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Research Article

Measuring the efficiency of bi-level structured systems based on directional distance function and network DEA model

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Abstract

In practical contexts, decision-making units (DMUs) are frequently organized hierarchically, encompassing various levels of functionality, multiple subunits across different tiers, or multiple stages of production. Traditional data envelopment analysis (DEA) models often neglect these hierarchical arrangements, potentially leading to assessments that do not accurately reflect actual unit performance. This study proposes a novel DEA model specifically designed to assess the efficiency of bi-level DMUs, consisting of a leader at the upper level and a follower at the lower level. By using a directional distance function, the proposed model simultaneously minimizes inputs and maximizes outputs, offering a more flexible and accurate evaluation compared to conventional methods. The model's effectiveness is demonstrated through its application in two case studies within the banking sector, showcasing its ability to provide deeper managerial insights and improve decision-making.

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1 Introduction

Traditional data envelopment analysis (DEA) models are often viewed as "black boxes," converting inputs to outputs without considering the internal configuration of evaluated units. These models rely on aggregating primary inputs and final outputs, potentially overlooking the complexities of unit structures. To address these shortcomings, network DEA models have emerged to integrate the internal structures of assessed units, offering a more comprehensive evaluation. Unlike conventional approaches, DEA network models are adaptable and can be tailored to different processes and various configurations of units.

Hierarchical structures within networks categorize organizations based on process intricacy and organizational scope. In large institutions like universities and banks, where hierarchical structures are prevalent, evaluating performance becomes crucial for managerial decision-making. Our study focuses on assessing bi-level units, which consist of a leader at the upper level and a follower at the lower level. The proposed model can be further expanded to evaluate various hierarchical unit typologies, thereby enhancing the applicability of DEA in organizational performance evaluations.

Some studies in the literature have investigated unit performance by examining their internal and hierarchical structures. A summary of the key literature related to hierarchical DEA is presented in Table 1. For example, Cook et al. [3] were pioneers in applying hierarchical DEA to evaluate the efficiency of power plants at various levels, a method that was later refined by Cook and Green [4] specifically for Canadian power plants. Kao [13] developed a DEA model for evaluating hierarchical units, where each level's efficiency is derived from the weighted average efficiency of its subunits at lower levels. However, these models overlook the independent performance of upper levels and intermediate outputs. Subsequent studies, such as those by Kashim, Kasim, and Abd Rahman [14], Gan et al. [8], Hendrawan, Purwanggono, and Suliantoro [12] have extended Kao's approach to various hierarchical structures. Ahmad et al. [1] expanded Kao's and Kashim's frameworks to evaluate faculty efficiency within the hierarchical structures of university faculties in Malaysia. Xiao et al. [30] extended Kao's method and utilized the cross-efficiency approach to assess the hierarchical network structures of 20 universities. They incorporated a secondary objective of maximizing-minimizing and employed the CRITIC method for determining criterion weights. Serkani et al. [25] combined Kao's method with intuitionistic fuzzy analytic network process to evaluate the efficiency of the faculty of basic sciences of Islamic Azad University in a hierarchical structure.

Ghasemi et al. [10] developed the approaches presented for evaluating two-stage structures proposed by Liang, Cook, and Zhu [18] and Du et al. [7] and applied them to evaluate the effectiveness of hierarchical structures in schools. Luban [19] introduced a fuzzy DEA model for hierarchical systems designed to determine the activity levels of decision-making units (DMUs) in order to meet both the global targets set by the coordinating unit and the individual targets of the DMUs. Deville, Ferrier, and Leleu [6] proposed an approach to measuring the performance of hierarchical organizations, such as bank branch networks, by decomposing overall performance into technical efficiency (TE) at the lower level (branch) and allocative efficiency at the upper level (top management). Ghasemi et al. [9] developed an incremental model for multi-stage structure proposed by Cook et al. [5] and combined it with fuzzy models to evaluate the hierarchical structure in Farhangian University campuses with imprecise data.

Wu [29] suggested a bi-level DEA model to assess cost efficiency in DMUs with one leader and one follower, while Zhou et al. [33] developed a similar model for systems with one leader and multiple followers. Yao, Shuai, and Chen [31] further extended Zhou's work to assess vulnerability in regional water systems. Omrani, Mohammadi, and Emrouznejad [22] developed a bi-level multi-objective DEA model to concurrently assess the performance of DMUs across two hierarchical dimensions, utilizing fuzzy goal programming to solve their model. Walheer and Hudik [27] proposed a nonparametric method to minimize costs within multidivisional firms, with applications in the Chinese manufacturing sector. Zhang and Chen [32] extended the idea of Kao [13] to a hierarchical system with two-stage processes and suggested additive and multiplicative network DEA models for evaluating their efficiency, employing semidefinite programming to solve these models and assess performance in China's high-technology sector. However, the nonlinear nature of their model limits its applicability.

Li et al. [17] proposed a hierarchical framework to assess power plant efficiency, using a DEA-like approach for sub-unit efficiencies and aggregating them into a two-stage process. They also introduced goal programming to minimize discrepancies between projected values of plant-level and corresponding total values of sub-unit projections. However, their model may not directly apply in cases where level 2 DMUs have distinct inputs and outputs. Pachar, Gupta, and Jha [23] introduced a DEA method to compute cost efficiency in multiple retail stores, within a Stackelberg dynamic network. Zhu et al. [34] evaluated the operational performance of hotels and their departments, such as catering and occupancy, within a hierarchical framework. Habibpoor, Alirezaee, and Rashidinia [11] proposed a DEA model for evaluating the performance of bi-level units, where lower level efficiency serves as an upper limit for higher-level efficiency. Muvingi et al.

[21] discussed a multi-function parallel system with hierarchical network structures, introducing additive and nonlinear multiplicative DEA models for operational performance assessment.

Muvingi, Peer, and Lotfi [20] proposed hierarchical non-Archimedean DEA models to assess the efficiency of DMUs with a hierarchical structure. Their model was applied to analyze the locations of mobile money agents, taking into account the variations in group sizes across different regions. Shafiee and Saleh [26] extended the DEA approach to accommodate tri-level structures and proposed a goal programming method for its resolution. Kweh et al. [16] investigated the impact of environmental, social, and governance factors on the efficiency of 29 suppliers, partners, and customers of Japan Airlines from 2013 to 2022, utilizing a two-stage hierarchical network DEA technique. Additionally, Muvingi, et al. [21] analyzed the performance of hierarchical units while considering fuzzy characteristics in input and output data based on TOPSIS rankings.

Much of the existing research in network DEA has concentrated on evaluating performance in two-stage or multi-stage structures, whereas hierarchical structures have received comparatively less attention [24]. In multi-stage models, it is typically assumed that the output of each stage serves as the input for the subsequent stage. However, in real-world settings, many organizations operate within hierarchical frameworks that involve multiple functional levels or consist of several subunits across different layers. A review of the literature on hierarchical structures reveals that while some studies have examined cost performance, others have focused on operational efficiency—primarily using the methodologies proposed by Cook and Kao. However, the approaches introduced by Cook et al [3, 4] and Kao [13] tend to disregard the independent performance of higher-level units and overlook outputs and intermediate activities, thereby limiting both the accuracy and applicability of these models. Addressing these challenges requires developing robust evaluation methods tailored to the hierarchical structure of organizations.

