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Research Article

A new exact solution method for bi-level linear fractional problems with multi-valued optimal reaction maps

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Abstract

In many practical applications, some problems are being modeled as bi-level programming problems where the upper and lower level objectives are linear fractional functions with polyhedral constraints. If the rational reaction set of (or the set of optimal solutions to) the lower level is not a singleton, then it is known that an optimal solution to the linear fractional bi-level programming problem may not occur at a boundary feasible extreme point. Hence, existing methods cannot solve such problems in general. In this article, a novel method is introduced to find the set of all feasible leader's variables that can induce multi-valued reaction map from

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the follower. The proposed algorithm combines the kth best procedure with a branch-and-bound method to find an exact global optimal solution for continuous optimistic bi-level linear fractional problems without assuming the lower level rational reaction map is single valued. The branching constraint is constructed depending on the coefficients of the objective function of the lower-level problem. The algorithm is shown to converge to the exact solution of the bi-level problem. The effectiveness of the algorithm is also demonstrated using some numerical examples.

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Keywords: Bi-level programming problem; Bi-level linear fractional programming problem; Multi-valued rational reaction map; kth best method; Branch-and-bound scheme.

1 Introduction

A bi-level problem is a constrained optimization problem where two optimization levels are involved, and one is considered as a constraint for the other. It models decentralized planning problems with two decision agents in two levels of hierarchy. Each decision maker is assumed to control a different set of variables, and the decisions are made sequentially according to a predefined order. The decision makers at the upper and lower levels are called, respectively, leader and follower. The leader and the follower each try to optimize their own objective functions, but the decision at one level affects the objective values and/or the choice of strategies of the other level.

Generally, a bi-level programming problem can be formulated as

$$\begin{aligned} \max_{x} \quad & F(x,y), \\ & \max_{y} \quad f(x,y), \\ & \qquad \text{s.t.} \quad (x,y) \in \Omega, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m$ is the variable vector controlled by the upper level decision maker, $y \in \mathbb{R}^n$ is the variable vector controlled by the lower level decision maker, $F, f : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are the objective functions of the leader and follower, respectively, and $\Omega \subseteq \mathbb{R}^m \times \mathbb{R}^n$ defines the common constraint region.

Let $\Omega_1 = \{x \in \mathbb{R}^m : \exists\ y \text{ such that } (x, y) \in \Omega\}$ be a projection of Ω onto the *Leader's decision space*. For a fixed choice $x \in \Omega_1$ of the leader, the follower is expected to react rationally by solving

$$\begin{aligned} &\max_{y} \quad f(x, y), \\ &\text{s.t.} \qquad y \in \Omega(x), \end{aligned} \tag{2}$$

where $\Omega(x) = \{y \in \mathbb{R}^n : (x, y) \in \Omega\}$ is the follower's feasible set for a given x, assuming that this problem has a solution. The set of optimal solution of (2) denoted by $R(x)$ is usually termed as the *rational reaction set* for the bi-level problem (1). For any decision (choice x) taken by the leader, we assume that the follower has some room to respond, that is, $R(x) \neq \emptyset$. The *inducible region*, which represents the set over which the leader may optimize his/her objective or the feasible region of the upper level decision maker, is given by $\mathcal{IR} = \{(x, y) \in \Omega : y \in R(x)\}$.

Thus, in terms of the inducible region, the bi-level problem can be equivalently [8] written as

$$\begin{aligned} &\max_{x,y} \quad F(x, y) \\ &\text{s.t.} \ \ (x, y) \in \mathcal{IR}. \end{aligned}$$

To assure the existence of the solution of bi-level problem, we may assume that the constraint set Ω is compact, and the inducible region $\mathcal{IR}$ is nonempty. When the rational reaction map $R(x)$ is not single-valued, difficulties may arise in finding a meaningful solution to the bi-level problem, and hence the problem become *not well-posed*. Various approaches have been proposed in literature to avoid this difficulty and to insure the well-posedness of the bi-level problem (see [8] and the references therein). Among the possible assumptions, the *optimistic* approach, where the leader assumes that the follower chooses a value that suits the choice of the leader, is more popular in application.

A linear fractional bi-level programming problem, which is a subclass of bi-level nonlinear problems, where the objective functions in both levels are linear fractional and the common constraint region is a polyhedron, can be given by the form:

$$\begin{aligned} \max_{x} \quad & F(x,y) = \frac{c_{11}^T x + c_{12}^T y + \alpha_{11}}{c_{21}^T x + c_{22}^T y + \alpha_{12}}, \\ & \max_{y} \quad f(x,y) = \frac{d_{11}^T x + d_{12}^T y + \alpha_{21}}{d_{21}^T x + d_{22}^T y + \alpha_{22}}, \\ & \text{s.t.} \quad A_1 x + A_2 y \le b, \\ & \qquad\qquad x, y \ge 0, \end{aligned} \tag{3}$$

where for $i, j \in \{1, 2\}$, α_{ij} are scalars, c_{ij}, d_{ij}, b are vectors, A_i's are matrices with appropriate dimensions, and with a common constraint region given by

$$\Omega = \{(x,y) : A_1 x + A_2 y \le b, x, y \ge 0\}.$$

Linear fractional bi-level programming problems appear in various areas of application, for instance in problems that optimize some efficiency measure of a system [5].

Given a feasible choice $x \in \Omega_1$ of the leader, the solution of the lower level problem:

$$\begin{aligned} \max_{y} \quad & f(x,y) = \frac{d_{11}^T x + d_{12}^T y + \alpha_{21}}{d_{21}^T x + d_{22}^T y + \alpha_{22}}, \\ \text{s.t.} \quad & y \in \Omega(x), \end{aligned} \tag{4}$$

where $\Omega(x) = \{y : A_2 y \le b - A_1 x, y \ge 0\}$, is the rational reaction set $R(x)$.

