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Research Article

Approximation of functions in Hölder's class and solution of nonlinear Lane–Emden differential equation by orthonormal Euler wavelets

H.C. Yadav*, A. Yadav and S. Lal

Abstract

In this article, a method has been developed to solve a nonlinear Lane–Emden differential equation based on the orthonormal Euler wavelet series. The orthonormal Euler wavelets are constructed by the dilatation and translation of orthogonal Euler polynomials. The convergence analysis of

*Corresponding author

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Harish Chandra Yadav
Department of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, India. e-mail: harishchandrayadav20395@gmail.com

Abhilasha Yadav
Department of Mathematics, Institute of Integrated and Honors Studies, Kurukshetra University, Kurukshetra, India. e-mail: yadavabhilasha1942@kuk.ac.in

Shyam Lal
Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India. e-mail: shyam_lal@rediffmail.com

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the orthonormal Euler wavelet series is studied in the Hölder's class. The orthonormal Euler wavelet approximations of solution functions of the nonlinear Lane–Emden differential equation in Hölder's class are determined by partial sums of their orthonormal Euler wavelet series. In concisely, two approximations $E^{(1)}_{2^{k-1},M}(f)$ and $E^{(2)}_{2^{k-1},M}(f)$ of solution functions of classes $H^{\alpha}_2[0,1)$ and $H^{\phi}_2[0,1)$ by $(2^k, M)^{th}$ partial sums of their orthonormal Euler wavelet expansions have been estimated. There are several applications of nonlinear differential equations, which include the nonlinear Lane–Emden differential equations. The solution of the nonlinear Lane–Emden differential equation obtained by the orthonormal Euler wavelets method is compared to its solution obtained by the Euler method and the ODE-45 method. It has been shown that the solutions produced by the orthonormal Euler wavelets are more accurate than those produced by the Euler method and the ODE-45 method. This is a result of the wavelet analysis research article.

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1 Introduction

The wavelet theory has acquired a lot of applications in recent times. Wavelets naturally adapt to irregular domains, simplifying computations and improving stability. There are various wavelet methods proposed for the approximation of functions and numerical solution of differential and integral equations, such as Legendre, Chebyshev, Gegenbauer, Genocchi, Vieta–Lucas, Euler, and sine-cosine wavelets. With the help of the orthogonal basis of those wavelets, it is possible to reduce the numerical problems of differential and integral equations to a system of linear and nonlinear algebraic equations. Many researchers like Chui [4], Debnath [5], Doha, Abd-Elhameed, and Youssri [6], Lal and Kumar [10], Lal and Patel [11], Meyer [14], and so on, are working in the direction of approximation of functions and solution of differential or integral equations. The present work introduces orthonormal

Euler wavelets (OEWs) as a novel basis for solving such equations. The key contributions include the construction of OEWs by combining Euler polynomials with wavelet theory, enabling efficient representation of solutions in Hölder spaces (Polat and Dincel [17]). Since the OEWs are generated by orthonormal Euler polynomials, the orthonormal Euler polynomials have fewer terms than other polynomials for generating their wavelets.

Various natural phenomena are studied and described using differential and integral equations [2, 3, 7, 8, 9, 1]. The nonlinear Lane–Emden differential equation is a fundamental model in astrophysics, describing phenomena such as stellar structure, isothermal gas spheres, and thermionic currents. Traditional numerical methods (e.g., finite difference, Runge–Kutta) often struggle with singularity at the origin and nonlinearity, leading to reduced accuracy and stability. To the best of our knowledge, there is no work related to the approximation of solution functions of the nonlinear Lane–Emden differential equation belonging to Hölder's class $H_2^{\alpha}[0,1)$ and $H_2^{\phi}[0,1)$ by the OEW expansion (Titchmarsh [16]). In Hölder's class $H_2^{\alpha}[0,1)$ and $H_2^{\phi}[0,1)$, the convergence analysis of the solution function f of the nonlinear Lane–Emden differential equation by the OEW series has been investigated. A method of the collocation has been proposed to find the numerical solution of the nonlinear Lane–Emden differential equations by the OEWs. Unlike other numerical methods, the collocation method easily transforms the differential equations into algebraic equations and can achieve high accuracy with relatively few collocation points. Rigorous convergence analysis in Hölder's class ensures that the wavelet approximations of solution functions are better and provide good results, demonstrating the method's effectiveness in handling singular and nonlinear terms.

The objective of this research paper are as follows:

(i) To define the Hölder's class $H_2^{\alpha}[0,1)$ and $H_2^{\phi}[0,1)$ in the interval $[0,1)$.

(ii) To define the orthonormal Euler polynomial and the OEW in the interval $[0,1)$.

(iii) To derive the approximation of the solution function f of the nonlinear Lane–Emden differential equations belonging to classes $H_2^{\alpha}[0,1)$ and $H_2^{\phi}[0,1)$.

(iv) To describe the procedure for calculating the numerical solution of the

nonlinear Lane–Emden differential equations and to provide examples to show the effectiveness of this procedure.
(v) To compare the exact solution of the nonlinear Lane–Emden differential equation using OEWs, the Euler method (EM), and the ODE-45 method.

The remaining parts of this paper are categorized as follows: In Section 2, some definitions and properties of the Hölder's class, the orthonormal Euler polynomial, and the OEW are mentioned. In Section 3, the convergence analysis of the solution function f of the nonlinear Lane–Emden differential equation by OEW series has been investigated. In Section 4, the definition of the OEW approximation of the solution function f of the nonlinear Lane–Emden differential equation and two estimators by OEW approximations and their proofs in $H_2^{\alpha}[0,1)$ and $H_2^{\phi}[0,1)$ class have been developed. In Section 5, the algorithm for the solution of the nonlinear Lane–Emden differential equation has been developed in the interval $[0,1)$, which is used to obtain the solution of the nonlinear Lane–Emden differential equation by the OEW. In Section 6, the solutions of the nonlinear Lane–Emden differential equation by OEWs, the EM, and the ODE45 method, and their absolute error have been obtained. Section 7 is designated for the conclusions of this research paper.

