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Research Article

Cutting-edge spectral solutions for differential and integral equations utilizing Legendre's derivatives

A.M. Abbas[1], Y.H. Youssri[2], M. El-Kady[1] and M. Abdelhakem[1,*]

Abstract

This research introduces a spectral numerical method for solving some types of integral equations, which is the pseudo-Galerkin spectral method. The presented method depends on Legendre's first derivative polynomials as basis functions. Subsequently, an operational integration matrix has been constructed to express integrals as a linear combination of these basis functions. This process transforms the given integral equation into a

*Corresponding author

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Ahmed M. Abbas, (ahmed.m.abbas.96@gmail.com; ahmedmabbas@science.helwan.edu.eg)
Youssri Hassan Youssri, (youssri@cu.edu.eg; youssri@aucegypt.edu)
Mamdouh El-Kady, (mamdouh_elkady@cic-cairo.com)
Mohamed Abdelhakem, (mabdelhakem@yahoo.com; mabdelhakem@science.helwan.edu.eg)

[1]Mathematics Department, Faculty of Science, Helwan University, Helwan 11795, Egypt, Helwan School of Numerical Analysis in Egypt (HSNAE).
[2]Mathematics Department, Faculty of Science, Cairo University, Giza 12613, Egypt.

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system of algebraic equations. The unknowns of the obtained system are the spectral expansion constants. Then, we solve the obtained algebraic system using the Gauss elimination method for linear systems or Newton's iteration method for nonlinear systems. This approach yields the desired semi-analytic approximate solution. Additionally, our method extends to the solution of ordinary differential equations, as every initial value problem can be equivalently represented as a corresponding Volterra integral equation. On the other hand, every boundary value problem can be transformed into a corresponding Fredholm integral equation. This transformation is achieved by incorporating the given conditions. Moreover, convergence and error analyses are thoroughly examined. Finally, to validate the efficiency and accuracy of the proposed method, we conduct numerical test problems.

AMS subject classifications (2020): 65R20, 65N35, 45L05, 33C45.

Keywords: Legendre polynomials; Pseudo-Galerkin spectral method; Integral equations; Lane–Emden equation; Stable population model.

	Abbreviations
BVPs:	Boundary Value Problems
f_n:	Function
IVPs:	Initial Value Problems
LGL:	Legendre Gauss Lobatto quadrature points
MAE:	Maximum Absolute Error
PGDL:	Pseudo-Galerkin method via Legendre's Derivative polynomials
RSE:	Root Square Error

1 Introduction

Integral and differential equations are found in numerous applications across various fields. In fluid mechanics, the modeling of fluid flow involves differential equations [28, 21, 18, 10, 14]. Electromagnetic wave-related problems in electromagnetism are often described using integral equations [22, 11]. Additionally, the behavior of heat flow and temperature distribution in different materials and shapes can be modeled effectively with differential and integral equations [26, 34]. Some other important applications, such as the Lane–Emden equation [8, 33, 54, 7], are discussed. Integral and differential equations stand as essential mathematical tools for problem modeling in science and engineering [29, 37, 46, 32, 30].

Often, obtaining the exact solution for these equations proves challenging, leading to the development of numerical, or approximation methods for introducing numerical and semi-analytic approximations as viable alternatives [24]. These methods include finite element [31, 25, 36], finite difference [23, 13], and spectral methods, among others [44, 41, 42, 45]. While finite element and finite difference methods yield numerical approximation solutions, providing a set of dependent variable values at independent variable values. On the other hand, spectral methods offer semi-analytic approximation solutions, representing smooth functions of the independent variable or variables.

The inner product space encompassing all polynomials is not a Cauchy space; nevertheless, any smooth function (f_n) serves as a limit point within this space. More precisely, any smooth f_n can be expressed as the limit of a sequence of polynomials. The fundamental concept underlying all spectral methods involves representing the dependent variable as a linear combination of a set of functions that constitute a basis for the polynomial space. In simpler terms, the unknown f_n is expanded as a series involving unknown constants multiplied by the basis functions. Crucially, these basis functions are required to be orthogonal concerning an inner product defined by a weight function $w_1(x)$ [12].

Expanding upon this idea, spectral methods leverage the orthogonality of basis functions to efficiently approximate complicated functions, even if they are two-dimensional [20]. Orthogonal basis functions contribute to simplifying the representation of the unknown function, leading to solutions that are both accurate and computationally efficient. The weight function is pivotal in establishing the inner product, impacting the orthogonality of the basis functions, and enhancing the effectiveness of spectral methods. In summary, the incorporation of orthogonal basis functions and a weight function in spectral methods enhances their ability to efficiently represent and approximate functions, making them effective tools for approximating the solution of differential and integral equations in various applications related to science and engineering.

The integral and differential equations can be formulated as an operator acting on the unknown function $y(x)$, equated to another given function. When the terms of the equation are rearranged such that they all reside on the left-hand side, and $y(x)$ is expressed as a finite series involving unknown constants multiplied by basis functions, the left-hand side of the equation is referred to as the residue function. The closer the residue function is to zero, the more accurate the approximate solution is to the exact one, and the value of the residue is termed the residual error. Another set of functions is defined, known as the set of test functions, and a distinct inner product is constructed using a suitable weight function $w_2(x)$. By setting the inner product of the residue function and the test functions equal to zero, a system of algebraic equations in the unknown expansion coefficients is obtained. Subsequently, the approximate solution is derived by solving this system of algebraic equations.

Spectral methods encompass three main types: the Galerkin [52, 9], Tau [19, 35, 38], and collocation [17, 27, 48] methods. In the Galerkin method, the test functions and the basis functions are identical, satisfying the initial/boundary conditions of the IVPs/BVPs. Additionally, the weight function $w_2(x)$ should match $w_1(x)$. For the collocation method, the weight function $w_2(x)$ is set to one, and the set of test functions comprises Dirac delta functions shifted to predetermined collocation points. In the Tau method, the test and basis functions differ, and the weight functions w_1 and w_2 may vary. Additionally, there is no requirement to satisfy any (initial/boundary) conditions.

A modified version of spectral methods, known as pseudo-spectral methods [1, 53, 47], employs the Gauss quadrature method to define an alternative inner product expressed as a summation over a set of quadrature points instead of integration. Moreover, the unknown expansion coefficients can be expressed in terms of the dependent variable evaluated at the quadrature points, aided by the orthogonality relation. Consequently, differentiation and integration matrices can be constructed.

At the heart of both spectral and pseudo-spectral methods lies the crucial decision regarding the choice of basis functions. In the work by the authors in [5], the first derivative of Legendre polynomials was chosen as the basis function. They employed both the tau and collocation methods and crafted the operational matrix of the differentiation of the chosen basis functions. Similarly, in the study by the authors in [4], the same basis functions were adopted, and the pseudo-spectral method was employed to formulate the differentiation and integration matrices. Higher derivatives of Legendre polynomials are utilized in [3, 15].

In this paper, we also employ the same basis functions; however, our focus shifts towards constructing the operational matrix of integration for these basis functions. We then utilize the pseudo-Galerkin spectral method [28, 16, 43] in our numerical approach, in which the system of algebraic equations can be formulated by substituting the Legendre Gauss–Lobattoen quadrature points into the residue function.

The paper consists as follows. Section 2 provides a comprehensive overview of Legendre and Legendre's derivative polynomials, establishing the foundational groundwork for the subsequent discussions. We delve into pertinent notations and historical relationships concerning the basis functions crucial to our methodology. The construction of the operational integration matrix, along with the introduction of key equations, is detailed in section 3. Section 4 encapsulates the algorithmic framework of our proposed approach. A meticulous exploration of convergence and error analysis is undertaken in section 5, shedding light on the method's reliability and precision. Section 6 is dedicated to the presentation of various test problems, strategically chosen to validate the accuracy, efficiency, and stability of our proposed methodology. Finally, section 7 synthesizes the overarching conclusions drawn from our study and outlines potential avenues for future research.

2 Preliminaries

Throughout this work, some needed properties of Legendre and Legendre's derivative polynomials are displayed, such as boundaries, upper bounds, orthogonality relations, and recurrence relations. They are essential to get some new desired properties of Legendre's derivative polynomials.

The Legendre polynomial of degree q, denoted by $\mathcal{L}_q(x)$, can be obtained using the following recurrence relations [5, 39]:

$$(q+1)\,\mathcal{L}_{q+1}(x) = (2q+1)\,x\mathcal{L}_q(x) - q\mathcal{L}_{q-1}(x)\,, \tag{1}$$

$$(2q+1)\,\mathcal{L}_q(x) = \mathcal{L}'_{q+1}(x) - \mathcal{L}'_{q-1}(x)\,, \tag{2}$$

$$\mathcal{L}'_q(x) = \sum_{j=0}^{\lfloor \frac{q-1}{2} \rfloor} \left[2\,(q-2j-1)+1\right]\mathcal{L}_{q-2j-1}(x)\,, \tag{3}$$

$$\left(1-x^2\right)\mathcal{L}'_q(x) = \frac{q\,(q+1)}{2q+1}\left[\mathcal{L}_q(x) - \mathcal{L}_{q+2}(x)\right], \tag{4}$$

according to the initials: $\mathcal{L}_0(x) = 1$ and $\mathcal{L}_1(x) = x$, where $q \geq 1$.

