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Research Article

A solution approach to the multilevel linear fractional programming problems

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Abstract

In this paper, we consider the multilevel linear fractional programming problems over a bounded polytope set. We present a characterization of the optimum solution to the n-level linear fractional programming problem for case $n > 2$. Then, we propose an extension of the Kth-best algorithm, for solving the n-level linear fractional programming problem with $n > 2$, and prove its convergence. Furthermore, we consider a previously published paper on such problems. It is shown that some results and proofs presented in that paper are incorrect by providing a counterexample. Finally, some numerical examples are presented, and the results are compared to those obtained from existing methods to show the accuracy and efficiency of the proposed algorithm.

AMS subject classifications (2020): Primary 90C26; Secondary 90C30, 90C32.

Keywords: Linear fractional programming; Multilevel programming; Vertex point; Kth-best algorithm.

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1 Introduction

Multilevel programming is a tool for modeling decision-making problems in systems where decision-makers (DMs) are arranged hierarchically. At each level, in such systems, DMs attempt to optimize their own goals while bearing in mind the decisions of all higher levels. Multilevel programming, in cases with only two or three levels, is referred to as bilevel or trilevel programming, respectively. Various multilevel programming models have appeared in different fields, such as planning [8, 10], security, accident management [1, 6], supply chain management [20, 23], economics [11], and decentralized inventory [9]. On the other hand, linear fractional programming (LFP) is appropriate for modeling various real-world problems, including efficiency measurement and financial planning [21]. The LFP problem is a particular class of nonlinear programming problems wherein the objective function is a ratio of two linear functions, but all constraints are linear. The LFP has attracted conspicuous attention in the literature [16, 21]. In this paper, we study the multilevel linear fractional programming (MLLFP) problems in which the objective functions of all levels are linear fractional, and the common constraint region is a bounded polytope. Some researchers have also studied the MLLFP problems and proposed some solution approaches [12, 22]. Fathy et al. [7] converted fully intuitionistic fuzzy linear fractional multilevel programming problem into a multilevel multiobjective linear programming. Then ϵ-constraint method is used to reduce the resulting problem to a single objective linear programming problem [7]. In [3], the authors have proved the existence of a border-feasible vertex of the inducible region (the feasible region of the first-level DM which is implicitly defined by constraints and the lower-levels DMs), that solves the bilevel linear fractional programming (BLLFP) problem. In this study, we extend this result to the MLLFP problems, including $n > 2$ levels. We then develop the Kth-best algorithm, one of the most well known methods for solving bilevel programming problems [5], to solve the MLLFP problems, and we discuss its convergence. Moreover, we consider a previously published article by Bhargava [2]. A counterexample is provided to demonstrate that some definitions and theorems in [2] are incorrect.

The structure of the paper is as follows. The mathematical formulation of the MLLFP problem under consideration, along with some relevant definitions, is presented in Section 2. The verification of some definitions and results given in [2] has been carried out in Section 3. In Section 4, some geometric properties of the optimum solution to the MLLFP problem have been proved. The Kth-best algorithm to solve the MLLFP problems and the proof of its convergency are developed in Section 5. A number of numerical examples to show the feasibility of the proposed Kth-best algorithm are presented in Section 6. Ultimately, the paper is finalized in Section 7.

2 Formulation of the problem

We address the MLLFP problem formulated as follows:

$$\begin{array}{cc}
\min\limits_{x_1\in X_1} & f_1(x_1,x_2,\ldots,x_n)=\frac{\theta_1+c_{11}x_1+\cdots+c_{1n}x_n}{\gamma_1+d_{11}x_1+\cdots+d_{1n}x_n},\\
\min\limits_{x_2\in X_2} & f_2(x_1,x_2,\ldots,x_n)=\frac{\theta_2+c_{21}x_1+\cdots+c_{2n}x_n}{\gamma_2+d_{21}x_1+\cdots+d_{2n}x_n},\\
 & \vdots\\
\min\limits_{x_n\in X_n} & f_n(x_1,x_2,\ldots,x_n)=\frac{\theta_n+c_{n1}x_1+\cdots+c_{nn}x_n}{\gamma_n+d_{n1}x_1+\cdots+d_{nn}x_n},\\
 & s.t.\quad \sum\limits_{i=1}^{n}A_ix_i\le b,
\end{array}\tag{1}$$

where n is a natural number, c_{ij}, d_{ij}, b, and A_i, for each $i,j\in\{1,2,\ldots,n\}$, are vectors and matrices of appropriate dimensions, respectively, θ_i, and γ_i are scalars, $n_1,\ldots,n_n$ are natural numbers whose sum is equal to $\overline{n}$, $x_i\in X_i\subset\mathbb{R}_+^{n_i}$ is the variable under the control of the ith-level DM, and the linear fractional function $f_i:\mathbb{R}^{\overline{n}}\to\mathbb{R}$, is well-defined and continuous on the constraint region C, defined below. We make assumption $\gamma_i+d_{i1}x_1+\cdots+d_{in}x_n>0$, for all $(x_1,\ldots,x_n)\in C$. If this assumption does not hold, then the linear fractional objective function f_i can be equivalently expressed as $\frac{-(\theta_i+c_{i1}x_1+\cdots+c_{in}x_n)}{-(\gamma_i+d_{i1}x_1+\cdots+d_{in}x_n)}$.

Now, we express some key definitions and concepts associated with the MLLFP problem [19].

