter, with the objective of maximizing the auctioneer's revenue both per auction and per unit of time. Through the development of a computational framework that reflects the intricacies of real-world auction environments, this study aims to advance auction theory and offer actionable guidance for the design of more effective DAs. This article reports simulation-based evidence; due to the unavailability of public transaction-level DA data, empirical alignment is provided through published field studies.

In conclusion, although substantial advancements have been achieved in the study of DDAs and revenue optimization, the incorporation of bidders' risk preferences and strategies for maximizing time-sensitive revenue has not been thoroughly investigated. This research seeks to address these deficiencies by presenting a comprehensive framework designed to improve both the efficiency and profitability of auctions.

3 Model development

This research investigates the impact of bidders' risk preferences on the revenue per unit of time in a DA featuring discrete bidding increments, within an independent private value (IPV) framework characterized by symmetric information. Under this setting, each bidder possesses knowledge solely of their own valuation for the auctioned item, which is independently drawn from a uniform distribution, and this information remains private and uninfluenced by the valuations of other participants [40, 34, 30]. The study considers scenarios in which bidders exhibit risk aversion, risk neutrality, or risk-loving behavior. In each case, a bidder is expected to place a bid when the asking price first drops to or below their valuation.

The discrete bid levels taken in this setting are $b_1 < b_2 < \cdots < b_m$, where $m \geq 1$. Initially, the auctioneer opens the bidding process at a very high bid level b_{m+1} where nobody is willing to bid, and then the price decreases to $b_m, b_{m-1}, \ldots, b_2, b_1$ after each preset interval of time until a bidder bids to buy the item at bid level b_i for any $i \in \{1, 2, \ldots, m\}$. In the DA setting, the item is sold at a price b_i if and only if there exist q number of bidders having their valuations in the interval $[b_i, b_{i+1})$ and nobody is willing to buy it for the price higher than b_{i+1}. Also, the remaining $n - q$ bidders' valuations lie below b_i, $i = 1, 2, \ldots, m$. If only one bidder has the valuation in the interval $[b_i, b_{i+1})$, then the object is sold to him/her and if there are two or more such bidders, the one who stops the clock first or calls out "mine" first will get the item.

If $n \geq 2$ participants take part in the auction, then the probability that the item is sold at the price level b_i, $i = 1, 2, \ldots, m$ is $P(b_i)$, which is given by [32, 34]:

$$P(b_i) = \sum_{q=1}^{n} \binom{n}{q} F(b_i)^{n-q} \left[F(b_{i+1}) - F(b_i)\right]^q , \\ = F(b_{i+1})^n - F(b_i)^n. \tag{1}$$

To account for the risk preferences of the bidders, whether they are risk-loving, risk-neutral, or risk-averse, their utility of accepting a bid at the price level b_i is represented using the CARA utility function $U(b_i) = \frac{1-e^{-\alpha b_i}}{\alpha}$, where α is the constant of absolute risk aversion [30, 3, 42, 35, 8]. Therefore, the expected revenue per unit of time by the auctioneer in a DDA considering the risk preferences is given by

$$\mathscr{L} = \frac{\sum_{i=1}^{m} U(b_i) P(b_i)}{\mathscr{D}}, \tag{2}$$

where $\mathscr{D}$ is the auction duration given as follows:

$$\begin{aligned} \mathscr{D} &= sE(m), \\ &= s\left[\sum_{i=1}^{m}(m+2-i)P(b_i) + (m+1)\left(1 - \sum_{i=1}^{m} P(b_i)\right)\right], \\ &= s\left[\sum_{i=1}^{m}(m+2-i)\left[F(b_{i+1})^n - F(b_i)^n\right] + (m+1)\left(1 - \sum_{i=1}^{m}\left[F(b_{i+1})^n - F(b_i)^n\right]\right)\right], \\ &= s\left[(1+m)\left(1 + F(b_1)^n - F(b_{m+1})^n\right) + \sum_{i=1}^{m}(2-i+m)\left(F(b_{i+1})^n - F(b_i)^n\right)\right]. \end{aligned} \tag{3}$$

In light of (1) and (3), (2) becomes

$$\mathscr{L} = \frac{\sum_{i=1}^{m} \frac{1-e^{-\alpha b_i}}{\alpha}\left[F(b_{i+1})^n - F(b_i)^n\right]}{s\left[(1+m)\left(1 + F(b_1)^n - F(b_{m+1})^n\right) + \sum_{i=1}^{m}(2-i+m)\left(F(b_{i+1})^n - F(b_i)^n\right)\right]}, \tag{4}$$

where α is the coefficient of constant absolute risk aversion, determining each bidder's risk attitude.

Here, a symmetric IPV setting is assumed; that is, the valuation of each bidder j is v_j, $j = 1, 2, \ldots, n$, which is drawn from a uniform distribution defined on $[0, \overline{v}]$ with cumulative distribution function (c.d.f.) $F(\cdot)$ and probability distribution function (p.d.f) $f(\cdot)$. In other words, all bidders share the same valuation distribution and private information. We also consider a DDA with a fixed number of price drop levels. The bid levels

$b_1 < b_2 < \ldots < b_m$ span from a minimum price $b_1 = 0$ (no reserve price) to a maximum $b_{m+1} = \overline{v}$ (starting price) partitioning the $[0, \overline{v}]$ valuation range. It follows that $F(b_1) = 0$, $F(b_{m+1}) = \overline{v}$ and $F(b_i) = \frac{b_i}{\overline{v}}$, $i = 1, 2, \ldots, m$ without any loss of generality. These modeling assumptions, consistent with prior literature [33, 36, 44], provide a tractable framework for our analysis. Hence, the seller's expected revenue per unit of time $\mathscr{X}$ can be expressed as follows:

$$\mathscr{X} = \frac{\sum_{i=1}^{m} \frac{1-e^{-\alpha b_i}}{\alpha} \left[\left(\frac{b_{i+1}}{\overline{v}}\right)^n - \left(\frac{b_i}{\overline{v}}\right)^n \right]}{s\left[(1+m)\left(1 + \left(\frac{b_1}{\overline{v}}\right)^n - \left(\frac{b_{m+1}}{\overline{v}}\right)^n\right) + \sum_{i=1}^{m} (2-i+m)\left(\left(\frac{b_{i+1}}{\overline{v}}\right)^n - \left(\frac{b_i}{\overline{v}}\right)^n\right)\right]},$$
$$= \frac{\sum_{i=1}^{m} \left(1 - e^{-\alpha b_i}\right)\left(b_{i+1}^n - b_i^n\right)}{\alpha s\left[(1+m)(\overline{v}^n + b_1^n - b_{m+1}^n) + \sum_{i=1}^{m} (2-i+m)(b_{i+1}^n - b_i^n)\right]}. \tag{5}$$

Therefore, the formulated model as a nonlinear program (NLP) in decision variables $b_1, b_2, \ldots, b_m$ and the parameters α, m, n, s, and $\overline{v}$ is given below:

Maximize

$$\mathscr{X} = \frac{\sum_{i=1}^{m} \left(1 - e^{-\alpha b_i}\right)\left(b_{i+1}^n - b_i^n\right)}{\alpha s\left[(1+m)(\overline{v}^n + b_1^n - b_{m+1}^n) + \sum_{i=1}^{m} (2-i+m)(b_{i+1}^n - b_i^n)\right]},$$

subject to:

$$\begin{aligned} & b_{i+1} \geq b_i, \quad i = 1, 2, \ldots, m, \\ & b_1 \geq 0, \\ & b_{m+1} = \overline{v}. \end{aligned} \tag{6}$$

In the above NLP (6), it is crucial to recognize that as α approaches 0, it signifies the risk-neutral case. This is due to the fact that $\lim_{\alpha \to 0} \frac{1-e^{-\alpha b_i}}{\alpha} = b_i$, which leads to the reduction of our NLP (6) to the model described by Li and Kuo [32], which does not account for the risk preferences of the bidders despite their claim, as that model lacks any parameters to define risk behaviors. Moreover, positive α indicates risk-averse bidders and negative α indicates risk-seeking behavior of the bidders [30, 3, 8].

