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Research Article

Utilizing the Hybrid approach of the Ramadan group transform and accelerated Adomian method for solving nonlinear integro-differential equations

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Abstract

In this paper, we investigate the application of the combination of the Ramadan group transform and the accelerated Adomian polynomial

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method for solving integro-differential equations. Integro-differential equations arise in various fields such as physics, engineering, and biology, often modeling complex phenomena. The Ramadan group transform, known for its transformation properties and its ability to simplify computational complexities, is coupled with the accelerated Adomian polynomial method, which is an effective series expansion technique. This combination enhances the convergence and efficiency of solving nonlinear integro-differential equations that are difficult to handle using traditional methods. The paper demonstrates the utility of this hybrid approach through several test cases, comparing it with existing methods in terms of accuracy, computational efficiency, and convergence rate.

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1 Introduction

An equation with the unknown function under the sign of integration and including the unknown function's derivatives is known as an integro-differential equation (IDE). It falls into one of two categories: Volterra equations or Fredholm equations. IDEs are one of the most important tools in mathematics [33].

Many researchers and scientists investigated IDEs while working on scientific applications such as heat transformers, neutron diffusion, and biological species coexisting with growing and decreasing rates of production and diffusion processes. Applications in physics, biology, and engineering, as well as models addressing complex integral equations like [14, 16], also use these kinds of equations. An IDE system can be solved using a variety of methods, such as the variational iteration method (VIM) [29], the rationalized Haar functions method [18], the Adomian decomposition method (ADM) [8, 26], and work by Younis and Al-Hayani [31, 3], the Galerkin method [19], and

He's homotopy perturbation method (HPM) [9, 32] and the work by Younis and Al-Hayani [30].

The analytical method known as ADM uses Adomian polynomials to evaluate the answer. Both linear and nonlinear issues can be solved using this method, which neither simplifies nor discretizes the provided problem. The Galerkin and rationalized Haar function methods are numerical techniques that can be used to solve IDEs in a variety of ways. The HPM, introduced by He in 1997 and further detailed in 2000, combines traditional perturbation techniques with the concept of Homotopy from topology [15]. He developed and extended this innovative method, which has since been applied to a wide range of linear and nonlinear problems.

Also, the use of the Laplace transform HPM by Al-Hayani [2]. Another analytical method, the VIM, is also capable of addressing various linear and nonlinear challenges. Additionally, Avudainayagam and Vani [5] explored the use of wavelet bases for solving IDEs. They proposed a method for computing a novel four-dimensional connection coefficient and validated their approach by solving two basic educational nonlinear IDEs [6].

Interest in linear and nonlinear Volterra integro-differential equations (VIDEs), which blend differential and integral components, has significantly increased in recent years [28]. Nonlinear VITEs are fundamental in various areas of nonlinear functional analysis and find widespread applications in engineering, mechanics, physics, electrostatics, biology, chemistry, and economics [7].

Recently, Ramadan et al. [21, 23, 22] have proposed the Ramadan group transform (RGT) and the accelerated Adomian method to address solutions for quadratic Riccati differential equations, the nonlinear Sharma–Tasso–Olver equation, and other forms of nonlinear partial differential equations.

In this paper, we present the RGT and accelerated Adomian method for solving the nonlinear VIDEs of the type:

$$y^{(i)}(x) = f(x) + \gamma \int_0^x K(x,t)G(y(x))dt \ ,$$

with the initial conditions

$$y^{(r)}(a) = b_r, \qquad r = 0, 1, 2, \ldots, \ (i-1),$$

where $y^{(i)}(x)$ is the ith derivative of the unknown function $y(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ is an analytic function, G is a nonlinear function of y, and a, b, γ and b_r are real finite constants. The main objective of this contribution is to present a comparative study of solving IDEs using the RGT method coupled with an accelerated Adomian method and solving them using other methods.

This paper is organized as follows:
Mathematical preliminaries and notions are stated in Section 2.
In Section 3, the analysis of the hybrid RGT accelerated Adomian method is explained thoroughly.
In Section 4, the proof of convergence of the hybrid RGT accelerated Adomian method when applied to a class of nonlinear Volterra-type IDEs, including the sufficient conditions guaranteeing existence and uniqueness are introduced.
To demonstrate the correctness and effectiveness of the suggested approach in comparison to the current one's numerical examples are solved in Section 5.
Concluding remarks are given in the last section.

2 Mathematical preliminaries and notions

We give the reader basic definitions and theorems in this section so they may comprehend RGT and its fundamental characteristics.

2.1 The Adomian polynomials [1]

A wide range of linear and nonlinear functional equations can be analytically approximated using the ADM.

The solution is defined by the infinite series in the standard ADM,

$$y = \sum_{n=0}^{\infty} y_n \ ,$$

after which the nonlinear term Ny is broken down into an infinite series. Moreover,

$$Ny = \sum_{n=0}^{\infty} A_n \ ,$$

where the regular Adomian polynomials are denoted by the A_n and are derived using the definitional formula for the nonlinearity $Ny = f(y)$. Also,

$$A_n = \frac{1}{\boldsymbol{n!}}\left(\frac{d^n}{d\lambda_n}[N(\sum_{i=0}^{n}\lambda^i y_i)]\right)_{\lambda=0} \ , \ n = 0, 1, 2, \ldots .$$

If $N(y) = \ y^2(x)$, then Adomian polynomials [1, 10] are

$$A_0 = {y_0}^2 \ ,$$
$$A_1 = 2y_0y_1,$$
$$A_2 = {y_1}^2 \ + 2y_0y_2 \ .$$

Ordinary and partial differential equations are solved by approximating the nonlinear term functions using the Adomian polynomials $\{A_n\}$.

2.2 Accelerated Adomian polynomials (El-Kalla Adomian polynomials) [27, 12, 13, 11]

The accelerated Adomian polynomials are given in the following form:

$$\overline{A}_n = N(s_n) - \sum_{i=0}^{n-1} \overline{A}_i,$$

where $\overline{A}_n$, are accelerated Adomian polynomials, $\overline{A}_0$, $\overline{A}_1$, $\overline{A}_2$,... and $N(s_n)$. Use the nonlinearity (n-times) to substitute the total of the responses.