Definition 1. The constraint region of the problem (1) is defined as
$C=\{(x_1,\ldots,x_n)\in X_1\times X_2\cdots\times X_n:\quad \sum\limits_{i=1}^{n}A_ix_i\le b\}$.

Definition 2. The image of C onto $X_1\times\cdots\times X_i$, for $i=2,\ldots,n-1$ is defined as

$$\begin{aligned}C_{X_1\times\cdots\times X_i}=\{(x_1,\ldots,x_i):\ &\text{there is } (x_{i+1},\ldots,x_n)\in X_{i+1}\times\cdots\times X_n\\ &\text{s.t. } (x_1,\ldots,x_i,x_{i+1},\ldots,x_n)\in C\}.\end{aligned}$$

Definition 3. The image C onto $X_i\times\cdots\times X_n$, for $i=2,\ldots,n-1$ is defined as

$$\begin{aligned}C_{X_i\times\cdots\times X_n}=\{(x_i,\ldots,x_n):\ &\text{there is } (x_1,\ldots,x_{i-1})\in X_1\times\cdots\times X_{i-1}\\ &\text{s.t. } (x_1,\ldots,x_i,\ldots,x_n)\in C\}.\end{aligned}$$

Definition 4. The point-to-set mapping $P_n: C_{X_1\times\cdots\times X_{n-1}}\to C_{X_n}$ is defined as

$$\begin{aligned} P_n(x_1,\dots,x_{n-1}) = \{x_n| \;\; & f_n(x_1,\dots,x_n) \\ &= \min\{f_n(x_1,\dots,x_{n-1},\hat{x}_n): \;\; (x_1,\dots,x_{n-1},\hat{x}_n) \in C\}\}. \end{aligned}$$

In actuality, P_n represents the logical reactions of the nth-level DM for the given $(x_1,\dots,x_{n-1})$.

Definition 5. The point-to-set mapping $P_k : C_{X_1\times\cdots\times X_{k-1}} \to C_{X_k\times\cdots\times X_n}$ for $k = 2,\dots,n-1$, is defined as

$$\begin{aligned} & P_k(x_1,\dots,x_{k-1}) \\ & = \big\{(x_k,\dots,x_n)|f_k(x_1,\dots,x_k,\dots,x_n) = \min\{\; f_k(x_1,\dots,x_{k-1},\hat{x}_k,\dots,\hat{x}_n) \\ & \qquad \text{s.t.}(x_1,\dots,x_{k-1},\hat{x}_k,\dots,\hat{x}_n) \in C \\ & \qquad \mathit{and}\ (\hat{x}_{k+1},\dots,\hat{x}_n) \in P_{k+1}(x_1,\dots,x_{k-1},\hat{x}_k)\}\big\}, \end{aligned}$$

where P_k represents the logical reactions of Kth-level DM for fixed values of $(x_1,\dots,x_{k-1})$.

Definition 6. The set of feasible solutions to the MLLFP problem (1) is termed inducible region (IR) and is defined as

$$IR := \{(x_1,\dots,x_n) \in C \mid (x_2,\dots,x_n) \in P_2(x_1)\}. \tag{2}$$

Thus, to obtain an optimum solution to the MLLFP problem (1), we can solve the following problem:

$$\min\{\; f_1(x_1,\dots,x_n)| \quad (x_1,\dots,x_n) \in IR\}. \tag{3}$$

Obviously, an optimum solution to this problem is also optimum for the original problem (1).

We will also use the following assumptions to come up with the existence of an optimal solution to the problem (1), [19].

Assumptions

1. The constraint region C is a nonempty, closed, and bounded polytope.

2. The mapping $P_k(x_1,\dots,x_{k-1})$ assigns a unique value to each $(x_1,\dots,x_{k-1}) \in C_{X_1\times\cdots\times X_{k-1}}$ and for each $2 \le k \le n$.

In the following section, we analyze some definitions, results, and proofs in [2].

3 Examining the definitions, results and proofs in [2]

As the major drawback, the author has mentioned the IR set in several places in [2], while the IR set is not defined throughout the article. Some of the cases where the IR set is used in [2] , are as follows:

1. On page 10, the statement "There exists an extreme point of IR, thus an extreme point of the polyhedron S, which is an optimal solution of the MLLFP problem" , without citing or proof.

2- In the last paragraph on page 14, the author express "Then, the extreme point in W which provides the best value of f_1 is selected to test whether it is a point of IR".

Actually, Bhargava [2] has defined $IR_1, \ldots, IR_{n-1}$ regions, that IR_1 exhibits the feasible region of the BLLFP problem, IR_2 exhibits the feasible region of the trilevel linear fractional programming (TLLFP) problem, and finally, IR_{n-1} exhibits the feasible region of the MLLFP problem with n levels. However, the definition of IR_i is only correct for the two-level case, that is, IR_1. The following TLLFP problem exemplifies this reality (note that any linear function can be considered as a linear fractional function with denominator one):

Example 1. Consider

$$\begin{aligned}
&\min_{x_1} \quad f_1(x_1,x_2,x_3) = -x_1 + 2x_2 - 4x_3,\\
&\min_{x_2} \quad f_2(x_1,x_2,x_3) = 3x_2 - 2x_3,\\
&\min_{x_3} \quad f_3(x_1,x_2,x_3) = 2x_3 - x_2,\\
&s.t. \begin{cases} x_1 + x_2 + x_3 \leq 2,\\ 0 \leq x_1, x_2, x_3 \leq 1. \end{cases}
\end{aligned} \tag{4}$$

By [18], $IR = \{(x_1, 0, 0) : \quad 0 \leq x_1 \leq 1\}$. According to the definition of IR_2 in [2], we have $IR_2 = \{(x_1, x_2, 0) : \quad x_1 + x_2 \leq 2\}$, which is clearly different from the inducible region of the problem.