This paper focuses on solving the optimization model (6) to find the revenue-maximizing set of bid levels and optimal revenue per unit of time under the given constraints. In mathematical terms, we tackle an NLP with m decision variables $b_1, b_2, \ldots, b_m$ (the bid levels). The objective function $\mathscr{X}(b_1, \ldots, b_m)$ is continuously differentiable but nonlinear and generally nonconvex, due to the combination of exponential utility terms and polynomial probability terms in (6). However, the structure of the problem offers some advantages: The feasible region is defined by simple linear inequalities $0 \leq b_1 \leq b_2 \leq \ldots \leq b_m \leq \overline{v}$, and we observed that increasing a bid level beyond its optimal point yields diminishing returns (suggesting a single prominent optimum in practice). This NLP is solved using a numerical optimization approach. Specifically, a program in R (using the `nloptr` package) is implemented to maximize (6) subject to the constraints. This solver employs an augmented Lagrangian method to handle the monotonicity constraints effectively, ensuring that the solution respects $b_1 \leq \ldots \leq b_m$. Each function evaluation of $\mathscr{X}$ involves summing over m terms and computing probabilities raised to the power n, which is an $O(m)$ computation. Thus, the computational complexity scales primarily with the number of bid levels m. In our study, we considered m up to 7, for which the solver finds solutions within a few hours on a standard PC. The number of bidders n influences the shape of the objective (larger n makes the revenue curve steeper) but does not increase the number of decision variables, so it has a minor impact on computation time. We also note that our model reduces to the known risk-neutral case when $\alpha \to 0$, for which analytical solution methods exist (see Li and Kuo [32]); but for arbitrary α, an analytical solution is intractable, validating our choice of a numerical solver. The use of a modern NLP solver is sufficient and efficient for the problem sizes in this study.

3.1 Risk preference asymmetry and utility curvature

We now formalize the distinction between risk-averse and risk-seeking bidders in our model. In the CARA utility framework $U(b_i) = \frac{1-e^{-\alpha b_i}}{\alpha}$, the sign of α governs the utility function's curvature and thereby the bidder's risk attitude. If $\alpha > 0$, then the second derivative $U''(b_i) < 0$, meaning $U(b_i)$ is concave, that is, the hallmark of risk aversion. The bidder derives diminishing marginal utility from monetary gains, preferring certain outcomes over gambles with the same expectation. Conversely, if $\alpha < 0$, then $U''(b_i) > 0$, making the utility convex. This corresponds to risk-seeking (risk-loving) behavior, where the bidder is inclined to gamble for higher returns, as the marginal utility of payoff increases with b_i. The boundary case $\alpha \to 0$ yields $U(b_i) = b_i$ (by L'Hopital's rule), a linear utility indicating risk neutrality. Thus, $\alpha \to 0$ is the cutoff point between two qualitatively different regimes of bidder behavior. We emphasize that positive α and negative α are not symmetric cases, rather they produce fundamentally different bidding incentives. A risk-averse bidder (positive α) is primarily concerned with avoiding high payments (losses), whereas a risk-loving bidder (negative α) focuses on the potential for paying very low prices (gains), even at the risk of possibly leaving empty-handed.

This asymmetry manifests in bidding strategies. A risk-averse bidder will tend to bid (stop the clock) earlier, at a higher price, to secure the item before the price falls too low and uncertainty increases. Their concave utility implies a high disutility for the "loss" incurred if the auction is lost or if the price drops further and someone else wins, hence they exit the auction sooner to minimize regret. In contrast, a risk-loving bidder gains extra utility from pushing their luck; the convex utility means the incremental utility of a lower price is high. Such a bidder is more willing to wait until the price has dropped significantly before bidding, even though waiting carries the risk of losing to a competitor. They effectively treat the prospect of getting a very cheap price as a gamble worth taking. Our model captures these tendencies via the parameter α. For example, in our simulations, a moderately risk-averse bidder ($\alpha = 0.2$) might stop the auction at a price around 80% of their private value, whereas a similarly strong risk-seeker ($\alpha = -0.2$) might hold out until the price is $50-60\%$ of their value, dramatically increasing variance in outcomes. Indeed, our computational results confirm this: Holding other parameters fixed, higher α leads to earlier bids and lower revenue, while more negative α leads to prolonged bidding and can raise revenue (see Tables 2 and 3). Formally, for any given number of bidders n and bid levels m, we find $\mathscr{Z}_{rl} > \mathscr{Z}_{rn} > \mathscr{Z}_{ra}$, where $\mathscr{Z}_{rl}$, $\mathscr{Z}_{rn}$, and $\mathscr{Z}_{ra}$ represent the auctioneer's expected revenue per unit of time for risk-loving, risk-neutral, and risk-averse bidders, respectively, underscoring how the auctioneer's expected revenue per unit time improves as bidders become more risk-seeking (refer to Tables 1–3). This is intuitive from the model: cautious bidders "quit" early, yielding higher prices but fewer active bidders at low price levels, whereas risk-seeking bidders stay longer in the game, driving the price lower and intensifying competition, which paradoxically can increase the auctioneer's time-adjusted revenue by shortening auction duration. The key point is that risk aversion versus risk seeking are asymmetrical in effect, they do not simply cancel out or mirror each other. The analysis and results of this study reflect the asymmetry clearly.

4 Results and discussion

In this section, a series of problem instances is examined to analyze the behavior of the proposed model under varying parameter configurations. The number of bid levels is represented as $m \in \{2, 3, \ldots, 7\}$, the number of bidders as $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$, and the risk parameter as $\alpha \in \{-0.5, -0.4, \ldots, 0.5\}$ with $\overline{v} = 1$ and $s = 1$ based on [34]. Using (6), NLPs are formulated and solved for different combinations of m, n, and α for $\overline{v} = 1$ and $s = 1$ by implementing a program in RStudio. In the subsequent discussion, $\mathscr{Z}^*_{m=\gamma}$ is used to denote the auctioneer's maximum expected revenue per unit of time when the number of bid levels is γ.

To facilitate further discussion, Table 1 provides a summary of the auctioneer's expected revenues per unit of time for all values of m specified above, under the assumption of risk-neutral bidders ($\alpha \to 0$). When bidders exhibit risk-neutral behavior ($\alpha \to 0$), the proposed model (6) reduces to the revenue model outlined by Li, Yue, and Kuo [34] for cases with zero salvage value. The results presented in Table 1 align with those reported by Li, Yue, and Kuo [34] when identical parameter values are used.

