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Research Article

Solving Bratu equations using Bell polynomials and successive differentiation

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Abstract

This paper uses transformations and recursive algebraic equations to obtain series expansions, utilizing Bell polynomials, to solve the one-dimensional Bratu problem and several Bratu-type equations. The central aim of this work is to compare this approach with the successive differentiation method (SDM) by using computer routines for the computation of Bell polynomials. The series expansion method is applied to these nonlinear ordinary differential equations, and the various aspects of computation are compared with those obtained by the SDM. The former method is effective in handling nonlinearity, especially those arising from exponential terms, and the complexity of computations involving exponentials is handled by readily available computer routines for Bell polynomials. On the other hand, the SDM needs to handle these complexities with each differentiation.

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1 Introduction

Wazwaz [16] introduced the successive differentiation method (SDM) for solving various types of ordinary differential equations (ODEs). To obtain the series solution of an initial value problem by using the SDM, one differentiates the associated ODE and evaluates the obtained derivatives by using the initial values. In the case of boundary value problems, one further uses the boundary values to determine a series solution for the problem. In [16], SDM was used to obtain series solutions for the Bratu equation; see [3], and a variety of Bratu-type equations; see [16, 14].

The Bratu boundary value problem, which is a one-dimensional version of the classical Bratu problem, see [3, 12, 10], is given by

$$\begin{aligned} u'' + \lambda e^{u} = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0, \end{aligned} \tag{1}$$

where λ is a parameter. The Bratu problem appears in the mathematical models of certain engineering problems including the fuel ignition model of thermal combustion, and radiative heat transfer; see [9, 12, 10, 13] and the references therein.

Following [16], we are further interested in Bratu type-I, type-II, and type-III equations. These boundary value problems can be expressed as

$$\begin{aligned} u'' - \pi^2 e^{u} = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} u'' + \pi^2 e^{-u} = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} u'' - e^{u} = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0, \end{aligned} \tag{4}$$

respectively. We refer the reader to [16] for detailed discussions of these equations.

A method for finding series solutions to differential equations was used by Zhou [18] in 1986 to solve linear and non-linear initial value problems in electric circuits. In 1996, Chen and Ho [5, 6] solved eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timeshenko beam under axial loading. Many researchers have used series expansions to investigate various problems.

Several techniques have been used to solve the Bratu problem. In [15], an Adomain decomposition method (ADM) was introduced in a framework to determine the exact solutions of Bratu-type equations. In [4], the series expansions were used to solve a particular case of a Bratu-type equation. In [10], similar methods were used for the Bratu boundary value problem. The method of weighted residual was used by Aregbesola to show the existence and multiplicity of solutions to the Bratu problem [1]. A method combining the Adomain decomposition method and the Laplace transforms was used by Syam and Hamdan to solve the Bratu equation [13]. In [9], high-order compact finite difference methods were used to numerically solve one-dimensional Bratu-type equations, and the analysis of convergence and their numerical rate of convergence were given. For the results related to other methods, see [2, 11, 12, 13].

Some of these methods can be further used to obtain series solutions of certain types of differential-algebraic systems whose solution is a vector-valued function admitting an analytical expansion with respect to a real variable. In this direction, we refer the reader to [8].

The structure of the present paper is similar to that of [16] and [4]. However, we use the notations of [17]. In Section 2, we present a lemma that allows us to compute transformations of nonlinear terms using Bell polynomials. The main tool, Lemma 1, utilizes Bell polynomials for this computational purpose. The use of Faà di Bruno's formula and Bell polynomials provides a method for the series expansion of composite nonlinear functions, as seen in [7], and will be applied throughout the subsequent sections. In Subsections 2.1 and 2.2, we focus on obtaining transformations for general first-order and second-order ODEs. In section 3, we address the Bratu boundary value problem, where a computational comparison with the SDM is provided to analyze the efficiency of the proposed method. Finally, in sections 4, 5, and 6, we respectively study the Bratu type-I, type-II, and type-III equations.