In [2, Theorem 1], the author claims that there is a border-feasible vertex (or boundary feasible extreme point) of C that solves the MLLFP problem. However, its proof is questionable. In the first paragraph of the proof of Theorem 1, the author asserted that "if $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is not a boundary feasible point, every extreme point adjacent to $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is in $(IR)_1$, and $f_1(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \leq f_1(x_1, x_2, \ldots, x_n)$ for all point $(x_1, x_2, \ldots, x_n)$ adjacent to $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$." Whereas according to [2, Definition 2], if $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is not a boundary feasible point, then every extreme point adjacent to $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is in $(IR)_{n-1}$. Also, the characterization of the optimal solution to the MLLFP problem is provided following Definition 2, and the definition of $(IR)_{n-1}$ on page 6, implies that $f_n(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \leq f_n(x_1, x_2, \ldots, x_n)$ for all point $(x_1, x_2, \ldots, x_n)$ adjacent to $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. Thus the inequality (5) in the proof of Theorem 1 is incorrect.

Bhargava [2] continued by introducing $(x_1^{[1]}, \ldots, x_n^{[1]})$ as the optimum solution to the relaxed problem (5), (defined in the next section). In[2, Theorem 2], the author stated that there is an edge path in C from any arbitrary vertex of C to $(x_1^{[1]}, \ldots, x_n^{[1]})$ that the value of $f_1, \ldots, f_n$ is nonincreasing along it. This conclusion does not always hold. As a counterexample, consider the vertex $(0, 1, 0)$ in the above example. Here, $(x_1^{[1]}, x_2^{[1]}, x_3^{[1]}) = (1, 0, 1)$. In Figure 1, it is easily seen that each edge path from F to C must include one of the edges from A to C, D to C, B to C, or E to C. However, the function f_3 is not nonincreasing along any of these paths.

The author then proposed a Kth-best algorithm to solve the MLLFP problem. Step 6 of the proposed algorithm [2, Page 15] begins with the phrase "on continuing the process" that is unclear for implementation. In what follows, we first present an extension of the Kth-best algorithm presented in [3] to solve the TLLFP problem, and then for the MLLFP problems with more than three levels.

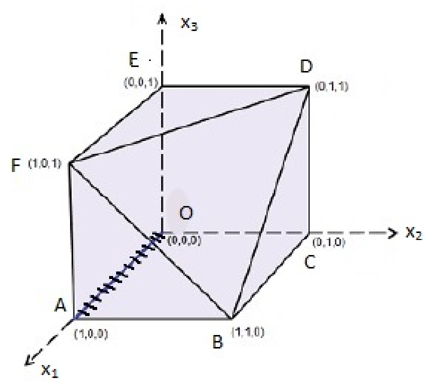

Figure 1: IR and the constraint region

4 Geometric characteristics

In this section, we first state several required definitions, and then we prove some main theorems on the optimum solution of the MLLFP problem (1).

Definition 7 (see [14]). Assume that g is a real-valued function defined on a convex subset X of R^n. Then

(1) g is quasi-concave on X if and only if $x^1, x^2 \in X, \lambda \in [0,1]$ and $g(x^1) \leq g(x^2) \Longrightarrow g(x^1) \leq$

$g[(1-\lambda)x^1+\lambda x^2]$.
(2) g is semi-strictly quasi-concave on X if and only if $x^1, x^2 \in X, x^1 \neq x^2, \lambda \in (0,1)$, and $g(x^1) < g(x^2) \Longrightarrow g(x^1) < g[(1-\lambda)x^1+\lambda x^2]$.
(3) g is explicitly quasi-concave on X if and only if it is quasi-concave and semi-strictly quasi-concave on X.
The function g is explicitly quasi-convex if and only if $-g$ is explicitly quasi-concave.
(4) g is explicitly quasi-monotonic on X if and only if it is explicitly quasi-concave and explicitly quasi-convex on X.

In the MLLFP problem (1), the linear fractional functions $f_1, f_2, \ldots, f_n$ are quasi-concave. Specifically, they are explicitly quasi-monotonic on C because $\gamma_i + d_{i1}x_1 + \cdots + d_{in}x_n \neq 0$ on S, for all $i = 1, \ldots, n$ (see [14, Theorem 3.53]). This establishes that the problem (1) belongs to a subclass of quasi-concave multilevel programming problems. In light of Assumption 2, it follows from [19] that

• the inducible region of the problem (1) is formed by the union of connected faces of the polytope C. Therefore, IR may not be a convex set.
• There exists a vertex of C (it also is a vertex of IR), which represents an optimum solution to the MLLFP problem (1).

Definition 8 (see [13])**.** A point $(x_1, \ldots, x_n) \in IR$ is termed a border-feasible vertex of C if there is an edge E of C in such a way that $(x_1, \ldots, x_n)$ is a vertex of E and another vertex of E does not lie in IR.