Table 1: Auctioneer's maximum expected revenue per unit of time for risk-neutral bidders (i.e., $\alpha \to 0$) for $\overline{v} = 1$, $s = 1$, $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$ and $m \in \{2, 3, \ldots, 7\}$.

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1671	0.1900	0.1932	**0.1934**	0.1934	0.1934
5	0.2717	0.2983	0.3016	**0.3019**	0.3019	0.3019
10	0.3445	0.3671	0.3691	**0.3693**	0.3693	0.3693
15	0.3798	0.3986	0.3999	**0.4000**	0.4000	0.4000
20	0.4011	0.4170	**0.4180**	0.4180	0.4180	0.4180
25	0.4155	0.4293	**0.4301**	0.4301	0.4301	0.4301
30	0.4259	0.4382	**0.4387**	0.4387	0.4387	0.4387
40	0.4402	0.4501	**0.4505**	0.4505	0.4505	0.4505
60	0.4562	0.4634	**0.4636**	0.4636	0.4636	0.4636
80	0.4651	0.4708	**0.4709**	0.4709	0.4709	0.4709
100	0.4708	0.4755	**0.4756**	0.4756	0.4756	0.4756

As shown in Table 1, the expected revenue per unit of time consistently increases with the number of bidders n for each value of bid levels m ranging from 2 to 7. Similarly, for any given n, the expected revenue per unit of time rises with an increasing number of bid levels m. However, this growth halts when m reaches 4 for $n \geq 20$ or 5 for $n < 20$. The highest optimal values for expected revenue per unit of time for each n are highlighted in bold in Table 1. This finding aligns with previously established results in the literature [32, 33, 34, 55]. For instance, Figure 1 illustrates the case where $n = 40$, showing that the optimum revenue per unit of time $\mathscr{Z}^*$ increases with m and reaches a peak value of 0.4505 when m is 4. These results suggest that the auctioneer can achieve maximum expected revenue per unit of time with no more than 5 bid levels, regardless of the value of n, consistent with the trends reported by Li, Yue, and Kuo [34].

It is important to note that this plateauing behavior occurs under our model's assumptions of symmetric bidders and uniform valuations. Intuitively, an auction's revenue per unit time cannot increase indefinitely with more competition, there is an upper limit. As n becomes very large, the highest bidder's valuation will likely be very close to the maximum value $\overline{v}$. (For instance, with valuations Uniform $[0, \overline{v}]$, the expected highest valuation among n bidders is $\frac{n}{n+1}\overline{v}$, which approaches $\overline{v}$ as $n \to \infty$.) This means adding more bidders beyond a certain point yields diminishing returns in expected revenue, causing the revenue curve to flatten out. Similarly, increasing the number of bid levels beyond about four or five yields negligible benefit because the auction outcome is then approaching that of a continuous DA. Additional intermediate price drops (bid levels) past this threshold do not significantly raise the winning price or reduce the selling time. Therefore, our model indicates that under standard conditions (uniform i.i.d. values and no reserve price) the optimal auction design need not exceed five price levels, a result that aligns with economic intuition and prior findings in the literature.

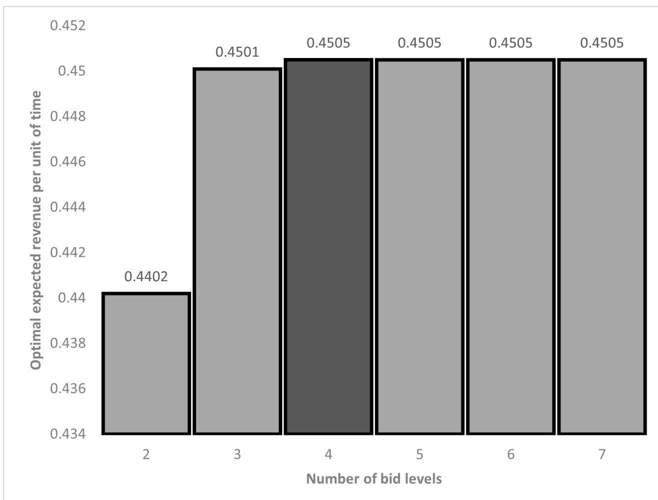


Figure 1: The auctioneer's maximum expected revenue per unit of time ($\mathscr{Z}^*$) versus number of bid levels m for $n = 40$ and risk-neutral bidders, that is, ($\alpha \to 0$).

Table 2 outlines the auctioneer's maximum expected revenue per unit of time across various bid levels (m) and numbers of bidders (n) under the condition of risk-averse bidders, characterized by $\alpha \in \{0.1, 0.2, \dots, 0.5\}$. A higher value of α indicates greater risk aversion [45, 11, 4]. The results demonstrate that as α increases, reflecting heightened risk aversion, the expected revenue per unit of time declines for every combination of m and n. This occurs because higher risk aversion leads bidders to adopt less aggressive bidding strategies, opting to wait for lower prices to mitigate potential losses. Consequently, the auctioneer's maximum expected revenue decreases with increasing α.

When comparing the results from Table 1 (risk-neutral bidders) and Table 2 (risk-averse bidders), it is evident that optimal expected revenue is higher under risk-neutral conditions. Risk-averse bidders, prioritizing loss minimization over potential gains, tend to bid conservatively, which negatively impacts the auctioneer's revenue [45, 53]. This strategic shift underscores how risk preferences influence auction dynamics, leading to lower bids and reduced revenue for the auctioneer [52, 7].

Furthermore, Tables 2a–2e highlight that for larger bidder groups, fewer bid levels are required to maximize the auctioneer's expected revenue per unit of time. Specifically, the optimal number of bid levels is typically fewer than five, dropping to four or even three in certain cases when the number of bidders is sufficiently high. For instance, in the scenario where $n = 100$ and $\alpha = 0.1$, four bid levels are sufficient to achieve maximum expected revenue. However, for the same bidding population and $\alpha = 0.5$, only three bid levels are required.

Table 3 highlights the auctioneer's maximum expected revenue per unit of time across various bid levels (m) and numbers of bidders (n) under the influence of risk-loving (or risk-seeking) bidders. Specifically, this analysis considers $\alpha \in \{-0.1, -0.2, \ldots, -0.5\}$, where more negative values of α indicate stronger risk-seeking behavior [45, 11, 4]. The table reveals that as α decreases (becomes more negative), reflecting heightened risk-seeking tendencies, the expected revenue per unit of time for the auctioneer increases consistently for all values of m and n. This behavior stems from the aggressive bidding strategies of risk-loving participants, who avoid delaying their bids for potential price drops, driven by their preference for higher risks. As a result, the auctioneer's maximum expected revenue increases as risk-seeking behavior intensifies.

A comparison between Table 1 (risk-neutral bidders) and Table 3 (risk-loving bidders) shows that the optimal revenue per unit of time generated from risk-neutral bidders is lower than that from risk-loving bidders. The propensity of risk-loving bidders to take bold risks results in more aggressive bidding behavior, which translates to higher maximum expected revenue per

Table 2: Auctioneer's maximum expected revenue per unit of time for risk-averse bidders (i.e., $\alpha \in \{0.1, 0.2, \ldots, 0.5\}$) for $\overline{v} = 1,\ s = 1,\ n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$ and $m \in \{2, 3, \ldots, 7\}$.