2 Series expansion using bell polynomials

In this section, we follow the conventions given in [17]. A general theory that is very close to the present discussion can be found in [4].

Let $u(x)$ be an analytic function in a domain D containing zero. The analytic expansion of the function $u(x)$ about an ordinary point x_i is of the form

$$u(x) = \sum_{s=0}^{\infty} \frac{(x-x_i)^s}{s!} \left(\frac{d^s u(x)}{dx^s} \right)_{x=x_i} \tag{5}$$

for x belonging to the domain D. Following [18, 17, 4], we put

$$U(s) = \frac{\mathcal{H}^s}{s!} \left(\frac{d^s u(x)}{dx^s} \right)_{x=0} \tag{6}$$

for $s \geq 0$, where $\mathcal{H} \neq 0$ is a constant. To simplify equations, we also write subscripted U_s for $U(s)$.

As an operator, the transformation mapping an analytic function $u(x)$ to $U(s)$ is linear, but certainly not invertible on the space of real sequences. An inversion formula transforming such $U(s)$ to an analytic function $u(x)$ is given by

$$u(x) = \sum_{s=0}^{\infty} \left(\frac{x}{\mathcal{H}} \right)^s U(s), \tag{7}$$

which allows the reconstruction of the original function $u(x)$ from $U(s)$ for $x \in D$.

In the following lemma, instead of $\exp\left(u(x)\right)$, we write $e^{u(x)}$. It describes the transformation of the exponential $e^{u(x)}$ of an analytic function $u(x)$ on D, in terms of the transformation of the function $u(x)$ itself, using Bell polynomials.

Bell polynomials, denoted as $B_{s,k}(x_1, x_2, \ldots, x_{s-k+1})$, are a family of polynomials that appear in various combinatorial problems. These polynomials provide a way to express the derivative of a composite function in terms of the derivatives of the individual functions; see [7] .

Lemma 1. The transformation of $e^{u(x)}$ is given by

$$N(s) = \frac{\mathcal{H}^s e^{U_0}}{s!} \sum_{k=1}^{s} B_{s,k}\left(\frac{U_1}{\mathcal{H}}, \frac{2U_2}{\mathcal{H}^2}, \frac{6U_3}{\mathcal{H}^3}, \ldots, \frac{(s-k+1)!U_{s-k+1}}{\mathcal{H}^{s-k+1}}\right) \tag{8}$$

for $s \geq 0$, where $B_{s,k}(x_1, x_2, \ldots, x_{s-k+1})$ denotes the Bell polynomials.

The proof of Lemma (1) follows from Faà di Bruno's formula, which gives a formula for the n-th derivative of the composition of two functions. Given the fact that there are well-developed computer routines available for the computation of Bell polynomials, Lemma (1) is very useful when solving nonlinear differential equations, particularly those involving exponential terms, such as the Bratu equation and Bratu-type equations. We remark that when $s = 0$ the equality (8) appearing in Lemma 1 should be understood as $N(0) = e^{U_0}$.

In the view of Lemma 1, the first few terms of the transformation of $e^{u(x)}$ can be written in terms of U_s as

$$\begin{aligned}
&N(0) = e^{U_0},\\
&N(1) = e^{U_0}U_1,\\
&N(2) = \frac{e^{U_0}}{2!}(U_1^2 + 2U_2),\\
&N(3) = \frac{e^{U_0}}{3!}(U_1^3 + 6U_1U_2 + 6U_3),\\
&N(4) = \frac{e^{U_0}}{4!}(U_1^4 + 12U_1^2U_2 + 24U_1U_3 + 12U_2^2 + 24U_4),\\
&N(5) = \frac{e^{U_0}}{5!}(U_1^5 + 20U_1^3U_2 + 60U_1^2U_3 + 60U_1U_2^2 + 120U_1U_4 + 120U_2U_3 + 120U_5),
\end{aligned}$$

which demonstrates the nonlinear dependence of $N(s)$ on $U_0, U_1, \ldots, U_s$. Furthermore, the equations for $N(s)$ show a recursive pattern; also, see [4, Eq.11]. Each subsequent term $N(s)$ in the sequence depends on $U_0, U_1, \ldots, U_n$.