In this study, we seek to overcome these challenges by explicitly incorporating the structural characteristics commonly observed in real-world hierarchical systems. This research offers a substantial contribution to the field of efficiency analysis within complex organizational environments—particularly in hierarchical institutions such as banks—by introducing a novel DEA approach. The proposed approach is designed to evaluate the performance of bi-level DMUs, explicitly accounting for inter-level interactions and the role of intermediate products transmitted between hierarchical layers. Our method leverages the distance function model, which offers greater flexibility compared to traditional models for measuring efficiency improvements. Specifically, this approach enables the simultaneous analysis of input contractions and output expansions, thereby supporting a more comprehensive evaluation of potential efficiency improvements across multiple dimensions. This model enables decision-makers to simultaneously evaluate input reductions and output increases, facilitating improvements in both dimensions. The distance function approach is especially effective in handling undesirable outputs, allowing

Table 1: Selected Literature on hierarchical network DEA Evaluation

Authors	**Methods**	**Application**	**Contributions**
Cook et al. [3]	Hierarchical DEA model	Power plants	Introduced hierarchical DEA to evaluate power plant efficiency across multiple levels
Cook & Green [4]	Hierarchical DEA model	Canadian power plants	Refined hierarchical DEA model for assessing power plant
Kao [13]	Hierarchical DEA model	Hierarchical units	Developed a DEA model for performance assessment at different hierarchical levels
Wu [29]	Bi-level DEA model	Cost efficiency in DMUs	Proposed a bi-level DEA model for assessing cost efficiency in systems with one leader and one follower
Zhou et al. [33]	Bi-level DEA model	Cost efficiency in Supply Chain	Developed a bi-level DEA model for systems with one leader and multiple followers
Omrani et al. [22]	Bi-level multi-objective DEA	profit and operational efficiency in Banking (Multi-dimensional performance)	Developed a multi-objective DEA model with fuzzy goal programming for hierarchical DMU performance
Walheer & Hudik [27]	Nonparametric method	Cost performance in Multidivisional firms in high-technology manufacturing industry	Proposed a method to minimize costs and reallocate resources across firms' divisions
Zhang & Chen [32]	Additive & multiplicative network DEA	TE of Hierarchical systems in high-technology industry	Suggested DEA model using semidefinite programming for hierarchical systems with two-stage processes
Li et al. [17]	DEA-like model with goal programming	Power plants	Combined DEA-like approach and goal programming to evaluate sub-unit performance and reduce discrepancies
Pachar et al. [23]	DEA with Stackelberg dynamic network	Cost efficiency in retail stores	Developed a Stackelberg-based DEA model for cost efficiency in retail stores
Zhu et al. [34]	hierarchical DEA	Operational performance in hotels and tourism	Evaluated operational performance of hotels and their department
Habibpoor et al. [11]	Bi-level DEA	Banking	Proposed a DEA model for evaluating two-level unit performance
Kremantzis et al. [15]	Additive & nonlinear multiplicative DEA	Operational performance in Higher education	Designed DEA models for operational performance in multi-function parallel systems

for their reduction alongside undesirable inputs, while potentially boosting desirable outputs. Furthermore, the direction of improvement can be customized according to specific contexts such as policy objectives or technological constraints. The results derived from this model are straightforward to interpret, representing the amount of adjustment required to reach the efficient frontier [28]. To demonstrate the practical applicability of our bi-level model, we apply it to two real-world case studies within the banking industry, providing empirical evidence of its effectiveness. Therefore, the subsequent sections of this paper are structured as follows: Section 2 introduces our proposed TE and pure efficiency (PE) models for evaluating bi-level unit performance. Section 3 presents two case studies illustrating the application and benefits of our proposed model. Finally, Section 4 provides concluding remarks.

2 Proposed models for bi-level units

The directional distance function proposed by Chambers, Chung, and Färe [2], inspired by Luenberger's benefit function, to assess TE through optimizing outputs and inputs concurrently. This method involves comparing input-output vectors $(\mathrm{X}_0, \mathrm{Y}_0)$ with a reference bundle $(\mathrm{g_X}, \mathrm{g_Y})$ against a production possibility set T. The directional distance function can be defined as

$$\overrightarrow{\mathrm{D}}\,(\mathrm{X}_0, \mathrm{Y}_0, \mathrm{g_X}, \mathrm{g_Y}) = \mathrm{Max}\ \varphi\text{:}\quad (\mathrm{X}_0{+}\varphi \mathrm{g_X}, \mathrm{Y}_0{+}\varphi \mathrm{g_Y}) \in \mathrm{T}. \tag{1}$$

The directional distance function's evaluation at a specific input-output bundle depends on $(\mathrm{g_X}, \mathrm{g_Y})$ and the chosen reference technology. The selected bundle $(\mathrm{g_X}, \mathrm{g_Y})$ determines the direction in which the observed bundle, if it is an interior point and is projected onto the efficient frontier of the production possibility set.

The choice of $(\mathrm{g_X}, \mathrm{g_Y})$ is arbitrary. We could opt select $(-\mathrm{X}_0,\ \mathrm{Y}_0)$ for $(\mathrm{g_X}, \mathrm{g_Y})$, in which case the directional distance function transforms into the following expression:

$$\overrightarrow{\mathrm{D}}\,(\mathrm{X}_0, \mathrm{Y}_0) = \mathrm{Max}\ \varphi\text{:}\{(1{-}\varphi)\mathrm{X}_0, (1{+}\varphi)\mathrm{Y}_0)\} \in \mathrm{T}. \tag{2}$$

In essence, this approach aims to increase output and decrease input in proportion to φ. For instance, if φ is 10%, then we aim to expand all outputs by 10% while simultaneously reducing all inputs by 10%.

Under the assumption of constant returns to scale (CRS), the production possibility set derived from a set of j DMUs is defined as

$$T_C = \left\{ (\mathrm{X},\mathrm{Y}): \quad \sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{X}_{\mathrm{j}} \leq \mathrm{X}, \quad \sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Y}_{\mathrm{j}} \geq \mathrm{Y}, \quad \lambda_{\mathrm{j}} \geq 0, \quad \mathrm{j} = 1,2,\ldots,\mathrm{J} \right\}. \tag{3}$$

The efficiency model formulated using the directional distance function for CRS production possibility sets, $\mathrm{T_C}$, can be expressed as

$$\begin{aligned} & TE_0 = Max \ \ \varphi \\ & s.t. \quad \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{X}_{\mathrm{j}} \leq (1-\varphi)\mathrm{X}_0, \\ & \qquad\ \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Y}_{\mathrm{j}} \geq (1+\varphi)\mathrm{Y}_0, \\ & \qquad\ \lambda_{\mathrm{j}} \geq 0, \quad j = 1,2,\ldots,J, \end{aligned} \tag{4}$$

where φ represents the measure of technical inefficiency of the $\mathrm{DMU_0}$. By implication, its efficiency equals $(1- \varphi)$. Hence, $\mathrm{DMU_0}$ is efficient if and only if $\varphi^*= 0$. In other words, we aim to increase the output and reduce the input by the proportion φ.