(a) For $\alpha = 0.1$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1628	0.1848	0.1879	**0.1881**	0.1881	0.1881
5	0.2629	0.2882	0.2912	0.2915	**0.2916**	0.2916
10	0.3318	0.3529	0.3548	**0.3549**	0.3549	0.3549
15	0.3648	0.3823	0.3835	**0.3836**	0.3836	0.3836
20	0.3846	0.3994	0.4003	**0.4004**	0.4004	0.4004
25	0.398	0.4108	**0.4115**	0.4115	0.4115	0.4115
30	0.4077	0.4190	**0.4195**	0.4195	0.4195	0.4195
40	0.4209	0.4300	**0.4304**	0.4304	0.4304	0.4304
60	0.4357	0.4423	**0.4425**	0.4425	0.4425	0.4425
80	0.4439	0.4491	**0.4492**	0.4492	0.4492	0.4492
100	0.4491	0.4534	**0.4535**	0.4535	0.4535	0.4535

(b) For $\alpha = 0.2$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1587	0.1799	0.1828	**0.1830**	0.1830	0.1830
5	0.2546	0.2786	0.2814	**0.2817**	0.2817	0.2817
10	0.3197	0.3395	0.3412	**0.3413**	0.3413	0.3413
15	0.3507	0.3669	0.3680	**0.3681**	0.3681	0.3681
20	0.3691	0.3829	0.3836	**0.3837**	0.3837	0.3837
25	0.3816	0.3934	**0.3940**	0.3940	0.3940	0.3940
30	0.3906	0.4010	**0.4014**	0.4014	0.4014	0.4014
40	0.4028	0.4112	**0.4115**	0.4115	0.4115	0.4115
60	0.4164	0.4225	**0.4226**	0.4226	0.4226	0.4226
80	0.4239	0.4287	**0.4288**	0.4288	0.4288	0.4288
100	0.4288	0.4327	**0.4327**	0.4327	0.4327	0.4327

(c) For $\alpha = 0.3$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1548	0.1752	0.1780	**0.1782**	0.1782	0.1782
5	0.2467	0.2694	0.2721	**0.2724**	0.2724	0.2724
10	0.3082	0.3268	0.3284	**0.3285**	0.3285	0.3285
15	0.3373	0.3524	0.3534	**0.3535**	0.3535	0.3535
20	0.3545	0.3672	0.3679	**0.3680**	0.3680	0.3680
25	0.3661	0.3770	**0.3775**	0.3775	0.3775	0.3775
30	0.3744	0.3840	**0.3844**	0.3844	0.3844	0.3844
40	0.3857	0.3934	**0.3937**	0.3937	0.3937	0.3937
60	0.3983	0.4038	**0.4040**	0.4040	0.4040	0.4040
80	0.4052	**0.4096**	0.4096	0.4096	0.4096	0.4096
100	0.4097	0.4132	**0.4133**	0.4133	0.4133	0.4133

(d) For $\alpha = 0.4$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1511	0.1706	0.1733	**0.1735**	0.1735	0.1735
5	0.2392	0.2607	0.2632	**0.2635**	0.2635	0.2635
10	0.2974	0.3147	0.3162	**0.3163**	0.3163	0.3163
15	0.3246	0.3387	**0.3396**	0.3396	0.3397	0.3397
20	0.3407	0.3525	**0.3531**	0.3531	0.3531	0.3531
25	0.3514	0.3615	**0.3620**	0.3620	0.3620	0.3620
30	0.3591	0.3680	**0.3684**	0.3684	0.3684	0.3684
40	0.3696	0.3767	**0.3769**	0.3769	0.3769	0.3769
60	0.3812	0.3863	**0.3864**	0.3864	0.3864	0.3864
80	0.3876	**0.3916**	0.3916	0.3916	0.3916	0.3916
100	0.3917	0.3949	**0.3950**	0.3950	0.3950	0.3950

(e) For $\alpha = 0.5$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1475	0.1663	0.1689	**0.1691**	0.1691	0.1691
5	0.232	0.2524	0.2548	**0.2550**	0.2550	0.2550
10	0.2871	0.3034	0.3047	**0.3048**	0.3048	0.3048
15	0.3126	0.3257	0.3265	**0.3266**	0.3266	0.3266
20	0.3276	0.3385	**0.3391**	0.3391	0.3391	0.3391
25	0.3376	0.3469	**0.3474**	0.3474	0.3474	0.3474
30	0.3448	0.3529	**0.3533**	0.3533	0.3533	0.3533
40	0.3544	0.3610	**0.3612**	0.3612	0.3612	0.3612
60	0.3652	0.3698	**0.3699**	0.3699	0.3699	0.3699
80	0.371	**0.3747**	0.3747	0.3747	0.3747	0.3747
100	0.3748	**0.3778**	0.3778	0.3778	0.3778	0.3778

unit of time for the auctioneer compared to their risk-neutral counterparts [45, 53]. In essence, risk-loving bidders focus on maximizing potential gains rather than minimizing losses. This leads to higher bids, directly enhancing the auctioneer's expected revenue per unit of time [52, 7].

Furthermore, Tables 3a–3e emphasize that as the number of bidders (n) increases, the number of bid levels required to maximize the auctioneer's expected revenue per unit of time generally decreases. The maximum number of bid levels required remains five or fewer in most cases, although it can reach six bid levels when n is relatively small. For instance, when $n = 100$, only four bid levels are sufficient to maximize expected revenue per unit of time, irrespective of the value of α.

Table 3: Auctioneer's maximum expected revenue per unit of time for risk-loving bidders (i.e., $\alpha \in \{-0.1, -0.2, \ldots, -0.5\}$) for $\overline{v} = 1$, $s = 1$, $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$ and $m \in \{2, 3, \ldots, 7\}$.

(a) For $\alpha = 0.1$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1716	0.1954	0.1987	**0.1990**	0.1990	0.1990
5	0.2808	0.309	0.3124	**0.3128**	0.3128	0.3128
10	0.358	0.3820	0.3843	**0.3844**	0.3844	0.3844
15	0.3956	0.4158	0.4173	**0.4174**	0.4174	0.4174
20	0.4185	0.4357	**0.4368**	0.4368	0.4368	0.4368
25	0.434	0.4490	**0.4498**	0.4498	0.4498	0.4498
30	0.4452	0.4585	0.4591	**0.4592**	0.4592	0.4592
40	0.4606	0.4715	**0.4719**	0.4719	0.4719	0.4719
60	0.478	0.4859	**0.4861**	0.4861	0.4861	0.4861
80	0.4877	0.4939	**0.4941**	0.4941	0.4941	0.4941
100	0.4939	0.4991	**0.4992**	0.4992	0.4992	0.4992

(b) For $\alpha = 0.2$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1763	0.201	0.2045	**0.2048**	0.2048	0.2048
5	0.2905	0.3202	0.3239	0.3242	**0.3243**	0.3243
10	0.3722	0.3979	0.4003	**0.4005**	0.4005	0.4005
15	0.4124	0.4341	0.4357	0.4358	**0.4359**	0.4359
20	0.4369	0.4555	0.4567	**0.4568**	0.4568	0.4568
25	0.4536	0.4699	0.4707	**0.4708**	0.4708	0.4708
30	0.4658	0.4802	**0.4809**	0.4809	0.4809	0.4809
40	0.4824	0.4942	**0.4947**	0.4947	0.4947	0.4947
60	0.5013	0.5099	**0.5102**	0.5102	0.5102	0.5102
80	0.5118	0.5186	**0.5188**	0.5188	0.5188	0.5188
100	0.5186	0.5243	**0.5244**	0.5244	0.5244	0.5244