If $N(y) = \ y^2(x)$, then accelerated Adomian polynomials are

$$\overline{A}_0 = {y_0}^2 \ ,$$
$$\overline{A}_1 = 2y_0y_1 + {y_1}^2 \ ,$$
$$\overline{A}_2 = 2y_0y_2 + 2y_1y_2 + {y_2}^2 \ ,$$

and if $N(y) = y^3(x)$, then accelerated Adomian polynomials are

$$\begin{aligned}
\overline{A}_0 &= {y_0}^3, \\
\overline{A}_1 &= 3{y_0}^2 y_1 + 3y_0{y_1}^2 + {y_1}^3, \\
\overline{A}_2 &= 3{y_0}^2 y_2 + 6y_0y_1y_2 + 3{y_1}^2 y_2 + 3y_0{y_2}^2 + 3y_1{y_2}^2 + {y_2}^3.
\end{aligned}$$

2.3 Ramadan group integral transform [24]

For exponentially ordered functions, a novel integral RGT was introduced. Functions in set A are examined, as defined by

$$A = \{f(t) : \exists M, t_1, t_2 > 0 \ s.t. \ |f(t)| < Me^{\frac{|t|}{t_n}}, \ if \ t \in (-1)^n \times [0, \infty)\}.$$

The RGT is defined by

$$\begin{aligned}
K(s,u) &= RG[f(t);(s,u)] \\
&= \begin{cases} \displaystyle\int_0^\infty e^{-st} f(ut)\,dt, & -t_1 < u \leq 0, \\ \displaystyle\int_0^\infty e^{-st} f(ut)\,dt, & 0 \leq u < t_2. \end{cases}
\end{aligned}$$

2.4 Ramadan group transform (RGT) convolution theorem

Definition 1 (Convolution of two functions [20]). The convolution of piecewise continuous functions $f(x), g(x) : R \to R$ is the function $f * g : R \to R$ and is determined by the integral

$$f * g = \int_0^x f(t)g(x-t)dt.$$

Theorem 1 (Convolution theorem of RGT). [20]

Let $f(x)$ and $g(x)$ be two functions with RGTs $K_1(s,u)$ and $K_2(s,u)$, respectively. Then

$$RG\left[(f*g)(s,u)\right] = uK_1(s,u)\,K_2(s,u)\,,$$

and

$$RG^{-1}\left[uK_1(s,u)\,K_2(s,u)\right] = f*g.$$

Proof. See [21] for theorem's proof. □

Table 1: Ramadan group transform (RGT) of some functions

$f(t)$	$RG[f(t)] = K(s,u))$
1	$\frac{1}{\mathrm{s}}$
t	$\frac{u}{s^2}$
$\frac{\mathrm{t}^{\mathrm{n}-1}}{(\mathrm{n}-1)!}$	$\frac{u^{n-1}}{s^n}$
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{su}}$
e^{at}	$\frac{1}{s-au}$
$\mathrm{te}^{\mathrm{at}}$	$\frac{u}{(s-au)^2}$
$\frac{\sin \mathrm{W}t}{\mathrm{W}}$	$\frac{u}{s^2+u^2w^2}$
$\cos \mathrm{W}t$	$\frac{s}{s^2+u^2w^2}$
$\frac{\sin at}{a}$	$\frac{u}{s^2+u^2a^2}$

3 Analysis of the Hybrid RGT accelerated Adomian method

This section outlines the steps of the suggested method for solving nonlinear IDEs where the accelerated Adomian polynomial appears in the estimated solution.

$$y^{(n)}(x) = f(x) + \int_0^x K(x-t)G(y(x))dt\ . \tag{1}$$

Applying the RGT for both sides, we get

$$\begin{aligned}&\frac{s^n}{u^n}RG\left[y(x)\right] - \frac{s^{n-1}}{u^n}y(0) - \frac{s^{n-2}}{u^{n-1}}y'(0) - \cdots - \frac{1}{u}y^{(n-1)}(0)\\&\quad = RG\left[f(x)\right] + RG\left[K(x)\bigotimes G(y(x))\right].\end{aligned} \tag{2}$$

The RGT of convolution term $K(x)\bigotimes G(y(x))$ can be written as a product of terms, so,

$$\begin{aligned}&\frac{s^n}{u^n}RG\left[y\left(x\right)\right]-\frac{s^{n-1}}{u^n}y\left(0\right)-\frac{s^{n-2}}{u^{n-1}}y'\left(0\right)-\cdots-\frac{1}{u}y^{(n-1)}\left(0\right)\\&\quad=RG\left[f\left(x\right)\right]+uRG\left[K\left(x\right)\right]RG\left[G\left(y\left(x\right)\right)\right].\end{aligned}\tag{3}$$

This can be reduced to

$$\begin{aligned}\frac{s^n}{u^n}RG\left[y\left(x\right)\right]=&\frac{s^{n-1}}{u^n}y\left(0\right)+\frac{s^{n-2}}{u^{n-1}}y'\left(0\right)+\cdots+\frac{1}{u}y^{(n-1)}\left(0\right)\\&+RG\left[f\left(x\right)\right]+uRG\left[K(x)\right]RG[G\left(y\left(x\right)\right)],\end{aligned}\tag{4}$$

$$\begin{aligned}RG\left[y\left(x\right)\right]=&\frac{u^n}{s^n}\left[\frac{s^{n-1}}{u^n}y\left(0\right)+\frac{s^{n-2}}{u^{n-1}}y'\left(0\right)+\cdots+\frac{1}{u}y^{(n-1)}\left(0\right)\right]\\&+\frac{u^n}{s^n}RG\left[f\left(x\right)\right]+\frac{u^{n+1}}{s^n}RG\left[K(x)\right]RG[G\left(y\left(x\right)\right)].\end{aligned}\tag{5}$$

Applying the inverse RGT for both sides, we get

$$\begin{aligned}y(x)=&RG^{-1}\left[\frac{u^n}{s^n}\left[\frac{s^{n-1}}{u^n}y\left(0\right)+\frac{s^{n-2}}{u^{n-1}}y'\left(0\right)+\cdots+\frac{1}{u}y^{(n-1)}\left(0\right)\right]\right]\\&+RG^{-1}\left[\frac{u^n}{s^n}RG\left[f\left(x\right)\right]\right]+RG^{-1}\left[\frac{u^{n+1}}{s^n}RG\left[K\left(x\right)\right]RG\left[G\left(y\left(x\right)\right)\right]\right].\end{aligned}\tag{6}$$

We represent the linear term $y(x)$ at the left side by an infinite series of components given by

$$y\left(x\right)=\sum_{n=0}^{\infty}y_n\left(x\right).\tag{7}$$

The nonlinear term $G\left(y\left(x\right)\right)$ at the right side of (6) will be represented by an infinite series of the accelerated Adomian polynomials $\overline{A}_n$

$$G\left(y\left(x\right)\right)=\sum_{n=0}^{\infty}\overline{A}_n\left(x\right).\tag{8}$$

3.1 Accelerated Adomian polynomials $\overline{A}_n$ formula

If the nonlinear function is $G\left(y\left(x\right)\right)=y^2\left(x\right)$, then the accelerated Adomian polynomials $\overline{A}_n$ are

$$\begin{aligned}\overline{A}_0 &= {y_0}^2 ,\\ \overline{A}_1 &= 2y_0y_1 + {y_1}^2 ,\\ \overline{A}_2 &= 2y_0y_2 + 2y_1y_2 + {y_2}^2 ,\end{aligned}$$

and $G(y(x)) = y^3(x)$, the accelerated Adomian polynomials $\overline{A}_n$ are

$$\begin{aligned}\overline{A}_0 &= {y_0}^3 ,\\ \overline{A}_1 &= 3{y_0}^2y_1 + 3y_0{y_1}^2 + {y_1}^3,\\ \overline{A}_2 &= 3{y_0}^2y_2 + 6y_0y_1y_2 + 3{y_1}^2y_2 + 3y_0{y_2}^2 + 3y_1{y_2}^2 + {y_2}^3,\end{aligned}$$

where $\overline{A}_n, n \geq 0$ can be obtained for all forms of nonlinearity. Substituting (7) and (8) into (5) leads to