Let $\mathrm{t} = 1 - \varphi$, the above model can be transformed into the following form:

$$\begin{aligned} & TE_0 = Min \ \ t \\ & s.t. \quad \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{X}_{\mathrm{j}} \leq \mathrm{t}\mathrm{X}_0 \\ & \qquad\ \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Y}_{\mathrm{j}} \geq (2-\mathrm{t})\mathrm{Y}_0, \\ & \qquad\ \lambda_{\mathrm{j}} \geq 0, \quad j = 1,2,\ldots,J, \end{aligned} \tag{5}$$

where t is the efficiency score of the $\mathrm{DMU_0}$. The dual problem of (5) is the following model:

$$\begin{aligned} & TE_0 = Max \ \ 2\ \mathrm{u}\mathrm{Y}_0 \\ & s.t. \quad \mathrm{u}\mathrm{Y}_{\mathrm{j}} - \mathrm{v}\mathrm{X}_{\mathrm{j}} \leq 0, \quad \mathrm{j} = 1,2,\ldots,\mathrm{J}, \\ & \qquad\ \mathrm{v}\mathrm{X}_0 + \mathrm{u}\mathrm{Y}_0 = 1, \\ & \qquad\quad u,\ v \geq 0. \end{aligned} \tag{6}$$

It is evident that model (6) is derived from the conversion of the following fractional model:

$$\begin{aligned} & TE_0 = Max \ \ \tfrac{2\ \mathrm{u}\mathrm{Y}_0}{\mathrm{v}\mathrm{X}_0 + \mathrm{u}\mathrm{Y}_0} \\ & s.t. \quad \tfrac{2\mathrm{u}\mathrm{Y}_{\mathrm{j}}}{\mathrm{v}\mathrm{X}_{\mathrm{j}} + \mathrm{u}\mathrm{Y}_{\mathrm{j}}} \leq 1, \quad \mathrm{j} = 1,2,\ldots,\mathrm{J}, \\ & \qquad\ u,\ v \geq 0. \end{aligned} \tag{7}$$

The above model measures the traditional TE of $\mathrm{DMU_0}$ by considering the aggregate inputs $\mathrm{X_0}$ and final outputs $\mathrm{Y_0}$, utilizing the directional distance function. In conventional model, DMUs are treated as black boxes, where their performance is evaluated by considering the primary inputs and final outputs, overlooking their internal structures. However, many DMUs exhibit hierarchical arrangements, with each unit containing lower level subunits or different functional levels.

Consider the bi-level structure depicted in Figure 1. In this framework, the leader at the upper level uses a portion of the inputs to produce independent final outputs and intermediate outputs for their subordinates. The remaining inputs and intermediate outputs are then allocated to the follower at the lower level. The follower uses both sets of inputs to generate their final outputs. Essentially, after receiving inputs from the leader, the followers act independently and demonstrate their performance based on these inputs. Thus, the leader's performance depends on the optimal performance of the follower at the lower level. As a result, a unit's overall performance is achieved based on the optimal performance of both the leader and the follower.

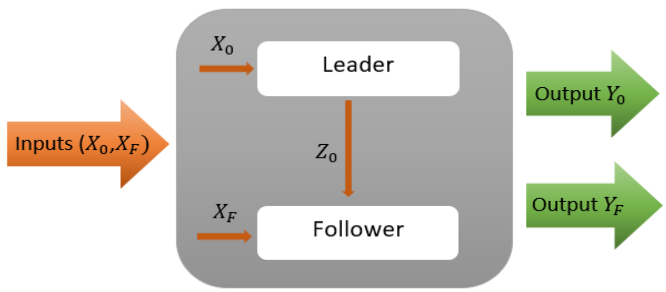

Figure 1: *The internal structure of* DMU_0.

Prior to formulating the model, we define the following symbols:

J: The total number of DMUs

j: The jth DMU possessing a bi-level structure

X_{Lj}: The input of the leader j

Y_{Lj}: The final output of the leader j

Z_{Lj}: The intermediate measure from leader j to follower j

X_{Fj}: The input of follower j

Y_{Fj}: The final output of follower j

v_1: The input weight vector of the leader

u_1: The output weight vectors of the leader

v_2: The input weight vector of the follower

u_2: The output weight vectors of the follower

w: The weight vector corresponding the intermediate measure from the leader to follower

2.1 Proposed TE model for bi-level units

The TE of the $\mathrm{DMU_0}$'s follower at the lower level based on directional distance function can be formulated as follows:

$$\begin{aligned} TE_{F0} = \quad & Max \ \ \frac{2\mathrm{u_2 Y_{F0}}}{\mathrm{v_2 X_{F0} + \ wZ_0 + u_2 Y_{F0}}} \\ & s.t. \quad \frac{2\mathrm{u_2 Y_{Fj}}}{\mathrm{v_2 X_{Fj} + wZ_j + u_2 Y_{Fj}}} \leq 1, \qquad j = 1,2,\ldots,J, \\ & \qquad \mathrm{u_2} \geq 0, \quad \mathrm{v_2} \geq 0, \quad \mathrm{w} \geq 0. \end{aligned} \tag{8}$$

This model features a bounded objective function with a value of one and also has a feasible solution, ensuring the existence of a finite optimal solution. Model (8) can be transformed into a linear model for easier solution. The equivalent linear model of model (8) is

$$\begin{aligned} & TE_{F0} = Max \ \ 2\mathrm{u_2 Y_{F0}} \\ & s.t. \quad \mathrm{u_2 Y_{Fj}} - \mathrm{v_2 X_{Fj}} - \mathrm{wZ_j} \ = \ 0, \qquad j = 1,2,\ldots,J, \\ & \qquad \mathrm{v_2 X_{F0} + \ wZ_0 + u_2 Y_{F0}} = 1, \\ & \qquad \mathrm{u_2} \geq 0, \quad \mathrm{v_2} \geq 0, \quad \mathrm{w} \geq 0. \end{aligned} \tag{9}$$

In the hierarchical structure, the follower at the lower level operates independently based on assigned input resources, while the leader's efficiency is assessed relative to the optimal performance of the follower under their supervision. Therefore, in terms of mathematical expression, the optimal solution of the upper level model should be feasible for both the upper and lower levels, and the lower level efficiency score should remain at $\mathrm{TE^*_{F0}}$, which indicates the optimal solution of the lower level problem (9). As a result, the proposed DEA model for evaluating the TE of the leader at the upper level can be stated as follows:

$$\begin{aligned} & TE_{L0} = Max \ \ \frac{2\mathrm{u_1 Y_{L0}} \ + 2\mathrm{wZ_0}}{\mathrm{v_1 X_{L0}} \ + \mathrm{u_1 Y_{L0}} + \ \mathrm{wZ_0}} \\ & s.t. \qquad \frac{2\mathrm{u_1 Y_{Lj}} \ + \ 2\mathrm{wZ_j}}{\mathrm{v_1 X_{Lj} + u_1 Y_{Lj}} \ + \ \mathrm{wZ_j}} \leq 1,, \qquad \mathrm{j} = 1,2,\ldots,\mathrm{J}, \\ & \qquad \qquad \frac{2\mathrm{u_2 Y_{Fj}}}{\mathrm{v_2 X_{Fj} + wZ_j + u_2 Y_{Fj}}} \leq 1, \qquad j = 1,2,\ldots,J, \\ & \qquad \qquad \frac{2\mathrm{u_2 Y_{F0}}}{\mathrm{v_2 X_{F0} + \ wZ_0 + u_2 Y_{F0}}} = \mathrm{TE^*_{F0}}, \\ & \qquad \qquad \mathrm{u_1} \geq 0, \quad \mathrm{v_1} \geq 0, \quad \mathrm{u_2} \geq 0, \quad \mathrm{v_2} \geq 0, \quad \mathrm{w} \geq 0. \end{aligned} \tag{10}$$