In addition, they can be expanded as follows [39]:

$$\mathcal{L}_q(x) = \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} (-1)^j \frac{(2q-2j)!}{2^q\, j!\,(q-j)!\,(q-2j)!} x^{q-2j}, \tag{5}$$

where $x \in [-1, 1]$, and $q \geq 0$.

The following definition describes the introduced basis functions used in this work for non-negative degree q [5].

Definition 1. Legendre's derivative polynomials of degree q, denoted by $DL_q(x)$, are defined to be the derivative of the Legendre polynomial that is one degree higher:

$$DL_q(x) = \frac{d}{dx}\mathcal{L}_{q+1}(x)\,. \tag{6}$$

Legendre's derivative polynomials can be expanded as follows [5]:

$$DL_q(x) = \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} (-1)^j \frac{(2q-2j+2)!}{2^{q+1}\,(j)!\,(q-j+1)!\,(q-2j)!} x^{q-2j}. \tag{7}$$

The boundary values for Legendre [39] and Legendre's derivative [5] polynomials are given by

$$\mathcal{L}_q(\pm 1) = (\pm 1)^q\,, \tag{8}$$

$$DL_q(\pm 1) = \frac{(\pm 1)^q\,(q+1)\,(q+2)}{2}. \tag{9}$$

Also, it is known that they are bounded as shown in the following relation [39, 5]:

$$|\mathcal{L}_q(x)| \leq 1, \tag{10}$$

$$|DL_q(x)| \leq (q+1)^2. \tag{11}$$

Legendre polynomials are orthogonal under the weight function $w(x) = 1$, and the orthogonality relation is given by

$$\int_{-1}^{1} \mathcal{L}_q(x)\,\mathcal{L}_r(x)\,\mathrm{dx} = \frac{2}{2q+1}\delta_{q,r}, \tag{12}$$

where $\delta_{q,r} = \begin{cases} 1, & q = r, \\ 0, & q \neq r, \end{cases}$ for nonnegative integers q and r.

The Legendre's derivative polynomials are orthogonal with respect to the weight function $1 - x^2$ such that [5]: [**?**])

$$\int_{-1}^{1} \left(1 - x^2\right) DL_q(x)\,DL_r(x)\,\mathrm{dx} = \frac{2\,(q+2)\,(q+1)}{2q+3}\delta_{q,r}. \tag{13}$$

To solve integral equations, it is necessary to expand the integration of the basis functions as a linear combination of the basis functions themselves. The essential relations and properties required are shown in the forward sections. The following section will introduce the derivation of some important relations and the operational integration matrix construction.

3 Legendre's derivative polynomials operational integration matrix

It is undeniable that integrals on the entire domain $[a, b]$ or a variable domain $[a, x]$ occur in integral equations. That is why we need to find a suitable representation for these integrals. Therefore, the integrals of our basis functions will be calculated in this section to help expand these integrals as summations. Hence, we can construct an integration operational matrix for Legendre's derivative polynomials.

At the beginning, some essential relations must be introduced. A recurrence relation of the used polynomials, $DL_q(x)$, will be constructed using recurrence relations (1) and (2):

$$x\,DL_q\left(x\right)=\frac{q+1}{2q+3}DL_{q+1}\left(x\right)+\frac{q+2}{2q+3}DL_{q-1}\left(x\right), \tag{14}$$

where $q\geq 1$.

The next lemma introduces the first moment equation.

Lemma 1. Consider Legendre's derivative polynomials column matrix $DL\left(x\right)$ of $N+1$ rows. Then the square matrix X of rank $N+1$ satisfies the following:

$$\mathrm{X}\cdot DL\left(x\right)=XDL\left(x\right), \tag{15}$$

where $DL\left(x\right)=\left[DL_0\left(x\right),DL_1\left(x\right),DL_2\left(x\right),\dots,DL_N\left(x\right)\right]^T$,
$XDL\left(x\right)=\left[0,x\,DL_0\left(x\right),x\,DL_1\left(x\right),\dots,x\,DL_{N-1}\left(x\right)\right]^T$, and
$\mathrm{X}=\left\{\mathrm{x}_{rc}\right\}_{(N+1)\times(N+1)}$, such that

$$\mathrm{x}_{rc}=\begin{cases}\frac{r}{2r+1}, & r=c,\\ \frac{r+1}{2r+1}, & c=r-2,\ r>1,\\ 0, & \text{otherwise},\end{cases} \tag{16}$$

where $r,c=0,1,2,\dots,N$.

Proof. By straight forward matrix multiplication with the help of (14), we have

$$\mathrm{X}\cdot DL\left(x\right)=\left\{\sum_{c=1}^{N+1}\mathrm{x}_{rc}DL_{c-1}\left(x\right)\right\}. \tag{17}$$

□

The next lemmas and theorems will be concerning different forms and techniques for the integration of the introduced basis function.

Lemma 2. Consider Legendre's derivative polynomials column matrix $DL\left(x\right)$ of $N+1$ rows and Legendre polynomials column matrix $\mathcal{L}\left(x\right)$ of $N+1$ rows. Then the square matrix M of rank $N+1$ satisfies the following:

$$\mathrm{M}\cdot DL\left(x\right)=\mathcal{L}\left(x\right), \tag{18}$$

where $DL(x) = [DL_0(x), DL_1(x), DL_2(x), \ldots, DL_N(x)]^T$,
$\mathcal{L}(x) = [\mathcal{L}_0(x), \mathcal{L}_1(x), \mathcal{L}_2(x), \ldots, \mathcal{L}_N(x)]^T$, and
$\mathrm{M} = \{\mathrm{m}_{rc}\}_{(N+1)\times(N+1)}$, such that

$$\mathrm{m}_{rc} = \begin{cases} \frac{1}{2r+1}, & r = c, \\ \frac{-1}{2r+1}, & c = r-2,\ r > 1, \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

where $r, c = 0, 1, 2, \ldots, N$.

Proof. The left-hand side of the relation (18) is expanded using the usual matrix multiplication as follows:

$$L.H.S. = \mathrm{M} \cdot DL(x) = \left[\sum_{c=0}^{N} \mathrm{m}_{0c}\, DL_c(x), \sum_{c=0}^{N} \mathrm{m}_{1c}\, DL_c(x), \right.$$
$$\left. \sum_{c=0}^{N} \mathrm{m}_{2c}\, DL_c(x), \ldots, \sum_{c=0}^{N} \mathrm{m}_{Nc}\, DL_c(x)\right]^T.$$

With the aid of relation (19), we get

$$L.H.S. = \left[DL_0(x), \frac{1}{3}DL_1(x), \frac{-1}{5}DL_0(x) + \frac{1}{5}DL_2(x), \ldots, \right.$$
$$\left. \frac{-1}{2N+1}DL_{N-2}(x) + \frac{1}{2N+1}DL_N(x)\right]^T.$$

Using relation (2), the proof is completed. □

Theorem 1. Integrals of the introduced basis functions can be presented as a linear combination of the introduced basis functions themselves according to

$$\int_{-1}^{x} DL_q(t)\,\mathrm{dt} = \sum_{k=0}^{2} \left[\frac{(-1)^k (1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}{2q+3} + (-1)^q \delta_{k,2}\right] \times DL_{(q+1-2k)(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}(x). \tag{20}$$

Proof. For $q = 0$, we get

$$\int_{-1}^{x} DL_0(t)\,\mathrm{dt} = \int_{-1}^{x} (1)\,\mathrm{dt} = x + 1 = \frac{1}{3}DL_1(x) + DL_0(x). \tag{21}$$

For $q = 1, 2, \ldots$, and by using Lemma 2 and (9), we can conclude that, for $N \geq q$,

$$\int_{-1}^{x} DL_q(t)\,\mathrm{dt} = \left[\sum_{c=0}^{N} \mathrm{m}_{qc}\, DL_c(t)\right]_{t=-1}^{t=x}$$
$$= \frac{1}{2q+3} DL_{q+1}(x) - \frac{1}{2q+3} DL_{q-1}(x) + (-1)^q DL_0(x), \tag{22}$$

which completes the proof for $q > 0$. □

As a direct result, we can deduce the following corollary.

Corollary 1. The integral on the entire domain $[-1, 1]$, can be calculated as follows:

$$\int_{-1}^{1} DL_q(t)\,\mathrm{dt} = 1 + (-1)^q, \tag{23}$$

where q is a nonnegative integer.