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) =& RG^{-1}\left[\frac{u^n}{s^n}\left[\frac{s^{n-1}}{u^n}y(0) + \frac{s^{n-2}}{u^{n-1}}y'(0) + \cdots + \frac{1}{u}y^{(n-1)}(0)\right]\right]\\ &+ RG^{-1}\left[\frac{u^n}{s^n}RG[f(x)]\right]\\ &+ RG^{-1}\left[\frac{u^{n+1}}{s^n}RG[K(x)]\,RG\left[\sum_{n=0}^{\infty}\overline{A}_n(x)\right]\right],\end{aligned}$$

where

$$\begin{aligned}y_0(x) =& RG^{-1}\left[\frac{u^n}{s^n}\left[\frac{s^{n-1}}{u^n}y(0) + \frac{s^{n-2}}{u^{n-1}}y'(0) + \cdots + \frac{1}{u}y^{(n-1)}(0)\right]\right]\\ &+ RG^{-1}\left[\frac{u^n}{s^n}RG[f(x)]\right],\end{aligned}$$

$$y_{n+1}(x) = RG^{-1}\left[\frac{u^{n+1}}{s^n}RG[K(x)]\,RG\left[\sum_{n=0}^{\infty}\overline{A}_n(x)\right]\right], \qquad n \geq 0.$$

4 Convergence of the proposed method

In this section, we present and prove a convergence theorem for the application of the hybrid RGT in combination with the accelerated Adomian polynomials.

Theorem 2. The solution of the nonlinear Volterra-type integro-differential equation

$$y^{(i)}(x) = f(x) + \int_{x_0}^{x} K(x,t)G(y(x))dt \ ,$$

using RGT converges if $G(y(x))$ satisfy Lipschitz condition in the interval of interest $J = [0,b]$ and this solution, is unique provided that $0 < MM_1 \dfrac{(x-x_0)^{i+1}}{(i+1)!} < 1$, for all $x \in J$, where M is the Lipschitz inequality constant.

Proof. Define a complete metric space $(C[0,b]\ d)$, the space of all continuous functions on J with the distance function

$$d\left(f_1(x), f_2(x)\right) = \max_{\forall x \in J} \left|f_1(x) - f_2(x)\right| \ .$$

Define the sequence $\{S_n\}$ such that $S_n = \sum_{i=0}^{n} y_i(x) = y_0 + y_1 + \cdots + y_n$ is a sequence of partial sums of the series solution $\sum_{i=0}^{\infty} y_i(x)$, since

$$f\left(\sum_{i=0}^{\infty} y_i(x)\right) = \sum_{i=0}^{\infty} \overline{A}_i\left(y_0, y_1, \ldots, y_i\right),$$

$$f\left(S_n\right) = \sum_{i=0}^{\infty} \overline{A}_i\left(y_0, y_1, \ldots, y_i\right).$$

Let, S_n and S_m be arbitrary partial sums with $n \geq m$. We prove that $\{S_n\}$ is a Cauchy sequence in this complete metric space:

$$\begin{aligned}
d\left(S_n \ , \ S_m\right) &= \max_{\forall x \in J} ||S_n - S_m|| \\
&= \max_{\forall x \in J} \left|\sum_{i=m+1}^{n} y_i(x)\right| \\
&= \max_{\forall x \in J} \left|\sum_{i=m+1}^{n} RG^{-1}[\int_{x_0}^{x} K(x,t)\,\overline{A}_{i-1}dt]\ \right| \\
&= \max_{\forall x \in J} \left|\sum_{i=m+1}^{n} RG^{-1}[\int_{x_0}^{x} K(x,t)\overline{A}_{i-1}dt]\ \right|
\end{aligned}$$

$$= \max_{\forall x \in J} \left| RG^{-1}[\int_{x_0}^{x} K(x,t) \sum_{i=m}^{n-1} \overline{A}_i dt] \right|$$
$$= \max_{\forall x \in J} \left| RG^{-1}[\int_{x_0}^{x} K(x,t)[f(S_{n-1}) - f(S_{m-1})]dt] \right|$$
$$\leq \max_{\forall x \in J} RG^{-1} \left[\int_{x_0}^{x} |K(x,t)| \, |f(S_{n-1}) - f(S_{m-1})| \, dt \right]$$
$$\leq M_1 \max_{\forall x \in J} |f(S_{n-1}) - f(S_{m-1})| \quad RG^{-1} \int_{x_0}^{x} dt.$$

Since $f(x)$ satisfy Lipschitz condition,

$$|f(S_{n-1}) - f(S_{m-1})| \leq M|S_{n-1} - S_{m-1}|,$$

so,

$$d(S_m , S_n) \leq MM_1 \max_{\forall x \in J} |S_{n-1} - S_{m-1}| \int_{x_0}^{x} \ldots (i+1) - fold \,.\,. \int_{x_0}^{x} dt \ldots dt \quad ,$$
$$\leq MM_1 \frac{(x - x_0)^{i+1}}{(i+1)!} d(S_{m-1} , S_{n-1}),$$
$$\leq \alpha d(S_{m-1} , S_{n-1}), \qquad \alpha = MM_1 \frac{(x - x_0)^{i+1}}{(i+1)!} \; .$$

Now, for $n = m + 1$,

$$d(S_{m+1} , S_m) \leq \alpha d(S_m , S_{m-1}) \leq \alpha^2 d(S_{m-1} , S_{m-2}) \leq \cdots \leq \alpha^m d(S_1 , S_0).$$

From the triangle inequality, we have

$$d(S_m , S_n) \leq \alpha [d(S_{m-1} , S_m) + d(S_m , S_{m+1}) + \cdots + d(S_{n-2} , S_{n-1})],$$
$$\leq \alpha \left[\alpha^{m-1} + \alpha^m + \cdots + \alpha^{n-2}\right] d(S_1 , S_0),$$
$$\leq \alpha^m \frac{1 - \alpha^{n-m}}{1 - \alpha} d(S_1 - S_0),$$
$$\leq \frac{\alpha^m}{1 - \alpha} d(S_1 , S_0).$$

Indeed, $d(S_1 , S_0) = \max_{\forall x \in J} |S_1 - S_0| = \max_{\forall x \in J} |y_1|$, which is bounded. As $m \to \infty$, $d(S_m , S_n) \to 0$ we conclude that $\{S_n\}$ is a Cauchy sequence in this complete metric space, so the series $\sum_{n=0}^{\infty} y_n(x)$ converges. For the uniqueness of the solution, assume that y and y^* are two different solutions. Then from (6), we have

$$\begin{aligned} d\left(y\ ,\ y^*\right) &= \max_{\forall x\in J}\left|RG^{-1}[\int_{x_0}^{x}K\left(x,t\right)[f\left(y\right)-f(y^*)]dt]\right|, \\ &\le \max_{\forall x\in J} RG^{-1}\left[\int_{x_0}^{x}\left|K\left(x,t\right)\right|\left|f\left(y\right)-f\left(y^*\right)\right|dt\right], \\ &\le M_1 \max_{\forall x\in J} \left|f\left(y\right)-f\left(y^*\right)\right| \quad RG^{-1}\int_{x_0}^{x}dt, \\ &\le \alpha d\left(y\ ,\ y^*\right). \end{aligned}$$

So, $(1-\alpha)d\left(y\ ,\ y^*\right) \le 0$ and $0 < \alpha <= 1$; then $d\left(y\ ,\ y^*\right) = 0$, which implies $y = y^*$.