(c) For $\alpha = 0.3$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1812	0.2070	0.2106	**0.2109**	0.2109	0.2109
5	0.3007	0.3320	0.3359	0.3363	**0.3364**	0.3364
10	0.3872	0.4147	0.4173	**0.4175**	0.4175	0.4175
15	0.4302	0.4535	0.4553	**0.4554**	0.4554	0.4554
20	0.4565	0.4766	0.4779	**0.4780**	0.4780	0.4780
25	0.4745	0.4921	0.4930	**0.4931**	0.4931	0.4931
30	0.4876	0.5032	0.5040	**0.5041**	0.5041	0.5041
40	0.5056	0.5185	**0.5190**	0.5190	0.5190	0.5190
60	0.5261	0.5355	**0.5358**	0.5358	0.5358	0.5358
80	0.5375	0.5450	**0.5452**	0.5452	0.5452	0.5452
100	0.5449	0.5511	**0.5512**	0.5512	0.5512	0.5512

(d) For $\alpha = 0.4$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1864	0.2132	0.2170	**0.2173**	0.2173	0.2173
5	0.3113	0.3445	0.3487	**0.3491**	0.3491	0.3491
10	0.4031	0.4324	0.4353	**0.4355**	0.4355	0.4355
15	0.4491	0.4741	0.4761	**0.4762**	0.4762	0.4762
20	0.4773	0.499	0.5004	**0.5005**	0.5005	0.5005
25	0.4966	0.5157	**0.5168**	0.5168	0.5168	0.5168
30	0.5108	0.5278	**0.5287**	0.5287	0.5287	0.5287
40	0.5303	0.5443	**0.5449**	0.5449	0.5449	0.5449
60	0.5525	0.5628	**0.5631**	0.5631	0.5631	0.5631
80	0.565	0.5731	**0.5733**	0.5733	0.5733	0.5733
100	0.573	0.5798	**0.5799**	0.5799	0.5799	0.5799

(e) For $\alpha = 0.5$

n	$\mathscr{Z}^*_{m=2}$	$\mathscr{Z}^*_{m=3}$	$\mathscr{Z}^*_{m=4}$	$\mathscr{Z}^*_{m=5}$	$\mathscr{Z}^*_{m=6}$	$\mathscr{Z}^*_{m=7}$
2	0.1918	0.2197	0.2237	**0.2240**	0.2240	0.2240
5	0.3226	0.3576	0.3621	0.3625	**0.3626**	0.3626
10	0.4199	0.4513	0.4543	**0.4546**	0.4546	0.4546
15	0.4691	0.496	0.4981	**0.4983**	0.4983	0.4983
20	0.4994	0.5228	0.5244	**0.5245**	0.5245	0.5245
25	0.5202	0.5408	**0.5421**	0.5421	0.5421	0.5421
30	0.5355	0.5539	0.5549	**0.5550**	0.5550	0.5550
40	0.5566	0.5718	**0.5725**	0.5725	0.5725	0.5725
60	0.5807	0.5920	**0.5923**	0.5923	0.5923	0.5923
80	0.5942	0.6032	**0.6034**	0.6034	0.6034	0.6034
100	0.603	0.6105	**0.6106**	0.6106	0.6106	0.6106

From Tables 1–3, it is evident that the inequality $\mathscr{Z}_{rl} > \mathscr{Z}_{rn} > \mathscr{Z}_{ra}$ consistently holds for all values of m and n. Here, $\mathscr{Z}_{rl}$, $\mathscr{Z}_{rn}$, and $\mathscr{Z}_{ra}$ represent the auctioneer's expected revenue per unit of time for risk-loving, risk-neutral, and risk-averse bidders, respectively. This relationship is further illustrated for $m = 5$ in Figure 2, which shows that the expected revenue per unit of time $\mathscr{Z}^*_{m=5}$ increases steadily as α decreases, thereby corroborating the stated inequality.

Moreover, Figure 2 reveals that as the number of bidders n increases, the revenue initially grows rapidly, but the rate of growth gradually slows down beyond a certain point for higher values of n. A similar trend can be observed for other values of m, indicating a consistent pattern across the auction's various configurations. The findings indicate that an increase in the number of bidders leads to a higher revenue per unit of time, as expected. However, beyond a certain threshold, adding more bidders has a diminishing impact on auction outcomes. This is due to the fact that while additional bidders contribute to increased competition, the marginal revenue gains become negligible. Moreover, excessively increasing the number of bidders results in significantly higher operational costs, including administrative expenses and auction management overhead, which may offset the benefits of increased participation.

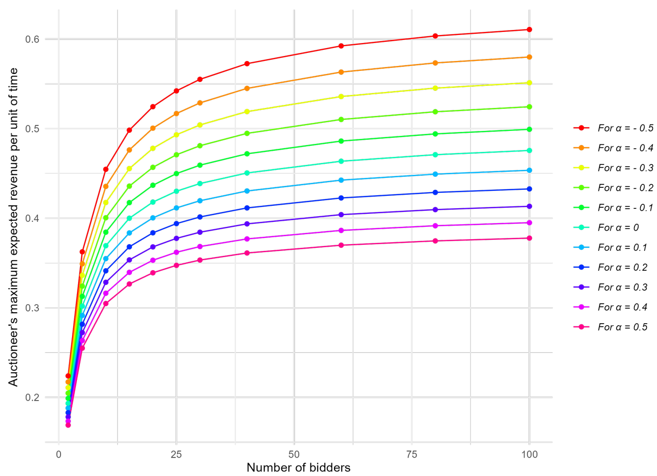


Figure 2: The auctioneer's maximum expected revenue per unit of time ($\mathscr{L}^*_{m=5}$) versus number of bidders (n) where $m = 5$, $s = 1$, $\overline{v} = 1$ and $\alpha \in \{-0.5, 0.4, \ldots, 0.5\}$.

Tables 4, 5, and 6 present the optimal bid levels for $m = 6$ with $\overline{v} = 1$, considering risk-neutral ($\alpha \to 0$), risk-averse ($\alpha > 0$), and risk-loving ($\alpha < 0$) bidders, respectively. In all cases, $b_1 = 0$ implies that the lowest bid level is zero, meaning that the item is given away for free if unsold by that point (as in [32]), an assumption in our developed model. Additionally, $b_{m+1} = 1$ represents the highest asking price, with all intermediate bid levels optimized using the NLP (6).

Figure 3 illustrates the relationship between the constant of absolute risk aversion α and the optimal bid levels b_i from Tables 4–6. Specifically, Figures 3a and 3b represent $n = 10$ and $n = 30$, showing that for a smaller number of bidders, the auctioneer must set distinct bid levels to maximize expected revenue per unit of time for each value of α. Conversely, Figures 3c and 3d demonstrate that as the number of bidders increases, the gap between the bid level curves b_i^* decreases, allowing the auctioneer to skip several intermediate bid levels while still maximizing the expected revenue per unit of time. These graphs confirm that fewer bid levels suffice to maximize expected revenue per unit of time as the number of bidders grows significantly.