2.1 The first-order ODEs

Following [16], we start by investigating the first order ODE

$$u'(x) - f(x)u(x) = g(x), u(0) = \alpha_0, \tag{9}$$

where the functions $f(x)$ and $g(x)$ are analytic on a domain containing zero. Equation (9) represents a general first-order linear ODE. We denote by $U(s), F(s)$ and $G(s)$ the transformations of the functions $u(x)$, $f(x)$ and $g(x)$, respectively. The transformation of (9) can be written as

$$\frac{s+1}{\mathcal{H}}U(s+1) - \frac{1}{\mathcal{H}^s}\sum_{k=0}^{s} F(s)U(s-k) = G(s) \tag{10}$$

for $s \geq 0$ with $U(0) = \alpha_0$. This transformation converts the differential equation into an algebraic equation. Rewriting (10) in the form

$$U(s+1) = \frac{\mathcal{H}}{s+1}\left[G(s) + \frac{1}{\mathcal{H}^s}\sum_{k=0}^{s} F(k)U(s-k)\right], \tag{11}$$

we obtain a recursive formula for $U(s+1)$. All values of $U(s+1)$ for $s \geq 0$ are completely determined by $U(0) = \alpha_0$. For instance, $U(1) = \mathcal{H}G(0) + \mathcal{H}F(0)\alpha_0$.

When we compare the SDM, see [16, Sec. 2.1], with the above result we see that the required number of differentiations and derivative evaluations are asymptotically equal to each other in both methods. The SDM involves differentiating the ODE and evaluating the derivatives at $x = 0$ to obtain a series solution. In the above, we use a transformation to convert the ODE into an algebraic equation which can then be solved recursively. We note that formula (11) is expressed in a concise mathematical form. The compactness of formula (11) simplifies the implementation of the method for computational purposes, as the number of steps needed to arrive at a solution is reduced. Therefore, the compact formula (11) can be further utilized for computational purposes. In addition, it is effective in handling nonlinearity, especially those arising from exponential terms, and the complexity of computations involving exponentials is handled by readily available computer routines for Bell polynomials. On the other hand, the SDM needs to handle these complexities with each differentiation.

2.2 The second order ODEs

We investigate the second-order ODE

$$u''(x) - f(x)u'(x) - h(x)u(x) = g(x), u(0) = \alpha_0, u'(0) = \alpha_1, \tag{12}$$

where the function $h(x)$ is analytic on a domain containing zero. We denote by $H(s)$ the transformation of the function $h(x)$. Similar to the previous analysis, the transformation of (12) is the algebraic equation

$$\begin{aligned} \frac{(s+1)(s+2)}{\mathcal{H}^2}U(s+2) - \frac{1}{\mathcal{H}^{s+1}}\sum_{k=0}^{s}(s-k+1)F(k)U(s-k+1) \\ - \frac{1}{\mathcal{H}^s}\sum_{k=0}^{s}H(k)U(s-k) = G(s) \end{aligned} \tag{13}$$

for $s > 0$ with $U(0) = \alpha_0$ and $U(1) = \mathcal{H}\alpha_1$. Equation (13) can be rearranged to find a recursive formula for U_{s+2} as

$$U_{s+2} = \frac{\mathcal{H}^2}{(s+1)(s+2)}\left[G_s + \frac{1}{\mathcal{H}^s}\sum_{k=0}^{s}H_kU_{s-k} + \frac{1}{\mathcal{H}^{s+1}}\sum_{k=0}^{s}(s-k+1)F_kU_{s-k+1}\right] \tag{14}$$

for $s \geq 0$. When $s = 0$, this equality reduces to

$$U(2) = \frac{\mathcal{H}^2}{2}G(0) + \frac{\mathcal{H}}{2}H(0)^2U(0)^2,$$

and hence, $U(0)$ and $U(1)$ determine $U(s+2)$ for $s \geq 0$ in the sense that (14) allows the computation of all $U(s)$ values for $s \geq 0$ using the initial values.