According to its first constraint ($\frac{2u_1Y_{Lj} + 2wZ_j}{v_1X_{Lj}+u_1Y_{Lj} + wZ_j} \leq 1, \ j=1,2,\ldots,J$), the objective function of the above model is constrained between zero and one. Additionally, if ($u_2, \ v_2, w$) is the optimal solution of the lower level problem, then ($\frac{Y_{F0}}{Y_{L0}} u_2, \frac{X_{Fj}}{X_{L0}} v_2, w$) serves as a feasible solution for the higher-level model. Therefore, the upper level model is feasible and has a finite optimal solution. Similarly, the equivalent linear form of model (10) can be expressed as follows:

$$\begin{aligned}
TE_{L0} = Max \ \ & 2u_1Y_{L0} + 2wZ_0 \\
s.t. \quad & u_1Y_{Lj} + wZ_j - v_1X_{Lj} \leq 0, \qquad j=1,2,\ldots,J, \\
& u_2Y_{Fj} - v_2X_{Fj} - wZ_j \leq 0, \qquad j=1,2,\ldots,J, \\
& v_1X_{L0} + u_1Y_{L0} + wZ_0 = 1, \\
& 2u_2Y_{F0} - TE^*_{F0}(v_2X_{F0} + wZ_0 + u_2Y_{F0}) = 0, \\
& u_1 \geq 0, \quad v_1 \geq 0, \quad u_2 \geq 0, \quad v_2 \geq 0, \quad w \geq 0.
\end{aligned} \tag{11}$$

After solving the model (11) and obtaining the upper level TE (TE_{L0}), we can determine the overall TE of bi-level unit DMU_0.

The overall TE of the DMU_0, $0 \epsilon \{1,2,\ldots,J\}$, can be expressed as follows:

$$\begin{aligned}
TE_0 &= \frac{2u_1Y_{L0}+2u_2Y_{F0}+2wZ_0}{u_1Y_{L0} + u_2Y_{F0}+v_1X_{L0} + v_2X_{F0}+2 wZ_0} \\
&= \frac{v_1X_{L0} + u_1Y_{L0} + wZ_0}{u_1Y_{L0} + u_2Y_{F0}+v_1X_{L0} + v_2X_{F0}+2 wZ_0} \times \frac{2u_1Y_{L0}+2wZ_0}{v_1X_{L0} + u_1Y_{L0} + wZ_0} \\
&+ \frac{v_2X_{F0} + wZ_0 + u_2Y_{F0}}{u_1Y_{L0} + u_2Y_{F0}+v_1X_{L0} + v_2X_{F0}+2 wZ_0} \times \frac{2u_2Y_{F0}}{v_2X_{F0} + wZ_0 + u_2Y_{F0}} \\
&= \frac{1}{1+t}TE_{L0} + \frac{t}{1+t}TE_{F0},
\end{aligned} \tag{12}$$

where $t = u_2^*Y_{F0} + v_2^*X_{F0} + w^*Z_0$ is obtained from solving model (11). Thus, the TE of the whole unit is computed as the weighted average of the upper and lower levels' technical efficiencies. A unit is deemed technically efficient if $TE_0 = 1$.

A notable advantage of our proposed approach is that if both the upper and lower levels are efficient, the entire unit will also be efficient. This feature distinguishes our method from previous research suggestions. Another significant advantage of our model is that we calculate the efficiencies of the upper and lower levels, considering their hierarchical relationship and internal interactions. Subsequently, we evaluate the overall efficiency of the bi-level system by integrating the impacts of both levels' performances. Furthermore, our model is adaptable to multi-level structures and can be customized for scenarios where units exhibit variable returns to scale (VRS).

2.2 Proposed PE model for bi-level units

The lower level PE model based on the directional distance function for DMU_0, $0 \in \{\ 1,\ 2,\ \ldots,\ \mathrm{J}\ \}$ can be represented as follows:

$$
\begin{aligned}
PE_{F0} = Max \ \ & \varphi_{\mathrm{F}} \\
s.t. \qquad & \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{X}_{\mathrm{Fj}} \leq (1-\varphi_{\mathrm{F}}) \mathrm{X}_{\mathrm{F0}}, \\
& \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Z}_{\mathrm{j}} \leq (1-\varphi_{\mathrm{F}}) \mathrm{Z}_0, \\
& \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Y}_{\mathrm{Fj}} \geq (1+\varphi_{\mathrm{F}}) \mathrm{Y}_{\mathrm{F0}}, \\
& \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} = 1, \\
& \lambda_{\mathrm{j}} \geq 0, \qquad j = 1, 2, \ldots, J,
\end{aligned} \tag{13}
$$

where φ_{F} is the measure of pure inefficiency of the DMU_0's follower. In fact, the follower of DMU_0 is purely efficient if and only if $\varphi_{\mathrm{F}}{}^{*} = 0$.

Now, let $\mathrm{t} = 1-\varphi_{\mathrm{F}}$, the above model can be converted into the following model:

$$
\begin{aligned}
PE_{F0} = Min \ \ & \mathrm{t} \\
s.t. \qquad & \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{X}_{\mathrm{Fj}} \leq \mathrm{t} \mathrm{X}_{\mathrm{F0}}, \\
& \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Z}_{\mathrm{j}} \leq \mathrm{t} \mathrm{Z}_0, \\
& \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} \mathrm{Y}_{\mathrm{Fj}} \geq (2-\mathrm{t}) \mathrm{Y}_{\mathrm{F0}}, \\
& \textstyle\sum_{\mathrm{j}=1}^{\mathrm{J}} \lambda_{\mathrm{j}} = 1, \\
& \lambda_{\mathrm{j}} \geq 0, \qquad j = 1, 2, \ldots, J,
\end{aligned} \tag{14}
$$

where t is the PE score of the follower in DMU_0. The dual model of (14) is the following model:

$$
\begin{aligned}
PE_{F0} = Max \ \ & 2\mathrm{u}_2 \mathrm{Y}_{\mathrm{F0}} - \mathrm{u}_{02} \\
s.t. \qquad & \mathrm{u}_2 \mathrm{Y}_{\mathrm{Fj}} - \mathrm{v}_2 \mathrm{X}_{\mathrm{Fj}} - \mathrm{w} \mathrm{Z}_{\mathrm{j}} - \mathrm{u}_{02} \leq 0, \qquad j = 1, 2, \ldots, J, \\
& \mathrm{v}_2 \mathrm{X}_{\mathrm{F0}} + \mathrm{w} \mathrm{Z}_0 + \mathrm{u}_2 \mathrm{Y}_{\mathrm{F0}} = 1, \\
& \mathrm{u}_2 \geq 0, \quad \mathrm{v}_2 \geq 0, \quad \mathrm{w} \geq 0,
\end{aligned} \tag{15}
$$

where u_{02} is free in sign variable corresponding to the lower level problem. It is evident that model (15) is derived from the conversion of the following fractional model:

$$
\begin{aligned}
PE_{F0} = Max \ \ & \tfrac{2\mathrm{u}_2 \mathrm{Y}_{\mathrm{F0}} - \mathrm{u}_{02}}{\mathrm{v}_2 \mathrm{X}_{\mathrm{F0}} + \mathrm{w} \mathrm{Z}_0 + \mathrm{u}_2 \mathrm{Y}_{\mathrm{F0}}} \\
s.t. \qquad & \tfrac{2\mathrm{u}_2 \mathrm{Y}_{\mathrm{Fj}} - \mathrm{u}_{02}}{\mathrm{v}_2 \mathrm{X}_{\mathrm{Fj}} + \mathrm{w} \mathrm{Z}_{\mathrm{j}} + \mathrm{u}_2 \mathrm{Y}_{\mathrm{Fj}}} \leq 1, \quad j = 1, 2, \ldots, J, \\
& \mathrm{u}_2 \geq 0, \quad \mathrm{v}_2 \geq 0, \quad \mathrm{w} \geq 0, \\
& \mathrm{u}_{02} \ \text{ free in sign.}
\end{aligned} \tag{16}
$$