Now, all the necessary relations that are needed to construct the operational matrix of integration have been introduced.

Theorem 2. Consider the column matrix $DL(x) = [DL_0(x), DL_1(x), DL_2(x), \ldots, DL_N(x)]^T$ and the column matrix $\int_{-1}^{x} DL(t)\,dt = \left[0, \int_{-1}^{x} DL_0(t)\,\mathrm{dt},\ \int_{-1}^{x} DL_1(t)\,\mathrm{dt}, \ldots, \int_{-1}^{x} DL_{N-1}(t)\,\mathrm{dt}\right]^T$ of $N+1$ rows. Then the square matrix $\mathcal{G}$ of rank $N+1$ satisfies (24)

$$\mathcal{G} \cdot DL(x) = \int_{-1}^{x} DL(t)\,dt, \tag{24}$$

where $\mathcal{G} = \{\eth_{rc}\}_{(N+1)\times(N+1)}$, such that

$$\eth_{rc} = \begin{cases} -\frac{6}{5}, & c = 0, r = 2, \\ (-1)^{r+1}, & c = 0, r \geq 1, r \neq 2, \\ \frac{1}{2r+1}, & c = r, r \geq 1, \\ \frac{-1}{2r+1}, & c = r-2, r \geq 3, \\ 0, & \text{otherwise}, \end{cases} \tag{25}$$

where $r, c = 0, 1, 2, \ldots, N$.

Proof. Multiply the matrix $\mathcal{G}$ defined in (25) by the column matrix $DL(x)$ using the matrix multiplication to get the left-hand side of (24) as follows:

$$L.H.S. = \mathcal{G} \cdot DL(x) = \left[\sum_{c=0}^{N} \eth_{0c}\, DL_c(x), \sum_{c=0}^{N} \eth_{1c}\, DL_c(x), \right.$$
$$\left. \sum_{c=0}^{N} \eth_{2c}\, DL_c(x), \ldots, \sum_{c=0}^{N} \eth_{Nc}\, DL_c(x) \right]^T .$$

Use Theorem 1 and the definition of $\eth_{rc}$ to conclude that

$$\begin{aligned} L.H.S. &= \left[0, \int_{-1}^{x} DL_0(t)\,\mathrm{dt}, \int_{-1}^{x} DL_1(t)\,\mathrm{dt}, \ldots, \int_{-1}^{x} DL_{N-1}(t)\,\mathrm{dt} \right]^T \\ &= \int_{-1}^{x} DL(t)\,dt = R.H.S. \end{aligned} \tag{26}$$

□

Remark 1. Note that the shift that occurs in the column matrix $\int_{-1}^{x} DL(t)\,dt$ is due to the integral operator producing a polynomial that is one degree higher than the input, which means that the polynomials in $\int_{-1}^{x} DL(t)\,dt$ should shift a row downward to align with the degree of the polynomials in the rows of the column matrix $DL(x)$.

Theorem 3. For all $q \geq 0$ and $m \geq 1$, the moment formula $x^m DL_q(x)$ can be represented as follows:

$$x^m DL_q(x) = \sum_{k=0}^{\min\left(m, \lfloor \frac{q+m}{2} \rfloor\right)} F_{m,k+1,q}\, DL_{q+m-2k}(x), \tag{27}$$

where

$$F_{1,1,q} = \frac{q+1}{2q+3},\ F_{1,2,q} = \frac{q+2}{2q+3},$$
$$F_{m,k,q} = \begin{cases} F_{1,1,q}\, F_{m-1,1,q+1}, & k = 1, \\ F_{1,1,0}\, F_{m-1,k,1}, & 1 < k < m+1,\ q = 0, \\ F_{1,1,q}\, F_{m-1,k,q+1} + F_{1,2,q}\, F_{m-1,k-1,q-1}, & 1 < k < m+1,\ q > 0, \\ F_{1,2,q}\, F_{m-1,m,q-1}, & k = m+1, \end{cases} \tag{28}$$

for $m > 1$.

Proof. Three cases will be studied, $q = 0$, $1 \leq q < m$, and $q \geq m$.

For $q = 0$:

We study this at $m = 1$ as the initial for the proof by mathematical induction:

$$R.H.S. = \sum_{k=0}^{0} F_{1,k+1,0}\, DL_{1-2k}\,(x) = x = x\, DL_0 = L.H.S.$$

Consider (27) be valid for some $m \geq 1$ and $q = 0$. Thus, for $m+1$, the right-hand side will be

$$\begin{aligned} R.H.S. &= \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} F_{m+1,k+1,0}\, DL_{m+1-2k}\,(x) \\ &= \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} F_{1,1,0}\, F_{m,k+1,1}\, DL_{m+1-2k}\,(x) \\ &= \frac{1}{3} x^m DL_1\,(x) = x^{m+1} DL_0\,(x) = L.H.S., \end{aligned}$$

which completes the proof of the first case.

Similarly, for $1 \leq q < m$, the initial step will be at $m = 2$, which is easy to be verified. Consider the induction step for some $m \geq 2$, so at $m+1$, the right-hand side will be

$$\begin{aligned} R.H.S. = &\sum_{k=0}^{\lfloor \frac{m+q+1}{2} \rfloor} F_{m+1,k+1,q}\, DL_{q+m+1-2k}\,(x) \\ &= F_{1,1,q}\, F_{m,1,q+1}\, DL_{q+m+1}\,(x) \\ &\quad + \sum_{k=1}^{\lfloor \frac{m+q+1}{2} \rfloor} \left(F_{1,1,q}\, F_{m,k+1,q+1} + F_{1,2,q}\, F_{m,k,q-1}\right) DL_{q+m+1-2k}\,(x)\,. \end{aligned}$$

$$
\begin{aligned}
L.H.S. &= x^{m+1} DL_q(x) = x^m \left(F_{1,1,q}\, DL_{q+1}(x) + F_{1,2,q}\, DL_{q-1}(x)\right) \\
&= F_{1,1,q} \sum_{k=0}^{\lfloor \frac{m+q+1}{2} \rfloor} F_{m,k+1,q+1}\, DL_{q+m+1-2k}(x) \\
&\quad + F_{1,2,q} \sum_{k=0}^{\lfloor \frac{m+q-1}{2} \rfloor} F_{m,k+1,q-1}\, DL_{q+m-1-2k}(x) \\
&= F_{1,1,q}\, F_{m,1,q+1}\, DL_{q+m+1}(x) \\
&\quad + \sum_{k=1}^{\lfloor \frac{m+q+1}{2} \rfloor} \left(F_{1,1,q}\, F_{m,k+1,q+1} + F_{1,2,q}\, F_{m,k,q-1}\right) DL_{q+m+1-2k}(x) \\
&= R.H.S.
\end{aligned}
$$

Finally, for $q \geq m$, the initial step is $m = 1$. So, the induction step for $m+1$, the right-hand side will be

$$
\begin{aligned}
R.H.S. = \sum_{k=0}^{m+1} F_{m+1,k+1,q}\, DL_{q+m+1-2k}(x) &= F_{1,1,q}\, F_{m,1,q+1}\, DL_{q+m+1}(x) \\
&+ \sum_{k=1}^{m} \left(F_{1,1,q}\, F_{m,k+1,q+1} + F_{1,2,q}\, F_{m,k,q-1}\right) DL_{q+m+1-2k}(x) \\
&+ F_{1,2,q}\, F_{m,m+1,q-1}\, DL_{q-m-1}(x).
\end{aligned}
$$

$$
\begin{aligned}
L.H.S. =& x^{m+1} DL_q(x) = x^m \left(F_{1,1,q}\, DL_{q+1}(x) + F_{1,2,q}\, DL_{q-1}(x)\right) \\
=& F_{1,1,q} \sum_{k=0}^{m+1} F_{m,k+1,q+1}\, DL_{q+m+1-2k}(x) \\
&+ F_{1,2,q} \sum_{k=0}^{m+1} F_{m,k+1,q-1}\, DL_{q+m-1-2k}(x) \\
=& F_{1,1,q}\, F_{m,1,q+1}\, DL_{q+m+1}(x) \\
&+ \sum_{k=1}^{m+1} \left(F_{1,1,q}\, F_{m,k+1,q+1} + F_{1,2,q}\, F_{m,k,q-1}\right) DL_{q+m+1-2k}(x) \\
&+ F_{1,2,q}\, F_{m,m+1,q-1}\, DL_{q-m-1}(x) = R.H.S.,
\end{aligned}
$$

which completes the proof. □

Theorem 4. For every $m, q \in \mathbb{N}$, we have

$$\int_{-1}^{1} t^m DL_q(t)\,\mathrm{dt} = \begin{cases} 1+(-1)^{q+m}, & q \geq m, \\ \sum\limits_{k=0}^{\min\left(m,\lfloor \frac{q+m}{2} \rfloor\right)} F_{m,k+1,q}\left(1+(-1)^{q+m-2k}\right), & 0 \leq q < m. \end{cases} \tag{29}$$

Proof. For $q \geq m$:

When $m = 0$, Corollary 1 is a special case of Theorem 4.