Since linear fractional problems are quasi-monotonic [5], their solutions are known to appear on a vertex of the inducible region. In terms of the inducible region, problem (3) can be equivalently written as

$$\begin{aligned} \max_{x,y} \quad & F(x,y) = \frac{c_{11}^T x + c_{12}^T y + \alpha_{11}}{c_{21}^T x + c_{22}^T y + \alpha_{12}}, \\ \text{s.t.} \quad & (x,y) \in \mathcal{IR}, \end{aligned}$$

and the relaxation for the upper level problem can be given by

$$\begin{aligned} \max_{x,y} \quad & F(x,y) \;=\; \frac{c_{11}^T x + c_{12}^T y + \alpha_{11}}{c_{21}^T x + c_{22}^T y + \alpha_{12}}, \\ \text{s.t.} \quad & (x,y) \in \Omega. \end{aligned} \tag{5}$$

Related Works – A theoretical framework for solving problem (3) was developed in [5] and is used to justify the use of the kth best algorithm to solve linear fractional bi-level problems when $R(x)$ is single-valued for each feasible x. This algorithm produces exact solution for a linear fractional bi-level programming problem. An enumerative method is further tuned in [7] by applying an upper bound filter scheme. Earlier studies [4] used parametric approach (which was introduced by [12]) to solve bi-level linear fractional programming problems.

A weighting method together with the analytic hierarchy process is used to convert the bi-level problem into a single level problem in [11] to solve a bi-level linear fractional programming problem, while Toksari [15] proposed the Taylor series approach to transform the bi-level linear fractional programming problem into equivalent linear objective functions by using first order approximation. A duality gap approach is used in [16] to transform the bi-level problem into an equivalent single-level one and used an enumerative scheme to search vertices that produce the best duality gap.

Vertex search methods, like the kth best solution approach in [5], upper bound filter scheme in [7], and the enumerative scheme used in [16], search over the vertex of the constraint region Ω, with the assumption that the set $R(x)$ is single-valued for any feasible x. In the case when $R(x)$ is a single-valued map, the set of vertices of the inducible region of the problem is shown in [5] to be the subset of the vertex set of the constraint region. However, when $R(x)$ is not single-valued the set of vertices of the inducible region is not necessarily a subset of the vertices of the polyhedral constraint region Ω, and the optimal solution for bi-level linear fractional problem does not necessarily occur at the vertices of Ω (for further discussion on this, interested readers may refer to [8].) That means, even if the optimistic approach is used, then vertex search methods cannot be applied in their usual sense unless all the vertices of the inducible region are known in advance.

Linear fractional optimization problems can also be equivalently converted to linear optimization problems by using either variable transformation approach [2, 14] or through the first order Taylor series approximation [1, 9, 13]. However, the resulting bi-level linear programming problem only locates its solutions if they are at the extreme points of the constraint region [3, 18, 17], which still fails to identify solutions that lie on the boundaries but not on the extreme points of the constraint region.

To the best knowledge of the authors, there is no exact method so far that can solve the general form of problem (3) if $R(x)$ is multi-valued for some feasible x. This is due to the fact that if $R(x)$ is multiple-valued for some feasible x, then the inducible region is not necessarily formed by the union of the faces of the polyhedral constraint region Ω as indicated in [10]. This implies that some vertices of the inducible region do not coincide with the vertices of the polyhedral region. Therefore, the methods that are reviewed above, including those described in [5, 7] cannot solve problem (3) when $R(x)$ is multi-valued for some feasible x as they miss some vertices of the inducible region that do not belong to the vertex set of the constraint region.

Contributions – The purpose of this article is to propose a procedure that can solve linear fractional bi-level problems by using the kth best solution technique together with the branch-and-bound method. A novel method is proposed in this article that helps to find the set of all feasible leader's variables that can induce multi-valued reaction map from the follower. Then, the coefficients of the objective function of the lower level problem are used to define the branching constraints, which contributes to formulation of an easily implementable solution algorithm for a general linear fractional bi-level programming problem. The proposed algorithm can also solve problems with single-valued reaction maps.

Outline – The paper is organized as follows: Section 2 provides review of some definitions and background concepts for the proposed method. Furthermore, the dependence of the actual relation between extreme points of the inducible region and extreme points of the constraint region, on the structure of the optimal solution set of the lower level are shown using examples. The proposed algorithm is presented in Section 3. Section 4 shows the effectiveness of the algorithm by giving illustrative examples. Finally, some limitations of the proposed method and their possible extensions are highlighted in the conclusion part, Section 5.

2 Background of the proposed method

Before we start the solution procedure for a bi-level linear fractional problem, let us consider the maximization form of a linear fractional problem:

$$\begin{aligned} \max_x h(x) = &\quad \frac{c_1^T x + \alpha_1}{c_2^T x + \alpha_2}, \\ &\text{s.t. } x \in S = \{Ax \leq b, x \geq 0\}. \end{aligned} \tag{6}$$

To assure existence of a solution, assume that the constraint set S is nonempty, closed, and bounded. Since the solution of a quasi-monotonic problem occurs at the extreme points of the feasible region and every linear fractional function is explicitly quasi-monotonic in its domain, the optimal solution of a linear fractional problem lies at some of the extreme points of the polyhedral constraint region [5, 6, 16]. Therefore, we search the optimal solution over extreme points of the constraint region. To do that, we start from one vertex of the constraint region, then move along a side adjacent to it such that the functional value increases. The process continues until an extreme point is obtained, where one cannot find a point at which the value of the function increases any more. The solution procedure is similar to the simplex method except for the formulation of the objective row. Since the objective function is linear fractional, it is a ratio of two linear functions. Then we can use a simplex-like method to solve the linear fractional problem by applying a few modifications as described in [2].