In the following theorem, we demonstrate that under some conditions, the optimum solution to the MLLFP problem (1) is a border-feasible vertex of IR. For this purpose, we introduce the following relaxed problem:

$$\begin{array}{ll} \min & f_1(x_1, x_2, \ldots, x_n) = \frac{\theta_1 + c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n}{\gamma_1 + d_{11}x_1 + d_{12}x_2 + \cdots + d_{1n}x_n} \\ s.t. & (x_1, x_2, \ldots, x_n) \in C. \end{array} \tag{5}$$

It is a well-established result from [14] that there exists a vertex of C that solves the problem (5). Denote this solution by $(x_{11}^{[1]}, \ldots, x_{1n}^{[1]})$. Note that f_1 is a quasi-concave function and that C is a nonempty, closed, and bounded polytope.

Theorem 1. If there exists a vertex of C outside IR that solves the problem (5), then an optimum solution to the MLLFP problem (1) lies on a border-feasible vertex of IR.

Proof. It is well known that an optimum solution to the MLLFP problem (1) lies on a vertex of C (see [19, Theorem 10]). We call it $(x_1^o, \ldots, x_n^o)$. Obviously, $(x_1^o, \ldots, x_n^o) \in IR$. If $(x_1^o, \ldots, x_n^o)$ is a border-feasible vertex of C, then the result follows, or else, all vertices adjacent to $(x_1^o, \ldots, x_n^o)$ are in IR and for every vertex adjacent to $(x_1^o, \ldots, x_n^o)$ in C, such as $(x_1, \ldots, x_n)$, we have

$$f_1(x_1^o, \ldots, x_n^o) \leq f_1(x_1, \ldots, x_n). \tag{6}$$

We now establish that there exists a vertex of C, as $(x_1', \ldots, x_n')$, adjacent to $(x_1^o, \ldots, x_n^o)$ satisfying

$$f_1(x_1', \ldots, x_n') = f_1(x_1^o, \ldots, x_n^o). \tag{7}$$

To demonstrate this, one can examine the relaxed problem (5). It yields from (6) that $(x_1^o, \ldots, x_n^o)$ is a local vertex-minimizer of f_1 in C. Because f_1 is quasi-concave, and explicitly quasi-convex on C, the point $(x_1^o, \ldots, x_n^o)$ can be considered as a global minimum solution to the problem (5) (see [14, Theorem 5.13]), that is,

$$f_1(x_1^o, \ldots, x_n^o) \leq f_1(x_1, \ldots, x_n), \quad \text{tfor all } (x_1, \ldots, x_n) \in C. \tag{8}$$

By hypothesis, a vertex of C, as $(x_1'', \ldots, x_n'')$, outside IR exists that is the optimum solution to the problem (5). Therefore, $f_1(x_1^o, \ldots, x_n^o) = f_1(x_1'', \ldots, x_n'')$. Notably, $(x_1'', \ldots, x_n'')$ is not adjacent to $(x_1^o, \ldots, x_n^o)$ because $(x_1^o, \ldots, x_n^o)$ is not a border-feasible vertex of C. Also, f_1 is continuous, quasi-convex, and explicitly quasi-concave on C. Thus, the convex hull of some vertices of C represents the set of optimum solutions to the problem (5) (see [14, Theorem 5.21]). So, the set of optimum solutions to the problem (5) is a polyhedron. Thus, there is an edge path in such polyhedron from $(x_1^o, \ldots, x_n^o)$ to $(x_1'', \ldots, x_n'')$. Then, there must be a vertex of C, as $(x_1', \ldots, x_n')$, adjacent to $(x_1^o, \ldots, x_n^o)$ remaining in the set of optimum solutions to the problem (5). Therefore, the validity of (7) is confirmed. If $(x_1', \ldots, x_n')$ is a border-feasible vertex of C, then the result follows, or else, we examine the vertex $(x_1', \ldots, x_n')$ instead of $(x_1^o, \ldots, x_n^o)$ and continue the aforementioned process. We obtain a new vertex of C adjacent to $(x_1', \ldots, x_n')$ for which (7) holds. If the new vertex is a border-feasible vertex of C, then the result follows. Else, by continuing the process, a border-feasible vertex of C will be obtained as an optimum solution to the MLLFP (1), after a finite number of stages (note that the number of vertices of the polytope C is finite). □

Theorem 2. Let $(x_1, \ldots, x_n)$ be a vertex of C. Then, an edge path in C from $(x_1, \ldots, x_n)$ to $(x_{11}^{[1]}, \ldots, x_{1n}^{[1]})$ exists so that along which the value of f_1 is nonincreasing.

Proof. See the proof of Theorem 6 in [3]. □

5 The Kth-best algorithm

In this section, we first present the Kth-best algorithm for TLLFP problem, that is, for case $n = 3$ and then for the MLLFP problems with $n > 3$ levels. Consider the following TLLFP problem:

$$\begin{aligned}
\min_{x_1\in X_1} \quad & f_1(x_1,x_2,x_3) = \frac{\theta_1 + c_{11}x_1 + c_{12}x_2 + c_{13}x_3}{\gamma_1 + d_{11}x_1 + d_{12}x_2 + d_{13}x_3},\\
\min_{x_2\in X_2} \quad & f_2(x_1,x_2,x_3) = \frac{\theta_2 + c_{21}x_1 + c_{22}x_2 + c_{23}x_3}{\gamma_2 + d_{21}x_1 + d_{22}x_2 + d_{23}x_3},\\
\min_{x_3\in X_3} \quad & f_3(x_1,x_2,x_3) = \frac{\theta_3 + c_{31}x_1 + c_{32}x_2 + c_{33}x_3}{\gamma_3 + d_{31}x_1 + d_{32}x_2 + d_{33}x_3},\\
s.t. \quad & \sum_{i=1}^{3} A_i x_i \le b. \qquad (9)
\end{aligned}$$