A series approximation to the solution of (12) can be obtained from the inversion formula (7) after evaluating $U(s+2)$ from (14). When we compare [16, Eq. 17] with (14) we see that the required number of derivatives and derivative evaluations are asymptotically equal to each other in both methods. However, the above method uses a recursive formula and avoids repeated differentiation of the original equation, which can be an advantage. Equation (14) is compact and avoids repeated differentiation of the original equation. In addition, the complexity of computations involving exponentials is handled by readily available computer routines for Bell polynomials.

3 The Bratu boundary value problem

The Bratu boundary value problem is given as

$$\begin{aligned} u'' + \lambda e^{u} = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0 \end{aligned} \tag{15}$$

where the parameter λ is a constant. It follows from (15) and Lemma 1 that

$$U_{s+2} = \frac{-\lambda \mathcal{H}^{s+2} e^{U_0}}{(s+2)!} \sum_{k=1}^{s} B_{s,k}\left(\frac{U_1}{\mathcal{H}}, \frac{2U_2}{\mathcal{H}^2}, \frac{6U_3}{\mathcal{H}^3}, \ldots, \frac{(s-k+1)!U_{s-k+1}}{\mathcal{H}^{s-k+1}}\right) \tag{16}$$

for $s \geq 0$ together with $U(0) = 0$. Hence, the transformation yields a recursive formula for U_{s+2} that involves the parameter λ, the constant $\mathcal{H}$, and the Bell polynomials. We note that Bell polynomials help in handling the nonlinearity introduced by the exponential term e^{u}.

Now, let us compute the first few terms of $U(s)$. For $s = 0$, we have

$$U(2) = \frac{-\lambda \mathcal{H}^2 e^{U_0}}{2!} = \frac{-\lambda \mathcal{H}^2}{2!}.$$

From the previous analysis we know that the values of $U(s)$ for $s \geq 1$ depend further on $U(1)$. Hence we have

$$\begin{aligned} U_3 &= -\frac{1}{6}\mathcal{H}^2 \lambda e^{U_0} U_1 = -\frac{1}{6}\mathcal{H}^2 \lambda U_1, \\ U_4 &= -\frac{1}{24}\mathcal{H}^4 \lambda e^{U_0}\left(\frac{U_1^2}{\mathcal{H}^2} + \frac{2U_2}{\mathcal{H}^2}\right) = \frac{1}{24}\mathcal{H}^2 \lambda\left(\mathcal{H}^2 \lambda - U_1^2\right), \\ U_5 &= -\frac{1}{120}\mathcal{H}^5 \lambda e^{U_0}\left(\frac{U_1^3}{\mathcal{H}^3} + \frac{6U_2 U_1}{\mathcal{H}^3} + \frac{6U_3}{\mathcal{H}^3}\right) = \frac{1}{120}\mathcal{H}^2 \lambda\left(4\mathcal{H}^2 \lambda U_1 - U_1^3\right). \end{aligned}$$