□

5 Numerical examples

In this numerical section, we apply the hybrid RGT and accelerated Adomian polynomials to solve several nonlinear IDEs. The results are compared with traditional methods, highlighting improvements in accuracy and computational efficiency, demonstrating the effectiveness of the proposed approach.

Example 1. Consider the nonlinear VIDE [25]

$$y'\left(x\right) = -1 + \int_{0}^{x} y^2(t)dt\ , \qquad y\left(0\right) = 0\ ,$$

whose exact solution takes the form

$$y\left(x\right) = \frac{-x + \dfrac{x^4}{28}}{1 + \dfrac{x^3}{21}}.$$

This example is solved by Rani and Mishra [25] where they used Laplace and a modification of ADM by computing the Adomian polynomials for the nonlinear term using the Newton-Raphson formula. We applied our hybrid method for combining RGT and the accelerated version of ADM. Four iterations are carried out and the approximate series solution is evaluated at the corresponding points as in [25].

Applying the RGT for both sides, we get

$$RG\left[\grave{y}\left(x\right)\right] = RG\left[-1\right] + RG\left[\int_0^x y^2(t)dt\right],$$

$$\frac{s}{u}RG\left[y(x)\right] - \frac{1}{u}y(0) = RG\left[-1\right] + RG\left[\int_0^x y^2(t)dt\right],$$

$$\frac{s}{u}RG\left[y(x)\right] = \frac{-1}{s} + RG\left[\int_0^x y^2(t)dt\right],$$

$$RG\left[y(x)\right] = \frac{-u}{s^2} + \frac{u}{s}RG\left[\int_0^x y^2(t)dt\right].$$

Applying the inverse RGT for both sides, we get

$$y\left(x\right) = RG^{-1}\left[\frac{-u}{s^2}\right] + RG^{-1}\left[\frac{u}{s}RG\left[\int_0^x y\left(t\right)^2 dt\right]\right],$$

$$y\left(x\right) = -x + RG^{-1}\left[\frac{u}{s}RG\left[\int_0^x y^2(t)dt\right]\right].$$

Let $y^2(t) = \sum\limits_{n=0}^{\infty} A_n$,

$$\sum_{n=0}^{\infty} y_n(x) = -x + RG^{-1}\left[\frac{u}{s}RG\left[\int_0^x \sum_{n=0}^{\infty} A_n \; dx\right]\right].$$

By comparing both sides, we get

$$y_0\left(x\right) = -x \; ,$$

$$y_{n+1}\left(x\right) = RG^{-1}\left[\frac{u}{s}RG\left[\int_0^x \sum_{n=0}^{\infty} A_n \; dx\right]\right].$$

Using accelerated Adomian polynomials, we have

$$\overline{A}_0 = {y_0}^2, \quad \overline{A}_1 = 2y_0y_1 + {y_1}^2, \qquad \overline{A}_2 = 2y_0y_2 + 2y_1y_2 + {y_2}^2, \quad \ldots.$$

Then

$$y_0(x) = -x \ , y_1(x) = \frac{x^4}{12}, y_2(x) = -\frac{x^7}{252} + \frac{x^{10}}{12960} \ ,$$

$$y_3(x) = \frac{11831339520x^{10} - 701537760x^{13} + 15992262x^{16} - 240240x^{19} + 1729x^{22}}{134167390156800}$$

$$\begin{aligned} y_4(x) = &-\frac{x^{13}}{884520} + \frac{89x^{16}}{849139200} - \frac{10757x^{19}}{2032839244800} + \frac{350993x^{22}}{1878343462195200} \\ &- \frac{4507x^{25}}{901604861853696} + \frac{24354871x^{28}}{221524314557453107200} \\ &- \frac{1312457x^{31}}{628869391142952960000} + \frac{253524431x^{34}}{7647700951798842654720000} \\ &- \frac{1709x^{37}}{4053164656463290368000} + \frac{241247x^{40}}{59943018919370607820800000} \\ &- \frac{x^{43}}{39132841298576965632000} + \frac{x^{46}}{12464483949901696204800000}. \end{aligned}$$

$$\begin{aligned} y(x)\,(approximate) &= y_0 + y_1 + y_2 + y_3 + y_4 \\ &= -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{157248} + \frac{2663x^{16}}{11887948800} + \cdots . \end{aligned}$$

Table 2: Comparison of the approximate solutions and absolute error against the method of Laplace Adomian using Newton–Raphson formula [25]

x	Exact solution	Approximate solution (presented method, 4 iterations)	Absolute error	Approximate solution [25] (4 iterations)	Absolute error
0.	0.	0.	0.	0.	0.
0.0625	-0.0624987284490275	-0.0624987284490275	2.082×10^{-17}	-0.062499682	3.1789×10^{-7}
0.125	-0.124979656839952	-0.124979656839974	2.2×10^{-14}	-0.124994914	1.4914×10^{-5}
0.1875	-0.187397035493881	-0.187397035495149	1.268×10^{-12}	-0.187474253	7.4253×10^{-5}
0.25	-0.249674721189591	-0.249674721212078	2.249×10^{-11}	-0.249918635	2.4863×10^{-4}
0.3125	-0.311706424640996	-0.311706424850075	2.091×10^{-10}	-0.31230139	5.9139×10^{-3}
0.375	-0.373356178680768	-0.373356179972127	1.291×10^{-9}	-0.374588271	1.2283×10^{-3}
0.4375	-0.434459096619924	-0.434459102631869	6.012×10^{-9}	-0.436737503	2.2775×10^{-3}
0.5	-0.494822485207101	-0.494822507955013	2.275×10^{-8}	-0.498699852	3.8799×10^{-3}

Table 2 and Figure 1 show that the proposed method achieves higher accuracy and improved computational efficiency, primarily because the accelerated Adomian polynomials eliminate the requirement to compute derivatives of the nonlinear functions. Another notable advantage of using the accelerated polynomial is its superior rate of convergence compared to the traditional polynomials.

Example 2. Consider the nonlinear VIDE [17]

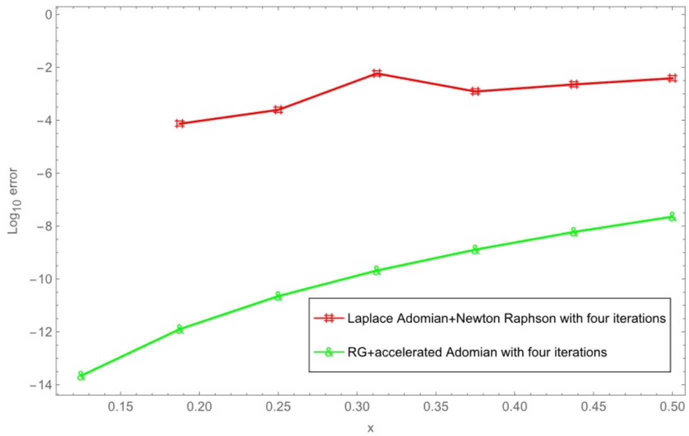

Figure 1: Comparison the absolute errors of the two approaches at four iterations.

$$y''(x) = 2 + 2x + x^2 - x^2 e^x - e^{2x} + \int_0^x e^{x-t} y^2(t)\, dt, \quad y(0) = 1, \quad y'(0) = 2\ ,$$

whose exact solution takes the form $y(x) = x + e^x$.