5.1 The Kth-best algorithm for the TLLFP problem

Step 1. Constitute the relaxed problem as follows:

$$\begin{aligned}
&\min \;\; f_1(x_1,x_2,x_3)\\
&s.t. \quad (x_1,x_2,x_3) \in C.
\end{aligned} \qquad (10)$$

Obtain an optimum solution to it and call it $(x_{11}^{[1]}, x_{12}^{[1]}, x_{13}^{[1]})$. Set $i = 1$, $U = \emptyset$, $V = \{(x_{11}^{[1]}, x_{12}^{[1]}, x_{13}^{[1]})\}$, and go to Step 2.

Step 2. Fix $x_1 = x_{11}^{[i]}$ and $x_2 = x_{12}^{[i]}$, and obtain an optimum solution to the following problem:

$$\begin{aligned}
\min \quad & f_3(x_{11}^{[i]}, x_{12}^{[i]}, x_3)\\
s.t. \quad & (x_{11}^{[i]}, x_{12}^{[i]},\; x_3) \in C.
\end{aligned} \qquad (11)$$

Call it $x_{33}^{[i]}$. If $x_{13}^{[i]} = x_{33}^{[i]}$, then go to Step 3. Else, go to Step 4.

• At this Step, it is tested whether point $(x_{11}^{[i]}, x_{12}^{[i]}, x_{13}^{[i]})$ belongs to the set IR of the bilevel problem, which consists of the second-level and third-level problems. If it is not, then Step 3 is not executed. Thus, the computational effort required to solve the TLLFP problem is reduced.

Step 3. Fix $x_1 = x_{11}^{[i]}$ and obtain an optimum solution to the following bilevel problem:

$$
\begin{aligned}
&\min_{x_2} \ f_2(x_{11}^{[i]}, x_2, x_3),\\
&\quad \min_{x_3} \ f_3(x_{11}^{[i]},\ x_2,\ x_3)\\
&\quad s.t. \quad (x_{11}^{[i]},\ x_2,\ x_3) \in C.
\end{aligned} \tag{12}
$$

Let $(x_{22}^{[i]}, x_{23}^{[i]})$ be an optimum solution to the bilevel problem (12). If $(x_{22}^{[i]},\ x_{23}^{[i]}) = (x_{12}^{[i]}, x_{13}^{[i]})$, then $(x_{11}^{[i]}, x_{12}^{[i]}, x_{13}^{[i]})$ is an optimum solution to the problem (9) and stop. Else, go to Step 4.

Step 4. Let $V^{[i]}$ be the set of all vertices adjacent to $(x_{11}^{[i]}, x_{12}^{[i]}, x_{13}^{[i]})$ in C. Set $U = U \cup \{(x_{11}^{[i]}, x_{12}^{[i]}, x_{13}^{[i]})\}$, $V = (V \cup V^{[i]}) \backslash U$, and go to Step 5.

- One can use the presented algorithm in [15, Section 3.7.1] to obtain all adjacent vertices.

Step 5. $i \longleftarrow i+1$ and choose $(x_{11}^{[i]}, x_{12}^{[i]}, x_{13}^{[i]}) \in V$ such that
$f_1(x_{11}^{[i]},\ x_{12}^{[i]},\ x_{13}^{[i]}) = \min_{(x_1,x_2,x_3)\in V} f_1(x_1,\ x_2,\ x_3)$,
and go to Step 2.

- To do this Step, one can sort the points of V in ascending order of f_1 objective value in a list from top to bottom and then select the vertex at the top of the list.

5.2 The *K*th-best algorithm for the MLLFP problem

Suppose that the Kth-best algorithm is available to solve the MLLFP problems with $n-1$ levels. The Kth-best algorithm to solve the MLLFP problems with n levels is as follows:

The algorithm

Step 1. Find an optimum solution to the relaxed problem (5) and call it $(x_{11}^{[1]}, \dots, x_{1n}^{[1]})$. Set $i = 1,\ U = \emptyset$, $V = \{(x_{11}^{[1]}, \dots, x_{1n}^{[1]})\}$, and go to Step 2.
Step 2. Fix $x_1 = x_{11}^{[i]}$. Solve the following linear fractional $(n-1)$-level programming problem:

$$
\begin{aligned}
&\min_{x_2} \ f_2(x_{11}^{[i]}, x_2, \dots, x_n),\\
&\vdots\\
&\min_{x_n} \ f_n(x_{11}^{[i]}, x_2, \dots, x_n),\\
&s.t. \quad (x_{11}^{[i]}, x_2, \dots, x_n) \in C.
\end{aligned} \tag{13}
$$

Call its optimum solution $(x_{11}^{[i]}, x_{n-1,2}^{[i]}, \dots, x_{n-1,n}^{[i]})$. If $(x_{11}^{[i]}, \dots, x_{1n}^{[i]}) = (x_{11}^{[i]}, x_{n-1,2}^{[i]}, \dots, x_{n-1,n}^{[i]})$, then $(x_{11}^{[i]}, \dots, x_{1n}^{[i]})$ is an optimum solution to the problem (1) and stop. Else, go to Step 3.