To formulate the appropriate modification, we consider the gradient of the objective function, which becomes

$$\bigtriangledown h = \frac{c_1(c_2^T x + \alpha_2) - c_2(c_1^T x + \alpha_1)}{(c_2 x + \alpha_2)^2}.$$

After rearranging, we get

$$\bigtriangledown h = \frac{1}{(c_2 x + \alpha_2)^2}\left(\alpha_2 c_1 - \alpha_1 c_2\right).$$

Since $\frac{1}{(c_2 x + \alpha_2)^2}$ is always positive for nonzero α_2, the sign of $\bigtriangledown h$ depends on the sign of $\alpha_2 c_1 - \alpha_1 c_2$, and hence it is usually called the reduced cost. At each iteration of the simplex method, the value of $\alpha_2 c_1 - \alpha_1 c_2$ determines the direction of increase or decrease of h. Therefore, depending on the value of the coefficient $\alpha_2 c_1 - \alpha_1 c_2$ corresponding to the nonbasic variables, we have three possibilities for the next move in solving problem (6). The first possibility is when $\alpha_2 c_1 - \alpha_1 c_2 > 0$ corresponding to some nonbasic variables. In this case, the current extreme point is not an optimal solution for problem (6). The second possibility is when $\alpha_2 c_1 - \alpha_1 c_2 < 0$ corresponding to all nonbasic variables. In this case, we cannot make any improvement on the value of h, which means the current extreme point is an optimal solution for problem (6). However, when $\alpha_2 c_1 - \alpha_1 c_2 = 0$ for some nonbasic variables while $\alpha_2 c_1 - \alpha_1 c_2 < 0$ for all other nonbasic variables, the current extreme point is an optimal solution for problem (6) and there is a possibility for another alternative optimal solution.

At a basic feasible solution x, let $z_1 = -(c_1^T x + \alpha_1)$ and $z_2 = -(c_2^T x + \alpha_2)$ be the numerator and the denominator functions, respectively, of the objective function h. Then the corresponding simplex tableau becomes like in Table 1.

Now, by using the above concept, we have the following properties.

Theorem 1. For any linear fractional problem (6), with objective function $h(x) = \frac{c_1^T x + \alpha_1}{c_2^T x + \alpha_2}$, the problem has multiple optimal solutions if and only if $(c_1\alpha_2 - c_2\alpha_1)_i = 0$ for some i, and $(c_1\alpha_2 - c_2\alpha_1)_j < 0$ for all $j \neq i$, where i and j are indices for the nonbasic variables which make the reduced cost to be zero and negative, respectively.

Proof. Let the problem have multiple optimal solutions, say x_1 and x_2, which are distinct. Then

Table 1: Simplex tableau for linear fractional problem given in (6)

h	$\alpha_2 c_1 - \alpha_1 c_2$	$\dfrac{c_1^T x + \alpha_1}{c_2^T x + \alpha_2}$
z_1	c_1^T	$-(c_1^T x + \alpha_1)$
z_2	c_2^T	$-(c_2^T x + \alpha_2)$
BV	x^T	RHS
x_B	A	b

1. the reduced costs for x_1 and x_2 satisfy $\bigtriangledown h(x_1) \leq 0$ and $\bigtriangledown h(x_2) \leq 0$, that is, $(c_1\alpha_2 - c_2\alpha_1)_k \leq 0, \forall k$, where k is the index for the nonbasic variables at the given iteration.

2. $h(x_1) = h(x_2)$,
   or equivalently,

$$(c_1^T x_1 + \alpha_1)(c_2^T x_2 + \alpha_2) = (c_1^T x_2 + \alpha_1)(c_2^T x_1 + \alpha_2).$$

   After rearranging the values in the equality, we get

$$(\alpha_1 c_2 - \alpha_2 c_1)^T (x_1 - x_2) = 0.$$

   Indeed since x_1 and x_2 are assumed to be distinct optimal solutions of the problem, $(x_1 - x_2)_i \neq 0$ for some i. Then we must have

$$(c_1\alpha_2 - c_2\alpha_1)_i = 0 \text{ for some } i,$$

   and

$$(c_1\alpha_2 - c_2\alpha_1)_j < 0 \quad \text{ for all other indices } j.$$

Conversely, let x_1 and x_2 be distinct feasible points that have different values corresponding to their ith components and the same values for each of their other components and both satisfy $(c_1\alpha_2 - c_2\alpha_1)_i = 0$ for some i and $(c_1\alpha_2 - c_2\alpha_1)_j < 0$ for all other indices j. Then since the components of the reduced cost of the problem (6) at both x_1 and x_2 are zero or negative, depending on their corresponding functional values, either x_1 or x_2 or both are optimal solutions.

Let us check which condition is satisfied. From the given conditions, we have

$$(\alpha_1 c_2 - \alpha_2 c_1)^T (x_1 - x_2) = 0.$$

Equivalently we can write it as

$$(c_1^T x_1 + \alpha_1)(c_2^T x_2 + \alpha_2) = (c_1^T x_2 + \alpha_1)(c_2^T x_1 + \alpha_2).$$

Rearranging this equations gives

$$\frac{c_1^T x_1 + \alpha_1}{c_2^T x_1 + \alpha_2} = \frac{c_1^T x_2 + \alpha_1}{c_2^T x_2 + \alpha_2}.$$

Hence $h(x_1) = h(x_2)$, which means both x_1 and x_2 are optimal solutions that the problem has at least two optimal solutions. □

When we return to the bi-level form of the problem, one searches the optimal solution over extreme points of the inducible region $\mathcal{IR}$. If $R(x)$ is single-valued, then the solution of (3) occurs at the extreme points of the constraint region Ω, because extreme points of the inducible region $\mathcal{IR}$ are also extreme points of Ω [5], but this may not be the case when $R(x)$ is a nonsingleton map for some x [10].

To see what $\mathcal{IR}$, may look like for a bi-level linear fractional problem, one may refer to Examples 1 and 2.