The inversion formula (7) can be used to approximate the solution $u(x)$ of (15). To see this, we first obtain

$$u^{(s)}(0) = \frac{s!U(s)}{\mathcal{H}^s}$$

from formula (7). Let $\mathcal{H} = 1$ and $u'(0) = \alpha$; see [16, Sec. 3]. Hence, we obtain

$$\begin{aligned}
u(0) &= 0,\\
u'(0) &= \frac{U(1)}{\mathcal{H}} = \alpha,\\
u''(0) &= \frac{-\lambda\mathcal{H}^2}{2!} = \frac{-\lambda}{2!},\\
u'''(0) &= \frac{3!U(3)}{\mathcal{H}^3} = -\alpha\lambda,\\
u^{(4)}(0) &= \frac{4!U(4)}{\mathcal{H}^4} = \lambda^2 - \lambda\alpha^2,\\
u^{(5)}(0) &= \frac{4!U(4)}{\mathcal{H}^4} = 4\lambda^2\alpha - \lambda\alpha^3,
\end{aligned}$$

for the values of derivatives of $u(x)$ at $x = 0$. The resulting series approximation to $u(x)$ is

$$u(x) = \alpha x - \frac{\lambda}{2!}x^2 - \frac{\alpha\lambda}{3!}x^3 - \frac{\alpha^2\lambda - \lambda^2}{4!}x^4 - \frac{\alpha^3\lambda - 4\lambda^2\alpha}{5!}x^5 + \cdots,$$

which is equal to the series approximation given in [16, Eq. 21]. The above series approximation matches the one obtained by SDM. Therefore the obtained result is consistent with approximate solutions of the Bratu boundary value problem.

The computation of α can be found in [16]. We note that neither the SDM nor the above method obtains the value of α from the boundary condition $u(1) = 0$ in a direct manner. Furthermore, neither of these methods distinguishes the critical value of λ for which the Bratu problem has no solution. Both methods generate a series solution, but the solution may be valid for a limited range of λ, or may not converge if λ is too large.

4 The Bratu-type equation I

The Bratu-type I equation is

$$\begin{aligned} u'' - \pi^2 e^u = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned} \tag{17}$$

This equation is a variation of the standard Bratu problem, and it is one of three Bratu-type equations that have been examined in the literature.

In this case, $U(0) = 0$. For $s \geq 3$, $U(s)$ can be expressed in terms of $U(1)$, for instance,

$$\begin{aligned} &U_0 = 0, \\ &U_1 = \alpha\mathcal{H}, \\ &U_2 = \frac{\pi^2\mathcal{H}^2}{2!}, \\ &U_3 = \frac{\pi^2\mathcal{H}^2}{3!}U_1 = \frac{\pi^2\mathcal{H}^3}{3!}\alpha, \\ &U_4 = \frac{\pi^2\mathcal{H}^2}{4!}\left(U_1^2 + 2U_2\right) = \frac{\pi^2\mathcal{H}^4}{4!}(\pi^2 + \alpha^2), \\ &U_5 = \frac{\pi^2\mathcal{H}^2}{5!}\left(U_1^3 + 6U_1U_2 + 6U_3\right) = \frac{\pi^2\mathcal{H}^5}{5!}(4\pi^2 + \alpha^2)\alpha. \end{aligned} \tag{18}$$

All the remaining terms can be obtained from the formula (16). Higher-order terms $U(s)$ for $s \geq 3$ are expressed in terms of previous terms, most importantly in terms of $U(1)$. Now, let us compute the values of the derivative of $u(x)$ at zero. We have

$$\begin{aligned} u'(0) &= \frac{U_1}{\mathcal{H}} = \alpha, \\ u''(0) &= \frac{2!U_2}{\mathcal{H}^2} = \pi^2, \\ u'''(0) &= \frac{3!U_3}{\mathcal{H}^3} = \pi^2\alpha, \\ u^{(4)}(0) &= \frac{4!U_4}{\mathcal{H}^4} = \pi^2(\pi^2 + \alpha^2), \\ u^{(5)}(0) &= \frac{5!U_5}{\mathcal{H}^5} = \pi^2(4\pi^2 + \alpha^2)\alpha, \end{aligned}$$

all of which follow from (18). These derivatives are consistent with those derived using the SDM. Hence, the series approximation of $u(x)$ is given by

$$u(x) = \alpha x + \frac{\pi^2}{2!}x^2 + \frac{\pi^2\alpha}{3!}x^3 + \frac{\pi^2(\pi^2+\alpha)}{4!}x^4 + \frac{\pi^2(4\alpha\pi^2+\alpha^3)}{5!}x^5 + \cdots ,$$

which is equal to the series approximation obtained in [16, Eq. 27]. We note that the computation of α can be found in [16]. The value of α which is equal to $u'(0)$, is not directly computed by either of the methods but can be found by applying the boundary condition $u(1) = 0$ to the series approximation.