Step 3. Let $V^{[i]}$ be the set of all vertices adjacent to $(x_{11}^{[i]}, x_{12}^{[i]}, \ldots, x_{1n}^{[i]})$ in C. Set $U = U \cup \{(x_{11}^{[i]}, x_{12}^{[i]}, \ldots, x_{1n}^{[i]})\}$, $V = (V \cup V^{[i]}) \backslash U$, and go to Step 4.
Step 4. $i \longleftarrow i+1$ and choose $(x_{11}^{[i]}, x_{12}^{[i]}, \ldots, x_{1n}^{[i]}) \in V$ such that
$f_1(x_{11}^{[i]},\ x_{12}^{[i]}, \ldots,\ x_{1n}^{[i]}) = \min_{(x_1,x_2,\ldots,x_n)\in V} f_1(x_1,\ x_2, \ldots,\ x_n)$,
and go to Step 2.

According to the above algorithm first, an optimum solution to the relaxed problem (5) is obtained, called $(x_{11}^{[1]}, \ldots, x_{1n}^{[1]})$. Step 2 tests whether the obtained point belongs to the IR of problem (1) or not. If the answer is positive, then it is an optimum solution to the problem (1), and the algorithm stops. If not, it will be eliminated from V and the set of all vertices adjacent to $(x_{11}^{[1]}, \ldots, x_{1n}^{[1]})$, that is, $V^{[i]}$ will be added to V. Then, the vertex in V that produces the second-best value of f_1, is picked out for testing whether it belongs to IR or not. The process stops when it reaches the first point in the IR of the problem (1).
Now, let $(x_{11}^{[1]}, \ldots, x_{1n}^{[1]}), (x_{11}^{[2]}, \ldots, x_{1n}^{[2]}), \ldots, (x_{11}^{[t]}, \ldots, x_{1n}^{[t]})$ be the t-ordered vertex solutions of the problem (5), which are computed successively using the described algorithm, that is,

$$f_1(x_{11}^{[i]}, \ldots, x_{1n}^{[i]}) \leq f_1(x_{11}^{[i+1]}, \ldots, x_{1n}^{[i+1]}), \quad i = 1, \ldots, t-1.$$

Clearly, $(x_{11}^{[k]}, \ldots, x_{1n}^{[k]})$ is a global optimum solution for the MLLFP problem (1) if

$$k = \min_{i\in\{1,\ldots,t\}} \{i \mid (x_{11}^{[i]}, \ldots, x_{1n}^{[i]}) \in IR\}.$$

Hence, based on Theorem 2, the Kth-best algorithm outputs a global optimum solution to the problem (1). However, if an optimum solution to the relaxed problem (5) lies out of IR, it can be inferred that the Kth-best algorithm identifies a border-feasible vertex of IR as an optimum solution to the problem (1). The following theorem highlights the relationship between the vertices generated by the proposed algorithm.

Theorem 3. The $(i+1)$st best vertex of C, $(x_{11}^{[i+1]}, \ldots, x_{1n}^{[i+1]})$, is adjacent to one of earlier vertices $(x_{11}^{[1]}, \ldots, x_{1n}^{[1]})$, $(x_{11}^{[2]}, \ldots, x_{1n}^{[2]})$ $\ldots$, or $(x_{11}^{[i]}, \ldots, x_{1n}^{[i]})$, $i < t$.

Proof. See the proof of Theorem 7 in [3]. □

Theorem 4. Assume that there is a vertex of C outside IR which solves the relaxed problem (5). Then, the Kth-best algorithm gets a border-feasible vertex of IR as an optimum solution to the MLLFP problem (1).

Proof. It follows from Theorem 1 that an optimum solution to the MLLFP problem (1) takes place at a border-feasible vertex of IR. On the other hand, the Theorem 3 implies that the ith best vertex of the relaxed problem (5) is adjacent to one of the 1st, 2nd, ... , or $(i-1)$st best vertex. Based on Theorem 2 and according to the termination condition, the Kth-best algorithm gets a global optimum solution to the problem (1) at a border-feasible vertex of IR. $\square$

6 Numerical examples

In this section, a number of numerical examples are solved by the presented Kth-best algorithm that show the efficiency of the algorithm.

Example 2. Consider

$$\begin{aligned}
&\min_{x_1} \quad f_1 = -8x_1 - 4x_2 + 4x_3 - 40x_4 - 4x_5\\
&\min_{x_2} \quad f_2 = \frac{-x_2 + 3x_3 + 4x_4 - x_5}{x_1 + x_2 + x_3 + x_4 + 2},\\
&\min_{(x_3,x_4,x_5)} f_3 = \frac{x_1 + x_2 + 2x_3 - x_4 + x_5 + 1}{2x_1 + x_3 + x_4 - 3x_5 + 6},\\
&s.t. \begin{cases} -x_3 + x_4 + x_5 \leq 1, \\ 2x_1 - x_3 + 2x_4 - 0.5x_5 \leq 1, \\ 2x_2 + 2x_3 - x_4 - 0.5x_5 \leq 1, \\ x_i \geq 0, \qquad\qquad i = 1,2,3,4,5. \end{cases}
\end{aligned}$$

First iteration of the Kth-best algorithm:

Step 1. By solving the corresponding problem (10), the point $(0, 0, \frac{3}{2}, \frac{3}{2}, 1)$ is obtained as the optimum solution. The optimum value of objective function f_1 is -58.