Example 1. Consider

$$\max_{x} \quad \frac{3x+2y}{4x+y+6},$$
$$\max_{y} \quad \frac{-5x-3y-9}{x+2y+3},$$
$$\text{s.t.}$$
$$x+y \leq 5,$$
$$x+3y \leq 10,$$
$$y \leq 3,$$
$$x, y \geq 0,$$

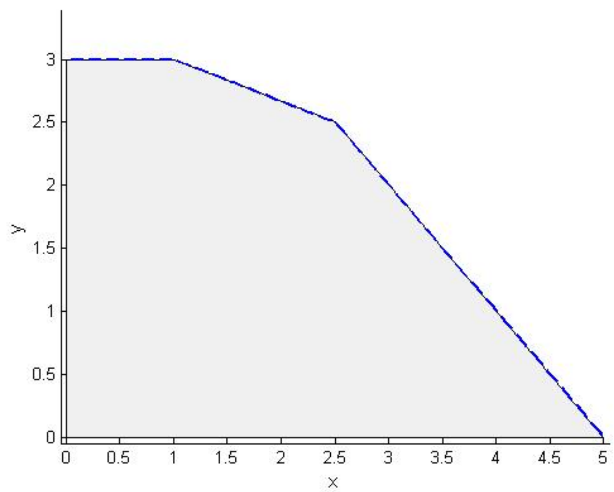

Figure 1: Constraint and inducible regions of Example 1

The common constraint region, Ω, and the inducible region $\mathcal{IR}$ of these examples are shown in Figure 1 and Figure 2, respectively. The hatched lines in these figures denote $\mathcal{IR}$. The lower-level problem of Example 1 has multiple optimal solutions corresponding to the point $x = -1.2857$, which is not part of the feasible region Ω_1. This means $R(x)$ of Example 1 is single-valued for all x in Ω_1; hence, $\mathcal{IR}$ is the union of faces of the polyhedron Ω as shown in Figure 1. That means, the set of extreme points of $\mathcal{IR}$ of Example 1 is $\{(5,0),(0,3),(1,3),(2.5,2.5)\}$, which is a subset of the set of extreme points of Ω, $\{(0,0),(5,0),(0,3),(1,3),(2.5,2.5)\}$. In this case, the optimal solution is $(1,3)$ found by using kth best or graphical method.

Example 2. Consider

$$\max_x \ \frac{3x+2y}{4x+y+6},$$
$$\max_y \ \frac{4x+3y}{4x+6y+3},$$
$$\text{s.t.}$$
$$x+y \le 5,$$
$$x+3y \le 10,$$
$$y \le 3,$$
$$x, y \ge 0,$$

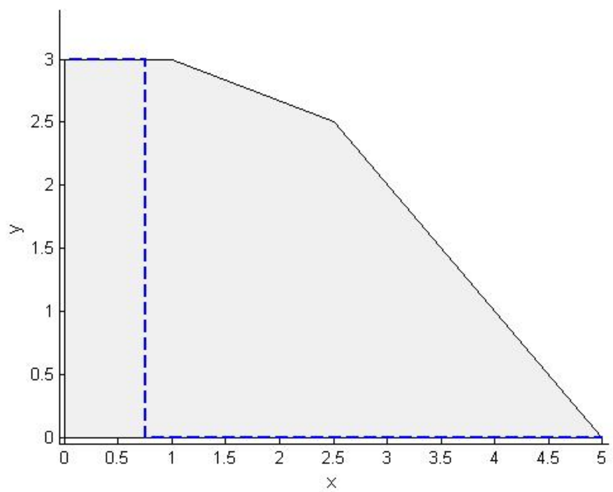

Figure 2: Constraint and inducible regions of Example 2

However, in Example 2, the lower level problem has multiple optimal solutions at $x = 0.75 \in \Omega_1$, which means $R(x)$ of Example 2 is not single-valued for at least one x in Ω_1. In this case, it can be observed that some elements of $\mathscr{IR}$ are in the interior of Ω, and $\mathscr{IR}$ is not the union of faces of the polyhedron Ω as shown in Figure 2. The set of extreme points of Ω of Example 2 is $\{(0,0),(5,0),(0,3),(1,3),(2.5,2.5)\}$ whereas the set of extreme points of $\mathscr{IR}$ of Example 2 is $\{(5,0),(0,3),(0.75,0),(0.75,3)\}$, which is not a subset of the set of extreme points of Ω. Using kth best method one can obtain $(0,3)$ as the maximum point. However, this point is not the optimal solution of the problem. The optimal solution is $(0.75, 3)$ found by inspection, and it is not part of the set of extreme points of Ω, rather, it is a boundary point of Ω.

Our main focus in this article is the case where $R(x)$ is a nonsingleton set for some x. To design a solution approach for such cases in general, we need to establish the following preliminary results.

Theorem 2. The optimal solution of the bi-level linear fractional problem (3) occurs generally at the boundary points of its constraint region.

Proof. For a fixed point x, we have $\mathcal{IR} = \{(x, y) \in \Omega : y \in R(x)\} \subseteq \{(x, y) : y \in \Omega(x)\} \subseteq \Omega$.

The intersections of the plane that contains x and the constraint set Ω is the set $\{(x, y) : y \in \Omega(x)\}$. The extreme points of the set $\{(x, y) : y \in \Omega(x)\}$ lie on the boundaries of Ω.

Since the objective functions are linear fractional, $R(x)$ is either an extreme point of $\Omega(x)$ (if it is single-valued), or it is a convex combination of some extreme points of $\Omega(x)$ (if it is multi-valued). In both cases, the set of extreme points of $R(x)$ is the subset of extreme points of $\Omega(x)$. Extreme points of $\mathcal{IR}$ are extreme points of the set $\{(x, y) \in \Omega : y \in R(x)\}$ and hence the subset of extreme points of $\{(x, y) \in \Omega : y \in \Omega(x)\}$. From these arguments, one can conclude that *extreme points* of $\mathcal{IR}$ lie on the boundaries of Ω. □

When $R(x)$ is nonsingleton, the difficulty in the use of the kth best algorithm (or any of the so far known methods, for that matter), is obtaining the extreme points of $\mathcal{IR}$, which are not part of the extreme points of Ω, but those are boundary points of Ω.

To address this difficulty, we first need to find all feasible variables of the leader that make the optimal reaction set of the follower multi-valued. The following theorem helps us to obtain those points.