So, we begin with $m = 1$. Using (14), integrating on $[-1, 1]$, and then using Corollary 1, we get

$$\begin{aligned} L.H.S. = \int_{-1}^{1} t\,DL_q(t)\,\mathrm{dt} &= \frac{q+1}{2q+3}\int_{-1}^{1} DL_{q+1}(t)\,\mathrm{dt} + \frac{q+2}{2q+3}\int_{-1}^{1} DL_{q-1}(t)\,\mathrm{dt} \\ &= 1+(-1)^{q+1} = R.H.S., \end{aligned} \tag{30}$$

for $q \geq 1$.

Let (29) be valid for some m, where $q \geq m$. Thus, for $m+1$, we have

$$\begin{aligned} L.H.S. &= \int_{-1}^{1} t^{m+1} DL_q(t)\,\mathrm{dt} = \int_{-1}^{1} t^m \left(t\,DL_q(t)\right)\mathrm{dt} \\ &= \int_{-1}^{1} t^m \left(\frac{q+1}{2q+3}DL_{q+1}(t) + \frac{q+2}{2q+3}DL_{q-1}(t)\right)\mathrm{dt} \\ &= \frac{q+1}{2q+3}\left[1+(-1)^{q+1+m}\right] + \frac{q+2}{2q+3}\left[1+(-1)^{q-1+m}\right] \\ &= 1+(-1)^{q+m+1} = R.H.S., \end{aligned}$$

for $q \geq m+1$.

The second case, for $0 \leq q < m$, use Theorem 3 to get

$$L.H.S. = \int_{-1}^{1} t^m DL_q(t)\,\mathrm{dt} = \int_{-1}^{1} \sum_{k=0}^{\min\left(m,\lfloor \frac{q+m}{2} \rfloor\right)} F_{m,k+1,q}\,DL_{q+m-2k}(t)\,\mathrm{dt}. \tag{31}$$

Using Corollary 1, we have

$$L.H.S. = \sum_{k=0}^{\min\left(m,\lfloor \frac{q+m}{2} \rfloor\right)} F_{m,k+1,q}\left(1+(-1)^{q+m-2k}\right) = R.H.S.. \tag{32}$$

□

4 Pseudo-Galerkin approach for solving integral equations

This section presents a method by which some types of integral equations can be solved. This method is based on Legendre's derivative polynomials as basis functions. The relations, lemmas, and theorems developed in section 3 are used while creating our solving technique.

4.1 Problem formulation

The introduced problem is a linear Fredholm–Volterra-type integral equation, which can be formulated as follows:

$$Q\left(x, y\left(x\right)\right) + \int_{-1}^{1} p_1\left(x,t\right) y\left(t\right) dt + \int_{-1}^{x} p_2\left(x,t\right) y\left(t\right) dt = 0, \tag{33}$$

where $p_1\left(x,t\right), p_2\left(x,t\right)$ are polynomials with respect to t and $Q\left(x, y\left(x\right)\right)$ is linear in $y\left(x\right)$, which means that $Q\left(x, y\left(x\right)\right) = f_1\left(x\right) + f_2\left(x\right) y\left(x\right)$ for any two arbitrary functions f_1 and f_2.

4.2 Presented method

The introduced method is the Pseudo-Galerkin method via Legendre's derivative polynomials (PGDL), which expands the unknown function y as a linear combination of the basis functions as follows:

$$y\left(x\right) \approx y_N\left(x\right) = \sum_{q=0}^{N} c_q\, DL_q\left(x\right), \tag{34}$$

for some $N \in \mathbb{N}$.

Thus, substitute into (33) to get the residue as follows:

$$\begin{aligned} R_N\left(x\right) = & f_1\left(x\right) + \sum_{q=0}^{N} c_q \Bigg[f_2\left(x\right) DL_q\left(x\right) + \int_{-1}^{1} p_1\left(x,t\right) DL_q\left(t\right) dt \\ & + \int_{-1}^{x} p_2\left(x,t\right) DL_q\left(t\right) dt \Bigg] \approx 0. \end{aligned} \tag{35}$$

Relations (27) and (28) are used to get the residue function. The integral equation can be written in the form:

$$\begin{aligned} R_N\left(x\right) = & f_1\left(x\right) + \sum_{q=0}^{N} c_q \Bigg[f_2\left(x\right) DL_q\left(x\right) + a_0\left(x\right)\left(1+(-1)^q\right) \\ & + \sum_{r=1}^{m_1} a_r\left(x\right) \sum_{k=0}^{\min\left(r,\lfloor \frac{q+r}{2} \rfloor\right)} F_{r,k+1,q} \left(1+(-1)^{q+r-2k}\right) \\ & + b_0\left(x\right) \sum_{k=0}^{2} \left[\mu(q,k) + (-1)^q\, \delta_{k,2}\right] DL_{z(q,k)}\left(x\right) \\ & + \sum_{r=1}^{m_2} b_r\left(x\right) \sum_{k=0}^{\min\left(r,\lfloor \frac{q+r}{2} \rfloor\right)} F_{r,k+1,q} \sum_{v=0}^{2} \left[\mu(q,v) + (-1)^q\, \delta_{v,2}\right] \times DL_{z(q,v)}\left(x\right) \Bigg] \\ = & 0, \end{aligned} \tag{36}$$

where $z(q,v) = (q+1-2v)\left(1-\delta_{v,2}\right)\left(1-\delta_{q,0}\,\delta_{v,1}\right),\ p_1\left(x,t\right) = \sum_{r=0}^{m_1} a_r\left(x\right) t^r$, $p_2\left(x,t\right) = \sum_{r=0}^{m_2} b_r\left(x\right) t^r$, for some positive integers $m_1,\ m_2,\ \{a_r\left(x\right),\, b_r\left(x\right)\}$ are real valued functions, and $\mu(q,v) = \frac{(-1)^v\left(1-\delta_{v,2}\right)\left(1-\delta_{q,0}\,\delta_{v,1}\right)}{2q+3}$.

Consequently, we substitute the independent variable with the set of $N+1$ Legendre–Gauss–Lobatto (LGL) quadrature points in the interval $[-1,1]$ into the residue function to get a system of algebraic equations. By solving the algebraic system to get the unknown coefficients' values c_q. Hence, the semi-analytic approximate solution is ready.

Remark 2. Differential equations can be transformed into corresponding integral equations depending on their types. For example, IVPs can be represented as Volterra integral equations. Meanwhile, BVPs can be transformed into Fredholm integral equations. This transformation involves incorporating the given conditions.

In the next section, we will study the error analysis for the PGDL method in detail to ensure that the presented method is accurate and efficient.

5 Convergence and error analysis

In this section, we will study the upper bound for the basis function, the upper bound for the expansion coefficients, and the uniform convergence of the presented method.

Lemma 3. The Legendre's derivative polynomials are bounded for $-1 \leq x \leq 1$ such that

$$|DL_q\,(x)| \leq \frac{(q+1)\,(q+2)}{2}. \tag{37}$$

Proof. Replace q by $q+1$ in (3), with the aid of Definition 1 to get

$$|DL_q(x)| \leq \text{‘}\sum_{j=0}^{\lfloor \frac{q}{2} \rfloor}[2(q-2j)+1]. \tag{38}$$

The left-hand side of the inequality (38) represents an arithmetic series with $\lfloor \frac{q}{2} \rfloor + 1$ terms, the first term $2q+1$, and the common difference -4. Thus the inequality (38) can be written as

$$|DL_q\,(x)| \leq \frac{\lfloor \frac{q}{2} \rfloor + 1}{2}\left[2\,(2q+1) - 4\lfloor \frac{q}{2} \rfloor\right]. \tag{39}$$

If q is even, then $\left\lfloor \frac{q}{2} \right\rfloor = \frac{q}{2}$, which implies

$$|DL_q\,(x)| \leq \frac{(q+2)\,(q+1)}{2}. \tag{40}$$

If q is odd, then $\left\lfloor \frac{q}{2} \right\rfloor = \frac{q-1}{2}$ and

$$|DL_q\,(x)| \leq \frac{(q+1)\,(q+2)}{2}, \tag{41}$$

which completes the proof. □

In [5], the authors proved that $|DL_q\,(x)| \leq (q+1)^2$. As a comparison between the two upper bounds, we found the that the upper bound obtained in this work is found to be better than the upper bound in [5].