Step 2. Fix $x_1 = x_2 = 0$ and solve the following problem:

$$\begin{aligned}
&\min \quad f_3 = \frac{2x_3 - x_4 + x_5 + 1}{x_3 + x_4 - 3x_5 + 6},\\
&s.t. \begin{cases} -x_3 + x_4 + x_5 \leq 1, \\ -x_3 + 2x_4 - 0.5x_5 \leq 1, \\ 2x_3 - x_4 - 0.5x_5 \leq 1, \\ x_i \geq 0, \qquad\qquad i = 3,4,5. \end{cases}
\end{aligned}$$

The optimum solution is $(x_3, x_4, x_5) = (0, \frac{1}{2}, 0)$. Since $(0, \frac{1}{2}, 0) \neq (\frac{3}{2}, \frac{3}{2}, 1)$, we go to Step 4.

Step 4. We obtain $V^{[1]} = \{(\frac{3}{2}, 0, 1, 0, 2), (0, \frac{9}{10}, 0, \frac{3}{5}, \frac{2}{5}), (0, 0, 1, 1, 0), (0, 0, 1, 0, 2), (0, 0, 0, \frac{3}{5}, \frac{2}{5})\}$

and $U = \{(0,0,\frac{3}{2},\frac{3}{2},1)\}, V = V^{[1]}$.

Step 5. We set $i = 2$ and $(x_{11}^{[2]},\dots,x_{15}^{[2]}) = (0,0,1,1,0)$ with $f_1 = -36$ and go to Step 2.

The output of the rest of iterations are displayed in Table 1. As seen in Table 1, in the third iteration of the algorithm, the point $(0,\frac{9}{10},0,\frac{3}{5},\frac{2}{5})$ is obtained as optimum solution. The optimum value of (f_1,f_2,f_3) equals $(-\frac{146}{5},\frac{11}{35},\frac{17}{54}) = (-29.2, 0.314, 0.315)$.

Table 1: The output of each iteration in Kth-best algorithm for Example 2

Iteration	$(x_{11}^{[i]},\dots,x_{15}^{[i]})$	V	
$i=1$	$(0,0,\frac{3}{2},\frac{3}{2},1) \notin IR$	$(\frac{3}{2},0,1,0,2)$	$f_1 = -16$
	$f_1 = -58$	$(0,\frac{9}{10},0,\frac{3}{5},\frac{2}{5})$	$f_1 = -29.2$
		$(0,0,1,1,0)$	$f_1 = -36$
		$(0,0,1,0,2)$	$f_1 = -4$
		$(0,0,0,\frac{3}{5},\frac{2}{5})$	$f_1 = -25.6$
$i=2$	$(0,0,1,1,0) \notin IR$	$(\frac{3}{4},0,\frac{1}{2},0,0)$	$f_1 = -4$
	$f_1 = -36$	$(0,\frac{3}{4},0,\frac{1}{2},0)$	$f_1 = -23$
		$(0,0,\frac{1}{2},0,0)$	$f_1 = 2$
		$(0,0,0,\frac{1}{2},0)$	$f_1 = -20$
$i=3$	$(0,\frac{9}{10},0,\frac{3}{5},\frac{2}{5}) \in IR$		
	$f_1 = \frac{-146}{5} = -29.2$		

Example 3. Consider

$$
\begin{aligned}
&\max_{x_1,x_2} \quad f_1 = \frac{7x_1+3x_2-4x_3+2x_4}{x_1+x_2+x_3+1},\\
&\max_{x_3} \quad f_2 = \frac{x_2+3x_3+4x_4}{x_1+x_2+x_3+2},\\
&\max_{x_4} \quad f_3 = \frac{2x_1+x_2+x_3+x_4}{x_1+x_2+x_3+3},\\
&s.t. \begin{cases}
x_1+x_2+x_3+x_4 \le 5, & x_1+x_2-x_3-x_4 \le 2,\\
x_1+x_2+x_3 \ge 1, & x_1-x_2+x_3+2x_4 \le 4,\\
x_1+2x_3+2x_4 \le 3, & x_4 \le 2,\\
x_i \ge 0, & i=1,2,3,4.
\end{cases}
\end{aligned}
$$

By applying the proposed Kth-best algorithm, the point $(\frac{7}{3},0,0,\frac{1}{3})$ is obtained as the optimum solution in the first iteration. The optimum value of (f_1,f_2,f_3) equals $(\frac{51}{10},\frac{11}{13},\frac{15}{16})$. However, Lachhwani and Poonia (2012) obtained the optimum solution $(x_1,x_2,x_3,x_4) = (1,0,0,1)$ with the optimum value of $(f_1,f_2,f_3) = (\frac{9}{2},\frac{4}{3},\frac{3}{4})$. Also, Lachwani and Nehra [12] using a modified fuzzy goal programming approach obtained the point $(2.33,0,0,2.37)$ as optimum solution, which is clearly infeasible because of the constraint $x_4 \le 2$.