Khanlari and Paripour [17] solved this example using a combination of Laplace and the homotopy analysis method (HAM) by computing the Adomian polynomials for the nonlinear term. We applied our hybrid method for combining the RGT and the accelerated version of ADM. Three iterations were carried out, and the approximate solution and absolute error were evaluated at the corresponding points as in [4].

We note that the integral term uses the RGT convolution theorem of the two functions e^x and $y^2(x)$ [13].

Applying the RGT for both sides, we get

$$RG[y''(x)] = RG\left[2 + 2x + x^2 - x^2 e^x - e^{2x}\right] + RG\left[e^x \bigotimes y^2(x)\ \right],$$

$$RG[y''(x)] = RG\left[2 + 2x + x^2 - x^2 e^x - e^{2x}\right] + uRG[e^x]\, RG\left[y^2(x)\right],$$

$$\frac{s^2}{u^2} RG[y(x)] - \frac{s}{u^2} y(0) - \frac{1}{u} y'(0) = \frac{2}{s} + \frac{2u}{s^2} + \frac{2u^2}{s^3} - \frac{2u^2}{(s-u)^3} - \frac{1}{-2u+s} + \frac{u}{s-u} RG\left[y^2(x)\right],$$

$$\frac{s^2}{u^2}RG\left[y\left(x\right)\right]=\frac{s}{u^2}+\frac{2}{u}+\frac{2}{s}+\frac{2u}{s^2}+\frac{2u^2}{s^3}-\frac{2u^2}{\left(s-u\right)^3}-\frac{1}{-2u+s}$$
$$+\frac{u}{s-u}RG\left[y^2\left(x\right)\right],$$

$$RG\left[y\left(x\right)\right]=\frac{1}{s}+\frac{2u}{s^2}+\frac{2u^2}{s^3}+\frac{2u^3}{s^4}+\frac{2u^4}{s^5}-\frac{2u^4}{s^2\left(s-u\right)^3}-\frac{u^2}{s^2(-2u+s)}$$
$$+\frac{u^3}{s^2(s-u)}RG\left[y^2\left(x\right)\right].$$

Applying the inverse RGT for both sides, we get

$$y\left(x\right)=RG^{-1}\left[\frac{1}{s}+\frac{2u}{s^2}+\frac{2u^2}{s^3}+\frac{2u^3}{s^4}+\frac{2u^4}{s^5}-\frac{2u^4}{s^2\left(s-u\right)^3}-\frac{u^2}{s^2\left(-2u+s\right)}\right]$$
$$+RG^{-1}\left[\frac{u^3}{s^2\left(s-u\right)}RG\left[y^2\left(x\right)\right]\right],$$

$$\sum_{n=0}^{\infty}y_n(x)=RG^{-1}\left[\frac{1}{s}+\frac{2u}{s^2}+\frac{2u^2}{s^3}+\frac{2u^3}{s^4}+\frac{2u^4}{s^5}-\frac{2u^4}{s^2\left(s-u\right)^3}-\frac{u^2}{s^2\left(-2u+s\right)}\right]$$
$$+RG^{-1}\left[\frac{u^3}{s^2\left(s-u\right)}RG\left[\sum_{n=0}^{\infty}A_n\right]\right].$$

By comparing both sides and using the Taylor series from 0 to 4, we get

$$y_0\left(x\right)=1+2x+\frac{x^2}{2}-\frac{x^4}{6}+\dots\ ,$$

$$y_{n+1}\left(x\right)=RG^{-1}\left[\frac{u^3}{s^2\left(s-u\right)}RG\left[\sum_{n=0}^{\infty}A_n\right]\right].$$

Using accelerated Adomian polynomials, we have

$$\overline{A}_0={y_0}^2,$$

$$\overline{A}_1=2y_0y_1+{y_1}^2,$$

$$\overline{A}_2=2y_0y_2+2y_1y_2+{y_2}^2,$$

$$\vdots$$

Then

$$y_0(x) = 1 + 2x + \frac{x^2}{2} - \frac{x^4}{6} + \dots \ ,$$

$$y_1(x) = \frac{x^3}{6} + \frac{5x^4}{24} + \frac{x^5}{8} + \frac{3x^6}{80} + \dots,$$

$$y_2(x) = \frac{x^6}{360} + \frac{x^7}{180} + \frac{89x^8}{20160} + \dots \ ,$$

$$y_3(x) = \frac{x^9}{90720} + \frac{29x^{10}}{907200} + \dots \ ,$$

$$y(x)\,(approximate) = y_0+y_1+y_2+y_3 = 1+2x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\cdots \ .$$

Table 3: Comparison of the approximate solutions and absolute error against the combination of the HAM and Laplace transform-Adomian method [4]

x	Exact solution	Approximate solution (presented method, 3 iterations)	Absolute error	Approximate solution [4] (4 iterations)	Absolute error
0.00	1.0000000000000	1.0000000000000	0.0000	1.00000	0.00000
0.02	1.0402013400268	1.0402013400268	7.105×10^{-15}	1.04042	2.16577×10^{-4}
0.04	1.0808107741924	1.0808107741933	9.064×10^{-13}	1.08175	9.37538×10^{-4}
0.06	1.1218365465454	1.1218365465611	1.573×10^{-11}	1.12412	2.28194×10^{-3}
0.08	1.1632870676750	1.1632870677947	1.197×10^{-10}	1.16767	4.38752×10^{-3}
0.10	1.2051709180756	1.2051709186553	5.796×10^{-10}	1.21259	7.41437×10^{-3}
0.12	1.2474968515794	1.2474968536878	2.108×10^{-9}	1.25905	1.15497×10^{-3}
0.14	1.2902737988572	1.2902738051525	6.295×10^{-9}	1.30729	1.70138×10^{-2}
0.16	1.3335108709918	1.3335108872586	1.627×10^{-8}	1.35758	2.40683×10^{-2}
0.18	1.3772173631218	1.3772174007597	3.764×10^{-8}	1.41024	3.30265×10^{-2}
0.20	1.4214027581602	1.4214028379772	7.982×10^{-8}	1.46567	4.42678×10^{-2}

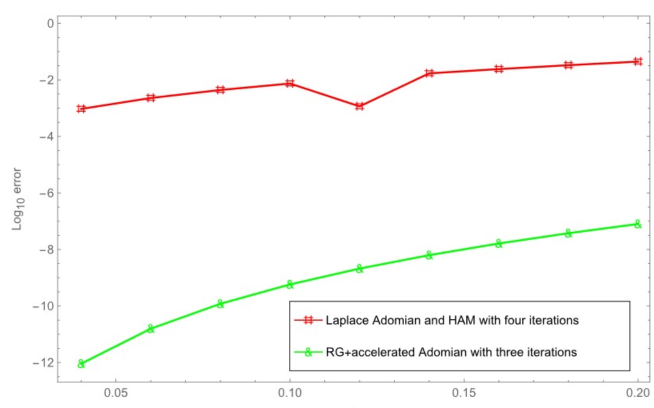

Figure 2: Comparison the approximate solutions of the two approaches.