In a similar manner, the optimal solution of the upper level model should be feasible for both the upper and lower levels, and the lower level efficiency score should remain at $\mathrm{PE}_{\mathrm{F0}}^{*}$, which indicates the optimal solution of the lower level problem (15). So, the upper level PE of the DMU_0, $0 \in \{1,\ 2,\ \ldots,\ \mathrm{J}\}$ can be represented as follows:

$$
\begin{aligned}
PE_{L0} = Max \quad & \frac{2\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + 2\mathrm{wZ}_0 - \mathrm{u}_{01}}{\mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{wZ}_0} \\
s.t. \quad & \frac{2\mathrm{u}_1\mathrm{Y}_{\mathrm{Lj}} + 2\mathrm{wZ}_{\mathrm{j}} - \mathrm{u}_{01}}{\mathrm{v}_1\mathrm{X}_{\mathrm{Lj}} + \mathrm{u}_1\mathrm{Y}_{\mathrm{Lj}} + \mathrm{wZ}_{\mathrm{j}}} \le 1, \qquad \mathrm{j} = 1, 2, \ldots, \mathrm{J}, \\
& \frac{2\mathrm{u}_2\mathrm{Y}_{\mathrm{Fj}} - \mathrm{u}_{02}}{\mathrm{v}_2\mathrm{X}_{\mathrm{Fj}} + \mathrm{wZ}_{\mathrm{j}} + \mathrm{u}_2\mathrm{Y}_{\mathrm{Fj}}} \le 1, \qquad j = 1, 2, \ldots, J, \\
& \frac{2\mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} - \mathrm{u}_{02}}{\mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + \mathrm{wZ}_0 + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}}} = \mathrm{PE}_{\mathrm{F0}}^{*}, \\
& \mathrm{u}_1 \ge 0, \quad \mathrm{v}_1 \ge 0, \quad \mathrm{u}_2 \ge 0, \quad \mathrm{v}_2 \ge 0, \quad \mathrm{w} \ge 0,
\end{aligned}
\tag{17}
$$

where u_{01} is free in sign variable corresponding to the upper level problem. The equivalent linear reformulation of the model (17) can be written as follows:

$$
\begin{aligned}
PE_{L0} = Max \quad & 2\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + 2\mathrm{wZ}_0 - \mathrm{u}_{01} \\
s.t. \quad & \mathrm{u}_1\mathrm{Y}_{\mathrm{Lj}} + \mathrm{wZ}_{\mathrm{j}} - \mathrm{v}_1\mathrm{X}_{\mathrm{Lj}} - \mathrm{u}_{01} \le 0, \qquad j = 1, 2, \ldots, J, \\
& \mathrm{u}_2\mathrm{Y}_{\mathrm{Fj}} - \mathrm{v}_2\mathrm{X}_{\mathrm{Fj}} - \mathrm{wZ}_{\mathrm{j}} - \mathrm{u}_{02} \le 0, \qquad j = 1, 2, \ldots, J, \\
& \mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{wZ}_0 = 1, \\
& 2\mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} - \mathrm{u}_{02} - \mathrm{PE}_{\mathrm{F0}}^{*}(\mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + \mathrm{wZ}_0 + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}}) = 0, \\
& \mathrm{u}_1 \ge 0, \quad \mathrm{v}_1 \ge 0, \quad \mathrm{u}_2 \ge 0, \quad \mathrm{v}_2 \ge 0, \quad \mathrm{w} \ge 0.
\end{aligned}
\tag{18}
$$

After solving the model (18) and obtaining the upper level PE ($\mathrm{PE}_{\mathrm{L0}}$), we can determine the overall PE of bi-level unit DMU_0. The overall PE of the DMU_0, $0 \epsilon \{1, 2, \ldots, \mathrm{J}\}$, can be expressed as follows:

$$
\begin{aligned}
PE_0 &= \frac{2\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + 2\mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} + 2\mathrm{wZ}_0 - \mathrm{u}_{01} - \mathrm{u}_{02}}{\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} + \mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + 2\mathrm{wZ}_0} \\
&= \frac{\mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{wZ}_0}{\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} + \mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + 2\mathrm{wZ}_0} \times \frac{2\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + 2\mathrm{wZ}_0 - \mathrm{u}_{01}}{\mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{wZ}_0} \\
&\quad + \frac{\mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + \mathrm{wZ}_0 + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}}}{\mathrm{u}_1\mathrm{Y}_{\mathrm{L0}} + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} + \mathrm{v}_1\mathrm{X}_{\mathrm{L0}} + \mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + 2\mathrm{wZ}_0} \times \frac{2\mathrm{u}_2\mathrm{Y}_{\mathrm{F0}} - \mathrm{u}_{02}}{\mathrm{v}_2\mathrm{X}_{\mathrm{F0}} + \mathrm{wZ}_0 + \mathrm{u}_2\mathrm{Y}_{\mathrm{F0}}} \\
&= \frac{1}{1+\mathrm{t}}\mathrm{PE}_{\mathrm{L0}} + \frac{\mathrm{t}}{1+\mathrm{t}}\mathrm{PE}_{\mathrm{F0}},
\end{aligned}
\tag{19}
$$

where $t = u_2^{*}Y_{F0}+v_2^{*}X_{F0}+ w^{*}Z_0$ is obtained from solving model (18). So, the overall PE is computed as the weighted average of the upper and lower levels' PEs. A unit is deemed PE if $PE_0 = 1$. If both the upper and lower levels are pure efficient, then the whole unit will also be pure efficient. Our suggested model can be further developed to accommodate different interactions that may occur between upper and lower levels.

3 Empirical examples and discussion

Case study 1. A case study of an Iranian commercial bank

The banking industry holds a critical position within any economy, fulfilling essential functions such as deposit acceptance, transaction intermediation, payment facilitation, and credit allocation. Moreover, banks play a pivotal role in maintaining economic stability through the implementation of monetary policies. Hence, ensuring the vitality and efficiency of the banking system is paramount, as poor performance can potentially trigger financial and economic crises. To avert such scenarios, bank managers must consistently assess the performance of their branches and devise strategies for improvement. Given the broad spectrum of banking operations across diverse economic sectors, it becomes crucial to comprehensively evaluate all dimensions of their performance.

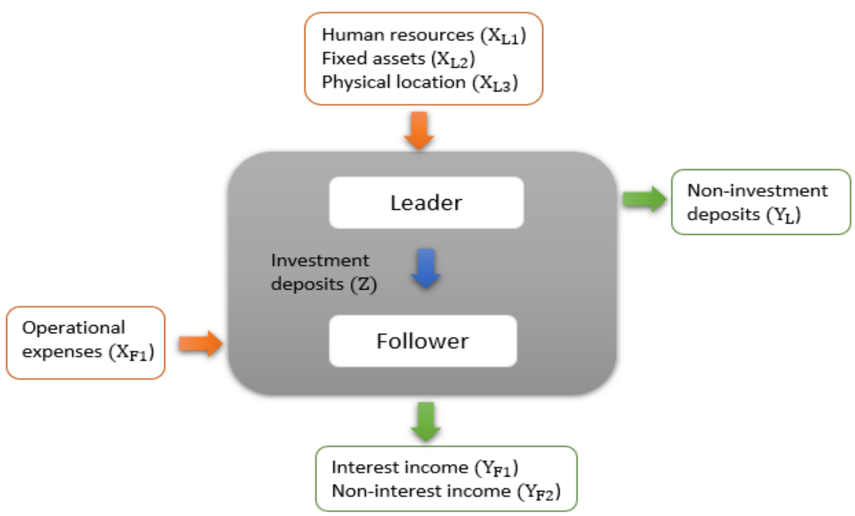

Figure 2: Bi-level structure in the case study of Iranian commercial bank.