Remark 3. For simplicity and compactness, some notations are introduced as

$$\Lambda_q(x) = \begin{cases} 1, & q = 0, \\ x, & q = 1, \\ \mathcal{L}_q(x) - \mathcal{L}_{q-2}(x), & q \geq 2, \end{cases} \tag{42}$$

$$\Delta_q(x) = \begin{cases} 1, & q = 0, \\ 3x, & q = 1, \\ DL_q(x) - DL_{q-2}(x), & q \geq 2. \end{cases} \tag{43}$$

In addition, we have the following:

$$\Delta_q(x) = \Lambda'_{q+1}(x), \tag{44}$$

and with the aid of (2), we get

$$\Lambda_{q+2}(x) = \eta_{q+2}\,\Delta_{q+2}(x) - \eta_q\,\Delta_q(x), \tag{45}$$

where

$$\eta_q = \frac{1}{q+1}. \tag{46}$$

Also, we need to define $\alpha_{m,k}$ as

$$\alpha_{m,k} = \sum_{i=0}^{m-1} (-1)^{\left\lfloor \frac{k}{2^{m-1-i}} \right\rfloor}, \tag{47}$$

where $k = 0, 1, 2, \ldots, 2^m - 1$, and $m \geq 1$.

Moreover, concerning $\alpha_{m,k}$, the following can be concluded:

$$\alpha_{m+1,2k} = \alpha_{m,k} + 1, \tag{48}$$

$$\alpha_{m+1,2k+1} = \alpha_{m,k} - 1, \tag{49}$$

$$|\alpha_{m,k}| \leq m, \tag{50}$$

precisely, $\alpha_{m,k} \in \{-m, 2-m, 4-m, \ldots, m-4, m-2, m\}$.

Lemma 4. For $x \in [-1, 1]$ and $m, q = 0, 1, 2, \ldots$, we have

$$\psi'_{m+1,q}(x) = -\psi_{m,q}(x), \tag{51}$$

where

$$\psi_{m,q}(x) = (-1)^{m+\lceil \frac{m}{2} \rceil} \frac{1}{2} \sum_{k=0}^{2^{m+1}-1} \Bigg((-1)^{\lceil \frac{1+\alpha_{m+1,k}}{2} \rceil} \Delta_{q+1+\alpha_{m+1,k}}(x) \times \prod_{j=1}^{m+1} \eta_{q+1+\alpha_{j,\lfloor \frac{k}{2^{m+1-j}} \rfloor}} \Bigg). \tag{52}$$

Proof. Equation (52) can be written as two collections of even terms and odd terms as follows:

$$\begin{aligned}\psi_{m,q}(x) = (-1)^{m+\lceil \frac{m}{2} \rceil} \frac{1}{2} \sum_{k=0}^{2^{m}-1} \Bigg[\Bigg((-1)^{\lceil \frac{1+\alpha_{m+1,2k}}{2} \rceil} \Delta_{q+1+\alpha_{m+1,2k}}(x) \\ \times \prod_{j=1}^{m+1} \eta_{q+1+\alpha_{j,\lfloor \frac{2k}{2^{m+1-j}} \rfloor}} \Bigg) \\ + \Bigg((-1)^{\lceil \frac{1+\alpha_{m+1,2k+1}}{2} \rceil} \Delta_{q+1+\alpha_{m+1,2k+1}}(x) \times \prod_{j=1}^{m+1} \eta_{q+1+\alpha_{j,\lfloor \frac{2k+1}{2^{m+1-j}} \rfloor}} \Bigg) \Bigg].\end{aligned} \tag{53}$$

Simplify it with the aid of (48, 49) as follows:

$$\begin{aligned}\psi_{m,q}(x) = (-1)^{m+\lceil \frac{m}{2} \rceil+1} \\ \times \frac{1}{2} \sum_{k=0}^{2^{m}-1} \Bigg[(-1)^{\lceil \frac{\alpha_{m,k}}{2} \rceil} \Big(\eta_{q+2+\alpha_{m,k}} \Delta_{q+2+\alpha_{m,k}}(x) \\ - \eta_{q+\alpha_{m,k}} \Delta_{q+\alpha_{m,k}}(x) \Big) \prod_{j=1}^{m} \eta_{q+1+\alpha_{j,\lfloor \frac{k}{2^{m-j}} \rfloor}} \Bigg].\end{aligned} \tag{54}$$

From (45) and using the fact that $\alpha_{n,k}$ and n are both even or both odd simultaneously, we have

$$\begin{aligned}\psi_{m,q}(x) = (-1)^{m+\lceil \frac{m-1}{2} \rceil} \frac{1}{2} \\ \times \sum_{k=0}^{2^{m}-1} \Bigg[(-1)^{\lceil \frac{1+\alpha_{m,k}}{2} \rceil} \Lambda_{q+2+\alpha_{m,k}}(x) \prod_{j=1}^{m} \eta_{q+1+\alpha_{j,\lfloor \frac{k}{2^{m-j}} \rfloor}} \Bigg]\end{aligned} \tag{55}$$

By replacing m with $m+1$ in (55) and differentiating with respect to x, we have

$$\psi'_{m+1,q}(x) = (-1)^{m+1+\lceil \frac{m}{2} \rceil} \frac{1}{2} \sum_{k=0}^{2^{m+1}-1} \Bigg[(-1)^{\left\lceil \frac{1+\alpha_{m+1,k}}{2} \right\rceil} \Lambda'_{q+2+\alpha_{m+1,k}}(x) \times \prod_{j=1}^{m+1} \eta_{q+1+\alpha_{j,\left\lfloor \frac{k}{2^{m+1-j}} \right\rfloor}} \Bigg], \tag{56}$$

which completes the proof. □

Lemma 5. For $x \in [-1,1]$ and $q = 0,1,2,\ldots$, we have

$$\left(1-x^2\right) DL_q(x) = \frac{2(q+1)(q+2)}{2q+3} \psi_{0,q}(x). \tag{57}$$

Proof. Equation (4) can be written as

$$\left(1-x^2\right) DL_q(x) = -\frac{(q+1)(q+2)}{2q+3} \Lambda_{q+2}(x), \tag{58}$$

where Λ_q is defined as in (42).

Use (45) to get

$$\left(1-x^2\right) DL_q(x) = -\frac{(q+1)(q+2)}{2q+3} \left[\eta_{q+2}\, \Delta_{q+2}(x) - \eta_q\, \Delta_q(x)\right]. \tag{59}$$

Then, (52) completes the proof. □

Lemma 6. For nonnegative integers q and r, we have

$$\left| \prod_{j=1}^{r+1} \eta_{q+1+\alpha_{j,\left\lfloor \frac{k}{2^{r+1-j}} \right\rfloor}} \right| \le \begin{cases} \frac{1}{2^{r+1}(q-r)^{r+1}}, & q \ge r+1, \\ \frac{1}{(2r+1)!!}, & q = r. \end{cases} \tag{60}$$

Proof. The first case, $q \ge r+1$, from (46), we have

$$|\eta_q| \le \frac{1}{2q}, q \neq 0, \tag{61}$$

while $\eta_0 = 1$.

Consequently,

$$\max\{\eta_{q_1}, \eta_{q_2}\} = \eta_{\min\{q_1,q_2\}}. \tag{62}$$

Those relations and inequalities, (50), (61), and (62), can be used to get

$$\left| \prod_{j=1}^{r+1} \eta_{q+1+\alpha_{j,\left\lfloor \frac{k}{2^{r+1-j}} \right\rfloor}} \right| \le \frac{1}{2^{r+1}(q-r)^{r+1}}. \tag{63}$$

While for the second case, $q = r$, we have

$$\left|\prod_{j=1}^{r+1} \eta_{r+1+\alpha_{j,\left\lfloor \frac{k}{2^{r+1-j}} \right\rfloor}}\right| \leq \left|\prod_{j=1}^{r+1} \eta_{r+1-j}\right| = \frac{1}{(2r+1)!!}, \tag{64}$$

which completes the proof. □

Lemma 7. For nonnegative integers q and r, we have

$$\left|\Delta_{q+1+\alpha_{r+1,k}}(x)\right| \leq \begin{cases} 31r^2(q-r)^2, & q \geq r+1, \\ 21r^2, & q = r. \end{cases} \tag{65}$$

where Δ_q is defined as (43).

Proof. Use (43) and Lemma 3 to get

$$\begin{aligned} |\Delta_q(x)| &= |DL_q(x) - DL_{q-2}(x)| \leq |DL_q(x)| + |DL_{q-2}(x)| \\ &\leq \frac{(q+1)(q+1)}{2} + \frac{(q-1)(q)}{2} = q^2 + q + 1. \end{aligned} \tag{66}$$

Thus,

$$\max\{|\Delta_{q_1}(x)|, |\Delta_{q_2}(x)|\} \leq q^2 + q + 1, \tag{67}$$

such that $q = \max\{q_1, q_2\}$.