Theorem 3. For a fixed $\bar{x}$ in problem (3), if

1. $(D\bar{x} + \beta)_i = 0$ for some i and $(D\bar{x} + \beta)_j \leq 0$ for all $i \neq j$, where

$$D = \begin{pmatrix} d_{12} & d_{22} \end{pmatrix} \begin{pmatrix} d_{21}^T \\ -d_{11}^T \end{pmatrix} \text{ and } \beta = \begin{pmatrix} d_{12} & d_{22} \end{pmatrix} \begin{pmatrix} \alpha_{22} \\ -\alpha_{21} \end{pmatrix},$$

2. $\bar{\alpha}_{22} = \alpha_{22} + d_{21}^T \bar{x} \neq 0$,

then the lower level problem has multiple optimal solutions.

Proof. The lower level problem (4) of (3) at a fixed point $\bar{x}$, can be rewritten as

$$\begin{aligned} & max_y \quad f(\bar{x}, y) = \frac{d_{11}^T \bar{x} + d_{12}^T y + \alpha_{21}}{d_{21}^T \bar{x} + d_{22}^T y + \alpha_{22}} = \frac{d_{12}^T y + \bar{\alpha}_{21}}{d_{22}^T y + \bar{\alpha}_{22}} \\ & \text{s.t.} \qquad y \in \Omega(\bar{x}), \end{aligned}$$

where

$$\bar{\alpha}_{21} = \alpha_{21} + d_{11}^T \bar{x} \text{ and } \bar{\alpha}_{22} = \alpha_{22} + d_{21}^T \bar{x},$$

and the problem is well defined for $\bar{\alpha}_{22} \neq 0$.

Let $(D\bar{x} + \beta)_i = 0$ for some i and $(D\bar{x} + \beta)_j \leq 0$ for all $j \neq i$.

Then

$$\begin{aligned} D\bar{x} + \beta = & \begin{pmatrix} d_{12} & d_{22} \end{pmatrix} \begin{pmatrix} d_{21}^T \\ -d_{11}^T \end{pmatrix} \bar{x} + \begin{pmatrix} d_{12} & d_{22} \end{pmatrix} \begin{pmatrix} \alpha_{22} \\ -\alpha_{21} \end{pmatrix} \\ = & \left(d_{12} d_{21}^T - d_{22} d_{11}^T \right) \bar{x} + d_{12}\alpha_{22} - d_{22}\alpha_{21} \\ = & d_{12} \left(d_{21}^T \bar{x} + \alpha_{22} \right) - d_{22} \left(d_{11}^T \bar{x} + \alpha_{21} \right) \\ = & d_{12}\bar{\alpha}_{22} - d_{22}\bar{\alpha}_{21}. \end{aligned}$$

Since $(D\bar{x} + \beta)_i = 0$, so $(d_{12}\bar{\alpha}_{22} - d_{22}\bar{\alpha}_{21})_i = 0$ for some i. Then by Theorem 1 the lower level problem (4) has multiple optimal solutions. □

In the following section, we shall formulate a solution procedure for bi-level linear fractional problems with possible multiple optimal reaction values from the lower level, based on the above preliminary results.

3 The proposed solution algorithm

It has been indicated in [5, 7] that bi-level linear fractional problems of type (3) can be solved by using the kth best (or vertex-searching) approach when the reaction set is a singleton for each feasible decision of the upper level. Since optimal solutions of problem (3) occur at the extreme points of the inducible region, the kth best solution approach cannot solve problem (3) when the rational reaction map is multi-valued for some feasible decision of the upper level. This is due to the fact that if the rational reaction map is multi-valued for some feasible points, then there are some extreme points of the inducible region, which are not part of extreme points of the constraint region, and they cannot be visited by the kth best solution approach. However, if we branch the problem at those feasible points, where the reaction map is multiple-valued, then we can make the kth best method to visit all extreme points of the inducible region. The branching constraints are formulated by using Theorem 3 and then incorporated into the relaxed problem (5).

To this end, let D have n rows. Then by Theorem 3, problem (4) has multiple solutions if $D_i x = -\beta_i$ and $D_j x \leq -\beta_j$ for all $j \neq i$. To get the branching constraint, we consider $D_i x \leq -\beta_i$ and $D_i x \geq -\beta_i$ in place of $D_i x = -\beta_i$. Therefore, for each $i \leq n$, we get two problems:

$$\begin{aligned} \max_{x,y} F(x,y) = &\ \frac{c_{11}^T x + c_{12}^T y + \alpha_{11}}{c_{21}^T x + c_{22}^T y + \alpha_{12}}, \\ \text{s.t.} & \\ & A_1 x + A_2 y \leq b, \\ & D_i x \leq -\beta_i, \\ & D_j x \leq -\beta_j, \\ & x, y \geq 0, \\ & i \neq j; \end{aligned} \tag{7}$$

and

$$\begin{aligned} \max_{x,y} F(x,y) = &\ \frac{c_{11}^T x + c_{12}^T y + \alpha_{11}}{c_{21}^T x + c_{22}^T y + \alpha_{12}}, \\ \text{s.t.} & \\ & A_1 x + A_2 y \leq b, \\ & D_i x \geq -\beta_i, \\ & D_j x \leq -\beta_j, \\ & x, y \geq 0, \\ & i \neq j. \end{aligned} \tag{8}$$

This branching procedure will result in $2 \times n$ problems in total. However, the first part (problem (7)) of the branching appears in all the cases. That means, the same problem is to be repeatedly solved in each case (n times). To avoid this repetition, we first consider

$$\begin{aligned} \max_{x,y} F(x,y) = \ & \frac{c_{11}^T x + c_{12}^T y + \alpha_{11}}{c_{21}^T x + c_{22}^T y + \alpha_{12}}, \\ \text{s.t.} & \\ & A_1 x + A_2 y \le b, \\ & Dx \le -\beta, \\ & x, y \ge 0, \end{aligned} \tag{9}$$

once, and then we solve the next branch for each i. Finally, we only have $n+1$ problems to be solved all together. In proposing the algorithm, we assume that the inducible region $\mathcal{IR}$ is nonempty and an *optimistic* version of the problem is considered.