2 Definitions and preliminaries

2.1 Function of Hölder's class $H_2^{\alpha}[0,1)$

A function f belongs to $H_2^{\alpha}[0,1)$, $\alpha \in (0,1]$, if f is continuous and satisfies the following condition:

$$\left(\int_0^1 (f(x+t)-f(x))^2 \, dx\right)^{\frac{1}{2}} = O(|t|^{\alpha}), \qquad \text{for all} \quad x, t, x+t \in [0,1).$$

2.2 Function of Hölder's class $H_2^{\phi}[0,1)$

Let $\phi(t)$ be positive monotonic increasing function of t such that $\phi(|t|) \to 0$ as $t \to 0$. A function f belongs to $H_2^{\phi}[0,1)$ if f is continuous and satisfies the following condition:

$$\left(\int_0^1 (f(x+t)-f(x))^2 \, dx\right)^{\frac{1}{2}} = O(\phi(|t|)), \qquad \text{for all } \ x, t, x+t \in [0,1).$$

If $\phi(t) = t^{\alpha}$, then $H_2^{\phi}[0,1)$ coincides with classical Hölder's class $H_2^{\alpha}[0,1)$.

2.3 Orthonormal Euler polynomial and OEW

The orthonormal Euler polynomials of order m is denoted by $E_m^{(O)}(t)$ and defined in the interval $[0,1)$ as

$$E_m^{(O)}(t) = \sqrt{2m+1}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{m+k}{k}t^k, \qquad m \in \{0,1,2,3,\ldots\}. \tag{1}$$

The orthonormality property for Euler polynomials is as follows:

$$\langle E_m^{(O)}, E_n^{(O)}\rangle = \begin{cases} 1, & \text{if } \ n=n', m=m'; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

By analyzing the integrals of these polynomials from 0 to t, we obtain

$$\int_0^t E_0^{(O)}(x)dx = \frac{1}{2}E_0^{(O)}(t) + \frac{1}{2\sqrt{3}}E_1^{(O)}(t), \tag{3}$$

and for $m \geq 1$,

$$\int_0^t E_m^{(O)}(x)dx = \frac{E_{m+1}^{(O)}(t)}{2\sqrt{(2m+1)(2m+3)}} - \frac{E_{m-1}^{(O)}(t)}{2\sqrt{(2m+1)(2m-1)}}. \tag{4}$$

Therefore,

$$2\sqrt{2m+1}E_m^{(O)}(t) = \frac{1}{\sqrt{2m+3}}(E_{m+1}^{(O)}(t))' - \frac{1}{\sqrt{2m-1}}(E_{m-1}^{(O)}(t))'. \qquad (5)$$

The OEWs denoted by $\psi_{n,m}^{(O)}$, are defined on $[0,1)$ by

$$\psi_{n,m}^{(O)}(t) = \begin{cases} 2^{\frac{k-1}{2}} E_m^{(O)}(2^{k+2}t - 4n + 2), & \text{if } t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}); \\ 0, & \text{otherwise,} \end{cases} \qquad (6)$$

where $n = 1, 2, 3, \ldots, 2^{k-1}$, $m = 0, 1, 2, \ldots, M-1$, m is the order of the orthonormal Euler polynomials, and $k = 1, 2, 3, \ldots$ is the level of resolution.

3 Convergence analysis of the OEW series

In this section, the convergence analysis of the solution function f, of the nonlinear Lane–Emden differential equations, in $L^2[0,1)$ by the OEW expansion has been described.

Theorem 1. If $f(t)$ is the exact solution of the nonlinear Lane–Emden differential equation, then its OEW series $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(O)}(t)$ converges uniformly to $f(t)$.

Proof.

$$\begin{aligned}
\text{Let} \quad f(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(O)}(t). \\
\text{Then } \langle f, f \rangle &= \left\langle \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(O)}(t), \sum_{n'=1}^{2^{k-1}} \sum_{m'=0}^{\infty} c_{n',m'}\psi_{n',m'}^{(O)}(t) \right\rangle \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{n'=1}^{2^{k-1}} \sum_{m'=0}^{\infty} c_{n,m}\overline{c_{n',m'}} \langle \psi_{n,m}^{(O)}, \psi_{n',m'}^{(O)} \rangle \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} |c_{n,m}|^2 ||\psi_{n,m}^{(O)}||_2^2 \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} |c_{n,m}|^2, \\
& \{\psi_{n,m}^{(O)}\} \text{ is an orthonormal basis of } L^2[0,1).
\end{aligned}$$

$$\text{Thus, } \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} |c_{n,m}|^2 = \langle f, f \rangle \ = \int_0^1 |f(t)|^2 dt < \infty, \ f \in L^2[0,1).$$

Therefore, the wavelet series $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(O)}(t)$ is convergent and by the Bessel's inequality, $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} |c_{n,m}|^2 \leq ||f||_2^2 < \infty, \quad \text{for all} \quad M > 2.$

For $N > M$ & $k > p$,

$$\text{let } (S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(O)}(t).$$

$$\begin{aligned} ||(S_{2^{k-1},N}f) - (S_{2^{p-1},M}f)||_2^2 &= \left|\left| \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{N-1} c_{n,m}\psi_{n,m}^{(O)}(t) - \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(O)}(t) \right|\right|_2^2 \\ &= \left|\left| \sum_{n=2^{(p-1)}+1}^{2^{k-1}} \sum_{m=M}^{N-1} c_{n,m}\psi_{n,m}^{(O)}(t) \right|\right|_2^2 \\ &= \sum_{n=2^{p-1}+1}^{2^{k-1}} \sum_{m=M}^{N-1} |c_{n,m}|^2 \to 0 \text{ as } M \to \infty, \ N \to \infty. \end{aligned}$$

Therefore, $||(S_{2^{k-1},N}f) - (S_{2^{p-1},M}f)||_2^2 \to 0$ as $M \to \infty, \ N \to \infty$. Hence, $(S_{2^{k-1},N}f)_{N=0}^{\infty}$ is a Cauchy sequence in $L^2[0,1)$. Since, $L^2[0,1)$ is a Banach space, the Cauchy sequence $(S_{2^{k-1},N}f)_{N=0}^{\infty}$ converges to a function $b(t)$, (say). Here, $b(t) = \lim_{N\to\infty} (S_{2^{k-1},N}f) = \lim_{N\to\infty} \sum_{n=1}^{2^k} \sum_{m=0}^{N-1} c_{n,m}\psi_{n,m}^{(O)}(t)$.