5 The Bratu-type equation II

The Bratu-type II equation is given by

$$\begin{aligned} u'' + \pi^2 e^{-u} &= 0 , \qquad 0 < x < 1, \\ u(0) = u(1) &= 0. \end{aligned} \tag{19}$$

Observe that $-u(x)$ appears as an exponent. We can still use Lemma 1 to find the transformation of $e^{-u(x)}$. Indeed, let $w(x) = e^{-u(x)}$ and denote by $W(s)$ the transformation of $w(x)$. It follows that

$$W(s) = \frac{\mathcal{H}^s e^{-U_0}}{s!} \sum_{k=1}^{s} B_{s,k}\left(-\frac{U_1}{\mathcal{H}}, -\frac{2U_2}{\mathcal{H}^2}, -\frac{6U_3}{\mathcal{H}^3}, \ldots, -\frac{(s-k+1)!U_{s-k+1}}{\mathcal{H}^{s-k+1}}\right), \tag{20}$$

which is expressed in terms of the values of U_s. In other words, we can obtain the transformation of $e^{-u(x)}$ from the formula given in Lemma 1 by replacing every occurrence of U_s with its negative. It follows that for (19) we have

$$U_{s+2} = \frac{-\pi^2\mathcal{H}^{s+2}e^{-U_0}}{(s+2)!} \sum_{k=1}^{s} B_{s,k}\left(-\frac{U_1}{\mathcal{H}}, -\frac{2U_2}{\mathcal{H}^2}, -\frac{6U_3}{\mathcal{H}^3}, \ldots, -\frac{(s-k+1)!U_{s-k+1}}{\mathcal{H}^{s-k+1}}\right) \tag{21}$$

for $s \geq 0$.

At this point, an informal remark regarding the sensitivity to changes in the ODE is being addressed. When we compare with the SDM of [16], we may conclude that the above method has the disadvantage that the transformation of an ODE may change drastically under slight changes of the ODE whereas SDM of [16] provides a uniform scheme in such situations. By the uniform scheme, we mean that SDM's approach to solving ODEs is more consistent and less prone to drastic changes in the solution process when the ODE is altered. However, if s is large enough, the computer routines related to Bell polynomials may balance the computation.

It follows from (21) that

$$\begin{aligned}
U_0 &= 0,\\
U_1 &= \alpha\mathcal{H},\\
U_2 &= \frac{-\pi^2\mathcal{H}^2}{2!},\\
U_3 &= \frac{\pi^2\mathcal{H}^2}{3!}U_1 = \frac{\pi^2\mathcal{H}^3}{3!}\alpha,\\
U_4 &= -\frac{\pi^2\mathcal{H}^2}{4!}\left(U_1^2 - 2U_2\right) = -\frac{\pi^2\mathcal{H}^4}{4!}(\pi^2+\alpha^2),\\
U_5 &= \frac{\pi^2\mathcal{H}^2}{5!}\left(U_1^3 - 6U_1U_2 + 6U_3\right) = \frac{\pi^2\mathcal{H}^5}{5!}(4\pi^2+\alpha^2)\alpha.
\end{aligned}$$

By using these, we obtain the values of the derivative of $u(x)$ at zero. In detail, we have

$$\begin{aligned}
u'(0) &= \frac{U_1}{\mathcal{H}} = \alpha,\\
u''(0) &= \frac{2!U_2}{\mathcal{H}^2} = -\pi^2,\\
u'''(0) &= \frac{3!U_3}{\mathcal{H}^3} = \pi^2\alpha,\\
u^{(4)}(0) &= \frac{4!U_4}{\mathcal{H}^4} = -\pi^2(\pi^2+\alpha^2),\\
u^{(5)}(0) &= \frac{5!U_5}{\mathcal{H}^5} = \pi^2(4\pi^2+\alpha^2)\alpha.
\end{aligned}$$