At each iterations, first, we solve either problem (9) (in the first iteration) or (8) by using the simplex procedure. There may be a solution to each of the branched bi-level problems or not. If we have a solution, then the next step is to find a bi-level solution by using the kth best approach. Indeed the obtained solution could be infeasible, or it may have appeared in one of the previous iterations, or the objective value at this iteration may not be better than those in the other branches.

Now, let us define some sets, which are to be used in Algorithm 1. Let N denote the set of bi-level infeasible points from among the extreme points of the feasible region, let S be the set of bi-level feasible points, let E^i be the set of extreme points, which are candidates of optimal solution at the ith iteration, and let A^i be the set of adjacent extreme points of (x^i, y^i) at the ith iteration. By making each of the nonbasic variables as an entering variable in the tableau corresponding to (x^i, y^i), we obtain elements of E^i and the set A^i at each iteration i. Let LB be a lower bound of problem (3), and its value can be updated if a bi-level feasible point with a better upper level objective value is obtained.

Algorithm 1 Algorithm for bi-level linear fractional problem with possible multiple optimal responses

Step 0. $i = 0, N = \emptyset, S = \emptyset, LB = -\infty$, and n is equal to the number of rows of D.

Step 1. Solve problem (9) by using the simplex method.

- If it has no solution, then go to Step 3.
- If it has a solution (x^i, y^i), then set $E^i = \{(x^i, y^i)\}$ and go to Step 2.

Step 2. Solve the lower-level problem (4) by fixing x^i, using the simplex procedure to get $\hat{y}$.

- If $\hat{y} = y^i$, then set $LB = F(x^i, y^i)$, $(x^*, y^*) = (x^i, y^i), S = S \cup \{(x^i, y^i)\}$, and go to Step 3.
- If not, then
  - find the set of adjacent extreme points, A^i, of (x^i, y^i) and $N = N \cup \{(x^i, y^i)\}$, $E^i = (E^i \cup A^i) \setminus N$.
  - solve $\max \left\{F(x, y) : (x, y) \in E^i\right\}$ to obtain $(\bar{x}^i, \bar{y}^i)$ and set $(x^i, y^i) = (\bar{x}^i, \bar{y}^i)$.
    - If $(x^i, y^i) \in S$, then go to Step 3.
    - Otherwise, repeat Step 2 with the updated values of x^i and y^i.

Step 3. Set $i = i + 1$

- If $i \leq n$, then go to Step 4.
- If $i > n$ (all the branching options are already explored), then stop, and set the optimal solution to be (x^*, y^*).

Step 4. Solve problem (8) by using the simplex procedure.

- If it has no solution, then go to Step 3.
- If the problem has a feasible solution, then let (x^i, y^i) be the solution and
  - if $F(x^i, y^i) < LB$ or $(x^i, y^i) \in S$ or $(x^i, y^i) \in N$, then go to Step 3,
  - otherwise let $E^i = \{(x^i, y^i)\}$, and go to Step 2.

Theorem 4. The solution procedure described in Algorithm 1 terminates to the solution of problem (3) after finite iterations.

Proof. In Algorithm 1, there are at most $n+1$ iterations, where n is the dimension of the lower level decision variable vector, and at each iteration the kth best algorithm was used to solve the problem. The convergence of kth best algorithm is proved in [5]. At each iteration if the problem has a solution, then we must check whether we need to further use kth best algorithm or not by using three conditions. The first one is comparing the value of the optimal solution with LB and if it has worst value, then we do not consider it any further. The second condition is existence of the solution in the nonfeasible set N. Again if the solution is in N, then we do not consider it further. The final condition is about the occurrence of the solution in the set S. If the solution is in S, then we do not consider it further as it was already considered in the previous steps and its value was compared with LB. These three conditions remove the unwanted repetition in the algorithm. After completing the $n+1$ iterations, the point corresponding to the LB becomes the solution of (3).

Since the branching constraints make the boundary points of Ω that coincide with extreme points of $\mathcal{IR}$ to be vertices of the branched region, all the feasible extreme points of the inducible region $\mathcal{IR}$ are visited by Algorithm 1. Hence the final solution is the global optimal solution of problem (3). □

4 Illustrative examples

In order to test our proposed Algorithm, we consider some numerical examples below, some taken from literature to check the validity of the output of the algorithm, and others are newly constructed to test for the additional conditions.

Here below, we present the solution of two examples by showing all the detailed procedures to demonstrate how the steps in the proposed algorithm work.

Example 3.

$$\max_x \frac{3x+2y}{4x+y+6},$$
$$\max_y \frac{4x+3y}{4x+6y+3},$$
$$\text{s.t.}$$
$$\begin{aligned} x+y &\leq 5,\\ x+3y &\leq 10,\\ y &\leq 3,\\ x,y &\geq 0. \end{aligned}$$

This is the problem presented in Example 2 above and the procedures of the solution are presented in the following detailed steps. Note that, existing methods cannot automatically address such a problem as it has a nonsingleton reaction map.

To check existence of multi-valued reaction, we first formulate

$$D = \begin{pmatrix} 3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix} = -12, \text{ and } \beta = \begin{pmatrix} 3 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 9.$$

Then we follow the steps below.

Step 0. $i = 0, N = \emptyset, S = \emptyset, LB = -\infty$ and $n = 1$ (as D has only 1 raw).

Step 1. Solve

$$\begin{aligned} \max_x \quad & \frac{3x+2y}{4x+y+6},\\ \text{s.t.} \quad & x+y \leq 5,\\ & x+3y \leq 10,\\ & y \leq 3,\\ & -12x \leq -9,\\ & x,y \geq 0. \end{aligned}$$

- When we solve this linear fractional problem using the simplex like method, we obtain a solution $(1,3)$ with $x^0 = 1, y^0 = 3$. Then we set $E^0 = \{(1,3)\}$ and go to Step 2.