Now, we need to show that $b(t) = f(t)$. For this, consider

$$\begin{aligned} \langle b(t) - f(t), \psi_{n,m}^{(O)}(t) \rangle &= \langle b(t), \psi_{n,m}^{(O)}(t) \rangle - \langle f(t), \psi_{n,m}^{(O)}(t) \rangle \\ &= \lim_{N\to\infty} \langle (S_{2^{k-1},N}f), \psi_{n,m}^{(O)}(t) \rangle - c_{n,m} \\ &= c_{n,m} - c_{n,m} = 0. \end{aligned}$$

Therefore, $b(t) = f(t) \quad \text{for all} \quad t \in [0,1)$. Hence, the OEW series $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{N-1} c_{n,m}\psi_{n,m}^{(O)}(t)$ converges uniformly to $f(t)$ as $N \to \infty$. □

4 Approximations and theorems

In this section, approximation and theorems based on the OEW have been established.

4.1 OEW approximation

Since, $\{\psi^{(O)}_{n,m}(t)\}$ forms an orthonormal basis for $L^2[0,1)$, a function $f \in L^2[0,1)$ can be expressed into the OEW series as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(O)}_{n,m}(t),\ c_{n,m} = \langle f, \psi^{(O)}_{n,m}\rangle. \tag{7}$$

The $(2^{k-1}, M)$th partial sum $(S_{2^{k-1},M}f)(t)$ of the OEW series (7) is given by

$$(S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi^{(O)}_{n,m}(t) = C^T\psi^{(O)}(t), \tag{8}$$

where $C = [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}; c_{2,0}, \dots, c_{2,M-1}, \dots, c_{2^{k-1},0}; \dots, c_{2^{k-1},M-1}]^T$ and
$\psi^{(O)}(t) = [\psi^{(O)}_{1,0}(t), \dots, \psi^{(O)}_{1,M-1}(t); \psi^{(O)}_{2,0}(t), \dots; \psi^{(O)}_{2^{k-1},0}(t), \dots, \psi^{(O)}_{2^{k-1},M-1}(t)]^T$.

The OEW approximation $E_{2^{k-1},M}(f)$ of f by $(2^{k-1}, M)^{\text{th}}$ partial sum $(S_{2^{k-1},M}f)$ of the OEW series (7), is defined by

$$E_{2^{k-1},M}(f) = \min_{(S_{2^{k-1},M}f)} ||f - (S_{2^{k-1},M}f)||_2. \tag{9}$$

Here, $E_{2^{k-1},M}(f)$ is said to be the best approximation of f by $(2^{k-1}, M)^{\text{th}}$ partial sum $(S_{2^{k-1},M}f)$, if $E_{2^{k-1},M}(f) \to 0$, as $k \to \infty$, $M \to \infty$ (Zygmund [18]).

4.2 Theorems

In this paper, the following theorems based on the OEW in the Hölder's class have been developed.

Theorem 2. If $f'' \in H_2^{\alpha}[0,1)$ class, then the OEW expansion of the solution function f of the nonlinear Lane–Emden differential equation is $f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(O)}(t)$ having $(2^{k-1}, M)^{\text{th}}$ partial sums,

$$(S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(O)}(t);$$

then, the OEW approximation of f by $(S_{2^{k-1},M}f)$, under $||\cdot||_2$, for $M > 2$ is given by

$$E_{2^{k-1},M}^{(1)}(f) = \min||f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(O)}(t)||_2 = O\left(\frac{1}{2^{(k-1)(\alpha+2)}(2M-3)^{\frac{3}{2}}}\right).$$

Proof. Consider

$$\begin{aligned}
f(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(O)}(t).\\
c_{n,m} &= \left\langle f, \psi_{n,m}^{(O)}\right\rangle\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,m}^{(O)}(t)dt\\
&= 2^{\frac{k-1}{2}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)E_m^{(O)}(2^{k-1}t-n+1)dt\\
&= 2^{\frac{k-1}{2}}\int_0^1 f\left(\frac{u+n-1}{2^{k-1}}\right)E_m^{(O)}(u)\frac{du}{2^{k-1}},\ 2^{k-1}t-n+1=u\\
&= \frac{1}{2^{\frac{k+1}{2}}\sqrt{2m+1}}\int_0^1 f\left(\frac{u+n-1}{2^{k-1}}\right)d\left(\frac{E_{m+1}^{(O)}(t)}{\sqrt{2m+3}} - \frac{E_{m-1}^{(O)}(t)}{\sqrt{2m-1}}\right) \quad (10)\\
&\quad \text{(by (5))}
\end{aligned}$$

Integrating (10) by parts, we have

$$
\begin{aligned}
c_{n,m} &= -\frac{1}{2^{\frac{3k-1}{2}}\sqrt{2m+1}} \int_0^1 f'\left(\frac{u+n-1}{2^{k-1}}\right)\left(\frac{E_{m+1}^{(O)}(t)}{\sqrt{2m+3}} - \frac{E_{m-1}^{(O)}(t)}{\sqrt{2m-1}}\right) dt \\
&= -\frac{1}{2^{\frac{3k-1}{2}}\sqrt{2m+1}} \int_0^1 f'\left(\frac{u+n-1}{2^{k-1}}\right) d\left(\frac{E_{m+2}^{(O)}(t)}{2(2m+3)\sqrt{2m+5}}\right. \\
&\quad \left. - \frac{\sqrt{2m+1}E_m^{(O)}(t)}{(2m+3)(2m-1)} + \frac{E_{m-2}^{(O)}(t)}{2(2m-1)\sqrt{2m+1}}\right), \text{ by } (5) \qquad (11)
\end{aligned}
$$