Although the optimal solutions of the NLP (6) for Li and Kuo's parameters [32] are not explicitly presented here, replicating their conditions with $\alpha \to 0$ validates our model against their findings. This validation underscores the robustness of our approach, which extends the existing literature by incorporating the impact of bidders' risk preferences on the auctioneer's expected revenue per unit of time in DDAs—a previously unexplored aspect.

Table 4: Risk-neutral (i.e., $\alpha \to 0$) optimal bid levels for $m = 6$, $\overline{v} = 1$, $s = 1$ and $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$.

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.00637	0.05086	0.16367	0.3564	0.63423
5	0	0.12656	0.22543	0.35649	0.52718	0.74089
10	0	0.29469	0.40417	0.52172	0.65895	0.81797
15	0	0.4047	0.50874	0.61167	0.72696	0.85617
20	0	0.43077	0.5754	0.66937	0.76961	0.87961
25	0	0.13506	0.61354	0.70967	0.79927	0.89571
30	0	0.03807	0.65014	0.74045	0.82134	0.90756
40	0	0.0032	0.4345	0.77603	0.85195	0.92399
60	0	0.00037	0.08696	0.82408	0.88799	0.94292
80	0	0.01177	0.0637	0.8562	0.90889	0.95373
100	0	0.02018	0.03763	0.87752	0.92267	0.96081

Beyond the numerical results, bidders' risk attitudes are shaped by behavioral factors. To capture this dimension, we complement the quantitative analysis with insights from the prospect theory (Section 4.1) and further relate our findings to published field evidence (Section 4.2).

Table 5: Risk-averse (i.e., $\alpha \in \{0.1, 0.2, \ldots, 0.5\}$) optimal bid levels for $m = 6$, $\overline{v} = 1$, $s = 1$, $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$.

(a) For $\alpha = 0.1$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006369	0.05019	0.160943	0.351108	0.628404
5	0	0.125464	0.223157	0.352824	0.522604	0.736984
10	0	0.292227	0.401038	0.518059	0.655308	0.815349
15	0	0.401718	0.505522	0.608329	0.723948	0.854164
20	0	0.217861	0.562987	0.665763	0.766988	0.87798
25	0	0.089728	0.609424	0.706885	0.79701	0.894332
30	0	0.031427	0.647208	0.737953	0.819334	0.906362
40	0	0.007614	0.457394	0.774618	0.850336	0.923038
60	0	0.041936	0.125849	0.822922	0.886796	0.942243
80	0	0.007262	0.062786	0.854846	0.90793	0.953198
100	0	0.006374	0.053867	0.87651	0.921869	0.960376

(b) For $\alpha = 0.2$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006361	0.049527	0.15829	0.34593	0.622612
5	0	0.124371	0.220902	0.349199	0.518056	0.733072
10	0	0.289806	0.397913	0.5144	0.651656	0.812708
15	0	0.398847	0.502304	0.604973	0.720908	0.852139
20	0	0.242129	0.560997	0.662805	0.764391	0.876328
25	0	0.110955	0.607069	0.704176	0.794732	0.892931
30	0	0.049844	0.644846	0.735459	0.817304	0.905144
40	0	0.008437	0.478437	0.773193	0.848701	0.92207
60	0	0.013095	0.100278	0.820966	0.88556	0.941551
80	0	0.000198	0.020311	0.853103	0.906942	0.952658
100	0	0.002268	0.037323	0.875249	0.921051	0.959933

(c) For $\alpha = 0.3$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006352	0.048875	0.155706	0.340861	0.61686
5	0	0.123276	0.218666	0.34561	0.513537	0.72915
10	0	0.287384	0.394792	0.510742	0.647993	0.810045
15	0	0.395954	0.499075	0.601603	0.717848	0.850091
20	0	0.249816	0.558328	0.659782	0.761765	0.874654
25	0	0.094469	0.603605	0.701375	0.792425	0.89151
30	0	0.054418	0.642144	0.732927	0.815248	0.903908
40	0	0.058763	0.5332	0.772527	0.847074	0.921089
60	0	0.009706	0.11239	0.819471	0.884319	0.940849
80	0	0.001578	0.03352	0.851851	0.90595	0.95211
100	0	0.023409	0.043642	0.874137	0.920222	0.959483

(d) For $\alpha = 0.4$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.00634	0.048234	0.153189	0.335902	0.611153
5	0	0.122181	0.216451	0.342061	0.50905	0.725222
10	0	0.284961	0.391676	0.507087	0.644322	0.807361
15	0	0.39306	0.49584	0.598221	0.714767	0.848021
20	0	0.21983	0.554073	0.656643	0.759109	0.872959
25	0	0.074269	0.599807	0.698539	0.790095	0.890071
30	0	0.087762	0.64014	0.730399	0.81317	0.902654
40	0	0.032483	0.505141	0.769579	0.845334	0.920089
60	0	0.012192	0.134555	0.818088	0.883061	0.940135
80	0	0.006956	0.025687	0.850382	0.904939	0.951553
100	0	0.031534	0.037506	0.872913	0.91938	0.959026

(e) For $\alpha = 0.5$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006326	0.047604	0.150739	0.33105	0.605494
5	0	0.121087	0.214256	0.33855	0.504596	0.721288
10	0	0.282548	0.388568	0.503437	0.640644	0.804655
15	0	0.390155	0.492596	0.594826	0.711667	0.845929
20	0	0.235182	0.551674	0.653599	0.756438	0.871244
25	0	0.145968	0.599328	0.695842	0.787747	0.888613
30	0	0.033733	0.635972	0.727758	0.811065	0.901384
40	0	0.005836	0.490001	0.767377	0.843606	0.919077
60	0	0.003413	0.122017	0.816238	0.881776	0.939411
80	0	0.016607	0.03418	0.849054	0.903916	0.950987
100	0	0.0000002	0.000859	0.871422	0.918521	0.958561

4.1 Behavioral perspective: Prospect theory and risk behavior

Growing research in behavioral economics shows that individuals' risk preferences are reference-dependent. In particular, Kahneman and Tversky's prospect theory posits that people evaluate outcomes relative to a reference point (status quo) and exhibit risk aversion for gains and risk seeking for losses, rather than a uniform risk attitude [27]. This behavior is captured by an S-shaped value function (refer to Figure 4): Concave in the gains region (implying diminishing sensitivity and risk-averse behavior) but convex in the losses region, reflecting risk-seeking tendencies to avoid sure losses. In the context of auctions, this means a bidder's inclination to take risks

Table 6: Risk-loving (i.e., $\alpha \in \{-0.1, -0.2, \dots, -0.5\}$) optimal bid levels for $m = 6$, $\overline{v} = 1$, $s = 1$, $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$.