Step 2. Solve the lower level problem (4) of Example 3 by fixing $x^0 = 1$, to get $\hat{y} = 0$. Then

- $\hat{y} \neq y^0$. Hence we obtain adjacent extreme points of (x^0, y^0):
- $A^0 = \{(0.75, 3), (2.5, 2.5)\}$,
  $N = N \bigcup \{(x^0, y^0)\} = \{(1, 3)\}$, $E^0 = (E^0 \bigcup A^0) \setminus N = A^0$.
- Solve $\max \left\{ F(x, y) : (x, y) \in E^0 \right\}$, to get $(0.75, 3) \notin S$. Then update $x^0 = 0.75, y^0 = 3$ and repeat Step 2.

Solve the lower level problem (4) of Example 3 by fixing $x^0 = 0.75$, to obtain $\hat{y} = 3$.

- Since $\hat{y} = y^0$, set $LB = F(0.75, 3) = 0.6875, S = S \bigcup \{(0.75, 3)\} = \{(0.75, 3)\}, (x^*, y^*) = (0.75, 3)$ and go to Step 3.

Step 3. $i = 0 + 1 = 1$

- Since i satisfies $i \leq n$, go to Step 4.

Step 4. Solve

$$
\begin{aligned}
\max_{x} \quad & \frac{3x + 2y}{4x + y + 6}, \\
\text{s.t.} \quad & x + y \leq 5, \\
& x + 3y \leq 10, \\
& y \leq 3, \\
& -12x \geq -9, \\
& x, y \geq 0.
\end{aligned}
$$

- Then we get a solution: $(0.75, 3)$, with $x^1 = 0.75, y^1 = 3$.
  - Since $(0.75, 3) \in S$, go to Step 3.

Step 3. $i = 1 + 1 = 2$

- Since i does not satisfy $i \leq n$, Stop.

Hence the optimal solution is $(x, y) = (0.75, 3)$ with the upper level optimal value $F = 0.6875$ and the lower level optimal value $f = 0.5$.

Example 4. A newly constructed problem with nonunique reaction set.

$$\max_x \quad \frac{-x_1 + x_2 + 2y_1 - 2y_2 - y_3 - 1}{-x_1 - 2y_1 + y_2 + 5y_3 + 8},$$
$$\max_y \quad \frac{x_1 + x_2 - 2y_1 + y_2 - y_3 - 2}{2x_1 + y_1 + y_2 + 3y_3 + 1},$$
$$\text{s.t.}$$
$$\begin{aligned} -y_1 + y_2 + y_3 &\leq 1, \\ 2x_1 - y_1 + 2y_2 + 0.5y_3 &\leq 3, \\ 2x_2 + 2y_1 - y_2 + 0.5y_3 &\leq 9, \\ x_1, x_2, y_1, y_2, y_3 &\geq 0, \end{aligned}$$

with $x = (x_1, x_2), y = (y_1, y_2, y_3)$.

Solution of Example 4: In this case we have

$$D = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ -1 & 3, \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ 1 & -1 \\ -5 & -3 \end{pmatrix}, \text{ and } \beta = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}.$$

Step 0. $i = 0, N = \emptyset, S = \emptyset, LB = -\infty$, and $n = 3$ (number of rows of D).

Step 1. Solve

$$\max_{x,y} \quad \frac{-x_1 + x_2 + 2y_1 - 2y_2 - y_3 - 1}{-x_1 - 2y_1 + y_2 + 5y_3 + 8},$$
$$\text{s.t.}$$
$$\begin{aligned} -y_1 + y_2 + y_3 &\leq 1, \\ 2x_1 - y_1 + 2y_2 + 0.5y_3 &\leq 3, \\ 2x_2 + 2y_1 - y_2 + 0.5y_3 &\leq 9, \\ -5x_1 - x_2 &\leq 0, \\ x_1 - x_2 &\leq -3, \\ -5x_1 - 3x_2 &\leq -5, \\ x_1, x_2, y_1, y_2, y_3 &\geq 0. \end{aligned}$$

- After solving, we obtain a solution: $(0, 3, 1.5, 0, 0)$ with $x^0 = (0, 3), y^0 = (1.5, 0, 0)$. Then set $E^0 = \{(0, 3, 1.5, 0, 0)\}$ and go to Step 2.

Step 2. Solve the lower level problem (4) of Example 4 by fixing $x^0 = (0, 3)$, to get $\hat{y} = (1.5, 0, 0)$.

- Since $\hat{y} = y^0$, set $LB = F(0, 3, 1.5, 0, 0) = 1$, $S = S \bigcup \{(0, 3, 1.5, 0, 0)\} = \{(0, 3, 1.5, 0, 0)\}$, $(x^*, y^*) = (0, 3, 1.5, 0, 0)$ and go to Step 3.

Step 3. $i = 1$

- Since i satisfies $i \le n$, go to Step 4.

Step 4. Solve the branched problem (complementing the condition: $-5x_1 - x_2 \le 0$):

$$\max_{x,y} \ \frac{-x_1 + x_2 + 2y_1 - 2y_2 - y_3 - 1}{-x_1 - 2y_1 + y_2 + 5y_3 + 8},$$

s.t.

$$\begin{aligned}
-y_1 + y_2 + y_3 &\le 1,\\
2x_1 - y_1 + 2y_2 + 0.5y_3 &\le 3,\\
2x_2 + 2y_1 - y_2 + 0.5y_3 &\le 9,\\
-5x_1 - x_2 &\ge 0,\\
x_1 - x_2 &\le -3,\\
-5x_1 - 3x_2 &\le -5,\\
x_1, x_2, y_1, y_2, y_3 &\ge 0.
\end{aligned}$$

- Since we have no solution for this problem, go to Step 3.

Step 3. $i = 2$

- Since i satisfies $i \le n$, go to Step 4.

Step 4. Solve the branched problem (complementing the condition that $x_1 - x_2 \le -3$):

$$\max_{x,y} \frac{-x_1+x_2+2y_1-2y_2-y_3-1}{-x_1-2y_1+y_2+5y_3+8},$$
s.t.
$$\begin{aligned} -y_1+y_2+y_3 &\le 1\\ 2x_1-y_1+2y_2+0.5y_3 &\le 3,\\ 2x_2+2y_1-y_2+0.5y_3 &\le 9,\\ -5x_1-x_2 &\le 0,\\ x_1-x_2 &\ge -3,\\ -5x_1-3x_2 &\le -5,\\ x_1,x_2,y_1,y_2,y_3 &\ge 0. \end{aligned}$$

- Then we obtain a solution, $(3.75,0,4.5,0,0)$ with $x^2=(3.75,0), y^2=(4.5,0,0)$.
- Since $F(3.75,0,4.5,0,0)=-0.89 \le LB$, go to Step 3.