For $q \geq r+1$, we can use (67) together with (50) to calculate

$$\begin{aligned} |\Delta_{q+1+\alpha_{r+1,k}}(x)| &\leq (q+r+2)^2 + (q+r+2) + 1 \\ &\leq (q-r)^2 + (4r+5)(q-r)^2 + \left(4r^2 + 10r + 7\right)(q-r)^2 \\ &\leq 31r^2(q-r)^2. \end{aligned} \tag{68}$$

While for $q = r$,

$$\left|\Delta_{r+1+\alpha_{r+1,k}}(x)\right| \leq (2r+2)^2 + (2r+2) + 1 \leq 21r^2, \tag{69}$$

which completes the proof. □

In Lemmas 6 and 7, there is no need to discuss the $q < r$ case, since the index will be a negative value, which is not defined and will not be used.

The next theorem shows that the spectral expansion's constants are bounded.

Theorem 5. Let $y\left(x\right)\in\mathcal{C}^{r}\left[-1,1\right]$ with bounded rth-derivative, where $r\geq 2$, and consider the expansion $y\left(x\right)=\sum\limits_{q=0}^{\infty}c_{q}\,DL_{q}\left(x\right)$. Then

$$\left|c_{q}\right|\leq\begin{cases}\frac{31Mr^{2}}{\left(q-r\right)^{r-1}}, & q\geq r+1,\\ \frac{21Mr^{2}2^{r+1}}{\left(2r+1\right)!!}, & q=r,\end{cases}\tag{70}$$

where $M\in\mathbb{R}$ such that $\left|y^{(r)}\left(x\right)\right|\leq M$.

Proof. Let $y\in C^{r}\left[-1,1\right]$, where $r\geq 2$ and $\left|y^{(r)}\left(x\right)\right|\leq M$ for some $M\in\mathbb{R}$.

Consider the expansion:

$$y\left(x\right)=\sum_{q=0}^{\infty}c_{q}\,DL_{q}\left(x\right).\tag{71}$$

From the orthogonality relation (13) and Lemmas 5 and 4, we get

$$\begin{aligned}c_{q}&=\frac{2q+3}{2\left(q+2\right)\left(q+1\right)}\int_{-1}^{1}\left(1-x^{2}\right)DL_{q}\left(x\right)y\left(x\right)dx\\&=-\int_{-1}^{1}\psi_{1,q}'\left(x\right)y\left(x\right)dx.\end{aligned}\tag{72}$$

Using integration by parts for (72) and applying Lemma 4, we get

$$c_{q}=M_{1,q}-\int_{-1}^{1}\psi_{2,q}'\left(x\right)y'\left(x\right)dx,\tag{73}$$

where $M_{1,q}=-\psi_{1,q}\left(1\right)y\left(1\right)+\psi_{1,q}\left(-1\right)y\left(-1\right).$

Apply integration by parts for the second time to get

$$c_{q}=M_{1,q}+M_{2,q}+\int_{-1}^{1}\psi_{2,q}\left(x\right)y''\left(x\right)dx,\tag{74}$$

where $M_{2,q}=-\psi_{2,q}\left(1\right)y'\left(1\right)+\psi_{2,q}\left(-1\right)y'\left(-1\right)$.

Repeat the above steps for $r-2$ times to get

$$c_{q}=\sum_{i=1}^{r}M_{i,q}+\int_{-1}^{1}\psi_{r,q}\left(x\right)y^{(r)}\left(x\right)dx,\tag{75}$$

where $M_{i,q}$ is can be estimated as

$$M_{i,q}=-\psi_{i,q}\left(1\right)y^{(i-1)}\left(1\right)+\psi_{i,q}\left(-1\right)y^{(i-1)}\left(-1\right).\tag{76}$$

Using (8), we have

$$\Lambda_q(\pm 1) = \mathcal{L}_q(\pm 1) - \mathcal{L}_{q-2}(\pm 1) = (\pm 1)^q - (\pm 1)^{q-2} = 0, \tag{77}$$

for $q \geq 2$.

Thus, from (55), we get

$$\psi_{m,q}(\pm 1) = \begin{cases} 0, & \text{for } q \geq m, \\ (-1)^{m+\left\lceil \frac{m-1}{2} \right\rceil + \left\lceil \frac{b_{m,q}-q-1}{2} \right\rceil} \Lambda_{b_{m,q}}(\pm 1)\ \frac{1}{2}\mathcal{A}, & \text{for } 0 \leq q < m, \end{cases} \tag{78}$$

where

$$b_{m,q} = \begin{cases} 0, & m-q \text{ is even}, \\ 1, & m-q \text{ is odd}, \end{cases} \tag{79}$$

and

$$\mathcal{A} = \sum_{\substack{k=0 \\ \alpha_{m,k}=-q-2+b_{m,q}}}^{2^m-1} \prod_{j=1}^{m} \eta_{q+1+\alpha_{j,\left\lfloor \frac{k}{2^{m-j}} \right\rfloor}}. \tag{80}$$

For $q \geq r+1$, use (76), (77), and (78) to conclude $\sum_{i=1}^{r} M_{i,q} = 0$.

To calculate the upper bound of $\psi_{r,q}(x)$ for $q \geq r+1$, we apply Lemmas 6 and 7 to (52):

$$\begin{aligned} |\psi_{r,q}(x)| &\leq \frac{1}{2} \sum_{k=0}^{2^{r+1}-1} \left| \Delta_{q+1+\alpha_{r+1,k}}(x) \right| \left| \prod_{j=1}^{r+1} \eta_{q+1+\alpha_{j,\left\lfloor \frac{k}{2^{r+1-j}} \right\rfloor}} \right| \\ &\leq \frac{1}{2} \sum_{k=0}^{2^{r+1}-1} 31 r^2 (q-r)^2 \frac{1}{2^{r+1}(q-r)^{r+1}} = \frac{31 r^2}{2(q-r)^{r-1}}. \end{aligned} \tag{81}$$

Hence,

$$|c_q| \leq \int_{-1}^{1} |\psi_{r,q}(x)| \left| y^{(r)}(x) \right| dx \leq \frac{31 M r^2}{(q-r)^{r-1}}. \tag{82}$$

For the second case, $q = r$, the upper bound for $\psi_{r,q}(x)$ can be calculated using Lemmas 6 and 7:

$$\begin{aligned}\left|\psi_{r,r}\left(x\right)\right| &\leq \frac{1}{2}\sum_{k=0}^{2^{r+1}-1}\left|\Delta_{r+1+\alpha_{r+1,k}}\left(x\right)\right|\left|\prod_{j=1}^{r+1}\eta_{r+1+\alpha_{j,\left\lfloor\frac{k}{2^{r+1-j}}\right\rfloor}}\right| \\ &\leq \frac{1}{2}\sum_{k=0}^{2^{r+1}-1}\frac{21r^2}{(2r+1)!!} = \frac{21r^2 2^r}{(2r+1)!!}.\end{aligned} \tag{83}$$

From (75) and (83), we have

$$\left|c_r\right| \leq \int_{-1}^{1}\left|\psi_r\left(x\right)\right|\left|y^{(r)}\left(x\right)\right|dx \leq \frac{21Mr^2 2^{r+1}}{(2r+1)!!}. \tag{84}$$

□

Theorem 6. Let $y\left(x\right)$ satisfy the conditions of Theorem 5. Then

$$\left|y\left(x\right)-y_N\left(x\right)\right| \lesssim \mathcal{O}\left(\frac{1}{N-r}\right)^{r-4}, \qquad N>r>4, \tag{85}$$

where $y_N\left(x\right)$ is shown in (34).

Proof. From (34), we have

$$\left|y\left(x\right)-y_N\left(x\right)\right| = \left|\sum_{q=N+1}^{\infty}c_q\, DL_q\left(x\right)\right| \leq \sum_{q=N+1}^{\infty}\left|c_q\right|\left|DL_q\left(x\right)\right|. \tag{86}$$

UseTheorem 5 for $q \geq r+1$ and Lemma 3 to get

$$\begin{aligned}\left|y\left(x\right)-y_N\left(x\right)\right| &\leq \frac{31Mr^2}{2}\sum_{q=N+1}^{\infty}\frac{\left(q+1\right)\left(q+2\right)}{\left(q-r\right)^{r-1}} \\ &= \frac{31Mr^2}{2}\sum_{q=N+1}^{\infty}\left(\frac{r^2+3r+2}{\left(q-r\right)^{r-1}}+\frac{2r+3}{\left(q-r\right)^{r-2}}+\frac{1}{\left(q-r\right)^{r-3}}\right) \\ &\leq \frac{31Mr^2}{2}\sum_{q=N+1}^{\infty}\frac{12r^2}{\left(q-r\right)^{r-3}}.\end{aligned} \tag{87}$$

For any decreasing positive function $F\left(q\right)$, we have $\left|\sum_{q=N+1}^{\infty}F\left(q\right)\right| \leq \int_N^{\infty}F\left(q\right)dq$, (see [40]). Hence,

$$\left|y\left(x\right)-y_N\left(x\right)\right| \leq \frac{31Mr^2}{2}\int_N^{\infty}\frac{12r^2}{\left(q-r\right)^{r-3}}\mathrm{dq} \leq \frac{186Mr^4}{\left(r-4\right)\left(N-r\right)^{r-4}}, \tag{88}$$

which completes the proof. □

Since $q > N > r$, which means $q \neq r$, so, only the case of $q \geq r+1$ is needed, and there is no need for the $q = r$ case.