(a) For $\alpha = 0.1$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006375	0.051546	0.166462	0.361795	0.640093
5	0	0.127647	0.227719	0.360184	0.531781	0.744772
10	0	0.29708	0.407293	0.525376	0.662573	0.820556
15	0	0.407425	0.511922	0.614991	0.729956	0.858142
20	0	0.482237	0.581117	0.672531	0.772176	0.881224
25	0	0.088777	0.615174	0.712301	0.801493	0.897075
30	0	0.063925	0.653424	0.742928	0.823323	0.908745
40	0	0.038886	0.554843	0.781179	0.853688	0.924935
60	0	0.012076	0.124245	0.826104	0.889195	0.943592
80	0	4.26E-05	0.026447	0.85719	0.909836	0.95425
100	0	0.030176	0.089937	0.878983	0.923466	0.96124

(b) For $\alpha = 0.2$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006373	0.052238	0.169329	0.367303	0.64598
5	0	0.128736	0.230026	0.363916	0.536405	0.748642
10	0	0.299522	0.410421	0.52903	0.666183	0.823122
15	0	0.410409	0.515113	0.618296	0.732924	0.860094
20	0	0.444641	0.581898	0.675348	0.77469	0.88281
25	0	0.105909	0.618475	0.714982	0.803695	0.898416
30	0	0.036579	0.655578	0.745332	0.825277	0.90991
40	0	0.000879	0.438112	0.77999	0.855147	0.925853
60	0	0.010616	0.098259	0.827382	0.890358	0.94425
80	0	0.00337	0.054132	0.85871	0.910774	0.954763
100	0	0.004399	0.005239	0.879533	0.924233	0.961661

(c) For $\alpha = 0.3$

n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006368	0.052938	0.172269	0.37292	0.651889
5	0	0.129817	0.232346	0.36768	0.541051	0.752494
10	0	0.301964	0.413547	0.532678	0.669776	0.82566
15	0	0.413297	0.518282	0.621579	0.735865	0.862021
20	0	0.437732	0.58437	0.678252	0.777185	0.884374
25	0	0.104765	0.621276	0.717632	0.80587	0.899737
30	0	0.082239	0.659187	0.747779	0.827208	0.911056
40	0	0.014346	0.477744	0.782869	0.856752	0.926763
60	0	0.013075	0.107012	0.829033	0.891514	0.944897
80	0	0.000748	0.034652	0.859841	0.911689	0.955267
100	0	0.001334	0.004129	0.880615	0.924997	0.962075

(d) For $\alpha = 0.4$

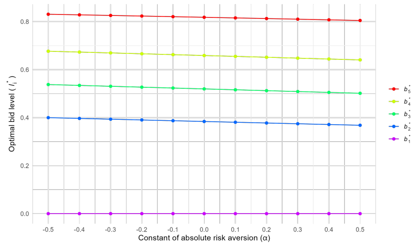
n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006358	0.053645	0.175284	0.378645	0.657814
5	0	0.130889	0.234679	0.371476	0.545715	0.756325
10	0	0.304386	0.416668	0.536319	0.673349	0.82817
15	0	0.416156	0.521432	0.624841	0.738779	0.863921
20	0	0.411012	0.585903	0.681063	0.779646	0.885914
25	0	0.094328	0.623848	0.720226	0.808014	0.901037
30	0	0.041685	0.66105	0.750112	0.829108	0.912183
40	0	0.001515	0.477023	0.784702	0.85829	0.927656
60	0	0.023113	0.116925	0.830673	0.892651	0.945533
80	0	0.018905	0.072682	0.861382	0.912598	0.955763
100	0	0.020446	0.04199	0.881898	0.925752	0.962482

(e) For $\alpha = 0.5$

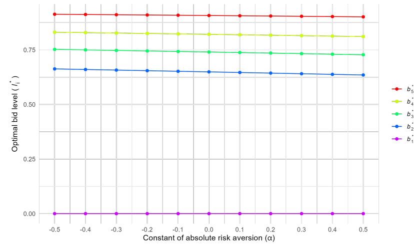
n	b_1	b_2	b_3	b_4	b_5	b_6
2	0	0.006345	0.054358	0.178373	0.384477	0.66375
5	0	0.131952	0.237024	0.375302	0.550396	0.760132
10	0	0.306795	0.419782	0.539951	0.676902	0.830652
15	0	0.419002	0.524563	0.62808	0.741664	0.865796
20	0	0.421518	0.589118	0.683961	0.782086	0.887431
25	0	0.128293	0.627457	0.722856	0.810134	0.902317
30	0	0.048893	0.663809	0.75246	0.830983	0.913292
40	0	0.001763	0.495012	0.787023	0.859826	0.928535
60	0	0.000512	0.087395	0.831879	0.893756	0.946157
80	0	0.005219	0.029883	0.862319	0.913477	0.956249
100	0	0.011548	0.026896	0.882862	0.926487	0.962881

may increase if they perceive themselves as “in the losses”—for example, if the current auction price exceeds their internal reference point (perhaps the price they initially hoped to pay). Conversely, if a bidder stands to obtain a item at a price well below their value (a perceived gain), then the prospect theory predicts more risk-averse behavior, that is, locking in the win rather than gambling further [31].

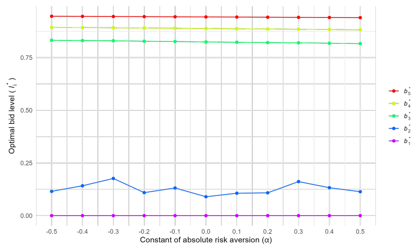
The Dutch auction format may interact with these behavioral tendencies. Bidders often set a mental reference price; dropping below it turns the potential purchase into a “gain” scenario, where they might become cautious and clinch the deal. If the price stays above that reference (a potential loss relative to their target), bidders might hold off (risk-seek) longer than standard risk-neutral models predict, hoping the price drops further. Empirical evi-



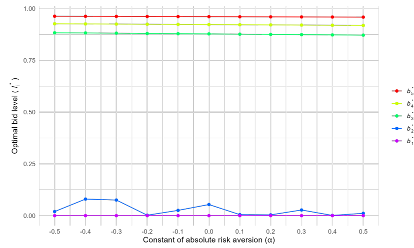
(a) For $n = 10$.



(b) For $n = 30$.



(c) For $n = 60$.



(d) For $n = 100$.

Figure 3: Constant of absolute risk aversion (α) versus optimal bid levels (b_i^*) when $m = 5$.

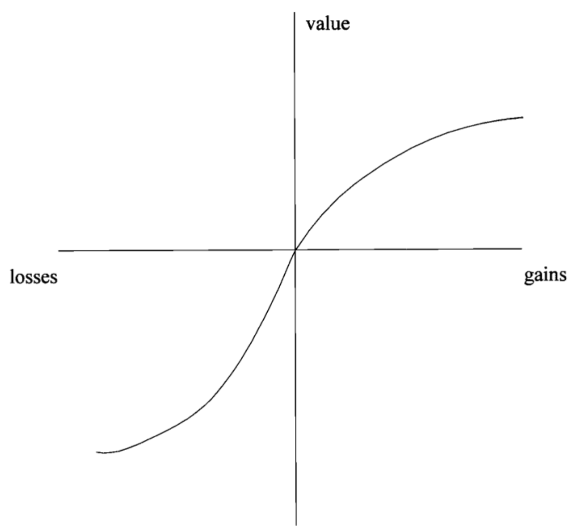


Figure 4: Value function (as in [31]).

dence of such behavior is noted in auction experiments and field data [18, 27]. For instance, experienced bidders sometimes "ride the clock"" longer when they feel they are “behind” (a form of loss-chasing), a behavior consistent with the prospect theory’s loss-domain risk seeking. This subsection bridges our model with real-world behavior: While our optimization assumes consistent risk preferences via CARA utilities, in practice a bidder’s risk posture might dynamically shift from conservative to bold depending on whether the current price is viewed as a gain or a loss. Incorporating such reference-dependent preferences formally is an interesting avenue for future extensions of our model.