Step 3. $i=3$

- Since i satisfies $i \le n$, go to Step 4.

Step 4. Solve the branched problem (complementing the condition that $-5x_1-3x_2 \le -5$):

$$\max_{x,y} \frac{-x_1+x_2+2y_1-2y_2-y_3-1}{-x_1-2y_1+y_2+5y_3+8},$$
s.t.
$$\begin{aligned} -y_1+y_2+y_3 &\le 1,\\ 2x_1-y_1+2y_2+0.5y_3 &\le 3,\\ 2x_2+2y_1-y_2+0.5y_3 &\le 9,\\ -5x_1-x_2 &\le 0,\\ x_1-x_2 &\le -3,\\ -5x_1-3x_2 &\ge -5,\\ x_1,x_2,y_1,y_2,y_3 &\ge 0. \end{aligned}$$

- Since this problem has no solution, go to Step 3.

Step 3. $i = 4$

- Since i dose not satisfy $i \leq n$, Stop.

Therefore, the optimal solution is $(x_1^*, x_2^*, y_1^*, y_2^*, y_3^*) = (0, 3, 1.5, 0, 0)$ with $F = 1$ is the upper level optimal value and $f = -0.8$ is the lower level optimal value.

The same algorithmic procedure can be used to solve linear fractional bi-level optimization problems with unique optimal response from the follower for each of the choices of variables of the leader. The examples below are taken from literature whose exact solutions were calculated; and we obtain the same result (shown in Table 2) for each one of them as in the references indicated.

Note that the purpose of the examples here below is not to compare the efficiency of the algorithm rather to show that the same exact solution can be obtained using the proposed algorithm as well, while it solve problems with multiple optimal response from the lower level. It is known that the methods given in each of the references for these problems fail to solve if the optimal response from the lower level is nonunique.

Example 5. Consider a bi-level problem from [5]

$$\begin{aligned} \max_x \quad & \frac{-x-3y-3}{x+y+5}, \\ \max_y \quad & \frac{x-2y-7}{x+y+2}, \\ \text{s.t.} \quad & x+2y \leq 20, \\ & x+y \leq 12, \\ & 2x+y \leq 20, \\ & 3x-4y \leq 19, \\ & x-4y \leq 5, \\ & x, y \geq 0, \end{aligned}$$

Example 6. Consider a bi-level problem from [10]

$$\begin{aligned}
\max_{x} \quad & \frac{-y+2}{x+y+1}, \\
\max_{y} \quad & \frac{-5x-4y-5}{5x+5y+10}, \\
\text{s.t.} \quad & 3x-2y \geq -5, \\
& 2x+9y \leq 69, \\
& 3x-2y \leq 26, \\
& x-6y \leq -2, \\
& x+y \geq 5, \\
& x, y \geq 0.
\end{aligned}$$

Example 7. Consider a bi-level problem from [6]

$$\begin{aligned}
\max_{x} \quad & \frac{-2x-3y_1-y_2-2}{x+6y_2+5}, \\
\max_{y} \quad & \frac{-3x-2y_1-y_2}{x+y_1+2y_2+1}, \\
\text{s.t.} \quad & x+y_2 \leq 1, \\
& y_1+y_2 \leq 1, \\
& x, y_1, y_2 \geq 0,
\end{aligned}$$

$y=(y_1,y_2)$.

Example 8. Consider a bi-level problem from [5]

$$\begin{aligned}
\max_{x} \quad & \frac{-x_1+x_2-2y_2-1}{-x_1-2y_1+y_2+5y_3+8}, \\
\max_{y} \quad & \frac{-x_1-x_2-2y_1+y_2-y_3-1}{2x_1+y_1+y_2-3y_3+6}, \\
\text{s.t.} \quad & -y_1+y_2+y_3 \leq 1, \\
& 2x_1-y_1+2y_2-0.5y_3 \leq 1, \\
& 2x_2+2y_1-y_2-0.5y_3 \leq 1, \\
& x_1, x_2, y_1, y_2, y_3 \geq 0,
\end{aligned}$$

$x=(x_1,x_2), y=(y_1,y_2,y_3)$.

Table 2: Summery of solutions for the problems in the examples.

Examples	optimal solution using the proposed algorithm
5	$(1, 0)$
6	$(3, 2)$
7	$(0.2, 0, 0.8)$
8	$(0.75, 0.75, 0, 0, 1)$

5 Conclusion

In this paper, we presented a vertex search method to find an exact global optimal solution to the continuous bi-level linear fractional programming problem. Our algorithm is a combination of the kth best method and a branch-and-bound procedure. The existing kth best method is known to find a global optimal solution for bi-level linear fractional problems with single valued reaction set for all upper level decisions. To overcome the limitations of the kth best method when there are nonsingleton optimal reaction sets for some upper level decisions, a new algorithm that combines the kth best method together with a branch-and-bound mechanism is proposed. In this algorithm, iterative solution procedure is applied, where the branch-and-bound method is used to branch the problem into two problems of the same type in each branching step. We implemented the algorithm using the MATLAB software and it can solve the optimistic version of any bi-level linear fractional problem. The algorithm can also be applied for solving bi-level problems when the objective functions are generally quasi-convex and the constraints are polyhedral.

The algorithm performs well in solving linear fractional bi-level programming problems of any kind. However, if the optimal response map of the lower-level is single valued for all feasible upper level variables, then some steps of the algorithm will still run to check if there are possible feasible solutions outside of the vertices of the constraint region. This might create unnecessary delay in the solution process. In the future one may try to develop a mechanism to avoid the process of execution of the unnecessary iterations within the framework of the proposed algorithm.

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