In the realm of banking industry performance analysis, two primary perspectives define inputs: The production perspective and the intermediation perspective. From the production perspective, banks are viewed as producers utilizing human and physical resources to generate various deposit accounts. Consequently, deposits are considered outputs within this framework. Conversely, the intermediation perspective portrays banks as facilitators of financial services. Here, banks gather deposits from customers and utilize operational expenses (e.g., deposit interest costs and other operational expenditures) to transform these funds into loans and services, thereby earning interest and non-interest income.

Recognizing the multifaceted activities of banks in both realms, managers often seek to evaluate branch performance by considering multiple facets simultaneously. This article adopts a bi-level framework that integrates both the production and intermediation approaches. In this study, we conceptualize the bank as a bi-level unit, functioning as a producer at the upper level and an intermediary at the lower level, as depicted in Figure 2. We apply this framework to assess the performance of 24 branches of a commercial bank in Iran within this bi-level structure.

Table 2: Data sets of the leaders and followers.

Branch	x_{L1} (Person)	x_{L2} (1 billion Rials)	x_{L3} (m∧2)	z_1 (10 billion Rials)	y_{L1} (100 million Rials)	x_{F1} (100 million Rials)	y_{F1} (10 billion Rials)	y_{F1} (10 billion Rials)
1	2.50	1.87	814.00	49.20	30.94	3.61	15.25	13.67
2	3.75	2.12	244.20	11.81	31.25	3.33	13.27	3.28
3	4.25	2.91	350.02	12.79	30.31	2.37	13.42	3.55
4	2.75	1.82	179.08	8.36	30.94	3.67	12.96	2.32
5	6.50	1.48	284.90	10.33	30.63	4.06	13.12	2.87
6	6.75	1.43	268.62	11.81	30.00	4.00	12.81	3.42
7	10.50	2.47	195.36	19.19	16.56	2.88	10.52	5.06
8	19.00	4.68	431.42	27.55	10.31	0.28	9.61	7.38
9	24.25	4.58	179.08	19.68	23.75	0.45	14.18	5.19
10	16.00	4.93	195.36	15.74	4.06	0.17	10.07	3.96
11	4.25	1.38	211.64	10.82	30.94	4.23	13.12	3.01
12	21.00	4.24	219.78	25.58	23.44	0.79	14.95	6.84
13	25.00	4.73	203.50	23.62	0.94	0.28	9.30	6.01
14	3.50	1.58	179.08	11.32	30.63	3.95	13.12	3.14
15	8.00	1.68	187.22	11.32	25.63	3.84	12.20	3.14
16	1.50	0.69	81.40	4.43	30.94	5.02	12.81	1.23
17	9.50	2.56	284.90	22.14	30.31	2.82	14.03	6.15
18	2.75	1.82	179.08	7.87	30.63	3.72	12.81	2.32
19	3.00	1.23	170.94	13.28	28.44	5.64	13.12	3.69
20	6.00	1.73	252.34	9.84	29.69	3.95	12.51	3.42
21	7.25	1.73	260.48	12.30	29.69	4.12	12.96	3.55
22	10.50	1.97	195.36	16.24	15.94	3.10	9.76	4.51
23	21.50	4.54	488.40	24.60	11.25	0.45	9.30	6.97
24	23.00	4.09	170.94	20.66	21.88	0.34	13.73	4.51

The leader's inputs at the production level include human resources (x_{L1}), fixed assets (x_{L2}), and physical locations (x_{L3}). The leader's outputs consist of investment deposits (z_1) as an intermediary output from leader to follower and non-investment deposits (y_{L1}) as an independent output. Investment deposits are the output for the production level and serve as input for the intermediation level. At the intermediary level, the inputs include invested deposits (z_1) and operational expenses (x_{F1}). The outputs include interest income (y_{F1}) and non-interest income (y_{F2}). Table 2 presents the datasets of the input and output indicators for the leaders and followers.

If the black box approach is utilized, then the bank's bi-level structure is disregarded. Figure 3 displays the black box structure for our scenario. Essentially, we will have a DMU with four inputs and three outputs. To assess the bank's performance in this manner, four inputs (human resources, fixed assets, physical location, and operational expenses) are employed to generate three outputs (non-investment deposits, interest income, and non-interest income).

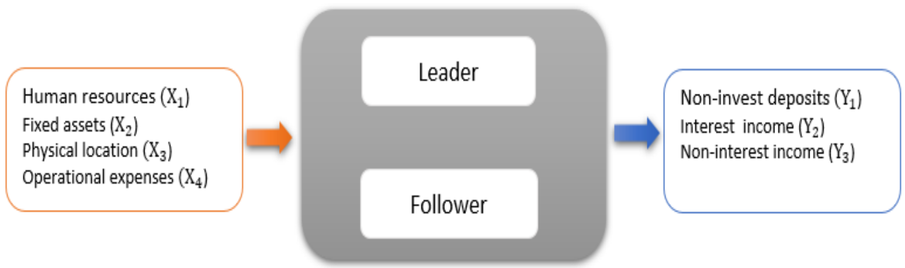

Figure 3: Black-box structure in the case study of an Iranian commercial bank.

The results of TE scores for the selected branches using both black-box and proposed models are presented in Table 3. Additionally, the upper level TE (TE_L) and lower level TE (TE_F) scores for the selected branches using the suggested model are also displayed in Table 3.

Table 4 displays the PE calculation results for all chosen branches using both black-box and suggested models. In Table 4, PEC_L and PEC_F represent the upper level and lower level PE scores using our model, respectively.

The following results were derived from the examination of the tables and the conduct of this study:

Increasing discrimination power in TE assessment:

The findings demonstrate that while the conventional model identifies 15 units as technically efficient, our model designates only 1 unit as efficient. Specifically, the black-box model categorizes regions into 10 groups, whereas the proposed model divides them into 24 categories.

Table 3: TE results of the black-box and proposed models.

DMU	Black-Box TE	Ranking	Proposed TE	TE_L	TE_F	Ranking
1	1	1	0.987	1	0.974	4
2	1	1	0.884	0.814	0.968	18
3	1	1	0.843	0.709	1	22
4	1	1	0.931	0.841	0.997	12
5	0.995	3	0.835	0.750	0.950	23
6	0.994	4	0.871	0.814	0.943	19
7	1	1	0.964	1	0.927	7
8	1	1	0.905	0.821	1	15
9	1	1	0.982	0.967	1	5
10	1	1	0.920	0.844	1	14
11	0.996	2	0.895	0.857	0.938	16
12	1	1	0.994	1	0.988	3
13	1	1	0.980	0.985	0.975	6
14	1	1	0.931	0.917	0.945	11
15	0.945	10	0.893	0.859	0.931	17
16	1	1	1	1	1	1
17	1	1	0.961	0.957	0.964	8
18	0.993	5	0.920	0.819	1	13
19	1	1	0.947	1	0.889	10
20	0.963	7	0.835	0.725	1	24
21	0.954	9	0.856	0.791	0.937	20
22	0.986	6	0.954	0.966	0.942	9
23	0.959	8	0.851	0.739	1	21
24	1	1	0.995	1	0.991	2

This results in a %41 improvement in the efficiency separation of units, highlighting the superior discrimination power of our model compared to the traditional approach.