4.2 Validation through published empirical evidence

Although this study is based on simulation results, it is important to consider whether the findings are consistent with empirical evidence. Access to detailed, transaction-level Dutch auction data remains highly restricted, since most fish and flower auction houses do not make such records publicly available. As a result, we validate our framework by drawing on published studies that have analyzed real auction data.

Fluvià et al. [16], using approximately 179,000 transactions from the Ancona fish market, reported the patterns that resonate strongly with our model: Prices decline substantially over the course of an auction, and auctioneer revenue per unit time reaches a plateau once bidder participation exceeds a certain threshold. Both results mirror the dynamics predicted in our simulations. Likewise, empirical evidence from the Dutch flower auctions (Royal FloraHolland) shows that transactions are typically concluded within seconds [51, 34], meaning that only a small number of bid decrements are actually employed. This observation supports our result that, beyond an optimal number of bid levels, additional increments contribute little to revenue performance. Taken together, these studies confirm that the main dynamics captured by our simulations are also observed in practice: (i) revenue per unit time increases with more bidders but eventually exhibits diminishing returns, and (ii) auction efficiency is achieved with only a limited number of bid decrements. While no new empirical dataset is introduced here, the consistency between the findings of the study and published evidence provides external validation of this research.

Another central contribution of this work is the explicit integration of bidders' risk preferences, modeled with the CARA utility function, into a framework for revenue per unit of time optimization. The simulations clearly indicate that risk-seeking behavior enhances auctioneer revenue, whereas risk aversion reduces it. This asymmetry follows from the curvature of the utility function: concavity (risk aversion) leads to cautious bidding, while convexity (risk seeking) encourages prolonged participation and more aggressive bids. These insights are consistent with prior laboratory and field studies. Kagel and Levin [26] demonstrated experimentally that risk-averse bidders shade their bids, lowering seller revenue. Hu, Matthews, and Zou [21] showed both analytically and experimentally that risk-averse participants produce lower expected revenues in Dutch auctions, while risk-loving bidders generate more aggressive competition and higher revenues. Evidence from fish auctions aligns with this as well: Fluvià et al. [16] found that cautious buyer strategies depressed prices, whereas aggressive bidding accelerated sales at higher prices, reflecting the same outcomes as our risk-seeking simulations.

By aligning these CARA-based results with established experimental and field evidence, this study demonstrates that bidder risk attitudes are not only theoretically significant but also observable in real markets. This strengthens the external validity of the proposed framework, even in the absence of new transaction-level data.

5 Conclusion

This study introduced a novel framework for modeling the DDA using a nonlinear programming approach to maximize the auctioneer's expected revenue per unit of time while explicitly accounting for bidders' risk preferences. By integrating the CARA utility function, the model extended existing research by incorporating α as a measure of bidders' risk attitudes. Results derived from extensive numerical experiments revealed several significant insights that enhance understanding of optimal auction design.

The findings demonstrated that the auctioneer's expected revenue per unit of time increases as the number of bidders n grows. Initially, the revenue per unit of time experiences a sharp rise with an increasing number of participants; however, as the number of bidders becomes sufficiently large, the rate of growth slows. This suggested diminishing returns in terms of revenue gains with further increases in bidder population. For auctions with smaller numbers of bidders, the auctioneer must set each bid level distinctly to achieve maximum revenue per unit of time. However, as the number of bidders increases, some bid levels can be omitted without affecting the optimal outcome. This observation implied that the auction design can be simplified for larger bidder populations without compromising efficiency of revenue per unit of time.

Furthermore, this study emphasized the influence of the number of bid levels (m) on the optimization of revenue per unit of time. The results indicated that, although an increase in bid levels initially enhances the auctioneer's maximum expected revenue per unit of time, this improvement eventually plateaus, suggesting that additional bid levels beyond a certain threshold do not yield further benefits. These observations substantiated that employing five or fewer bid levels is sufficient for optimizing revenue per unit of time, regardless of the risk preferences of the bidders.

The implications of these findings are significant for auctioneers seeking to forecast potential revenues per unit of time and enhance the efficiency of their auction operations. By taking into account both the number of participants and the optimal structuring of bid levels, auctioneers can effectively balance operational complexity with the goal of maximizing revenue per unit of time. This research not only corroborates previous studies but also deepens the understanding of the role that risk preferences play in shaping auction outcomes, thereby providing valuable, practical recommendations for the implementation of auctions in real-world settings.

As the risk aversion coefficient (α) increases, there is a corresponding decrease in the auctioneer's expected revenue per unit of time. This reduction is primarily due to the conservative bidding behavior exhibited by risk-averse participants, who prioritize minimizing potential losses over seeking additional gains. In contrast, when α assumes more negative values, reflecting stronger risk-loving tendencies, the auctioneer's expected revenue per unit of time rises. This outcome is attributable to the assertive bidding strategies adopted by risk-loving bidders, who focus on maximizing potential gains rather than limiting losses. Throughout this study, it is consistently observed that $\mathscr{Z}_{rl} > \mathscr{Z}_{rn} > \mathscr{Z}_{ra}$, where $\mathscr{Z}_{rl}$, $\mathscr{Z}_{rn}$, and $\mathscr{Z}_{ra}$ denote the auctioneer's expected revenue for risk-loving, risk-neutral, and risk-averse bidders, respectively. These findings underscored the significant impact of bidders' risk preferences on auction outcomes.

Although the model developed in this study advanced prior research by explicitly integrating risk preferences, it also demonstrated the capacity to reproduce earlier results when $\alpha \to 0$, thus affirming its validity and wider applicability. This dual functionality highlighted the model's robustness and adaptability, establishing it as a noteworthy contribution to the literature on DDA.

Beyond the numerical optimization results, this work underscored the fundamental role of risk attitudes in shaping auctioneer revenue per unit time. The consistent ordering of revenues across risk profiles, the behavioral justification provided by the prospect theory, and the alignment with empirical patterns reported in the literature together validated the robustness of the proposed CARA-based approach. While direct access to proprietary datasets remains limited, the convergence of our simulation outcomes with documented field evidence ensured that the framework not only advances theoretical auction design but also offers insights that are credible and transferable to real-world market settings.

The results of this study contributed to a deeper understanding of DDA and provided valuable guidance for enhancing auction design. Nevertheless, several limitations should be recognized. This research does not utilize real-world data for empirical validation, presumes a zero minimum selling price, and relies exclusively on the CARA utility function in conjunction with uniformly distributed bidder valuations. These simplifying assumptions suggested avenues for future inquiry. Subsequent studies could address these constraints by investigating the impact of nonzero minimum selling prices, employing alternative probability distributions for bidder valuations, and exploring different utility functions to model risk preferences. Furthermore, empirical validation of the model, where feasible, would strengthen its practical significance and applicability.

In summary, this study enhances the comprehension of DDA by incorporating bidders' risk preferences into a computational optimization framework. Through the application of nonlinear programming methods to examine the effects of these preferences, the research advanced auction theory and illustrated the utility of mathematical computing in the formulation of effective auction mechanisms. The findings presented herein established a basis for the development of more efficient and practical auction models, with relevance extending across a variety of economic and computational contexts.

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