Integrating (11) by parts, we have

$$
\begin{aligned}
c_{n,m} &= \frac{1}{2^{\frac{5k-3}{2}}\sqrt{2m+1}} \int_0^1 f''\left(\frac{u+n-1}{2^{k-1}}\right) \\
&\quad \times \left(\frac{E_{m+2}^{(O)}(t)}{(2m+3)\sqrt{2m+5}} - \frac{\sqrt{2m+1}E_m^{(O)}(t)}{2(2m+3)(2m-1)} + \frac{E_{m-2}^{(O)}(t)}{2(2m-1)\sqrt{2m+1}}\right) dt \\
&= \frac{1}{2^{\frac{5k-3}{2}}\sqrt{2m+1}} \int_0^1 \left(f''\left(\frac{u+n-1}{2^{k-1}}\right) - f''\left(\frac{n-1}{2^{k-1}}\right)\right) \\
&\quad \times \left(\frac{E_{m+2}^{(O)}(t)}{(2m+3)\sqrt{2m+5}} - \frac{\sqrt{2m+1}E_m^{(O)}(t)}{2(2m+3)(2m-1)} + \frac{E_{m-2}^{(O)}(t)}{2(2m-1)\sqrt{2m+1}}\right) dt \\
&\quad + \frac{1}{2^{\frac{5k-3}{2}}\sqrt{2m+1}} f''\left(\frac{n-1}{2^{k-1}}\right) \int_0^1 \left(\frac{E_{m+2}^{(O)}(t)}{(2m+3)\sqrt{2m+5}}\right. \\
&\quad \left. - \frac{\sqrt{2m+1}E_m^{(O)}(t)}{2(2m+3)(2m-1)} + \frac{E_{m-2}^{(O)}(t)}{2(2m-1)\sqrt{2m+1}}\right) dt \\
&\leq \frac{1}{2^{\frac{5k-3}{2}}\sqrt{2m+1}} \left(\int_0^1 \left(f''\left(\frac{u+n-1}{2^{k-1}}\right) - f''\left(\frac{n-1}{2^{k-1}}\right)\right)^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^1 \left(\frac{E_{m+2}^{(O)}(t)}{(2m+3)\sqrt{2m+5}} - \frac{\sqrt{2m+1}E_m^{(O)}(t)}{2(2m+3)(2m-1)}\right.\right. \\
&\quad \left.\left. + \frac{E_{m-2}^{(O)}(t)}{2(2m-1)\sqrt{2m+1}}\right)^2 dt\right)^{\frac{1}{2}} \\
&\leq \frac{r}{2^{\frac{5k-3}{2}}\sqrt{2m+1}} \left(\frac{1}{2^{k-1}}\right)^{\alpha} \left(\frac{1}{(2m+3)^2(2m+5)}\right. \\
&\quad \left. - \frac{2m+1}{4(2m+3)^2(2m-1)^2} + \frac{1}{4(2m-1)^2(2m+1)}\right)^{\frac{1}{2}} \\
&\quad \text{(by (2) and } f'' \in H_2^{\alpha}[0,1),\ r \text{ be a positive constant)}
\end{aligned}
$$

$$|c_{n,m}| \leq \frac{r}{2^{(k-1)(\alpha+\frac{5}{2})}(2m-3)^2}. \tag{12}$$

$$\begin{aligned}
\text{Then, } (E^{(1)}_{2^{k-1},M}(f))^2 &= ||f(t)-(S_{2^{k-1},M}f)(t)||_2^2 = \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2 \\
&\leq \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left(\frac{r}{2^{(k-1)(\alpha+\frac{5}{2})}(2m-3)^2}\right)^2 \\
&= \frac{r^2 2^{k-1}}{2^{(k-1)(2\alpha+5)}}\sum_{m=M}^{\infty}\frac{1}{(2m-3)^4} \\
&\leq \frac{r^2 2^{k-1}}{2^{(k-1)(2\alpha+5)}}\left(\frac{1}{(2M-3)^4}+\int_M^{\infty}\frac{dm}{(2m-3)^4}\right) \\
&\quad \text{(by Cauchy's integral test)} \\
&\leq \frac{r^2}{2^{(k-1)(2\alpha+4)}}\left(\frac{7}{6(2M-3)^3}\right). \\
E^{(1)}_{2^{k-1},M}(f) &\leq \frac{r\sqrt{7}}{2^{(k-1)(\alpha+2)}\sqrt{6}(2M-3)^{\frac{3}{2}}}. \\
\text{Therefore, } E^{(1)}_{2^{k-1},M}(f) &= O\left(\frac{1}{2^{(k-1)(\alpha+2)}(2M-3)^{\frac{3}{2}}}\right),\ M>2.
\end{aligned}$$

□

Theorem 3. If $f'' \in H_2^{\phi}[0,1)$ class, such that $\phi(|t|) \to 0$ as $t \to 0$, then the OEW approximation of f by $(S_{2^{k-1},M}f)$ satisfies

$$\begin{aligned}
E^{(2)}_{2^{k-1},M}(f) &= \min||f-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c'_{n,m}\psi^{(O)}_{n,m}(t)||_2 \\
&= O\left(\frac{1}{2^{(k-1)}(2M-3)^{\frac{3}{2}}}\phi\left(\frac{1}{2^{k-1}}\right)\right),\ M>2.
\end{aligned}$$

Proof. Following the proof of Theorem 2 and for $f'' \in H^{\phi}[0,1)$ class, we have

$$\begin{aligned}
c'_{n,m} \leq \frac{1}{2^{\frac{5k-3}{2}}\sqrt{2m+1}}&\left(\int_0^1\left(f''\left(\frac{u+n-1}{2^{k-1}}\right)\right.\right. \\
&\left.\left.-\ f''\left(\frac{n-1}{2^{k-1}}\right)\right)^2\right)^{\frac{1}{2}}\left(\int_0^1\left(\frac{E^{(O)}_{m+2}(t)}{(2m+3)\sqrt{2m+5}}\right.\right.
\end{aligned}$$

$$
- \frac{\sqrt{2m+1}E_m^{(O)}(t)}{2(2m+3)(2m-1)} + \frac{E_{m-2}^{(O)}(t)}{2(2m-1)\sqrt{2m+1}} \Bigg)^2 dt \Bigg)^{\frac{1}{2}}
$$

$$
\leq \frac{q}{2^{\frac{5k-5}{2}}(2m-3)^2}\phi\left(\frac{1}{2^{k-1}}\right) \tag{13}
$$