Increasing discrimination power in PE assessment:

Our proposed model shows a significant increase in the discrimination of PEs when compared to the black-box model. As shown in Table 3, while the black-box model, assuming VRS, groups regions into 8 categories, our model classifies them into 20 categories. This enhanced discrimination offers a more nuanced perspective on unit rankings, providing greater differentiation between units. Thus, our model delivers a more precise evaluation of unit performance than conventional methods.

Increasing discrimination compared to traditional approaches:

For units deemed inefficient by both methods, the differences in their efficiency scores and rankings highlight the fundamental differences between the black-box and bi-level approaches. This comparison illustrates how evaluating units by considering their various functional aspects simultaneously produces different results than conventional methods, which fail to account for interactions and internal structures. A key factor contributing to this enhanced discrimination is the integration of the leader-follower relationship inherent in bi-level structures. Addition-

Table 4: PE results of the black-box and proposed models.

DMU	Black-Box PE	Ranking	Proposed PE	PE_L	PE_F	Ranking
1	1	1	1	1	1	1
2	1	1	1	1	0.993	1
3	1	1	0.970	0.969	1	11
4	1	1	0.994	0.994	1.000	4
5	0.996	2	0.985	0.985	0.990	8
6	1	1	0.965	0.965	0.977	13
7	1	1	0.965	1	0.931	12
8	1	1	0.925	0.857	1	17
9	1	1	0.991	0.985	1	6
10	1	1	0.924	0.866	1	18
11	0.996	3	0.996	0.996	0.988	3
12	1	1	1	1	1	1
13	1	1	0.984	0.993	0.976	10
14	1	1	0.993	0.994	0.986	5
15	0.970	6	0.893	0.870	0.942	19
16	1	1	1	1	1	1
17	1	1	0.997	1	0.993	2
18	0.995	4	0.984	0.984	1	9
19	1	1	0.990	1	0.980	7
20	0.966	7	0.951	0.951	1	16
21	0.960	8	0.952	0.952	0.983	15
22	1	1	0.956	0.967	0.944	14
23	0.980	5	0.861	0.751	1	20
24	1	1	1	1	1	1

ally, the inclusion of intermediate outputs, often overlooked in traditional models, acts as logical constraints that further improve the model's discriminative ability. Another strength of our approach is its ability to simultaneously reduce input consumption and increase output production, reinforcing its effectiveness in assessing unit efficiency.

Identify sources of inefficiency:

The results from our model offer a deeper understanding of the sources of inefficiency in branches compared to traditional models. For example, Table 3 shows that branches 20 and 23 are deemed technically inefficient by both models. However, while the black-box model does not pinpoint the source of inefficiency, our model reveals that the inefficiency stems from the first level (production). This insight suggests that managers at these branches should focus on improving performance at the upper level, reducing costs, and attracting more deposits. Furthermore, applying our model enables a detailed evaluation of bank branches' performance across both production and intermediation levels. The results indicate which branches, at the production level, could attract more deposits by reducing costs like human resource expenses, and which branches, at the intermediation level, have the potential to increase their income. Analyzing both functional levels provide actionable insights into how branches can improve by optimizing both costs and income. A key innovation of our model is that a unit is only considered efficient if both its upper and lower levels are efficient. This criterion emphasizes the need to address deficiencies at all levels to achieve optimal performance, a perspective often overlooked in previous studies.

Managerial guidelines:

Based on the findings of this case study, decision-makers can consider the following recommendations to improve branch efficiency:

Targeted allocation of input resources at upper and lower levels: Using insights from the bi-level approach, managers can allocate resources more effectively. Addressing inefficiencies at the resource absorption level can lead to optimized input usage and improved cost management. For instance, reallocating resources from less efficient branches to those that are more effective at marketing and intermediation can enhance overall operational efficiency.

Modifying strategic plans to attract more revenue: This could include revising operational processes, improving customer engagement, using new technologies to streamline financial processes, or creating programs to attract more customers and increase non-interest income. Strategic changes like these can help improve both non-interest income and overall branch performance.

In summary, these findings highlight the superiority of our approach in accurately classifying and evaluating units, uncovering inefficiencies often overlooked by traditional black-box models. The comprehensive evaluation offered by our model underscores the importance of adopting advanced DEA techniques for more accurate performance assessment in the banking sector. By enhancing discrimination and considering critical structural aspects, our approach provides a robust framework for evaluating and improving the efficiency of organizations, particularly those with hierarchical structures.

Case study 2. A case study of US Commercial Banks

To demonstrate the capabilities and practical utility of our proposed model, we apply it to analyze US commercial banks, a topic previously investigated by Liang, Cook, and Zhu [18]. Their study assesses the performance of these banks using a two-stage production process that focuses on profitability and marketability (see Figure 4). They assumed that all outputs from the first stage serve as inputs to the second stage, implying that the upper level does not generate independent outputs (i.e., $\mathrm{Y_L}$=0). Additionally, the inputs at the lower level are solely intermediate outputs of the leader, without utilizing independent inputs (i.e., $\mathrm{X_F}$=0). They introduced a non-cooperative model for two scenarios: One where the first stage acts as the leader and another where the second stage assumes leadership.

We focus on the first scenario to compare our model with theirs. Here, the first stage is the leader, with inputs including the number of employees, assets ($millions), and equity ($million), while the second stage involves three outputs: Market value ($millions), earnings per share ($),

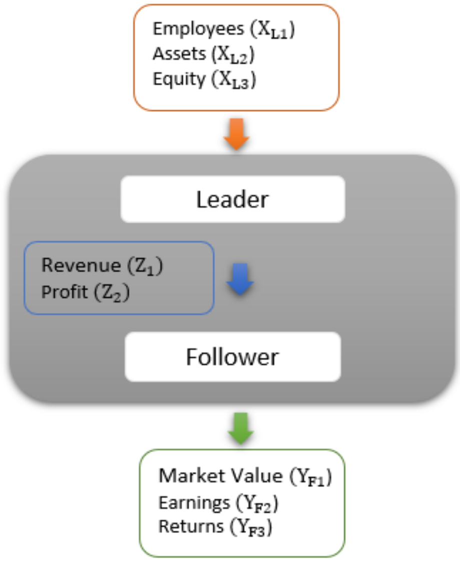

Figure 4: Bi-level structure in the case study of US commercial banks.

and returns to investors (%). Additionally, there are two intermediate measures between the stages: Profit ($millions) and revenue ($millions). The data statistics are presented in Table 5.

Table 5: Data statistics of US commercial bank.

Data	Min	Max	Ave	STD
Employees (x_{L1})	4900.00	95288.00	28945.70	20872.11
Assets (x_{L2})	13228.90	256853.00	85019.90	64686.99
Equity (x_{L3})	1265.10	20222.00	6348.16	4717.13
Revenue (z_1)	2565.40	31690.00	7865.10	6418.33
Profit (z_2)	215.00	3464.00	942.30	732.58
Market Value (y_{F1})	3218.00	33221.70	10888.15	6820.65
Earnings (y_{F2})	1.19	20.37	4.88	3.60
Returns (y_{F3})	28.30	108.50	58.37	17.35

Table 6 compares the TE results obtained from the black-box model, Liang's model, and our proposed model. It's important to note that in the conventional model, DEA scores for the overall process are determined by considering employees, assets, and equity as inputs, and market value, earnings, and returns as outputs.