In the next section, some test problems will be solved to clarify the accuracy and efficiency of the presented method.

6 Examples

This section will apply the presented method to approximate some types of integral equations. Also, a physical application, Lane–Emden, modeled by a value problem, has been approximated. Three error metrics have been calculated to show the accuracy and efficiency of the presented methods: the maximum absolute error (MAE), point-wise absolute error, and root square error (RSE_N).

Test Problem 1. Consider the following linear Fredholm integral equation [6]:

$$y(x) = \frac{1}{4} - x + \int_0^1 \left(3t - 6x^2\right) y(t)\,\mathrm{d}t, \tag{89}$$

where $x \in [0,1]$, with the exact solution $y = x^2 - x$. Apply the PGDL method by expanding the unknown function to be

$$y_2(x) = \sum_{k=0}^{2} c_k\, DL_k(x). \tag{90}$$

Shift the given problem to $[-1,1]$ to get the residual as follows:

$$\begin{aligned} \frac{1}{4} + \frac{1}{2}x + \sum_{k=0}^{2} c_k \Bigg(DL_k(x) - \frac{3}{4}\int_{-1}^{1} t\, DL_k(t)\,\mathrm{d}t \\ + \frac{3}{4}\left(x^2 + 2x\right)\int_{-1}^{1} DL_k(t)\,\mathrm{d}t \Bigg) = 0. \end{aligned} \tag{91}$$

From (23) and Theorem 4, we get

$$\sum_{k=0}^{2} c_k \left(DL_k(x) - \frac{3}{4}\left(1 + (-1)^{k+1}\right) + \frac{3}{4}\left(x^2 + 2x\right)\left(1 + (-1)^k\right)\right) = -\frac{1}{2}x - \frac{1}{4}. \tag{92}$$

Collocate the above residue by the three LGL points, $x_0 = -1, x_1 = 0, x_2 = 1$ to get

$$\begin{aligned} -\frac{1}{2}c_0 - \frac{9}{2}c_1 + \frac{9}{2}c_2 &= \frac{1}{4}, \\ c_0 - \frac{3}{2}c_1 - \frac{3}{2}c_2 &= -\frac{1}{4}, \\ \frac{11}{2}c_0 + \frac{3}{2}c_1 + \frac{21}{2}c_2 &= -\frac{3}{4}. \end{aligned} \tag{93}$$

Solving the system (93) to get $c_0 = -\frac{1}{5}, c_1 = 0, c_2 = \frac{1}{30}$, which is the exact solution in the domain $[-1, 1]$.

Test Problem 2. Consider the following linear Volterra–Fredholm integral equation [49]:

$$2y(x) - \int_0^1 (x+t)\, y(t)\, \mathrm{dt} - \int_0^x (x-t)\, y(t)\, \mathrm{dt} = \frac{1}{12}x^4 - \frac{1}{6}x^3 - \frac{5}{2}x^2 + \frac{5}{6}x + \frac{17}{12}, \tag{94}$$

where $x \in [0, 1]$ and the exact solution is $y = -x^2 + x + 1$.

Applying the PGDL method with $N = 2$, after shifting the domain of (94) will be expanded as follows:

$$\begin{aligned} \sum_{q=0}^{2} c_q \bigg[2\, DL_q(x) - \frac{x+2}{4} \int_{-1}^{1} DL_q(t)\, \mathrm{dt} - \frac{1}{4} \int_{-1}^{1} t\, DL_q(t)\, \mathrm{dt} \\ - \frac{x}{4} \int_{-1}^{x} DL_q(t)\, \mathrm{dt} + \frac{1}{4} \int_{-1}^{x} t\, DL_q(t)\, \mathrm{dt} \bigg] \\ = \frac{1}{12}\left(\frac{x+1}{2}\right)^4 - \frac{1}{6}\left(\frac{x+1}{2}\right)^3 - \frac{5}{2}\left(\frac{x+1}{2}\right)^2 \\ + \frac{5}{6}\left(\frac{x+1}{2}\right) + \frac{17}{12}. \end{aligned} \tag{95}$$

There are four integrals; the first integral can be computed easily from (23) as

$$\int_{-1}^{1} DL_q(t)\, \mathrm{dt} = 1 + (-1)^q. \tag{96}$$

The second one is from Theorem 4 to be

$$\int_{-1}^{1} t\, DL_q(t)\, \mathrm{dt} = 1 + (-1)^{q+1}. \tag{97}$$

While the third integral can be computed from (20) to get

$$\int_{-1}^{x} DL_q(t)\,\mathrm{dt} = \sum_{k=0}^{2}\left(\frac{(-1)^k(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}{2q+3} + (-1)^q\,\delta_{k,2}\right) \times DL_{(q+1-2k)(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}(x). \tag{98}$$

Finally, for the last one, using (27) at $m=1$, hence the integration can be determined as the previous one to get the following:

$$\int_{-1}^{x} t\,DL_q(x) = \sum_{j=1}^{\min\left(2,\lfloor\frac{q+3}{2}\rfloor\right)} F_{1,j,q} \times \sum_{k=0}^{2}\left(\frac{(-1)^k(1-\delta_{k,2})(1-\delta_{q+3-2j,0}\,\delta_{k,1})}{2q+9-4j} + (-1)^{q+3-2j}\,\delta_{k,2}\right) \times DL_{(q+4-2k-2j)(1-\delta_{k,2})(1-\delta_{q+3-2j,0}\,\delta_{k,1})}(x). \tag{99}$$

Thus, (95) takes the form

$$\begin{aligned}
&\sum_{q=0}^{2} c_q\Bigg[2\,DL_q(x) - \frac{x+2}{4}\left(1+(-1)^q\right) - \frac{1}{4}\left(1+(-1)^{q+1}\right)\\
&- \frac{x}{4}\sum_{k=0}^{2}\left(\frac{(-1)^k(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}{2q+3} + (-1)^q\,\delta_{k,2}\right)\\
&\times DL_{(q+1-2k)(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}(x)\\
&+ \frac{1}{4}\sum_{j=1}^{\min\left(2,\lfloor\frac{q+3}{2}\rfloor\right)} F_{1,j,q}\sum_{k=0}^{2}\left(\frac{(-1)^k(1-\delta_{k,2})(1-\delta_{q+3-2j,0}\,\delta_{k,1})}{2q+9-4j}\right.\\
&\left.+(-1)^{q+3-2j}\,\delta_{k,2}\right) DL_{(q+4-2k-2j)(1-\delta_{k,2})(1-\delta_{q+3-2j,0}\,\delta_{k,1})}(x)\Bigg]\\
&\quad = \frac{1}{12}\left(\frac{x+1}{2}\right)^4 - \frac{1}{6}\left(\frac{x+1}{2}\right)^3 - \frac{5}{2}\left(\frac{x+1}{2}\right)^2 + \frac{5}{6}\left(\frac{x+1}{2}\right) + \frac{17}{12}.
\end{aligned} \tag{100}$$

Substitute by three LGL points $x_0=-1, x_1=0, x_2=1$ to get a system of algebraic equations and get the values of the spectral constants as $c_0 = \frac{6}{5}, c_1 = 0, c_2 = -\frac{1}{30}$. So, the solution is

$$y_2(x) = \frac{6}{5}(1) - \frac{1}{30}\left(\frac{15}{2}x^2 - \frac{3}{2}\right) = \frac{5-x^2}{4}, \tag{101}$$

which is the exact solution on the domain $[-1,1]$.

Test Problem 3. Consider the stable population model, which is a Volterra integral equation that describes the number of female births [4]:

$$y\left(x\right)=e^{x}-\int_{0}^{x}\left(x-t\right)y\left(t\right)dt, \tag{102}$$

where $x\in[0,1]$, $(x-t)$ is the net maternity function of females class age t at time x, and e^{x} is the contribution of birth due to female already present at time x. The exact solution is $y\left(x\right)=\frac{1}{2}\left[e^{x}+\cos x+\sin x\right]$.

Table 1 presents the best maximum absolute error for PGDL method compared to other methods. Table 2 shows MAE and $\mathrm{RSE_N}$ for different values of N, where

$$\mathrm{RSE_N}=\sqrt{\sum_{i=0}^{N}\left(y_{\text{exact}}\left(x_i\right)-y_{\text{approximate}}\left(x_i\right)\right)^2}, \tag{103}$$

such that the points x_i's have been chosen to be LGL quadrature points.

Table 1: MAE for Example 3 compared to other methods.