(by (2) and $f'' \in H_2^{\phi}[0,1)$)

$$
\begin{aligned}
\text{Then, } (E_{2^{k-1},M}^{(2)}(f))^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c'_{n,m}|^2 \\
&\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} \left(\frac{q}{2^{\frac{5k-5}{2}}(2m-3)^2}\phi\left(\frac{1}{2^{k-1}}\right)\right)^2 \\
&= \frac{q^2 2^{k-1}}{2^{5(k-1)}}\phi^2\left(\frac{1}{2^{k-1}}\right) \sum_{m=M}^{\infty} \frac{1}{(2m-3)^4} \\
&\leq \frac{q^2 2^{k-1}}{2^{5(k+1)}}\phi^2\left(\frac{1}{2^{k-1}}\right)\left(\frac{7}{6(2M-3)^3}\right).
\end{aligned}
$$

$$
\text{Therefore, } E_{2^k,M}^{(2)}(f) = O\left(\frac{1}{2^{(k-1)}(2M-3)^{\frac{3}{2}}}\phi\left(\frac{1}{2^{k-1}}\right)\right), \ M > 2.
$$

This completes the proof of the Theorem 3. □

5 Algorithm for the nonlinear Lane–Emden differential equation

This section contains the procedure for solving the nonlinear Lane–Emden differential equations by the OEW. The five basis functions of the OEW for $k = 1$ and $M = 5$ are as follows:

$$
\left.
\begin{aligned}
&\psi_{1,0}^{(O)}(t) = 1 \\
&\psi_{1,1}^{(O)}(t) = \sqrt{3}(2t-1) \\
&\psi_{1,2}^{(O)}(t) = \sqrt{5}(6t^2-6t+1) \\
&\psi_{1,3}^{(O)}(t) = \sqrt{7}(20t^3-30t^2+12t-1) \\
&\psi_{1,4}^{(O)}(t) = \sqrt{9}(70t^4-140t^3+90t^2-20t+1
\end{aligned}
\right\}, \qquad t \in [0,1). \tag{14}
$$

Let $y(t)$ be the solution of the nonlinear Lane–Emden differential equation:

$$y''+\frac{2}{t}y'+y^{\gamma}=0, \gamma\geq 2,\ t\in[0,1],\ y(0)=1,\ y'(0)=0 \text{ (Mukherjee et al.[15])}. \tag{15}$$

$$\text{Then, } y(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}^{(O)}(t) \tag{16}$$

and $(2^{k-1}, M)^{th}$ partial sum of series (16) is

$$y(t)=(S_{2^{k-1},M}f)(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{(O)}(t)=C^T\psi^{(O)}(t). \tag{17}$$

By initial conditions (15), (17) reduces to

$$y(0)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{(O)}(0)=1,\ y'(0)=\frac{d}{dt}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{(O)}(t)\right)_{t=0}=0.$$

In (17), C^T contains $2^{k-1}M$ unknown coefficients. Hence, excluding initial conditions of (15), $2^{k-1}M-2$ extra conditions are needed for the solution of the nonlinear Lane–Emden differential equation (15). For determining the values of $2^{k-1}M$ unknown coefficients $c_{n,m}$, collocation points $t_i=\frac{i-1}{2^{k-1}M}, i=2,\dots,2^{k-1}M$, are substituted in (17) to obtain $2^{k-1}M-2$ system of algebraic equations. Hence, the values of unknown coefficients $c_{n,m}$ are obtained by solving these $2^{k-1}M$ system of algebraic equations. This algorithm is also applicable to the solution of higher-order linear and nonlinear differential equations (Lal and Yadav [12], Yogit, Scindia, and Kumar [13]).

6 Results and discussion

The applicability of the suggested approach for numerical solution of the nonlinear Lane–Emden differential equation and its error analysis has been covered in this section. The solutions obtained are also compared in the suggested way, the EM, and the ODE-45 method with their exact solutions for $\gamma=5$.

Example 1. Consider the nonlinear Lane–Emden differential equation for $\gamma=2$

$$y''+\frac{2}{t}y'+y^2=0,\ y(0)=1,\ y'(0)=0. \tag{18}$$

By the algorithm of the OEW approach described in section 5, the nonlinear Lane–Emden differential equations have been solved. For the approximate solution of (18), take $M = 5$ and $k = 1$. Then $y(t)$ will be

$$\begin{aligned} y(t) &= \sum_{m=0}^{4} c_{1,m}\psi^{(O)}_{1,m}(t) \\ &= c_{1,0} + \sqrt{3}(2t-1)c_{1,1} + \sqrt{5}(6t^2-6t+1)c_{1,2} + \sqrt{7}(20t^3 - 30t^2 \\ &\quad +12t-1)c_{1,3} + \sqrt{9}(70t^4 - 140t^3 + 90t^2 - 20t + 1)c_{1,4}, \quad t \in [0,1)\,. \end{aligned} \tag{19}$$

For calculating the unknown values $c_{1,0}$, $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, and $c_{1,4}$, we collocate (18) and (19) at $t = 0.25, 0.5, 0.75$ and using the initial condition in (19), system of five equations is obtained. Solving this system of equations, the values of the unknowns are as follows:

$$\begin{aligned} &c_{1,0} = 0.947558069231224,\ c_{1,1} = -0.044519188708113, \\ &c_{1,2} = -0.010463291917928,\ c_{1,3} = 0.000547078923993, \\ &c_{1,4} = 0.000058833597300. \end{aligned} \tag{20}$$

Putting the values of $c_{1,0}$, $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, and $c_{1,4}$ from (20) into (19), we obtain

$$\begin{aligned} y(t) &= 0.947558069231224 - 0.044519188708113\sqrt{3}(2t-1) \\ &\quad -0.010463291917928\sqrt{5}(6t^2-6t+1) + 0.000547078923993\sqrt{7}(20t^3 \\ &\quad -30t^2+12t-1) + 0.0000588335973\sqrt{9}(70t^4-140t^3+90t^2-20t+1). \end{aligned}$$

The approximate solution of the nonlinear Lane–Emden differential equation (18) is obtained by the OEW method for $M = 5, 10,$ and 15 in the interval $[0, 1)$. Also, this solution has been compared with the ODE-45 method and the EM is shown in Table 1.