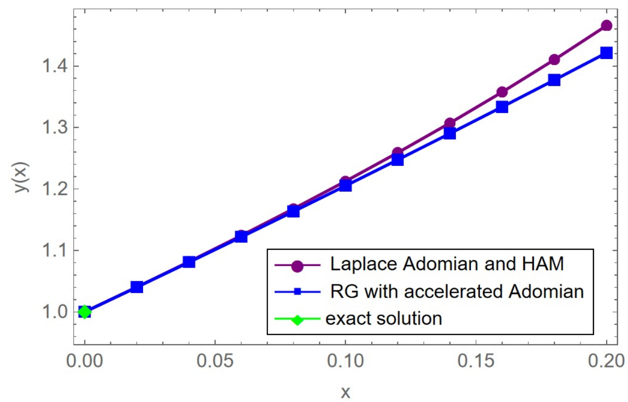

Figure 3: Comparison the absolute errors of the two approaches.

Based on Table 3 and Figures 2 and 3, the RGT combined with the accelerated Adomian method yields higher accuracy compared to the hybrid approach of the HAM and the Laplace transform-Adomian method. Notably, this improved performance is achieved using only three iterations, whereas HAM requires four.

Example 3. Consider the nonlinear VIDE [4]

$$y'''(x) = \frac{-2}{3} - \frac{5}{3}\cos(x) + \frac{4}{3}\cos^2(x) + \int_0^x \cos(x-t)\, y^2(t)\, dt,$$
$$y(0) = y'(0) = 1, \quad y''(0) = -1,$$

whose exact solution takes the form

$$y(x) = \sin(x) + \cos(x) .$$

Almousa et al. [4] approached this example by employing a combination of Laplace transform and HAM. They calculated the Adomian polynomials for the nonlinear term. In our study, we utilized a hybrid method that combines the RGT with an accelerated Adomian polynomial. We conducted two iterations and evaluated both the approximate solutions and absolute error at the corresponding points, as described in [20].

We note that the integral term uses the RGT convolution theorem of two functions cos(x) and $y^2(x)$ [10].

Applying the RGT for both sides, we get

$$RG\left[y'''(x)\right] = RG\left[\frac{-2}{3} - \frac{5}{3}\cos(x) + \frac{4}{3}\cos^2(x)\right] + RG\left[\cos(\mathrm{x}) \bigotimes y^2(x)\right],$$

$$RG\left[y'''(x)\right] = RG\left[\frac{-2}{3} - \frac{5}{3}\cos(x) + \frac{4}{3}\cos^2(x)\right] + uRG\left[\cos(\mathrm{x})\ \right]\ RG[y^2(x)],$$

$$\begin{aligned}&\frac{s^3}{u^3}RG\left[y(x)\right] - \frac{s^2}{u^3}y(0) - \frac{s}{u^2}y'(0) - \frac{1}{u}y''(0)\\ &\qquad = -\frac{s^3+6su^2}{s^4+5s^2u^2+4u^4} + \frac{us}{s^2+u^2}RG[y^2(x)],\end{aligned}$$

$$\frac{s^3}{u^3}RG\left[y(x)\right] = \frac{s^2}{u^3} + \frac{s}{u^2} - \frac{1}{u} - \frac{s^3+6su^2}{s^4+5s^2u^2+4u^4} + \frac{us}{s^2+u^2}RG[y^2(x)],$$

$$RG\left[y(x)\right] = \frac{1}{s} + \frac{u}{s^2} - \frac{u^2}{s^3} - \frac{u^3(s^3+6su^2)}{s^3(s^4+5s^2u^2+4u^4)} + \frac{u^4}{s^2(s^2+u^2)}RG[y^2(x)].$$

Applying the inverse RGT for both sides, we get

$$\begin{aligned}y(x) =& RG^{-1}\left[\frac{1}{s} + \frac{u}{s^2} - \frac{u^2}{s^3} - \frac{u^3\left(s^3+6su^2\right)}{s^3\left(s^4+5s^2u^2+4u^4\right)}\right]\\ &+ RG^{-1}\left[\frac{u^4}{s^2\left(s^2+u^2\right)}RG\left[y^2(x)\right]\right],\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) =& RG^{-1}\left[\frac{1}{s} + \frac{u}{s^2} - \frac{u^2}{s^3} - \frac{u^3\left(s^3+6su^2\right)}{s^3\left(s^4+5s^2u^2+4u^4\right)}\right]\\ &+ RG^{-1}\left[\frac{u^4}{s^2\left(s^2+u^2\right)}RG\left[\sum_{n=0}^{\infty} A_n\right]\right].\end{aligned}$$

By comparing both sides and using the Taylor series from 0 to 9, we get

$$y_0(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{560} - \frac{41x^9}{362880} + \dots,$$

$$y_{n+1}(x) = RG^{-1}\left[\frac{u^4}{s^2\left(s^2+u^2\right)}RG\left[\sum_{n=0}^{\infty} A_n\right]\right].$$

Using accelerated Adomian polynomials

$$\overline{A}_0 = {y_0}^2, \overline{A}_1 = 2y_0y_1 + {y_1}^2, \overline{A}_2 = 2y_0y_2 + 2y_1y_2 + {y_2}^2,$$

$$\vdots$$

Then

$$y_0(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{560} - \dots,$$

$$y_1(x) = \frac{x^4}{24} + \frac{x^5}{60} - \frac{x^6}{720} - \frac{x^7}{504} - \frac{x^8}{40320} + \dots,$$

$$y_2(x) = \frac{x^8}{20160} + \dots,$$

$$y(x)\,(approximate) = y_0 + y_1 + y_2.$$

Table 4: Comparison of the approximate solutions and absolute error against the method of (HAM) [4]

x	Exact solution	Approximate solution (presented method, 2 iterations)	Absolute error	Approximate solution [4] (4 iterations)	Absolute error
0.00	1.0000000000000	1.0000000000000	0	1.00000	0
0.02	1.019798673359911	1.019798673359911	0	1.01980	1.37991×10^{-6}
0.04	1.0391894408476121	1.0391894408476123	2.22×10^{-16}	1.03920	1.14101×10^{-5}
0.06	1.0581645464146487	1.0581645464146487	0	1.05820	3.97525×10^{-5}
0.08	1.0767164002717922	1.0767164002717917	4.441×10^{-16}	1.07681	9.71538×10^{-5}
0.10	1.094837581924854	1.0948375819248513	2.665×10^{-15}	1.09503	1.95418×10^{-4}
0.12	1.1125208431427855	1.1125208431427716	1.399×10^{-14}	1.11287	3.47374×10^{-4}
0.14	1.1297591108568736	1.1297591108568175	5.618×10^{-14}	1.13033	5.66843×10^{-4}
0.16	1.1465454899898728	1.1465454899896865	1.863×10^{-13}	1.14741	8.68596×10^{-4}
0.18	1.1628732662139456	1.1628732662134090	5.367×10^{-13}	1.16414	1.26832×10^{-3}
0.20	1.1787359086363027	1.1787359086349205	1.382×10^{-12}	1.18052	1.78257×10^{-3}

From Table 4, RGT with accelerated Adomian gives better accuracy compared with the HAM and Laplace transform-Adomian method. Although, RGT with accelerated Adomian polynomials uses less iterations than HAM.