Hence, the series approximation of $u(x)$ is given by

$$u(x) = \alpha x - \frac{\pi^2}{2!}x^2 + \frac{\pi^2\alpha}{3!}x^3 - \frac{\pi^2(\pi^2+\alpha)}{4!}x^4 + \frac{\pi^2(4\alpha\pi^2+\alpha^3)}{5!}x^5 + \cdots,$$

which is identical to the series approximation obtained using the series approximation obtained in [16, Eq. 34]. When we compare the SDM with the above result we see that the required number of derivatives and derivative evaluations are asymptotically equal to each other in both methods.

6 The Bratu-type equation III

The Bratu-type III equation is given by

$$\begin{aligned} u'' - e^{u} = 0\,, \qquad 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned} \tag{22}$$

Since (16) depends linearly on λ, the computation of $U(s)$ for (22) is similar to that of the Bratu type-I equation. In detail, values of $U(s)$ for the Bratu type-III equation can be obtained from (18) by multiplying both sides by $1/\pi^2$. Hence, in the present case, we have

$$\begin{aligned} U_0 &= 0, \\ U_1 &= \alpha\mathcal{H}, \\ U_2 &= \frac{\mathcal{H}^2}{2!}, \\ U_3 &= \frac{\mathcal{H}^2}{3!}U_1 = \frac{\mathcal{H}^3}{3!}\alpha, \\ U_4 &= \frac{\mathcal{H}^2}{4!}\left(U_1^2 + 2U_2\right) = \frac{\mathcal{H}^4}{4!}(1+\alpha^2), \\ U_5 &= \frac{\mathcal{H}^2}{5!}\left(U_1^3 + 6U_1U_2 + 6U_3\right) = \frac{\mathcal{H}^5}{5!}(4+\alpha^2)\alpha. \end{aligned}$$

All the remaining terms can be obtained from the formula (16) similarly. Let us compute the values of the derivative of $u(x)$ at zero. We have

$$\begin{aligned} u'(0) &= \frac{U_1}{\mathcal{H}} = \alpha, \\ u''(0) &= \frac{2!U_2}{\mathcal{H}^2} = 1, \\ u'''(0) &= \frac{3!U_3}{\mathcal{H}^3} = \alpha, \\ u^{(4)}(0) &= \frac{4!U_4}{\mathcal{H}^4} = (1+\alpha^2), \\ u^{(5)}(0) &= \frac{5!U_5}{\mathcal{H}^5} = (4+\alpha^2)\alpha. \end{aligned}$$

Hence, the series approximation of $u(x)$ is given by

$$u(x) = \alpha x + \frac{1}{2!}x^2 + \frac{\alpha}{3!}x^3 + \frac{(1+\alpha^2)}{4!}x^4 + \frac{\alpha(4+\alpha^2)}{5!}x^5 + \cdots ,$$

which is identical to the series approximation obtained using the series approximation obtained in [16, Eq. 38].

7 Conclusion

We used transformations and recursive algebraic equations to obtain the series of solutions to the Bratu problem and Bratu-type equations. In all cases, the obtained series solutions agree with the series solutions given in [16]. Because there are well-developed computer routines for the computation of Bell polynomials, it can be compared with the alternative method of successive differentiation method. We use Bell polynomials to handle the nonlinear terms and, with readily available computer routines for computing these polynomials. Neither methods do not directly compute the value of $u'(0)$ using the boundary condition $u(1) = 0$, but instead uses it to solve for the unknown value, α. The methods also do not distinguish a critical value of λ for which the Bratu problem has no solution.

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