The graph between the ODE-45, EM and OEW of solution of the nonlinear Lane–Emden differential equation (18) is shown in Figure 1.

Example 2. Consider the nonlinear Lane–Emden differential equation for $\gamma = 4$ as

$$y'' + \frac{2}{t}y' + y^4 = 0,\ y(0) = 1,\ y'(0) = 0. \tag{21}$$

Table 1: Comparison between ODE-45, OEW and EM for Example 1

t	ODE-45	OEW ($M = 5$)	OEW ($M = 10$)	OEW ($M = 15$)	EM ($M = 10$)	EM ($M = 15$)
0.1	0.9983349952	0.9983262964	0.9983349986	0.9983349985	0.9985010028	0.9984456377
0.2	0.9933599136	0.9933369662	0.9933599072	0.9933599071	0.9936874525	0.9935781792
0.3	0.9851339505	0.9851019189	0.9851339470	0.9851339469	0.9856141179	0.9854538721
0.4	0.9737541183	0.9737207165	0.9737541164	0.9737541163	0.9743738441	0.9741669514
0.5	0.9593527169	0.9593225730	0.9593527158	0.9593527158	0.9600952381	0.9598472632
0.6	0.9420940363	0.9420663544	0.9420940358	0.9420940352	0.9429394773	0.9426570314
0.7	0.9221703488	0.9221405789	0.9221703486	0.9221703485	0.9230963905	0.9227869095
0.8	0.8997973703	0.8997634170	0.8997973703	0.8997973702	0.9007799757	0.9004514781
0.9	0.8752093703	0.8751826912	0.8752093704	0.8752093703	0.8762235317	0.8758843691

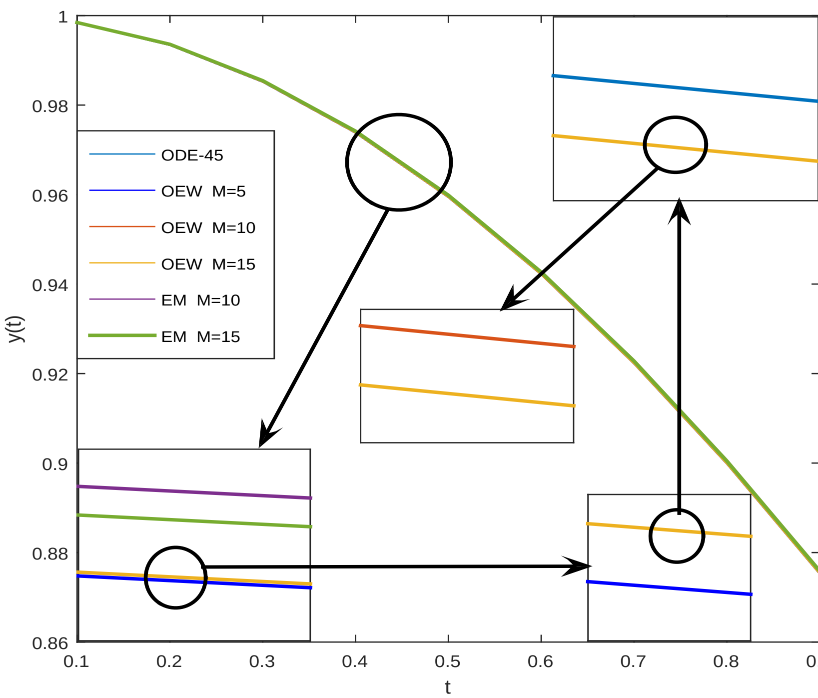

Figure 1: The graphs between ODE-45, OEW and EM for Example 1

The approximate solution of the nonlinear Lane–Emden differential equation (21) obtained by OEW is given in Table 2, for $M = 5, 10,$ and 15 in the interval $[0, 1)$. Also, a comparison among this solution, the ODE-45, and EM is shown in Table 2.

The graph between ODE-45, EM and OEW solutions of the nonlinear Lane–Emden differential equation (21) is shown in Figure 2.

Table 2: Comparison between ODE-45, OEW and EM for Example 2

t	ODE-45	OEW ($M=5$)	OEW ($M=10$)	OEW ($M=15$)	EM ($M=10$)	EM ($M=15$)
0.1	0.9983366618	0.9983025751	0.9983366543	0.9983366595	0.9985020041	0.9984468288
0.2	0.9933862156	0.9932972736	0.9933862071	0.9933862135	0.9937080741	0.9936006083
0.3	0.9852648948	0.9851421997	0.9852648879	0.9852648944	0.9857260484	0.9855719731
0.4	0.9741584084	0.9740323168	0.9741584026	0.9741584089	0.9747346018	0.9745419648
0.5	0.9603109012	0.9601994473	0.9603108961	0.9603109023	0.9609727347	0.9607513270
0.6	0.9440112896	0.9439122728	0.9440112841	0.9440112908	0.9447263563	0.9444869990
0.7	0.9255783519	0.9254763341	0.9255783457	0.9255783526	0.9263133809	0.9260672071
0.8	0.9053459236	0.9052340309	0.9053459167	0.9053459238	0.9060687187	0.9058265206
0.9	0.8836493241	0.8835646224	0.8836493175	0.8836493239	0.8843303463	0.8841020405