Table 6 illustrates the rankings of TE scores based on our new model, the black-box model, and Liang's model, revealing non-identical rankings among them. The black-box model recognizes five units (5, 16, 24, 29, and 30) as efficient units. However, our proposed model and Liang's model identify units 16 and 30 as efficient and rank them first. Following these units, our

Table 6: TE results comparison between our model, Liang's model, and the black-box model.

Bank	Black-box TE	Ranking	Liang 's TE	Ranking	Proposed TE	TE_L	TE_F	Ranking
1	0.749	14	0.449	18	0.808	0.912	0.654	16
2	0.622	25	0.363	28	0.730	0.745	0.705	27
3	0.694	18	0.422	22	0.779	0.814	0.724	22
4	0.762	12	0.427	21	0.813	0.900	0.686	14
5	1	1	0.606	10	0.891	1	0.756	7
6	0.725	17	0.418	23	0.787	0.816	0.742	21
7	0.688	20	0.394	27	0.765	0.796	0.714	25
8	0.738	15	0.475	16	0.793	0.798	0.785	20
9	0.681	21	0.449	19	0.794	0.800	0.786	19
10	0.668	22	0.339	29	0.639	0.470	1	29
11	0.691	19	0.416	25	0.722	0.642	0.865	28
12	0.629	24	0.454	17	0.804	0.802	0.806	18
13	0.733	16	0.446	20	0.732	0.578	1	26
14	0.652	23	0.417	24	0.774	0.789	0.750	23
15	0.891	6	0.642	8	0.808	0.730	0.925	15
16	1	1	1	1	1	1	1	1
17	0.985	2	0.714	6	0.892	0.922	0.855	6
18	0.938	4	0.797	4	0.942	0.890	1	3
19	0.797	9	0.584	12	0.854	0.827	0.891	10
20	0.854	7	0.592	11	0.834	0.797	0.886	11
21	0.796	10	0.567	13	0.830	0.754	0.938	12
22	0.759	13	0.520	15	0.823	0.775	0.892	13
23	0.575	26	0.398	26	0.771	0.760	0.788	24
24	1	1	0.826	2	0.918	0.855	0.992	5
25	0.954	3	0.740	5	0.919	0.850	1	4
26	0.770	11	0.531	14	0.804	0.733	0.910	17
27	0.895	5	0.655	7	0.882	0.789	1	8
28	0.833	8	0.620	9	0.868	0.852	0.889	9
29	1	1	0.809	3	0.945	0.895	1	2
30	1	1	1	1	1	1	1	1

proposed model ranks DMUs 29, 18, and 25 as the top units, respectively, whereas Liang's model ranks units 24, 29, and 18 after units 16 and 30. Notably, significant disparities in rankings are observed for certain DMUs, including 4, 5, 8, 11, 13, 15 and 24, between our proposed model and Liang's model.

This comparison underlines the applicability of our proposed model for evaluating two-stage structures. The findings demonstrate improved separability relative to the traditional model and reveal differences in the ranking of inefficient units when compared to the Liang model. It is important to note that the Liang model is input-oriented and focuses on reducing input indicators, while our approach employs a directed distance function that simultaneously considers input reduction and output enhancement.

This feature makes our efficiency scores higher than Liang's efficiency scores because our directed function approach seeks a more accessible and appropriate set of benchmarks for inefficient

Table 7: The PE results of our proposed model and black-box model.

Bank	Black-box PE	Ranking	Proposed PE	PE_L	PE_F	Ranking
1	1	1	1	1	1	1
2	0.933	6	0.956	0.905	1	4
3	0.909	9	0.933	0.886	0.973	9
4	0.929	8	0.929	0.962	0.901	10
5	1	1	0.976	1	0.956	2
6	0.870	10	0.861	0.815	0.898	18
7	0.810	15	0.851	0.843	0.859	19
8	0.937	5	0.935	0.869	0.992	8
9	0.969	2	0.941	0.910	0.966	5
10	0.695	20	0.706	0.620	1	27
11	0.784	18	0.720	0.563	0.933	26
12	0.931	7	0.925	0.901	0.943	11
13	0.818	13	0.800	0.664	1	24
14	0.730	19	0.820	0.842	0.794	23
15	1	1	0.900	0.803	1	14
16	1	1	1	1	1	1
17	1	1	0.940	1	0.874	6
18	1	1	0.962	0.917	1	3
19	0.957	3	0.913	0.899	0.924	13
20	0.858	11	0.880	0.874	0.888	17
21	0.814	14	0.840	0.771	0.942	20
22	0.788	17	0.828	0.778	0.907	22
23	0.588	21	0.797	0.787	0.816	25
24	1	1	0.936	0.888	0.992	7
25	1	1	0.921	0.862	1	12
26	0.804	16	0.837	0.774	0.953	21
27	0.951	4	0.899	0.816	1	15
28	0.848	12	0.884	0.843	0.967	16
29	1	1	1	1	1	1
30	1	1	1	1	1	1

units. Another usefulness of our model is that it can be generalized to evaluate the performance of organizations that have undesirable outputs. By selecting an appropriate direction and identifying effective criteria, it becomes possible to assess the performance of these organizations in a more accurate and realistic manner.

Liang's model assumes CRS, whereas our proposed model can be designed for VRS. The efficiency results under the VRS assumption using our PE model and the black-box PE model are reported in Table 7.

The results reveal that the black-box PE identifies 10 units as purely efficient, while the bi-level model designates only 4 units as such. This demonstrates the enhanced resolution of our proposed model compared to the traditional model.

4 Conclusions

This paper introduced a new DEA model specifically designed to evaluate bi-level units, characterized by a leader at the upper level and followers at the lower level. By incorporating the interactions between different hierarchical levels and intermediate outputs, our model provided a detailed and comprehensive evaluation of DMU performance in hierarchical settings. It established a logical evaluation framework based on the leader-follower dynamic; an aspect previously underexplored in research. The proposed model can be easily extended for evaluating various hierarchical setups.

Another distinguishing feature of our model is its evaluation criterion based on simultaneously reducing input consumption and increasing output production. This criterion enhances the precision and reliability of performance assessments compared to traditional methods, marking a substantial advancement in achieving precise assessments within hierarchical organizations.

We applied our model to assess the efficiency of banking units from two different performance aspects and examined the effect of each aspect in evaluating the overall branch performance. The results showed that our approach provides a more accurate and meaningful assessment of branches compared to traditional models by creating more discrimination power in branch ranking. We found that conventional DEA models often fail to evaluate all functional aspects. The conventional DEA models integrate all functional aspects of a system as the sum of its initial inputs and final outputs, which leads to ignoring some intermediate activities. This oversight leads to ambiguous performance results and an inability to identify inefficiencies. Furthermore, our model demonstrated its effectiveness and comprehensiveness in evaluating units with two-stage processes. Consequently, our approach offers managers a deeper insight for decision-making within hierarchical structures. Overall, our findings represent a meaningful advancement in the field of performance evaluation, for complex hierarchical organizations.

Future research could further develop this approach by applying it across diverse industries and organizational structures. Additionally, addressing scenarios involving undesirable outputs or data uncertainty would be valuable directions for extending the model's applicability.

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Conflicts of Interest

The authors declare no conflicts of interest.

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