Example 4. Consider the following nonlinear VIDE [4]:

$$y'(x) = \frac{3}{2}e^{x} - \frac{1}{2}e^{3x} + \int_0^x e^{x-t}\, y^3(t)\; dt, \quad y(0) = 1\ ,$$

whose exact solution takes the form $y(x) = e^x$.

This example is solved by Almousa et al. [4], and they used a hybrid ADM with Modified Bernstein Polynomials by using the ADM for the nonlinear term. We applied our hybrid method for combining the RGT and the accelerated Adomian polynomial. Four iterations are carried out, and the approximate series solution and absolute error are evaluated at the corresponding points as in [4].

We note that the integral term uses the RGT convolution theorem of two functions e^x and $y^3(x)$ [12].

Applying RGT for both sides, we get

$$RG\left[\ y'(x)\ \right] = RG\left[\ \frac{3}{2}e^x\ \right] - RG\left[\ \frac{1}{2}e^{3x}\ \right] + RG\left[\ e^x \bigotimes y^3(x)\right]\ ,$$

$$RG\left[\ y'(x)\ \right] = RG\left[\ \frac{3}{2}e^x\ \right] - RG\left[\ \frac{1}{2}e^{3x}\ \right] + uRG\left[\ e^x\ \right] RG\left[y^3(x)\right],$$

$$\frac{s}{u}RG\left[y(x)\right] - \frac{1}{u}y(0) = \frac{3}{2s-2u} - \frac{1}{2s-6u} + \frac{u}{s-u}\ RG\left[y^3(x)\right],$$

$$\frac{s}{u}RG\left[y(x)\right] = \frac{1}{u} + \frac{3}{2s-2u} - \frac{1}{2s-6u} + \frac{u}{s-u}\ RG\left[y^3(x)\right],$$

$$RG\left[y(x)\right] = \frac{1}{s} + \frac{3u}{2s(s-u)} - \frac{u}{2s(s-3u)} + \frac{u^2}{s(s-u)}\ RG\left[y^3(x)\right].$$

Applying the inverse RGT for both sides, we get

$$y(x) = \frac{1}{6}\left(-2+9e^x-e^{3x}\right) + RG^{-1}\left[\frac{u^2}{s(s-u)}\ RG\left[y^3(x)\right]\right],$$

Let $y(x) = \sum_{n=0}^{\infty} y_n(x), \quad y^3(x) = \sum_{n=0}^{\infty} A_n(x)$,

$$\sum_{n=0}^{\infty} y_n(x) = \frac{1}{6}\left(-2+9e^x-e^{3x}\right) + RG^{-1}\left[\frac{u^2}{s(s-u)}\ RG\left[\sum_{n=0}^{\infty} A_n(x)\ \right]\right].$$

By comparing both sides and using the Taylor series from 0 to 6, we get

$$y_0(x) = 1 + x - \frac{x^3}{2} - \frac{x^4}{2} - \frac{13x^5}{40} - \frac{x^6}{6} + \ldots,$$

$$y_{n+1}(x) = RG^{-1}\left[\frac{u^2}{s(s-u)}\ RG\left[\sum_{n=0}^{\infty} A_n(x)\ \right]\right].$$

Using accelerated Adomian polynomials, we have

$$\begin{aligned}
\overline{A}_0 &= {y_0}^3 \ , \\
\overline{A}_1 &= 3{y_0}^2 y_1 \ + 3y_0 {y_1}^2 \ + {y_1}^3, \\
\overline{A}_2 &= 3{y_0}^2 y_2 \ + 6y_0 y_1 y_2 \ + 3{y_1}^2 y_2 + 3y_0 {y_2}^2 \ + 3y_1 {y_2}^2 \ + {y_2}^3, \\
\overline{A}_3 &= 3{y_0}^2 y_3 \ + 6y_0 y_1 y_3 \ + 3{y_1}^2 y_3 \ + 6y_0 y_2 y_3 \ + 6y_1 y_2 y_3 \ + 3{y_2}^2 y_3 \\
&\quad + 3y_0 {y_3}^2 \ + 3y_1 {y_3}^2 \ + 3y_2 {y_3}^2 \ + {y_3}^3, \\
&\vdots
\end{aligned}$$

Then

$$y_0(x) = 1 + x - \frac{x^3}{2} - \frac{x^4}{2} - \frac{13x^5}{40} - \frac{x^6}{6} + \dots,$$

$$y_1(x) = \frac{x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{12} + \frac{7x^5}{120} - \frac{101x^6}{720} + \dots,$$

$$y_2(x) = \frac{x^4}{8} + \frac{11x^5}{40} + \frac{71x^6}{240} + \dots,$$

$$y_3(x) = \frac{x^6}{80} + \dots \ ,$$

$$y_4(x) = \frac{3x^8}{4480} + \frac{x^9}{896} + \dots,$$

$$y(x)\,(approximate) = y_0 + y_1 + y_2 + y_3 + y_4.$$

Table 5: Comparison of the approximate solutions and absolute error against Hybrid ADM with modified Bernstein polynomials [4]

x	Exact solution	Approximate solution (presented method, 4 iterations)	Absolute error	Approximate solution [4] (4 iterations)	Absolute error
0.0	1.000000000000000	1.000000000000000	0	1.000000	0
0.1	1.105170918075648	1.105170918063368	1.228×10^{-11}	1.105170917	1.333×10^{-9}
0.2	1.221402758160170	1.221402757841270	3.189×10^{-10}	1.221402667	9.133×10^{-8}
0.3	1.349858807576003	1.349858828402902	2.083×10^{-8}	1.349857750	1.058×10^{-6}
0.4	1.491824697641270	1.491825086984127	3.893×10^{-7}	1.491818667	6.031×10^{-6}
0.5	1.648721270700128	1.648724413674975	3.143×10^{-6}	1.648697917	2.335×10^{-5}
0.6	1.822118800390509	1.822135294857143	1.649×10^{-5}	1.822048000	7.080×10^{-5}
0.7	2.013752707470477	2.013818459141493	6.575×10^{-5}	2.013571417	1.813×10^{-4}
0.8	2.225540928492468	2.225756899555555	2.16×10^{-4}	2.225130667	4.103×10^{-4}
0.9	2.459603111156950	2.460217010731026	6.139×10^{-4}	2.458758250	8.449×10^{-4}
1.0	2.718281828459045	2.719841269841270	1.559×10^{-3}	2.716666667	1.615×10^{-3}

According to Table 5, the proposed method is both more accurate and computationally simpler than the Adomian hybrid decomposition method with modified Bernstein polynomials, which involves extensive calculations when the same number of iterations is used.