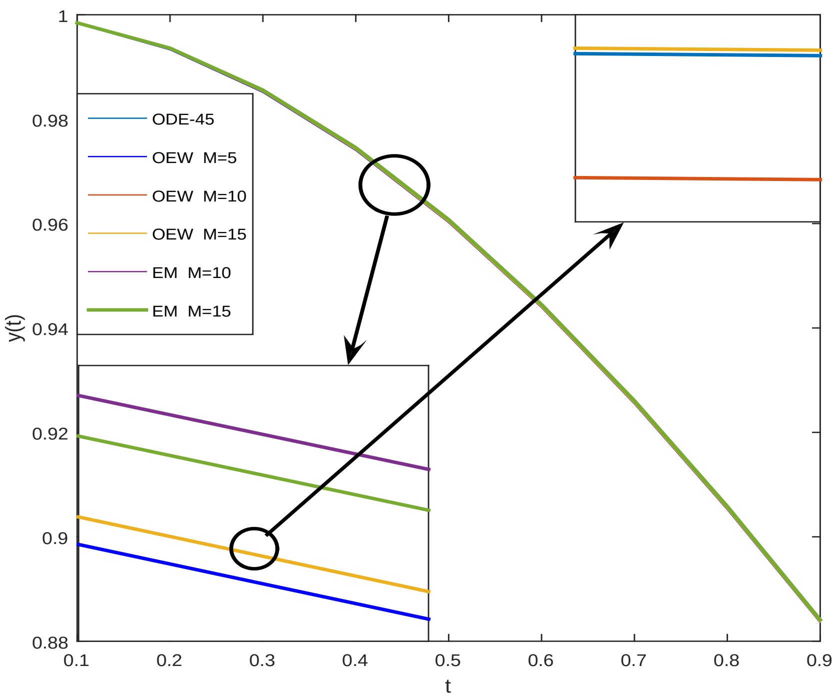

Figure 2: The graphs between ODE-45, OEW and EM for Example 2

Example 3. Consider the nonlinear Lane–Emden differential equation for $\gamma = 5$ as

$$y'' + \frac{2}{t}y' + y^5 = 0, \ y(0) = 1, \ y'(0) = 0. \tag{22}$$

The exact solution of (22) is $y(t) = \frac{1}{\sqrt{1+\frac{t^2}{3}}}$.

Consider $\quad y''(x+t) - y''(x) = \beta_2|t|.$

By Lagrange's mean value theorem, $|y'''(c_1)| \leq \beta_1,\ c_1 \in (0,1)$.

$$\left(\int_0^1 |y''(x+t) - y''(x)|^2 dx\right)^{\frac{1}{2}} \leq \beta_1 \left(\int_0^1 |t|^2 dx\right)^{\frac{1}{2}} = O(|t|^{\alpha}),\ \alpha \in (0,1].$$

Hence, $y''(t) \in H_2^{\alpha}[0,1)$.

The approximate solution of the nonlinear Lane–Emden differential equation (22) obtained by OEW for $M = 10,$ and 15 in the interval $[0,1)$ is given in Table 3. This solution has also been compared to the exact solution (ES), the solution obtained by the ODE-45 and EM is shown in Table 3.

Table 3: Comparison between ES, ODE-45, OEW and EM for Example 3

t	ES	ODE-45	OEW ($M = 10$)	OEW ($M = 15$)	EM ($M = 10$)	EM ($M = 15$)
0.1	0.9983374884	0.9983374920	0.9983374705	0.9983374884	0.9985025042	0.9984474235
0.2	0.9933992677	0.9933992691	0.9933992459	0.9933992677	0.9937183239	0.9936117508
0.3	0.9853292781	0.9853292778	0.9853292561	0.9853292781	0.9857812146	0.9856301345
0.4	0.9743547036	0.9743547022	0.9743546823	0.9743547036	0.9749103063	0.9747244246
0.5	0.9607689228	0.9607689208	0.9607689022	0.9607689228	0.9613937542	0.9611845569
0.6	0.9449111825	0.9449111809	0.9449111607	0.9449111825	0.9455686216	0.9453483586
0.7	0.9271455408	0.9271455401	0.9271455187	0.9271455408	0.9277996158	0.9275803477
0.8	0.9078412990	0.9078412992	0.9078412772	0.9078412990	0.9084590067	0.9082518193
0.9	0.8873565094	0.8873565100	0.8873564897	0.8873565094	0.8879094562	0.8877239019

The graph between exact solution, ODE-45, EM and OEW solutions of the nonlinear Lane–Emden differential equation (22) is shown in Figure 3.
By Table 3 and Figure 3, it is clear that the exact and the OEW solutions of the nonlinear Lane–Emden equation (22) coincide almost everywhere for $M = 15$.

6.1 Absolute error

The absolute error in the approximate solution of the nonlinear Lane–Emden differential equation (22) by the ODE-45 OEW method and the EM is given in Table 4. The absolute error is negligible in the solution obtained by the OEW method for $M = 15$.
The graphs of the absolute error in the solution of the nonlinear Lane–Emden differential equation (22) by the OEW method for $M = 10, 15$ and ODE-45 method are shown in Figure 4.