Method	Best MAE
[50]	2.14E-14
[51]	1.25E-15
[2]	4.44E-16
PGDL Method	7.13E-18

Table 2: MAE and RSE_N at different values of N for Example 3.

N	MAE	RSE
2	6.52E-03	6.23E-04
4	2.55E-05	2.97E-07
6	2.36E-08	7.70E-11
8	3.18E-11	4.95E-14
10	1.11E-14	9.72E-18
12	7.13E-18	3.97E-21

Figure 1 shows the log error, which confirms the stability of the presented method.

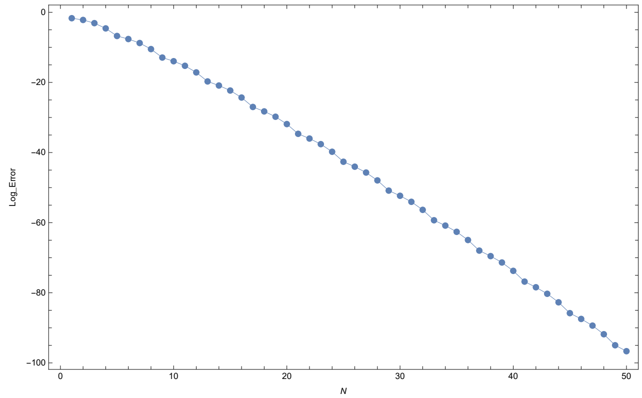

Figure 1: Log error graph for Example 3.

Test Problem 4. Consider the following linear Fredholm type integral equation [49]:

$$y\left(x\right)+\int_{0}^{1}e^{x-t}y\left(t\right)\mathrm{dt}=e^{x}, \tag{104}$$

whose exact solution is given by $y\left(x\right)=\frac{1}{2}e^{x}$, where $x\in[0,1]$. In this case, the exponential within the integral part will be expanded using Taylor's expansion.

Using a similar procedure for using PGDL to get the following residual:

$$\begin{aligned}R_{N}\left(x\right)=\sum_{q=0}^{N}c_{q}\Bigg[&DL_{q}\left(x\right)+\frac{1}{2}e^{\frac{x}{2}}\sum_{k=0}^{13}\frac{\left(-1\right)^{k}}{2^{k}k!}\\ &\times\sum_{j=1}^{\min\left(k+1,\lfloor\frac{q+k+2}{2}\rfloor\right)}F_{k,j,q}\left(1+\left(-1\right)^{q+k+2-2j}\right)-e^{\frac{x+1}{2}}\Bigg].\end{aligned} \tag{105}$$

Table 3 shows $\mathrm{RSE_N}$ and MAE for various values of N.

In Figure 2, we can track the log error from $N=1$ to $N=15$, which shows the stability of the presented approximate solution.

Table 3: RSE_N and MAE for Example 4 at different values of N.

N	[49]	PGDL Method	
	RSE	RSE	MAE
5	1.59E-07	6.97E-11	8.58E-07
6	2.29E-09	1.28E-13	3.03E-08
7	2.09E-10	1.03E-16	9.36E-10
8	2.65E-12	7.20E-17	2.58E-11
9	1.98E-13	7.57E-17	6.42E-13
10	2.27E-15	7.93E-17	1.45E-14

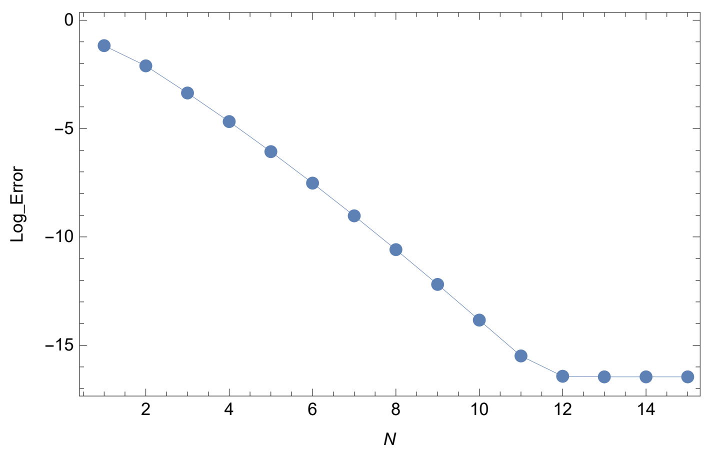

Figure 2: Log error graph for Example 4.

The next example will attempt to approximate the solution of the IVP.

Test Problem 5. Consider the following 2^{nd} order Lane–Emden differential equation [2]:

$$y''(x) + \frac{2}{x} y'(x) + y^m(x) = 0, \quad 0 \le m \le 5, \quad x \in [0, 3.1], \tag{106}$$

with the following initial conditions, $y(0) = 1$, and $y'(0) = 0$. While the exact solution at $m = 1$ is $y(x) = \frac{\sin x}{x}$.

Integrate the IVP (106) over the interval $[0, x]$ transforms into the following Volterra integral equation:

$$x\,y\,(x) - x + \int_0^x (x-t)\,t\,y^m\,(t)\,\mathrm{dt} = 0. \tag{107}$$

Table 4 shows the point-wise error compared to the method in [2], which ensures the accuracy and efficiency of our method. On the other hand, Table 5 presents the MAE and $\mathrm{RSE_N}$ for various values of N.

Table 4: Point-wise absolute error for Example 5.

x	[2]	PGDL Method	
	$N=15$	$N=14$	$N=15$
0.0	4.44E-16	1.11E-13	5.97E-16
0.1	2.88E-15	1.33E-14	5.42E-17
0.2	5.77E-15	6.38E-15	3.45E-17
0.3	6.77E-15	2.09E-15	6.17E-18
0.4	6.99E-15	4.33E-15	1.67E-17
0.5	7.54E-15	1.92E-15	4.26E-18
0.6	7.66E-15	2.38E-15	1.12E-17
0.7	7.32E-15	2.67E-15	1.26E-18
0.8	7.43E-15	1.06E-16	6.77E-18
0.9	7.32E-15	2.20E-15	4.69E-18
1.0	7.10E-15	1.72E-15	1.55E-18
1.5	5.88E-15	1.33E-15	8.32E-19
2.0	4.05E-15	1.71E-16	2.16E-19
2.5	2.30E-15	5.73E-16	1.17E-18
3.0	6.45E-16	4.44E-16	1.07E-18
3.1	4.59E-16	3.99E-28	7.85E-31

Table 5: MAE and RSE_N at different values of N for Example 5.

N	MAE	RSE
5	2.63E-04	2.94E-04
7	2.70E-06	3.04E-06
9	1.74E-08	1.96E-08
11	7.64E-11	8.62E-11
13	2.45E-13	2.76E-13
15	5.97E-16	6.75E-16

Figure 3 represents the log error from $N = 1$ to $N = 30$, which verifies the stability of PGDL method.

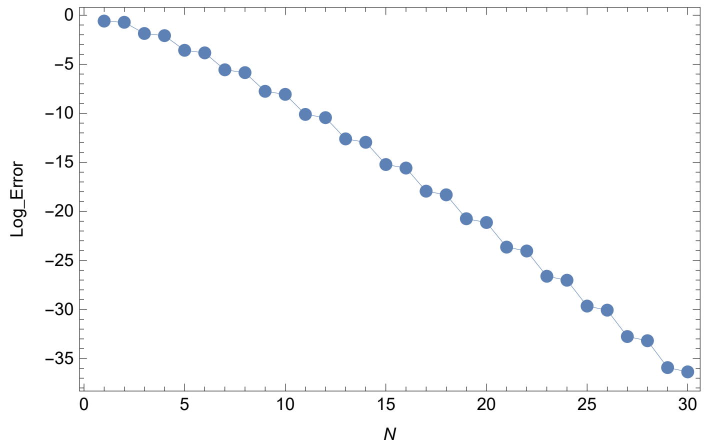

Figure 3: Log error graph for Example 5.

7 Concluding remarks

Through this work, we introduced the operational integration matrix for Legendre's derivative polynomials. Essential relations have been investigated, including the moment relation. Also, a technique that solves some types of integral equations has been presented without the need to do any actual integration. Furthermore, we have studied the error analysis and achieved a better upper bound for Legendre's derivative polynomials. In addition, the investigated matrix and technique are applied to approximate some types of differential equations. Some numerical test problems have been solved, and they show the accuracy, efficiency, and stability of the presented method. As a future direction, our results encourage applying our method to more complicated problems, such as nonlinear integral equations, systems of integral equations, and two-dimensional integral equations.

Conflict of interest

The authors declare that they have no conflict of interest.

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Author Contributions Statement

YHY, AMA, and MA conducted the mathematical analysis, developed the methodology, verified the results, wrote the initial draft, and reviewed the final version. AMA contributed to the original manuscript, software development, and methodology. ME reviewed and edited the final version of the manuscript.

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