Example 5. Consider the nonlinear VIDE [28]

$$y^{(4)}(x) = e^{-3x} + e^{-x} - 1 + 3\int_0^x y^3(t)\,dt\ ,$$

with the conditions

$$y(0) = y''(0) = 1, \qquad y'(0) = y'''(0) = -1\ ,$$

whose exact solution takes the form $y(x) = e^{-x}$.

This example is solved by Sharif, Hamoud, and Ghadle [28] using a Laplace and modified homotopy perturbation method (MHPM) by using the ADM for the nonlinear term. We applied our hybrid method for combining the RGT and the accelerated Adomian polynomial. Three iterations are carried out, and the approximate series solution and absolute error are evaluated at the corresponding points as in [28].

Applying RGT for both sides, we get

$$RG[y^{(4)}(x)] = RG[e^{-3x}] + RG[e^{-x}] - RG[1] + 3RG[\int_0^x y^3(t)\,dt\ ],$$

$$\frac{s^4}{u^4}RG[y(x)] - \frac{s^3}{u^4}y(0) - \frac{s^2}{u^3}y'(0) - \frac{s}{u^2}y''(0) - \frac{1}{u}y'''(0) = \frac{1}{(s+3u)} + \frac{1}{(s+u)} - \frac{1}{s} + 3RG[\int_0^x y^3(t)\,dt\ ],$$

$$\frac{s^4}{u^4}RG[y(x)] = \frac{s^3}{u^4} - \frac{s^2}{u^3} + \frac{s}{u^2} - \frac{1}{u} - \frac{1}{s} + \frac{1}{(s+3u)} + \frac{1}{(s+u)} + 3RG[\int_0^x y^3(t)\,dt\ ],$$

$$RG[y(x)] = \frac{1}{s} - \frac{u}{s^2} + \frac{u^2}{s^3} - \frac{u^3}{s^4} - \frac{u^4}{s^5} + \frac{u^4}{s^4(s+3u)} + \frac{u^4}{s^4(s+u)} + \frac{3u^4}{s^4}RG[\int_0^x y^3(t)\,dt\ ],$$

Applying the inverse RGT for both sides, we get

$$y(x) = -\frac{1}{81} + \frac{e^{-3x}}{81} + e^{-x} + \frac{x}{27} - \frac{x^2}{18} + \frac{x^3}{18} - \frac{x^4}{24} + RG^{-1}[\ \frac{3u^4}{s^4}RG[\int_0^x y^3(t)\,dt\ ]\ ]\ ,$$

$$y^3(t) = \sum_{n=0}^{\infty} A_n\ ,$$

$$\sum_{n=0}^{\infty} y_n(x) = -\frac{1}{81} + \frac{e^{-3x}}{81} + e^{-x} + \frac{x}{27} - \frac{x^2}{18} + \frac{x^3}{18} - \frac{x^4}{24}$$
$$+ RG^{-1}\left[\ \frac{3u^4}{s^4} RG\left[\int_0^x \sum_{n=0}^{\infty} A_n\ dx\ \right]\ \right],$$

By comparing both sides and using the Taylor series from 0 to 5, we get

$$y_0(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \dots ,$$

$$y_{n+1}(x) = RG^{-1}\left[\ \frac{3u^4}{s^4} RG\left[\int_0^x A_n\ dx\ \right]\right].$$

Using accelerated Adomian polynomials, we have

$$\overline{A}_0 = {y_0}^3 ,\ \ \overline{A}_1 = 3{y_0}^2 y_1 + 3y_0{y_1}^2 + {y_1}^3,$$

$$\overline{A}_2 = 3{y_0}^2 y_2 + 6y_0y_1y_2 + 3{y_1}^2 y_2 + 3y_0{y_2}^2 + 3y_1{y_2}^2 + {y_2}^3,$$

$$\vdots$$

Then

$$y_0(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \dots ,$$
$$y_1(x) = \frac{x^5}{40} - \frac{x^6}{80} + \frac{3x^7}{560} - \frac{9x^8}{4480} + \dots ,$$
$$y_2(x) = \frac{x^{10}}{134400} - \frac{x^{11}}{98560} + \frac{3x^{12}}{394240} + \dots ,$$
$$y_3(x) = \frac{x^{15}}{5381376000} + \dots ,$$
$$y(x)\,(approximate) = y_0 + y_1 + y_2 + y_3.$$

Based on Table 6, the proposed method outperforms the others at both two and three iterations specifically when two iterations are used in the MHPM and three iterations in the LADM [15].

Table 6: Comparison of the approximate solution against LADM [17] and MHPM [7]

x	Exact solution	Approximate solution for presented method using three iterations	Absolute Error for presented method using three iterations	LADM [17]	Absolute Error for LADM [17]	MHPM [7]	Absolute Error for MHPM [7]
0.00	1.0000000000	1.0000000000	0.0000×10^{0}	1.0000000000	0.0000×10^{0}	1.0000000000	0.0000×10^{0}
0.04	0.9607894392	0.9607894391	5.5992×10^{-11}	0.9607895450	1.0580×10^{-7}	0.9608106692	2.1230×10^{-5}
0.08	0.9231163464	0.9231163429	3.5278×10^{-9}	0.9231180530	1.7066×10^{-6}	0.9232854120	1.6906×10^{-4}
0.12	0.8869204367	0.8869203971	3.9569×10^{-8}	0.8869290770	8.6403×10^{-6}	0.8874885866	5.6815×10^{-4}
0.16	0.8521437890	0.8521435700	2.1898×10^{-7}	0.8521710940	2.7305×10^{-5}	0.8534850921	1.3413×10^{-3}
0.20	0.8187307531	0.8187299301	8.2298×10^{-7}	0.8187974190	6.6670×10^{-5}	0.8213405980	2.6098×10^{-3}
0.24	0.7866278611	0.7866254393	2.4218×10^{-6}	0.7867661000	1.3824×10^{-4}	0.7911217470	4.4938×10^{-3}
0.28	0.7557837415	0.7557777214	6.0201×10^{-6}	0.7560398470	2.5611×10^{-4}	0.7628963355	7.1125×10^{-3}
0.32	0.7261490371	0.7261358094	1.3228×10^{-5}	0.7265859450	4.3691×10^{-4}	0.7367334755	1.0584×10^{-2}
0.36	0.6976763261	0.6976498733	2.6453×10^{-5}	0.6983761680	6.9984×10^{-4}	0.7127037390	1.5027×10^{-2}

6 Conclusions

In this study, a hybrid approach combining the RGT with the accelerated Adomian polynomial is introduced to solve IDEs numerically. The resulting method is straightforward and efficient, as demonstrated by the numerical results presented in the tables.

These results highlighted the improved accuracy achieved through this combination, outperforming other existing methods. An important advantage and as a key contribution of the proposed convergence analysis is the use of the classical fixed-point theorem in conjunction with accelerated polynomials, rather than traditional polynomials.

This approach enhanced the robustness and efficiency of analysis. All computations were carried out using MATHEMATICA 12.

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