Table 2: The output of each iteration in Kth-best algorithm for Example 4

Iteration	$(x_{11}^{[i]},\dots,x_{15}^{[i]})$	V
$i=1$	$(0,\frac{3}{4},0,0,1)\notin IR$ $f_1=\frac{1}{52}$	$(\frac{3}{4},\frac{3}{4},0,0,1), f_1=\frac{4}{49}$ $(0,0,1,0,2), f_1=\frac{1}{16}$ $(0,\frac{9}{10},0,\frac{3}{5},\frac{2}{5}), f_1=\frac{13}{106}$ $(0,\frac{1}{2},0,0,0), f_1=\frac{1}{16}$ $(0,0,0,0,1), f_1=\frac{1}{13}$
$i=2$	$(0,\frac{1}{2},0,0,0)\notin IR$ $f_1=\frac{1}{16}$	$(\frac{1}{2},\frac{1}{2},0,0,0), f_1=\frac{2}{15}$ $(0,0,\frac{1}{2},0,0), f_1=\frac{1}{7}$ $(0,\frac{5}{16},0,\frac{1}{2},0), f_1=\frac{109}{272}$ $(0,0,0,0,0), f_1=\frac{1}{8}$
$i=3$	$(0,0,1,0,2)\notin IR$ $f_1=\frac{1}{16}$	$(\frac{3}{2},0,1,0,2), f_1=\frac{5}{29}$ $(0,0,\frac{3}{2},\frac{3}{2},1), f_1=\frac{8}{23}$ $(0,0,\frac{1}{2},0,0), f_1=\frac{1}{7}$ $(0,0,0,0,1), f_1=\frac{1}{13}$
$i=4$	$(0,0,0,0,1)\notin IR$ $f_1=\frac{1}{13}$	$(0,0,\frac{1}{2},0,0), f_1=\frac{1}{7}$ $(0,0,0,\frac{3}{5},\frac{2}{5}), f_1=\frac{11}{53}$ $(0,0,0,0,0), f_1=\frac{1}{8}$
$i=5$	$(\frac{3}{4},\frac{3}{4},0,0,1)\in IR$ $f_1=\frac{4}{49}$	

Example 4. Consider

$$\begin{aligned}
&\min_{(x_1,x_2)} \quad f_1=\frac{1+x_1-x_2+2x_4}{8-x_1-2x_3+x_4+5x_5},\\
&\min_{(x_3,x_4)} \quad f_2=\frac{1+x_1+x_2+2x_3-x_4+x_5}{6+2x_1+x_3+x_4-3x_5},\\
&\min_{x_5} \quad f_3=\frac{2+x_1+x_2-x_3+x_4+x_5}{1-x_1+x_2-x_3+2x_4+x_5},\\
&s.t.\begin{cases}
-x_3+x_4+x_5\le 1,\\
2x_1-x_3+2x_4-\frac{1}{2}x_5\le 1,\\
2x_2+2x_3-x_4-\frac{1}{2}x_5\le 1,\\
x_i\ge 0, \qquad\qquad i=1,\dots,5.
\end{cases}
\end{aligned}$$

The result of the proposed Kth-best algorithm is shown below. As seen in Table 2, the optimum solution $(\frac{3}{4},\frac{3}{4},0,0,1)$ is obtained in the fifth iteration with the optimum value of $(f_1,f_2,f_3)=(\frac{4}{49},\frac{7}{9},\frac{9}{4})=(0.0816,0.777,1.119)$.

Example 5. Consider

$$
\begin{aligned}
&\min_{x_1} \quad f_1 = \frac{-x_1 - 5x_2 - x_3 + x_4 - 4}{x_1 + 12x_2 + x_3 + 2x_4},\\
&\min_{x_2} \quad f_2 = \frac{-8x_1 - 7x_2 - 2x_3 - x_4 - 4}{9x_1 + 11x_2 + x_3 + x_4 + 5},\\
&\min_{x_3} \quad f_3 = \frac{-12x_1 - 3x_2 - x_3 + x_4 - 4}{12x_1 + 5x_2 + x_3 + 2x_4 + 3},\\
&\min_{x_4} \quad f_4 = \frac{-4x_1 - 2x_2 - x_3 - x_4 - 4}{5x_1 + 3x_2 + x_3 + x_4 + 5},\\
&s.t. \begin{cases}
x_1 + x_2 + x_3 - x_4 \leq 3, \quad x_1 + x_2 + x_3 - x_4 \geq 1, \\
x_1 + x_2 - x_3 - x_4 \leq 7, \quad -x_1 + x_2 + x_3 - x_4 \leq 1, \\
x_2 \geq 0.25, \quad x_3 \leq 1, \quad x_4 \leq 3, \quad x_i \geq 0, \qquad i = 1, 2, 3, 4.
\end{cases}
\end{aligned}
$$

Using the Kth-best algorithm, we obtain the optimum solution $(0, \frac{1}{4}, \frac{3}{4}, 0)$ in the first iteration. The optimum value of $(f_1, \dots, f_4)$ is equal to $(-1.6, -0.88, -1.1, -0.808)$. However, Pal, Kumar, and Sen [17] found the optimum solution $(\frac{1}{2}, \frac{1}{4}, 1, \frac{1}{4})$ with the optimum value of $(f_1, \dots, f_4) = (-1.33, -0.96, -0.89, -0.82)$.

7 Conclusions

In this study, an MLLFP problem has been considered wherein common constraint region is a closed, and bounded polytope. Under mild conditions, it is proved that an optimum solution to the MLLFP problem occurs at a border-feasible vertex of the inducible region. Also, a former published paper on the MLLFP problems has been considered, and some drawbacks in that paper have been stated. Moreover, the Kth-best algorithm has been developed for solving the MLLFP problem in question, and its convergence is proved. The efficiency of the proposed algorithm has been demonstrated by solving several numerical examples and comparing the results with those obtained by some existing methods. The study of the MLLFP problems over an unbounded constraint region will be addressed in our future research.

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Conflicts of Interest

The authors declare no conflicts of interest.

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