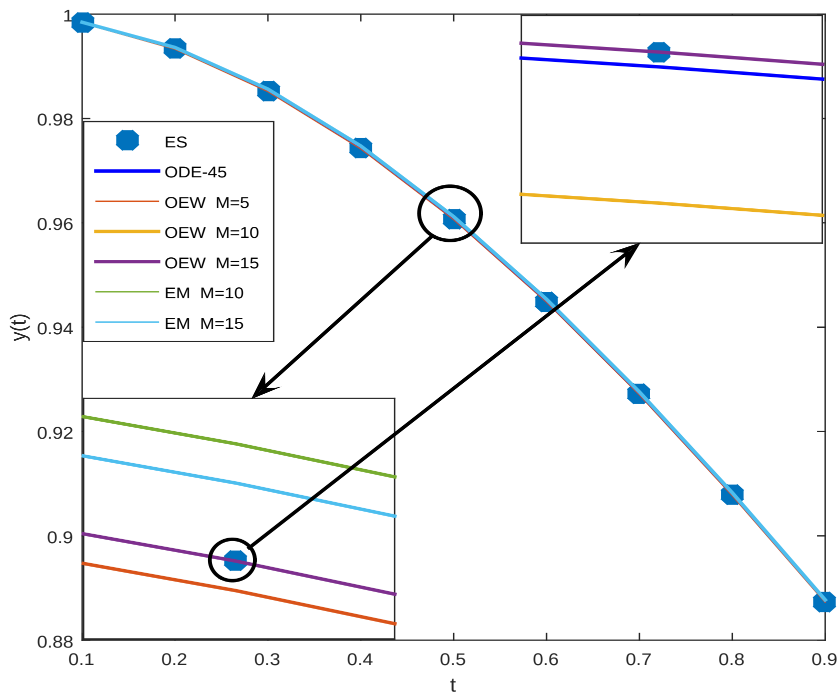

Figure 3: The graphs between ES, ODE-45, OEW and EM for Example 3

Table 4: Absolute error between ES, ODE-45, OEW and EM of Example 3

t	ODE-45 ($\times 10^{-8}$)	OEW $M=5$ ($\times 10^{-3}$)	OEW $M=10$ ($\times 10^{-7}$)	OEW $M=15$ ($\times 10^{-11}$)	EM $M=10$ ($\times 10^{-3}$)	EM $M=15$ ($\times 10^{-3}$)
0.1	0.3610796328	0.0495610203	0.1794897497	0.2246203223	0.1650157999	0.1099351000
0.2	0.1358930190	0.1286525500	0.2188676739	0.2629008122	0.3190562000	0.2124830999
0.3	0.0307614156	0.1764263294	0.2203716298	0.2814748434	0.4519364999	0.3008563999
0.4	0.0257128496	0.1800378634	0.2136527910	0.2774225293	0.5556026999	0.3697209999
0.5	0.1931504844	0.1575967430	0.2060257175	0.2659761300	0.6248313999	0.4156341000
0.6	0.1528948967	0.1378408315	0.2174169055	0.2488009798	0.6574391000	0.4371761000
0.7	0.0634760466	0.1390788755	0.2205584404	0.2272737553	0.6540750000	0.4348068999
0.8	0.0205243488	0.1496226164	0.2179250213	0.2095879025	0.6177077000	0.4105202999
0.9	0.0616085071	0.1112946509	0.1962926043	0.1538325022	0.5529467999	0.3673924999

6.2 Physical interpretation

The solution of the Lane–Emden equation represents the dimensionless density profile of a polytropic stellar model. These solutions predict the stability conditions for polytropic stars, influencing theories of stellar evolution. The Lane–Emden equation has a singularity at $t = 0$, but wavelets resolve this

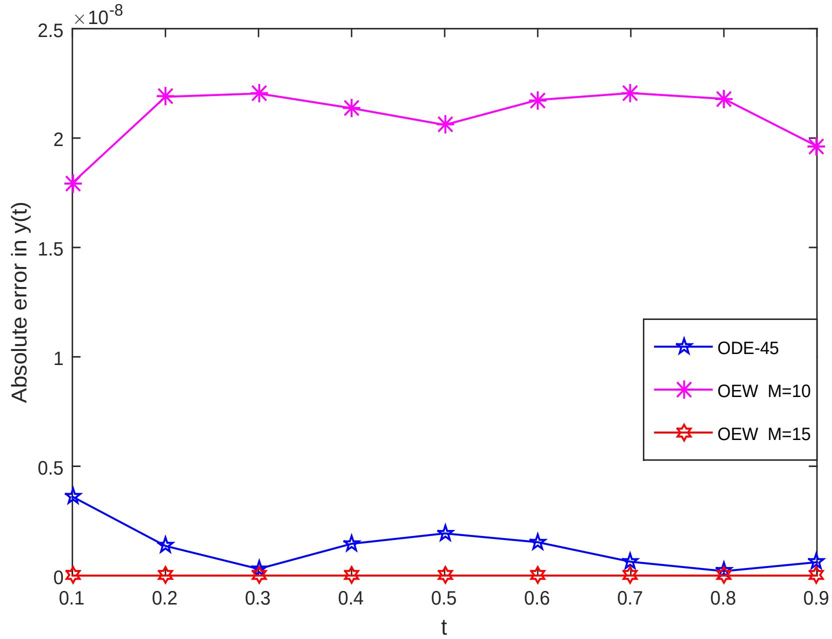

Figure 4: The graphs of the absolute error between ES, ODE-45, OEW and EM for Example 3

singularity by adjusting local resolution, leading to more stable and physically meaningful solutions as shown in the above examples.

7 Conclusion

1. The OEW approximation of solution functions of the nonlinear Lane–Emden differential equation of Theorems 2 and 3 is given by
$E^{(1)}_{2^{k-1},M}(f) = O\left(\frac{1}{2^{(k-1)(\alpha+2)}(2M-3)^{\frac{3}{2}}}\right)$, $M > 2$. $(E^{(1)}_{2^{k-1},M}(f)) \to 0$ as $k \to \infty$ and $M \to \infty$;
$E^{(2)}_{2^{k-1},M}(f) = O\left(\frac{1}{2^{(k-1)}(2M-3)^{\frac{3}{2}}}\phi\left(\frac{1}{2^{k-1}}\right)\right)$, $M > 2$. $(E^{(2)}_{2^{k-1},M}(f)) \to 0$ as $k \to \infty$ and $M \to \infty$.
Therefore, the approximations $E^{(1)}_{2^{k-1},M}(f)$ and $E^{(2)}_{2^{k-1},M}(f)$ of the solution functions of the nonlinear Lane–Emden differential equation belonging to the classes $H_2^{\alpha}[0,1)$ and $H_2^{\phi}[0,1)$ are the best possible in wavelet analysis.

2. By Tables 1, 2, and 3 and Figures 1, 2, and 3, it is shown that the OEW solutions for $M = 15$ of the nonlinear Lane–Emden differential equation coincide almost everywhere in the interval $[0, 1)$.

3. It is clear from Table 4 and Figure 4 that the OEW approach for $M = 15$ has significantly lower absolute error than the ODE-45 method and EM.

4. The examples provided illustrate the applicability and precision of the algorithm in Section 6 for solving the nonlinear Lane–Emden differential equation.

5. Future research directions using this approach can be generalized to solve the higher-order nonlinear differential equations and extended to multi-dimensional linear Partial differential equations.

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